

New Fourth-Order Grayscale Indicator-Based Telegraph Diffusion Model for Image Despeckling

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Abstract

Second-order PDE models have been widely used for suppressing multiplicative noise, but they often introduce blocky artifacts in the early stages of denoising. To resolve this, we propose a fourth-order nonlinear PDE model that integrates diffusion and wave properties. The diffusion process, guided by both the Laplacian and intensity values, reduces noise better than gradient-based methods, while the wave part keeps fine details and textures. The effectiveness of the proposed model is evaluated against two second-order anisotropic diffusion approaches using the Peak Signal-to-Noise Ratio (PSNR) and Mean Structural Similarity Index (MSSIM) for images with available ground truth. For SAR images, where a noise-free reference is unavailable, the Speckle Index (SI) is used to measure noise reduction. Additionally, we extend the proposed model to study color images by applying the denoising process independently to each channel, preserving both structure and color consistency. The same quantitative metrics PSNR and MSSIM are used for performance evaluation, ensuring a fair comparison across grayscale and color images. In all the cases, our computed results produce better results compared to existing models in this genre.

1 Introduction

Image restoration plays a crucial role in image processing by reconstructing original images from noisy observations. In real-world applications, noise can degrade images in additive, multiplicative, or mixed forms, making denoising an essential preprocessing step for tasks like segmentation and pattern recognition. This study focuses on multiplicative speckle noise, a common issue in synthetic aperture radar (SAR) and ultrasound imaging. The multiplicative noise is mathematically represented as:

$$N = I\eta, \tag{1}$$

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where I is the original image, η denotes noise, and N is the noisy image observed. In the case of multiplicative speckle noise, η follows a gamma distribution with mean 1 and variance $1/L$, where L represents the number of observations:

$$p(\eta) = \begin{cases} \frac{1}{\Gamma(L)} L^L \eta^{L-1} e^{-L\eta}, & \eta > 0, \\ 0, & \eta = 0. \end{cases} \quad (2)$$

Unlike additive noise, multiplicative noise varies with intensity, making brighter regions more vulnerable to degradation. This complexity makes speckle noise reduction particularly challenging. Over the years, variational and PDE-based methods have been extensively explored to address image denoising [2–5]. The AA model by Aubert and Aujol [3] introduced a regularization-based approach, but its non-convex nature posed computational difficulties. To resolve this, Shi and Osher [2] transformed the multiplicative noise into an additive form using logarithmic functions, leading to a globally convex model. Further advancements, such as the novel total variation model (NTV) [9] and the convex variational model (CV) [11], improved edge preservation and denoising efficiency. In PDE-based methods, Zhou et al. [4] introduced a Doubly Degenerate (DD) diffusion model, which used gray-level indicators in the diffusion coefficient but lacked well-posedness. Later, Shan et al. [17] refined this model to ensure stability. Recent approaches have combined diffusion and wave dynamics to better preserve high-frequency details while reducing noise. However, second-order PDEs tend to produce blocky artifacts in denoised images. Majee et al. [26] have proposed effective methods for color image denoising, addressing both additive Gaussian noise and multiplicative speckle noise. Their approaches demonstrate significant improvements in noise reduction while preserving essential image details. More recently, Roy et al. [25] developed a Bayesian MAP framework integrating total variation regularization to effectively model speckle noise while preserving edges. Li et al. [24] proposed a hybrid PDE model, blending total variation filtering with anisotropic diffusion, achieving better despeckling performance in SAR images. Cuomo et al. [23] introduced a learned variational model leveraging deep learning to suppress adaptive noise in ultrasound and satellite imaging. Building on these developments, we propose a novel fourth-order hyperbolic-parabolic PDE model designed specifically for multiplicative noise suppression in SAR images. Our approach balances noise removal and texture preservation, retains fine image details while significantly reduces noise.

1.1 Diffusion Models

Perona and Malik [16] introduced the first anisotropic diffusion model in 1990 to address Gaussian additive noise. The governing equation of the Perona-Malik model is given by:

$$\frac{\partial I}{\partial t} = \text{div} (C(|\nabla I|) \nabla I), \quad (3)$$

where I represents the image, t denotes time, and $C(|\nabla I|)$ is the diffusion coefficient that depends on the gradient magnitude. This function, which is smooth and non-increasing, satisfies $C(0) = 1$ and $\lim_{x \rightarrow \infty} C(x) = 0$. The model effectively preserves edges but may retain or amplify high-frequency noise, limiting its denoising capabilities. To enhance its

performance, several extensions have been proposed. Dong et al. [11] highlighted the significance of image brightness for multiplicative noise reduction. Zhou et al. [4] introduced a Doubly Degenerate (DD) model that incorporates both grayscale indicators and gradients, improving image denoising compared to purely gradient-based models. The governing equation is:

$$\frac{\partial I}{\partial t} = \operatorname{div} (C(I, |\nabla I|) \nabla I), \quad \Omega_T = \Omega \times (0, T), \quad (4)$$

with the boundary and initial conditions:

$$\frac{\partial I}{\partial n} = 0, \quad \text{on } \partial\Omega_T = \partial\Omega \times (0, T), \quad (5)$$

$$I(a, 0) = f(a), \quad \text{in } \Omega. \quad (6)$$

Here, $C(I, |\nabla I|)$ incorporates intensity variations, $\frac{\partial I}{\partial n}$ denotes the normal derivative, and $f(a)$ represents the noisy image. The diffusion coefficient is given by:

$$C(I, |\nabla I|) = \frac{2|I|^\alpha}{M^\alpha + |I|^\alpha} \cdot \frac{1}{(1 + |\nabla I|^2)^{(1-\beta)/2}}, \quad (7)$$

where β regulates gradient sensitivity. Zhou et al. [19] later refined this model with the ZZDB model, written as:

$$\frac{\partial I}{\partial t} = \operatorname{div} \left(\frac{\nabla I}{1 + \left(\frac{|\nabla I_\xi|}{K} \right)^{\beta(I)}} \right), \quad \text{in } \Omega_T, \quad (8)$$

where $\beta(I)$ dynamically adjusts based on intensity variations:

$$\beta(I) = 2 - \frac{2|I|^\alpha}{M^\alpha + |I|^\alpha}. \quad (9)$$

subject to the boundary and initial conditions:

$$\frac{\partial I}{\partial n} = 0, \quad \text{on } \partial\Omega_T, \quad (10)$$

$$I(a, 0) = f(a), \quad \text{in } \Omega. \quad (11)$$

Shan et al. [17] introduced a regularized version of the DD model to ensure well-posedness:

$$\frac{\partial I}{\partial t} = \nabla \cdot (C(I_\xi, |\nabla I_\xi|) \nabla I), \quad (12)$$

with boundary and initial conditions:

$$\frac{\partial I}{\partial n} = 0, \quad \text{on } \partial\Omega_T, \quad (13)$$

$$I(a, 0) = f(a), \quad \text{in } \Omega. \quad (14)$$

where I_ξ is a Gaussian-smoothed image, and the diffusion coefficient is defined as:

$$C(I_\xi, |\nabla I_\xi|) = \left(\frac{I_\xi}{M_\xi} \right)^\alpha \cdot \frac{1}{1 + |\nabla I_\xi|^\nu}. \quad (15)$$

