A Note on the Distribution of the Sum of Lengths of the Initial Longest Increasing Sequences in Cycles of a Random Permutation

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Abstract

Let S_n be the set of all permutations of $\{1, 2, \ldots, n\}$ and let $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n) \in S_n$. The *initial longest increasing sequence* (ILIS) in σ has length m if, for $1 \le m \le n-1$, $\sigma_1 < \sigma_2 <, \ldots, < \sigma_m, \sigma_m > \sigma_{m+1}$, and has length n if $\sigma = (1, 2, \ldots, n)$. Let $l(\sigma)$ be the length of the ILIS in σ. We assume that σ is represented in cycle notation, so that the first number of each cycle is the minimum number of this cycle. We also assume that σ is chosen uniformly at random from S_n, i.e., with probability $1/n!$. Let $C_n(\sigma)$ be the set of all cycles of σ. In [9], T. Mansour investigated enumerative properties related to lengths of the ILIS in random permutations represented by the cycle notation. In particular, he studied the sum of the ILIS' lengths defined by $s_n = \sum_{c \in C_n(\sigma)} l(c)$ and derived exact and asymptotic expressions for its expectation and variance. In this note, we supplement Mansour's results on s_n with a limit theorem. We show that s_n, appropriately normalized, converges weakly to a standard normal random variable as $n \to \infty$.

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1 Introduction and Statement of the Main Results

We start with some notation that will be used further. Let S_n be the set of all permutations of $[n] := \{1, 2, \ldots, n\}$. We introduce the uniform probability measure $\mathbb{P}$ on the set S_n, that is, we assign the probability $1/n!$ to each $\sigma \in S_n$.

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The notation $\mathbb{E}$ stands for the expectation with respect to $\mathbb{P}$. Clearly, any permutation

$$\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n) \in S_n \tag{1}$$

corresponds to a directed graph $G_\sigma = ([n], \{(i, \sigma_i) : i \in [n]\})$, which is a union of disjoint directed cycles. All numerical characteristics of $\sigma \in S_n$ (and G_σ) become random variables (permutation statistics) on the finite probability space defined above. Some of them, like number of cycles, length of the longest cycle, number of cycles of length k, $1 \le k \le n$, do not depend on the vertex labeling in G_σ. We are interested here in permutation statistics, which depend on the labels of the vertices in G_σ. A typical statistic of this type is the length $L_n(\sigma)$ of the longest increasing subsequence in a random permutation given by (1). It is defined by

$$\begin{aligned} L_n = L_n(\sigma) := \max\{m : \text{there exist} \quad & 1 \le i_1 < i_2 < \ldots < i_m, \\ \text{such that} \quad & \sigma_{i_1} < \sigma_{i_2} < \ldots < \sigma_{i_m}\}. \end{aligned}$$

Baik, Deift and Johansson [3] proved a remarkable limit theorem for L_n. They found an appropriate normalization for L_n and showed that it converges weakly to a random variable having the Tracy-Widom distribution function. It has been recognized for a long time that, together with the enumerative combinatorics, this problem also concerns several disparate mathematical areas (e.g., random matrix theory, representation theory, integrable systems and statistical mechanics). For more details, we refer the reader to [2, 4, 11].

In this note, we focus on the length of the *initial longest increasing sequence* (ILIS) of a random permutation, represented as a union of cycles. This problem was recently studied by Mansour [9]. Our work supplements in a probabilistic aspect one of his main results and may be considered as a continuation of his study.

To introduce the concept of the ILIS, we consider again the permutation given by (1). The ILIS in σ has length m if, for $1 \le m \le n-1$, $\sigma_1 < \sigma_2 <, \ldots, < \sigma_m, \sigma_m > \sigma_{m+1}$, and has length n if $\sigma = (1, 2, \ldots, n)$. For $\sigma \in S_n$, we define $l = l(\sigma)$ to be the length of the ILIS in σ. To illustrate, consider the permutation

$$\tilde{\sigma} = (1, 3, 5, 4, 7, 6, 2) \in S_7. \tag{2}$$

Obviously, $l(\tilde{\sigma}) = 3$ since the ILIS in $\tilde{\sigma}$ is $1, 3, 5$. For the longest subsequence in $\tilde{\sigma}$, we have $L_7(\tilde{\sigma}) = 4$, but note that there are two subsequences of length 4 in $\tilde{\sigma}$: $1, 3, 5, 7$ and $1, 3, 5, 6$. Also note that the longest increasing subsequence in a permutation is in general not unique.

In [9], Mansour showed that if σ is written in the one-line form (1), then the probability mass function and the expectation of $l(\sigma)$ can be easily computed, using a direct counting argument. The main results of [9] are devoted to permutations that are represented as unions of cycles, where each cycle is considered as a separate indecomposable (one-cycle) permutation. To introduce the statistics studied in [9], for $\sigma \in S_n$, we denote by $C_n(\sigma)$ the set of all cycles of σ. We suppose that the numbers in each cycle of $C_n(\sigma)$ are written in a linear order, so that its minimum number is the first number in that order. In addition, if $C_n(\sigma) = \{c_1, c_2, \ldots, c_d\}$, where $1 \le d \le n$, we also suppose that the cycles c_j are arranged in the following way: $\min\{i \in c_1\} < \min\{i \in c_2\} < \ldots < \min\{i \in c_d\}$. Consider now the following two statistics of a random permutation $\sigma \in S_n$:

$$\mathcal{L}_n = \mathcal{L}_n(\sigma) := \max\{l(c) : c \in C_n(\sigma)\}$$

and

$$s_n = s_n(\sigma) := \sum_{c \in C_n(\sigma)} l(c). \tag{3}$$

The cycle decomposition of the permutation $\tilde{\sigma}$ given by (2) is $\tilde{\sigma} = (1)(2357)(4)(6)$. Hence $\mathcal{L}_7(\tilde{\sigma}) = 4$ and $s_7(\tilde{\sigma}) = 1 + 4 + 1 + 1 = 7$. For fixed m, Mansour [9] obtained an expression for the exponential generating function of the probabilities $\mathbb{P}(\mathcal{L}_n \le m)$, from which he deduced the exact and asymptotic distributions of $\mathcal{L}_n$. Mansour has also obtained the exponential generating function of the expectations $\mathbb{E}(y^{s_n})$ and derived exact and asymptotic expressions for the expected value and the variance of s_n.

Our main goal in this note is to find a proper normalization for s_n, which yields convergence to the standard normal distribution as $n \to \infty$. In the next section we shall prove the following

Theorem 1 . *Let*

$$s'_n := \frac{s_n - e\log n}{\sqrt{3e\log n}}, \quad n \ge 1. \tag{4}$$

The random variable s'_n converges in distribution to a standard normal random variable as $n \to \infty$, i.e.,

$$\lim_{n\to\infty} \mathbb{P}(s'_n \le u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{u} e^{-v^2/2} dv, \quad -\infty < u < \infty. \tag{5}$$

In the proof of Theorem 1, we apply:

- Mansour's identity for the exponential generating function of the expectations $\mathbb{E}(y^{s_n})$ (see [9, Theorem 2.3]);
- Darboux's theorem for asymptotic estimates of the coefficients of generating functions with algebraic singularities on the circle of convergence [6];
- Curtiss' continuity theorem for moment generating functions [5].

2 Proof of Theorem 1

We start by considering Mansour's identity. We give it as a separate lemma.