To further enhance despeckling performance, Majee et al. [12] proposed the TDE model, which combines the diffusion and wave equations. The governing equation for the TDE model is:

$$\frac{\partial^2 I}{\partial t^2} + \gamma \frac{\partial I}{\partial t} = \nabla \cdot (C(I_\xi, |\nabla I_\xi|) \nabla I), \quad (16)$$

where γ regulates wave behavior. The initial and boundary conditions are:

$$I(a, 0) = f(a), \quad \frac{\partial I}{\partial t}(a, 0) = 0, \quad \text{in } \Omega, \quad (17)$$

$$\frac{\partial I}{\partial n} = 0, \quad \text{on } \partial\Omega_T. \quad (18)$$

The diffusion coefficient in the TDE model is expressed as:

$$C(I_\xi, |\nabla I_\xi|) = \frac{2|I_\xi|^\alpha}{M_\xi^\alpha + |I_\xi|^\alpha} \cdot \frac{1}{1 + (|\nabla I_\xi|/k)^2}, \quad (19)$$

where $M_\xi = \max_{x \in \Omega} |I_\xi(x, t)|$, α is a parameter that regulates intensity adaptation, and k controls the gradient-based diffusion. The inclusion of wave dynamics in the TDE model helps in maintaining fine details and providing improved image restoration results. Higher-order PDEs have been explored to overcome the limitations of second-order models. You and Kaveh [18] replaced the gradient with the Laplacian, resulting in the fourth-order equation:

$$\frac{\partial I}{\partial t} = -\Delta (C(\Delta I) \Delta I), \quad (20)$$

with boundary and initial conditions:

$$\frac{\partial I}{\partial n} = 0, \quad \frac{\partial(\Delta I)}{\partial n} = 0 \quad \text{on } \partial\Omega_T, \quad (21)$$

$$I(a, 0) = f(a), \quad \text{in } \Omega. \quad (22)$$

where the diffusion coefficient is:

$$C(|\Delta I|) = \frac{1}{1 + (|\Delta I|/k)^2}. \quad (23)$$

This approach improves noise reduction while preserving edges. To further refine speckle noise removal, an energy-based approach was introduced:

$$\arg \min E(I) = \int_{\Omega} |\nabla^2 I| d\Omega + \lambda \frac{1}{|\Omega|} \int_{\Omega} \left(\frac{I - f}{I} \right)^2 d\Omega. \quad (24)$$

Applying Euler-Lagrange minimization results in the fourth-order PDE:

$$\frac{\partial I}{\partial t} = -\nabla^2 (C(\nabla^2 I) \nabla^2 I) - \lambda \int_{\Omega} \left(\frac{I - f}{I} \right)^2, \quad (25)$$

subject to boundary and initial conditions:

$$\frac{\partial I}{\partial n} = 0, \quad \frac{\partial(\Delta I)}{\partial n} = 0 \quad \text{on } \partial\Omega_T, \quad (26)$$

$$I(a, 0) = f(a), \quad \text{in } \Omega. \quad (27)$$

We propose a novel fourth-order nonlinear PDE model tailored for general multiplicative speckle noise, achieving superior results in real SAR images.

2 Proposed Model

We propose a nonlinear fourth-order model that effectively integrates wave dynamics and diffusion to balance noise reduction with structure preservation. The governing equation is:

$$\frac{\partial^2 I}{\partial t^2} + \gamma \frac{\partial I}{\partial t} = -\Delta (C(I_{\xi}, |\Delta I_{\xi}|) \Delta I) - \lambda \left(\frac{I - f}{I} \right)^2, \quad \forall (x, t) \in \Omega \times (0, T), \quad (28)$$

with initial and boundary conditions:

$$I(a, 0) = f(a), \quad I_t(a, 0) = 0, \quad \forall x \in \Omega, \quad (29)$$

$$\frac{\partial I}{\partial n} = 0, \quad \frac{\partial(\Delta I)}{\partial n} = 0, \quad \forall (x, t) \in \partial\Omega \times (0, T).$$

Here, f is the noisy image, and t represents time.

Diffusion Coefficient

The diffusion coefficient is defined as:

$$C(I_{\xi}, |\Delta I_{\xi}|) = \frac{2|I_{\xi}|^{\alpha}}{M_{\xi}^{\alpha} + |I_{\xi}|^{\alpha}} \cdot \frac{1}{1 + (|\Delta I_{\xi}|/k)^2}, \quad (30)$$

where $M_{\xi} = \max_{x \in \Omega} |I_{\xi}(x, t)|$ represents the maximum intensity, and α and k are parameters. The term $\frac{2|I_{\xi}|^{\alpha}}{M_{\xi}^{\alpha} + |I_{\xi}|^{\alpha}}$ adapts diffusion to intensity variations, reducing smoothing in regions with significant features to preserve edges. The factor $\frac{1}{1 + (|\Delta I_{\xi}|/k)^2}$ further modulates diffusion based on the Laplacian, slowing diffusion in high-gradient regions. Gaussian smoothing, $I_{\xi} = J_{\xi} * I$, stabilizes the model by reducing noise while retaining fine details.

3 Numerical Implementation

3.1 Numerical Scheme

We consider the proposed fourth-order PDE as given in equation (28):

$$\frac{\partial^2 I}{\partial t^2} + \gamma \frac{\partial I}{\partial t} = -\Delta (C(I_\xi, |\Delta I_\xi|) \Delta I) - \lambda \left(\frac{I - f}{I} \right)^2, \quad \forall (x, t) \in \Omega \times (0, T), \quad (31)$$

The 2D domain Ω is discretized into a grid with spatial step sizes δx , δy , and a temporal step size δt . Let $I_{i,j}^n$ represent the numerical approximation of I at grid point (x_i, y_j, t_n) .

$$\frac{I_{i,j}^{n+1} - 2I_{i,j}^n + I_{i,j}^{n-1}}{(\delta t)^2} + \gamma \frac{I_{i,j}^{n+1} - I_{i,j}^n}{\delta t} = -\Delta (C(I_{\xi,i,j}^n, |\Delta I_{\xi,i,j}^n|) \Delta I_{i,j}^n) - \lambda \left(\frac{I_{i,j}^n - f_{i,j}}{I_{i,j}^n} \right)^2.$$

Here temporal discretization employs a second-order central difference for the second derivative,

$$I_{tt}(t_n) \approx \frac{I_{i,j}^{n+1} - 2I_{i,j}^n + I_{i,j}^{n-1}}{(\delta t)^2},$$

and a first-order forward difference for the first derivative,

$$I_t(t_n) \approx \frac{I_{i,j}^{n+1} - I_{i,j}^n}{\delta t}.$$

For spatial derivatives, second-order central differences are used for both the gradient,

$$\nabla_x I_{i,j}^n \approx \frac{I_{i+1,j}^n - I_{i-1,j}^n}{2\delta x}, \quad \nabla_y I_{i,j}^n \approx \frac{I_{i,j+1}^n - I_{i,j-1}^n}{2\delta y},$$

and the Laplacian,

$$\Delta I_{i,j}^n \approx \frac{I_{i+1,j}^n - 2I_{i,j}^n + I_{i-1,j}^n}{(\delta x)^2} + \frac{I_{i,j+1}^n - 2I_{i,j}^n + I_{i,j-1}^n}{(\delta y)^2}.$$

The laplacian magnitude($|\Delta I|$) is computed as:

$$|\Delta I_{i,j}^n| = \sqrt{(\Delta_x I_{i,j}^n)^2 + (\Delta_y I_{i,j}^n)^2}.$$

Rearranging terms to solve for $I_{i,j}^{n+1}$, we obtain:

$$I_{i,j}^{n+1} = \frac{1}{(\delta t)^2 + \gamma \delta t} \left[2I_{i,j}^n - I_{i,j}^{n-1} - (\delta t)^2 \Delta (C(I_{\xi,i,j}^n, |\Delta I_{\xi,i,j}^n|) \Delta I_{i,j}^n) - (\delta t)^2 \lambda \left(\frac{I_{i,j}^n - f_{i,j}}{I_{i,j}^n} \right)^2 \right].$$

Here, λ represents the fidelity term's weight, ensuring that the solution stays close to the clean image. The discretized fidelity term $\lambda \frac{(I_{i,j}^n)^2 - (f_{i,j})^2}{(I_{i,j}^n)^2}$ gives a balance between preservation of the image's original features and noise reduction.

For color images, the restoration process is applied separately to each channel (Red, Green, and Blue). Let $I_{i,j,s}^n$ denote the intensity at pixel (i, j) in channel s at time step n , where $s \in \{R, G, B\}$. The discretized equation for each channel follows:

$$\frac{I_{i,j,s}^{n+1} - 2I_{i,j,s}^n + I_{i,j,s}^{n-1}}{(\delta t)^2} + \gamma \frac{I_{i,j,s}^{n+1} - I_{i,j,s}^n}{\delta t} = -\Delta \left(C(I_{\xi,i,j,s}^n, |\Delta I_{\xi,i,j,s}^n|) \Delta I_{i,j,s}^n \right) - \lambda \left(\frac{I_{i,j,s}^n - f_{i,j,s}}{I_{i,j,s}^n} \right)^2. \quad (32)$$

Each channel is processed independently using the same numerical scheme as in the grayscale case. Spatial derivatives $(\nabla I, \Delta I)$ and the fidelity term are computed separately for each channel. The final restored color image is obtained by merging the individually denoised channels:

$$I_{\text{denoised}} = \text{cat}(I_{\text{denoised},R}, I_{\text{denoised},G}, I_{\text{denoised},B}). \quad (33)$$

3.2 Numerical Experiments and Performance Evaluation

This section evaluates the proposed model in comparison with existing second-order models using natural grayscale images (Boat, Texture), five images from the BSD68 dataset, and color images (Baboon, Peppers), all corrupted with multiplicative speckle noise at varying intensity levels ($L = 1, 3, 5, 10$). The noise was generated using MATLAB's `gamrnd` function as $\eta = \text{gamrnd}(L, 1/L, A, B)$, where $A \times B$ represents the image dimensions.

Experiments were conducted in MATLAB on an Intel Core i7 (3.60 GHz) with 8 GB RAM. The proposed model was evaluated across five images, ensuring consistency by applying the same finite-difference discretization across all methods.

For numerical stability, the explicit scheme adhered to the CFL condition [20–22], given by $\tau \leq \frac{h}{\max g(x,t)}$, where h is the spatial step size. Time steps of $\tau = 0.2$ and $\xi = 2$ were used. The stopping criteria were set to a tolerance of 0.0001 for SAR images, while PSNR served as the convergence criterion for natural images, ensuring optimal restoration quality.

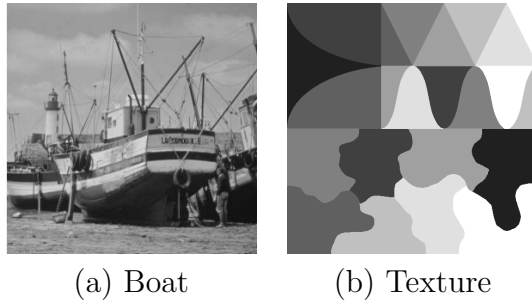


Figure 1: Grayscale images: (a) Boat, (b) Texture.



Figure 2: Grayscale images: (a) test_011, (b) test_015, (c) test_019, (d) test_022, (e) test_037.



Figure 3: color Images: (a) Baboon, (b)Pepper.

Image Quality Measurement

This section evaluates the proposed model's performance in noise reduction and edge preservation. For comparison, we use the following test images: a grayscale Boat image of size 292×292 and a Texture image of size 512×512 (Figure 1); two color images, namely Baboon of size 500×480 and Peppers of size 512×512 (Figure 3); and five images from the BSD68 dataset, each of size 321×481 . (Figure 2) Two widely used metrics, i.e. PSNR and MSSIM, are computed to assess image quality.

For synthetic aperture radar (SAR) images, the Speckle Index is used to measure noise reduction. A higher PSNR and MSSIM indicate better restoration, while a lower SI suggests effective despeckling. The SI is given by:

$$SI = \frac{S.D.(I)}{M(I)}, \quad (34)$$

where $S.D.(I)$ and $M(I)$ represent the standard deviation and mean of pixel intensities, respectively.

The PSNR is calculated as:

$$PSNR = 10 \cdot \log_{10} \left(\frac{MAX^2}{MSE} \right), \quad (35)$$

where MSE represents the Mean Squared Error, and MAX is the maximum pixel value.

The Structural Similarity Index (SSIM) is defined as:

$$SSIM(a, b) = \frac{(2\mu_a\mu_b + d_1)(2\sigma_{ab} + d_2)}{(\mu_a^2 + \mu_b^2 + d_1)(\sigma_a^2 + \sigma_b^2 + d_2)}, \quad (36)$$

where μ_a, μ_b are mean values, σ_a^2, σ_b^2 are variances, and σ_{ab} is the covariance. The constants d_1 and d_2 stabilize the calculation. The MSSIM over N iterations is:

$$MSSIM = \frac{1}{N} \sum_{i=1}^N SSIM_i. \quad (37)$$

For SAR image denoising, we apply specific norm-based criteria to evaluate the convergence of the proposed model. The convergence is generally assessed using the relative error norm and the Speckle Index. When the ground truth image is unavailable, the relative error norm serves as the stopping criterion. It is given by the formula:

$$\text{Relative Error} = \frac{\|I^{k+1} - I^k\|_2}{\|I^k\|_2} \leq \epsilon, \quad (38)$$

where I^k and I^{k+1} are the images in the iterations k -th and $(k+1)$ -th. $\|\cdot\|_2$ denotes the Euclidean norm. ϵ is a small positive threshold, often set to 10^{-4} . This criterion ensures that the algorithm is terminated when the changes between successive iterations become negligibly small. These norm-based criteria ensure the robustness and precision of the proposed model's performance during the image restoration process. A thorough investigation consisting of both visual and quantitative experiments was carried out to rigorously evaluate the despeckling capabilities of the proposed fourth-order telegraph diffusion model. The evaluation employed a diverse set of benchmark images, spanning both grayscale and color modalities, corrupted by multiplicative speckle noise at different noise levels ($L = 1, 3, 5, 10$). Figures 4 5 6 7 8 9 10 11 systematically present the denoising results across representative grayscale images such as Boat, Texture, and a typical sample from the BSD68 dataset, as well as widely used color images Baboon and Peppers. Across all these examples, qualitative assessment reveals that the proposed method attains superior image restoration quality compared to classical second-order PDE-based despeckling methods. The denoised images exhibit significantly enhanced visual clarity, characterized by notable noise suppression, sharper edge preservation, and faithful maintenance of intricate texture details. Along with visual comparison, quantitative measures were used to check the quality of restoration. We calculated Peak Signal-to-Noise Ratio (PSNR) and Mean Structural Similarity Index (MSSIM) for each restored image. The results are listed in Tables 1 and 2 for grayscale images, and in Table 4 for color images. These numerical results consistently demonstrate that the fourth-order telegraph diffusion model significantly outperforms benchmark models such as the Shan and TDM methods. The improvements in PSNR and MSSIM values indicate the model's enhanced ability to suppress noise while preserving essential image structures.