Lemma 1 *[9, Theorem 3.2]. Let*

$$f_n(y) := \sum_{\sigma \in S_n} y^{s_n(\sigma)}, \quad n \geq 1, \quad f_0(y) := 1, \tag{6}$$

and let

$$H(x, y) := \sum_{n \geq 0} \frac{f_n(y)}{n!} x^n$$

be the exponential generating function of the functions $f_n(y)$. Then we have

$$H(x, y) = \frac{1}{(1-x)^{1-(1-y)e^y}} e^{(1-y)e^y \int_0^1 \frac{e^{-y(1-x)u} - e^{-yu}}{u} du}. \tag{7}$$

It is easily seen that Lemma 1 provides a formula for the exponential generating function of the expectations $\mathbb{E}(y^{s_n})$. In fact, from the decomposition $S_n = \cup_{j=1}^n \{\sigma : s_n(\sigma) = j\}$ and (6) it follows that

$$\frac{f_n(y)}{n!} = \sum_{j=1}^n \mathbb{P}(s_n = j) y^j = \mathbb{E}(y^{s_n}), \quad n \geq 1.$$

Hence (7) becomes

$$H(x, y) = \sum_{n \geq 0} \mathbb{E}(y^{s_n}) x^n = \frac{1}{(1-x)^{1-(1-y)e^y}} e^{(1-y)e^y \int_0^1 \frac{e^{-y(1-x)u} - e^{-yu}}{u} du}. \tag{8}$$

Further on, we make a convention that x is a variable from the complex plane and that y assumes real values in a neighborhood of $y = 1$, say,

$$1 - \delta < y < 1 + \delta, \quad 0 < \delta < e^{-2}. \tag{9}$$

The function in the right-hand side of (8) is of the following general form:

$$G(x) := (1-x)^\alpha g(x). \tag{10}$$

We shall introduce now the concept of an *algebraic singularity* on the circle of convergence of a function of a complex variable. Let us assume that α and $g(x)$ from the right-hand side of (10) satisfy the following three conditions:

(i) $\alpha \notin \{0, 1, \ldots\}$;

(ii) $g(x)$ is a function analytic in a neighborhood of $x = 1$;

(iii) $g(1) \neq 0$.

A singularity of a certain function $F(x)$ is called *algebraic* at $x = 1$ if $F(x)$ can be written as the sum of a function analytic in a neighborhood of 1 and a finite number of terms of the form (10). Darboux's theorem [6] gives asymptotic expansions of the Taylor coefficients for functions with algebraic singularities on their circle of convergence. For more details on singularity analysis of generating functions, we also refer the reader to [10, Section 11] and [7, Chapter VI]. In (8), we encounter an algebraic singularity containing only a single term. Let

$x^n[F(x)]$ denote the coefficient of x^n in the power series expansion of $F(x)$. Darboux's theorem, as given by Odlyzko in [10, Theorem 11.7], then yields

$$x^n[G(x)] = g(1)\frac{n^{-\alpha-1}}{\Gamma(-\alpha)} + o(n^{-\alpha-1}), \tag{11}$$

where $G(x)$ is given by (10) and Γ denotes Euler's gamma function. The function from the right-hand side of (8), written in terms of notation (10), has

$$\alpha = (1-y)e^y - 1 \quad \text{and} \quad g(x) = e^{(1-y)e^y \int_0^1 \frac{e^{-y(1-x)u} - e^{-yu}}{u} du}. \tag{12}$$

Here the variable y may be viewed as a parameter satisfying the inequalities (9). It can be easily verified that conditions (i)–(iii) hold for α and $g(x)$, defined by (12). In fact, using the first equality of (12) and both inequalities in (9), we see that $\alpha < e^{\delta-1} - 1 < 0$, which proves condition (i). To prove (ii), we note that in a neighborhood of $x = 1$, the integral

$$\int_0^1 \frac{e^{-y(1-x)u} - e^{-yu}}{u} du \tag{13}$$

has an expansion as a series of powers of $x - 1$ with coefficients, which are functions of y and are obtained after term-by-term integration of the power series expansion of the integrand in (13). Therefore $g(x)$, defined by the second equality of (12), has also such an expansion in powers of $x - 1$. This implies that $g(x)$ is analytic in a neighborhood of $x = 1$ and thus (ii) is proved. The verification of condition (iii) is straightforward and follows from the exponential representation of $g(x)$ given by (12).

Our next goal is to apply the general formula (11) for the nth coefficient of $G(x) = H(x, y)$. From (8) it follows that

$$x^n[H(x,y)] = \mathbb{E}(y^{s_n}). \tag{14}$$

In the right-hand side of (11) we have to deal with α and $g(x)$ defined by (12). To get a convenient expression for $g(1)$, we set $x = 1$ in (12), expand the integrand in a power series, and then integrate with respect to u. We obtain

$$g(1) = e^{(1-y)e^y h(y)}, \tag{15}$$

where

$$h(y) = \sum_{j=1}^{\infty} (-1)^{j-1} \frac{y^j}{j j!}. \tag{16}$$

We are now ready to apply a particular case of Darboux's theorem in the form given by (11). Combining (11) with (12), (14) and (15), we observe that

$$\mathbb{E}(y^{s_n}) = \frac{n^{(y-1)e^y}}{\Gamma(1 + (y-1)e^y)} e^{-(y-1)e^y h(y)}(1 + o(1)), \quad n \to \infty, \tag{17}$$

where $h(y)$ is defined by (16) and $o(1) \to 0$ uniformly for all y satisfying (9).

Further on, we shall use the moment generating function $M_n(t)$ of the random variable s'_n, defined by (4). We have

$$M_n(t) := \mathbb{E}(e^{ts'_n}) = e^{-t\sqrt{(e\log n)/3}}\mathbb{E}(e^{ts_n/\sqrt{3e\log n}}). \tag{18}$$

We intend to apply Curtiss' continuity theorem [5] to the sequence $(M_n(t))_{n\geq 1}$ (the statement and proof of this theorem may be also found in [12, Chapter I, Section 2]). Roughly speaking, Curtiss' theorem asserts that the uniform convergence in a neighborhood of the origin of the moment generating functions of a sequence of random variables yields the convergence in distribution (weak convergence) of this sequence. Consequently, we can obtain the convergence in distribution (5) of the sequence $(s'_n)_{n\geq 1}$ provided we prove that, for a certain $t_0 > 0$,

$$\lim_{n\to\infty} M_n(t) = e^{t^2/2}, \quad \text{for all } t \text{ with } |t| < t_0, \tag{19}$$

where $M_n(t)$ is given by (18). We also recall that $e^{t^2/2}$ is the moment generating function of the standard normal distribution (see, e.g., [8, Section 5.8]).

The proof of (19) is based on an asymptotic estimate for the expectation $\mathbb{E}(e^{ts_n/\sqrt{3e\log n}})$ given in the right-hand side of (18). Our analysis stems from (17), in which we set

$$y = e^{t/\sqrt{3e\log n}}. \tag{20}$$

After this change of the variable y, (17) remains true since by (19), $|t|$ is bounded and thus, for large n (say, $n \geq n_0(t_0, \delta)$), the inequalities from (9) are again satisfied with y given by (20). In what follows next, using (20), we shall deduce suitable asymptotic expansions for the terms from the right-hand side of (17) as $n \to \infty$. We begin by considering $y - 1$ and e^y. Expanding the exponent in (20) in powers of t, where t is inherited from (19), we get

$$y - 1 = \frac{t}{\sqrt{3e\log n}} + \frac{t^2}{6e\log n} + O(\log^{-3/2} n), \tag{21}$$

$$\begin{aligned} e^y &= \exp\left(1 + \frac{t}{\sqrt{3e\log n}} + \frac{t^2}{6e\log n} + O(\log^{-3/2} n)\right) \\ &= ee^{t/\sqrt{3e\log n}}e^{t^2/6e\log n}(1 + O(\log^{-3/2} n)) \\ &= e\left(1 + (1 + \frac{t}{\sqrt{3e\log n}} + \frac{t^2}{6e\log n} + O(\log^{-3/2} n)\right) \\ &\times \left(1 + \frac{t^2}{6e\log n} + O(\log^{-2} n)\right)(1 + O(\log^{-3/2} n)) \\ &= e\left(1 + \frac{t}{\sqrt{3e\log n}} + \frac{t^2}{3e\log n} + O(\log^{-3/2} n)\right). \end{aligned} \tag{22}$$