For fairness, all models parameter optimization tailored to grayscale and color image scenarios; optimal parameters utilized in the comparisons are summarized in Tables 3 and 5. The robustness of the proposed model is particularly evident in its application to Synthetic Aperture Radar (SAR) images (Figure 12), where it effectively reduces challenging speckle noise while preserving crucial geometric shapes and textural patterns that often degrade under conventional techniques. Further validation is provided in Figures 13 and 14, which show two-dimensional contour profiles of the Texture image ($L = 3$) and the Peppers image ($L = 5$), respectively. The noisy images exhibit strong and irregular intensity fluctuations, while the proposed model produces much smoother profiles. The fluctuations are significantly

reduced compared to the SHAN and TDM models. Figure 15 offers a comparative overview of PSNR and MSSIM across various methods and noise levels, underscoring the consistent dominance of the fourth-order model, especially in difficult scenarios such as the Baboon image. Further Speckle Index measurements in Table 6 for SAR imagery experiments, where the notably lower speckle index values achieved by the proposed method affirm its superior despeckling efficiency. Finally, Figure 16 provides a zoomed-in visual comparison on a region of interest within the *Peppers* color image at $L = 3$. The visual details reveal that the proposed model delivers markedly cleaner restorations, with crisper edges and more accurate color fidelity relative to existing models, reinforcing its practical utility in real applications.

For the BSD68 Dataset experimental comparisons employed each method’s individually optimized parameter settings to ensure unbiased evaluation. Specifically, Shan’s model was run with $\alpha = 0.1$, $\beta = 1$; the TDM model with $\gamma = 5$, $\alpha = 1$, $k = 2$; and the proposed model with $\gamma = 5$, $\alpha = 2$, $k = 2$, and $\lambda = 0.08$.

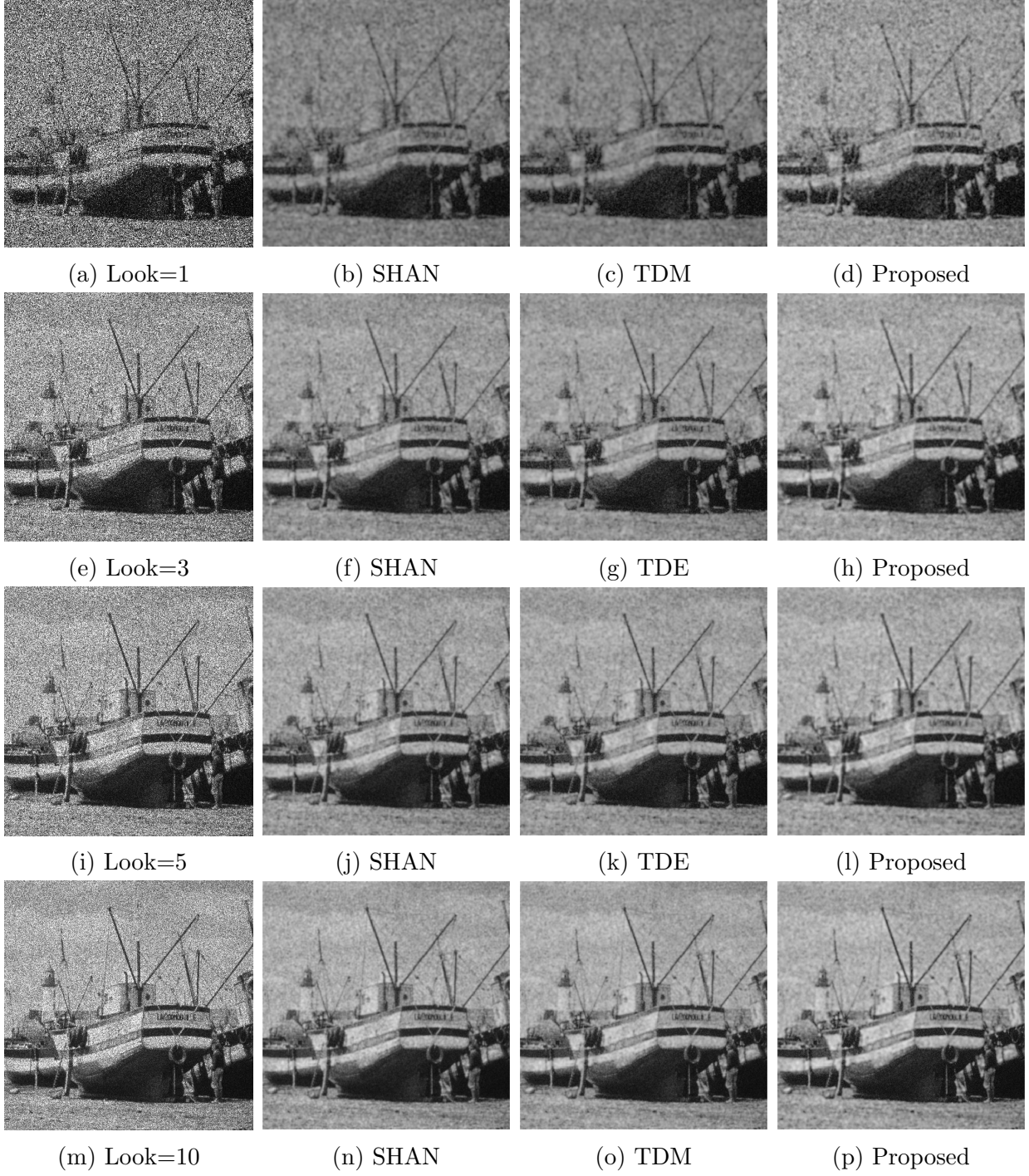


Figure 4: The first column contains noisy boat images with noise level Look = 1, 3, 5, 10. Subsequent columns: Restored boat images using different models (SHAN Model, TDM Model, Proposed Model).

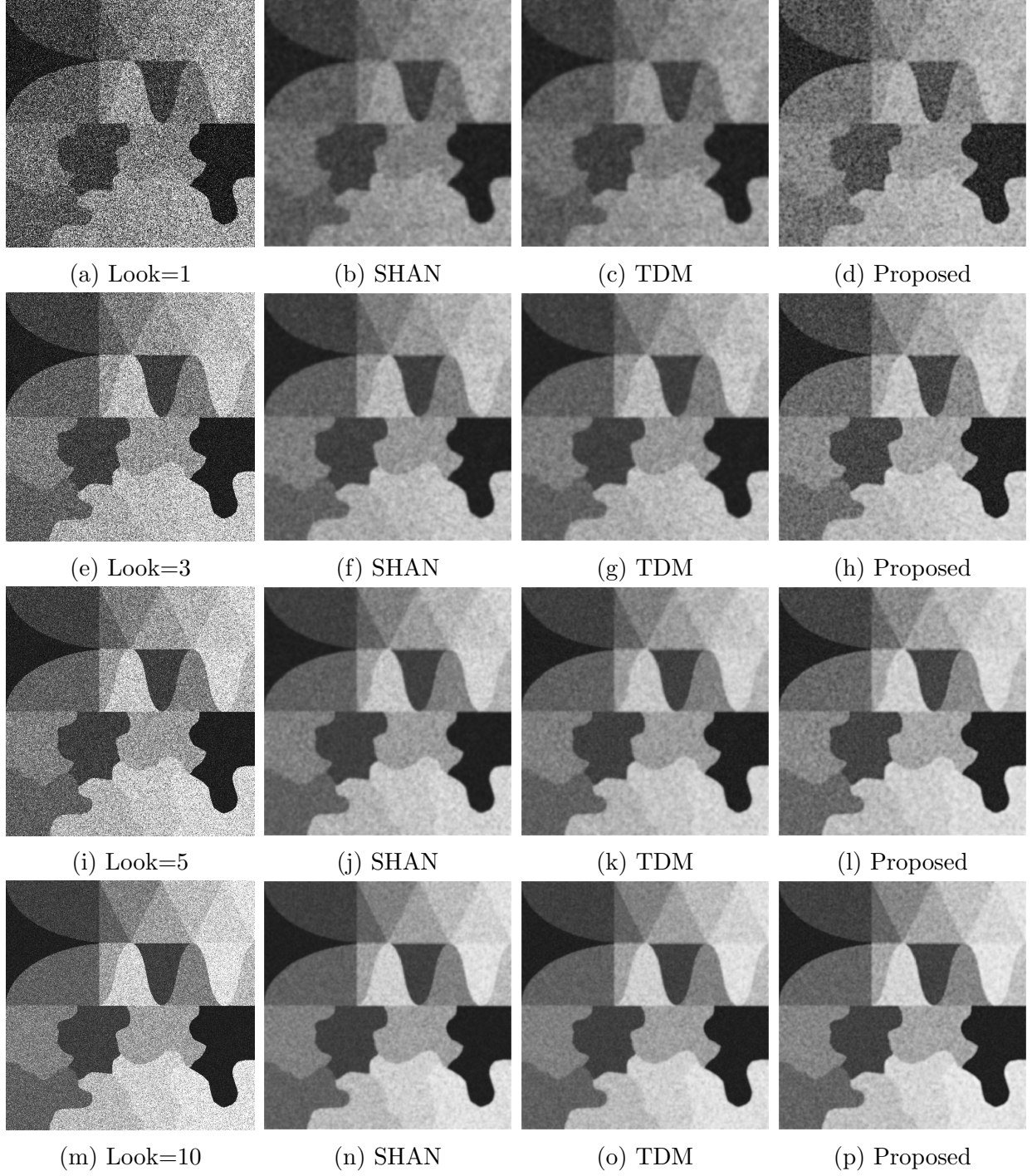


Figure 5: The first column contains noisy texture images with noise levels $\text{Look} = 1, 3, 5, 10$. Subsequent columns: Restored texture images using different models (SHAN, TDM, and Proposed Model).

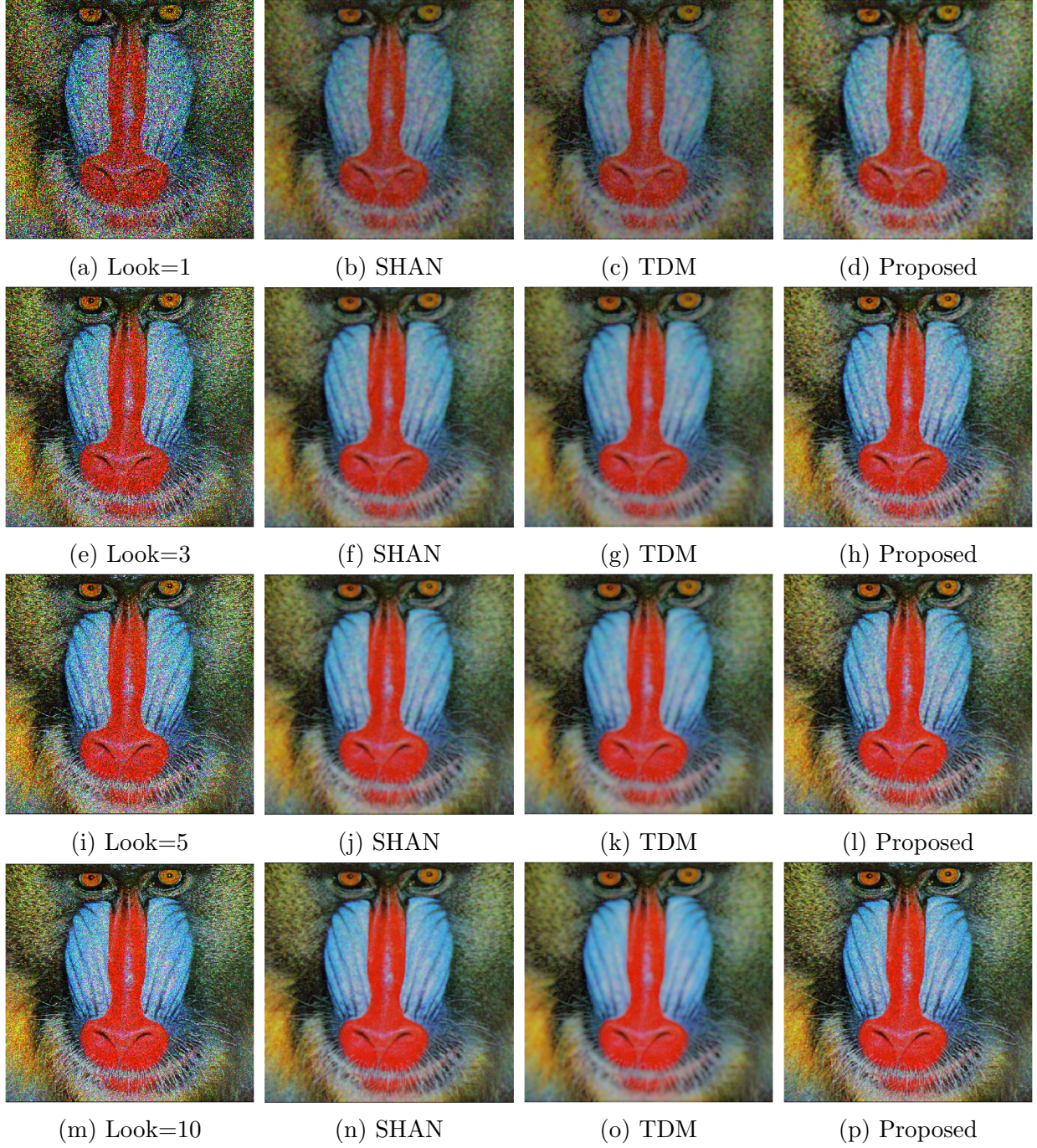


Figure 6: The first column contains noisy baboon images with noise levels Look = 1, 3, 5, 10. Subsequent columns: Restored texture images using different models (SHAN, TDM, and Proposed Model).