Combining (21) and (22), we obtain

$$(y-1)e^y = e\left(\frac{t}{\sqrt{3e\log n}} + \frac{t^2}{2e\log n} + O(\log^{-3/2} n)\right). \tag{23}$$

This formula implies the less precise estimate $(y-1)e^y = O(\log^{-1/2} n)$. Hence, by the polynomial approximation formula for the gamma function [1, formula 6.1.35], we have

$$\Gamma(1 + (y-1)e^y) = 1 + O(\log^{-1/2} n). \tag{24}$$

We return next to the power series $h(y)$, defined by (16). It is clear that $h(y)$ is bounded for all y from a finite interval. Combining this with (23) gives

$$(y-1)e^y h(y) = O(\log^{-1/2} n),$$

and thus

$$e^{-(y-1)e^y h(y)} = 1 + O(\log^{-1/2} n). \tag{25}$$

Finally, note that (23) also implies

$$\begin{aligned} n^{(y-1)e^y} &= \exp\left((y-1)e^y \log n\right) \\ &= \exp\left(t\sqrt{\frac{e\log n}{3}} + \frac{t^2}{2} + O(\log^{-1/2} n)\right). \end{aligned} \tag{26}$$

Putting (24)–(26) into (17) with y taken from (20), we get

$$\mathbb{E}(e^{ts_n/\sqrt{3e\log n}}) = (1 + o(1)) \exp\left(t\sqrt{\frac{e\log n}{3}} + \frac{t^2}{2}\right),$$

which by (18) implies (19). The proof of Theorem 1 is now completed after an application of Curtiss' continuity theorem.

References

[1] M. Abramovitz, I. A. Stegun, Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables, Dover Publ. Inc., New York, 1965.

[2] D. Aldous and P. Diaconis, Longest increasing subsequences: from patience sorting to the Baik-Deift-Johansson theorem, Bull. Amer. Math. Soc. (N. S.) 36(1999), 413–432.

[3] J. Baik, P. Deift and K. Johansson, On the distribution of the lrngth of the longest increasing subsequence of random permutations, J. Amer. Math. Soc. 12(1999), 1119–1178.

[4] I. Corwin, Commentary on "Longest increasing subsequences: from patience sorting to the Baik-Deift-Johansson theorem" by David Aldous and Persi Diaconis, Bull. Amer. Math. Soc. (N. S.) 55(2018), 363–374.

[5] I. H. Curtiss, A note on the theory of moment genrating functions, Ann. Math. Statist. 13(1942), 430–433.

[6] G. Darboux, Mémoire sur l'approximation des fonctions de trés-grands nombres, et sur une classe étendue de développements en série, J. Math. Pures Appl. 4(1878), 5–56, 377–416.

[7] P. Flajolet and R. Sedgewick, Analytic Combinatorics, Cambridge Univ. Press, Cambridge, 2009.

[8] G. Grimmett and D. Stirzaker, Probability and Random Processes, Oxford Univ. Press, Oxford, 2001.

[9] T. Mansour, The length of the initial longest increasing sequence in a permutation, Art Discr. Appl. Math. 6(2023), #P3.05.

[10] A. M. Odlyzko, Asymptotic enumeration methods. In: Handbook of Combinatorics (R. Graham, M. Grötschel and L. Lovász, Eds.), volume II, Elsevier, Amsterdam, 1995, pp. 1063-1229.

[11] D. Romik, The Surprising Mathematics of Longest Increasing Subsequences, Vol. 4 of Institute of Mathematical Statistics Textbooks, Cambridge Univ. Press, New York, 2015.

[12] V. N. Sachkov, Probabilistic Methods in Combinatorial Analysis, Encyclopedia Math. Appl., Vol. 56, Cambridge Univ. Press, Cambridge, 1997.