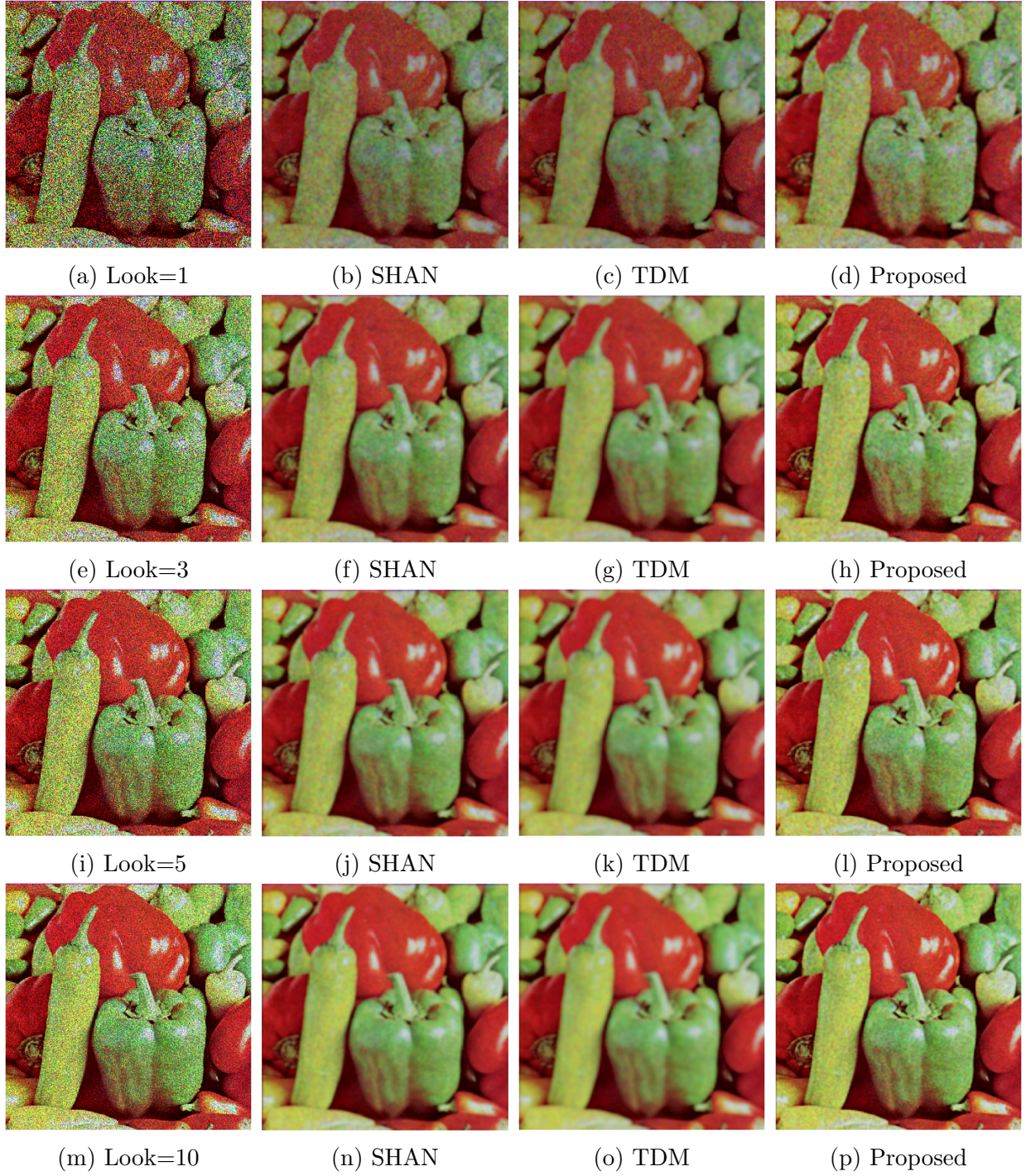


Figure 7: The first column contains noisy pepper images with noise levels Look = 1, 3, 5, 10. Subsequent columns: Restored texture images using different models (SHAN, TDM, and Proposed Model).

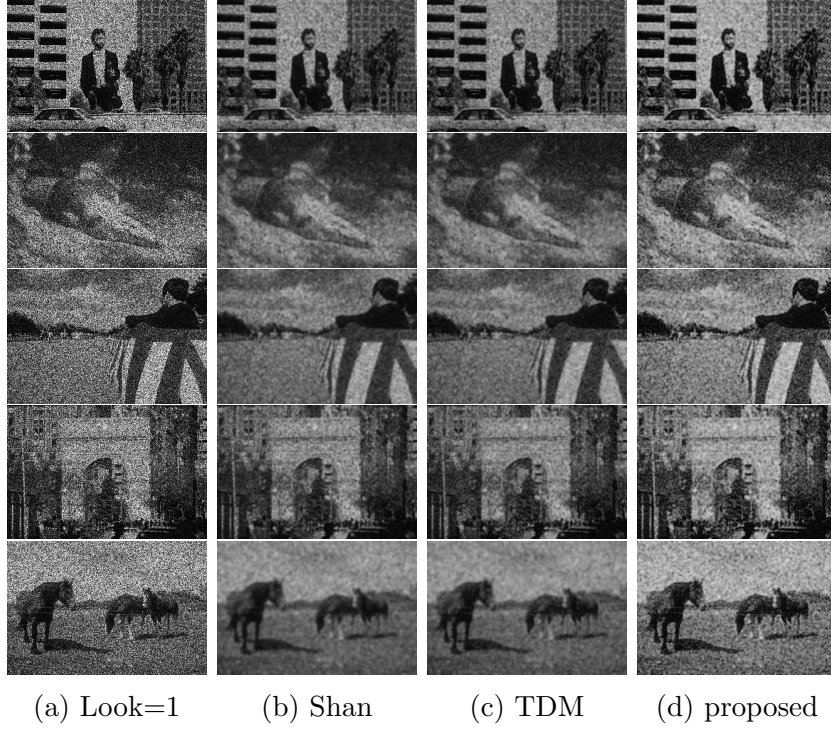


Figure 8: Restoration results for BSD68 grayscale images (Look = 1). For each image, the first column shows the noisy input, followed by restorations from SHAN, TDM, and the proposed fourth-order approach, displayed in successive columns.

Table 1: Comparison of MSSIM and PSNR values for boat and texture images using Shan, TDE, and the Proposed model at different noise levels (Look values).

images	look	shan		TDM		Proposed	
		MSSIM	PSNR	MSSIM	PSNR	MSSIM	PSNR
boat	1	0.4000	17.06	0.4116	17.06	0.4524	20.86
	3	0.4614	22.29	0.4803	22.30	0.5726	23.79
	5	0.4951	24.12	0.5208	24.17	0.5992	24.65
	10	0.5434	25.89	0.5702	25.91	0.6109	25.94
texture	1	0.5427	14.71	0.5453	14.71	0.5463	17.20
	3	0.5585	19.28	0.5729	19.28	0.6092	23.79
	5	0.5724	21.47	0.5874	21.47	0.6454	24.65
	10	0.5862	24.07	0.6032	24.11	0.7730	25.64

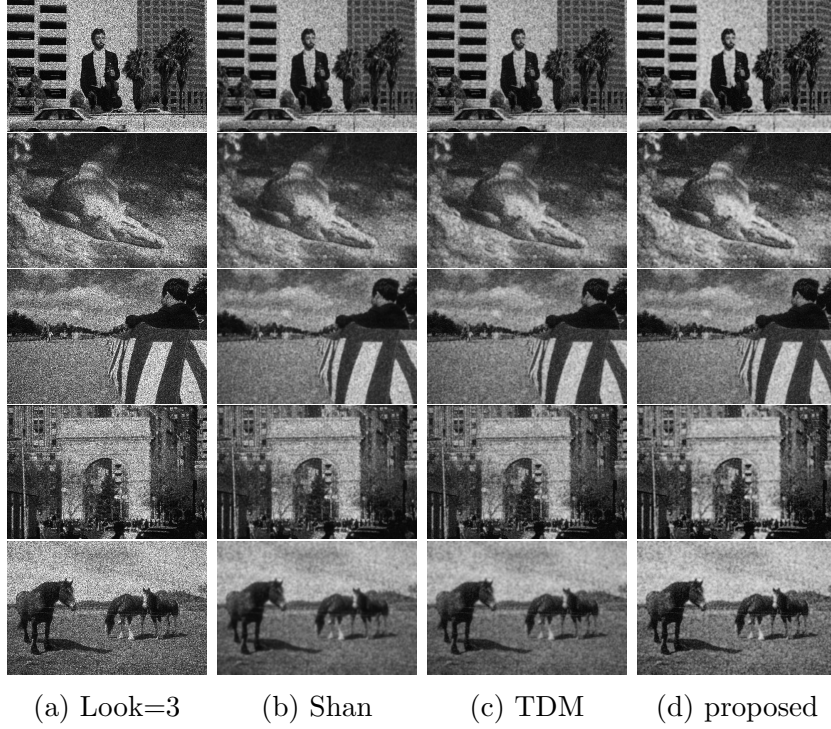


Figure 9: Restoration results for BSD68 grayscale images (Look = 3). For each image, the first column shows the noisy input, followed by restorations from SHAN, TDM, and the proposed fourth-order approach, displayed in successive columns.

Table 3: Optimum parameter values for different models on boat and texture images

Image	Look	SHAN		TDM			Proposed			
	L	α	β	γ	α	k	γ	α	k	λ
Boat	1	1	1	5	2	2	4	1	2	0.03
	3	1.1	1	5	2.5	2	4	1	2	0.03
	5	1.1	1	5	2.5	2	4	1	2	0.03
	10	1.2	1	5	2	2	4	3	2	0.03
Texture	1	0.01	0.01	4	0.1	2	4	0.4	2	0.03
	3	0.1	0.01	4	1	2	4	1	2	0.03
	5	0.1	0.01	2	1	2	4	0.5	2	0.03
	10	0.4	0.01	2	1	2	4	0.5	2	0.05

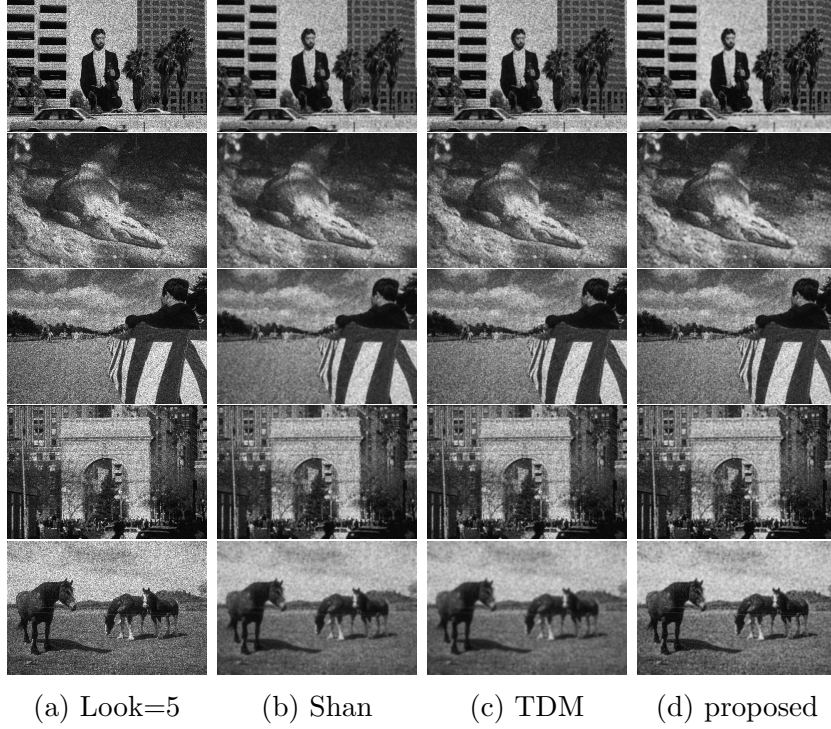


Figure 10: Restoration results for BSD68 grayscale images (Look = 5). For each image, the first column shows the noisy input, followed by restorations from SHAN, TDM, and the proposed fourth-order approach, displayed in successive columns.

Table 4: Comparison of MSSIM and PSNR values for Baboon and Peppers images using Shan, TDE, and the Proposed model at different noise levels (Look values).

Image	Look	Shan Model		TDE Model		Proposed Model	
		MSSIM	PSNR	MSSIM	PSNR	MSSIM	PSNR
<i>Baboon</i>	1	0.5287	16.69	0.5386	16.71	0.5682	17.91
	3	0.6221	19.07	0.6470	19.15	0.6833	19.51
	5	0.6572	19.80	0.6722	19.89	0.7366	20.32
	10	0.7438	21.11	0.7540	21.13	0.7998	21.60
<i>Peppers</i>	1	0.8155	17.55	0.8188	17.55	0.8515	20.47
	3	0.9185	22.51	0.9187	22.51	0.9218	24.07
	5	0.9401	24.52	0.9407	24.57	0.9408	25.51
	10	0.9563	26.36	0.9577	26.56	0.9578	27.15

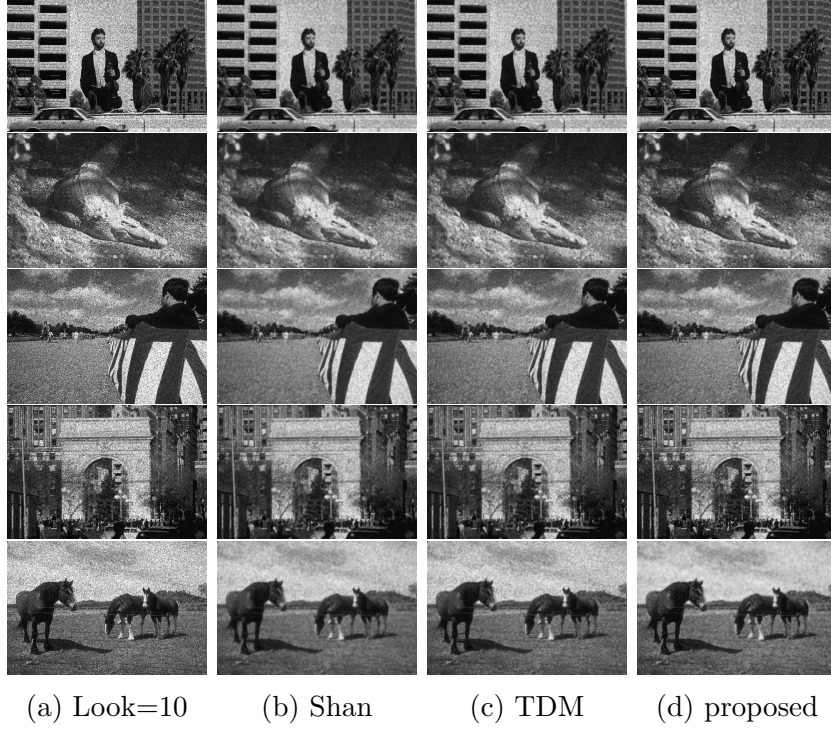


Figure 11: Restoration results for BSD68 grayscale images (Look = 10). For each image, the first column shows the noisy input, followed by restorations from SHAN, TDM, and the proposed fourth-order approach, displayed in successive columns.

Table 5: Optimum parameter table for the Baboon and Pepper images

images	L	Shan		TDE			Proposed			
		α	β	α	K	γ	α	K	γ	λ
baboon	1	1	1	2.5	2	9	1	2	5	0.1
	3	1	1	3	2	5	1.1	2	5	0.1
	5	1.1	1	3.1	2	5	1.2	2	5	0.1
	10	1.2	1	3.1	2	5	1.2	2	5	0.1
pepper	1	1	1	2	2	5	1	2	5	0.1
	3	1	1	2.1	2	9	1	2	5	0.1
	5	1.1	1	2.5	2	9	1	2	5	0.1
	10	1.2	1	2.5	2	9	2	2	5	0.08

Table 6: S.I. for the Synthetic Aaprature Resonance Imags

Image	Noisy	Parameters				Proposed Method
		γ	α	λ	K	
Image_1	1.02	5	0.1	0.007	4	0.2911
Image_2	0.92	2	0.1	0.001	4	0.3275

Table 2: Comparison of PSNR and MSSIM values for different models on five BSD68 images under various Look levels.

Image	Look	SHAN		TDM		Proposed	
		MSSIM	PSNR	MSSIM	PSNR	MSSIM	PSNR
test_11	1	0.3885	16.26	0.3918	16.33	0.4819	18.44
	3	0.4531	19.95	0.5002	20.18	0.6017	21.79
	5	0.4616	20.45	0.4721	20.95	0.6448	21.65
	10	0.5123	23.12	0.5376	23.55	0.7143	24.31
test_15	1	0.3374	17.93	0.3439	17.89	0.3847	18.13
	3	0.4192	21.08	0.4535	21.99	0.5293	22.10
	5	0.4278	22.25	0.4289	21.68	0.6435	22.08
	10	0.5207	23.69	0.5326	23.76	0.7140	23.98
test_19	1	0.3129	16.75	0.3204	16.80	0.3736	18.20
	3	0.3910	20.20	0.4136	20.23	0.4964	21.04
	5	0.4247	21.57	0.4340	21.87	0.5471	22.21
	10	0.4636	23.09	0.4748	23.11	0.7141	23.94
test_22	1	0.3722	17.56	0.3864	17.58	0.4446	18.24
	3	0.4603	20.70	0.5114	20.86	0.5741	21.11
	5	0.4824	22.14	0.5059	21.62	0.5506	21.97
	10	0.5336	23.29	0.5848	23.97	0.7145	24.32
test_37	1	0.3763	16.21	0.3785	16.23	0.3894	18.01
	3	0.3728	20.81	0.4295	20.89	0.5126	21.17
	5	0.3889	22.60	0.3984	22.69	0.5763	23.31
	10	0.3633	23.07	0.3795	23.32	0.6560	23.91

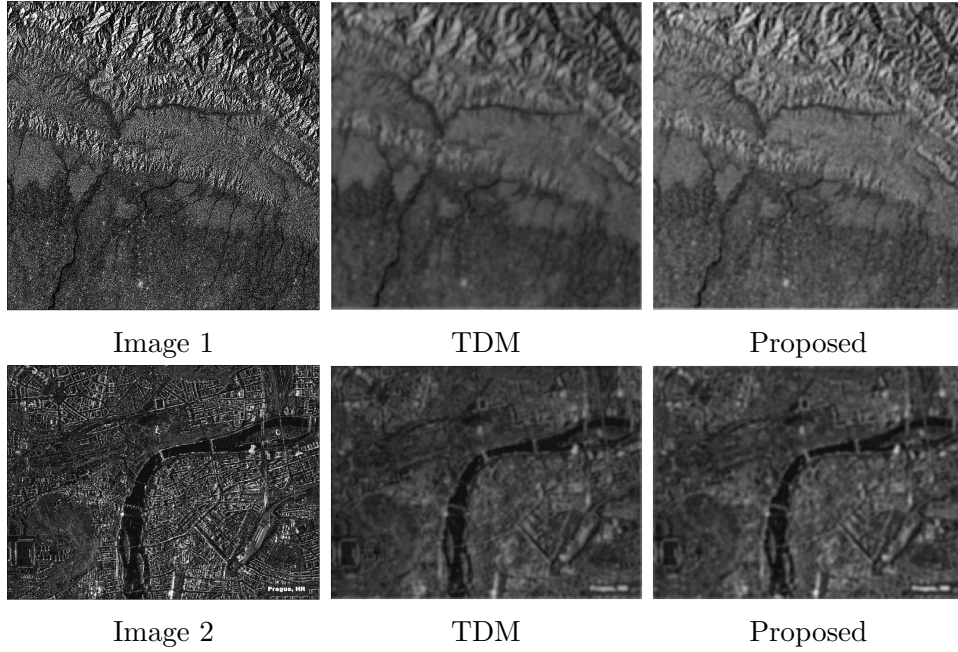
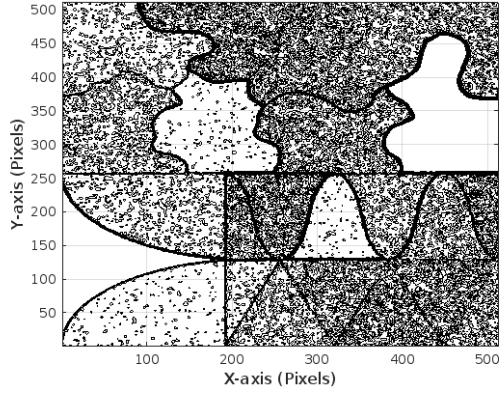


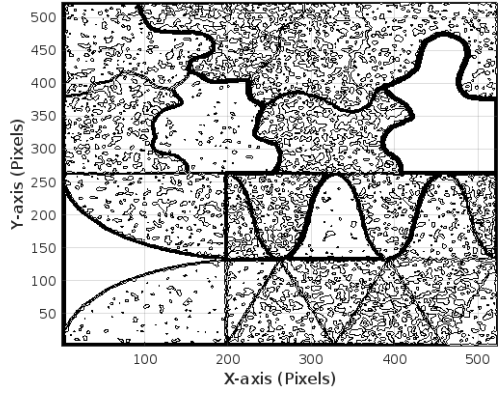
Figure 12: Comparison of SAR Image Restoration. Each row presents (1) Noisy Image (2) TDM Restored (3) Proposed Model Restored.

Black & White 2D Contour Plot of Grayscale Image



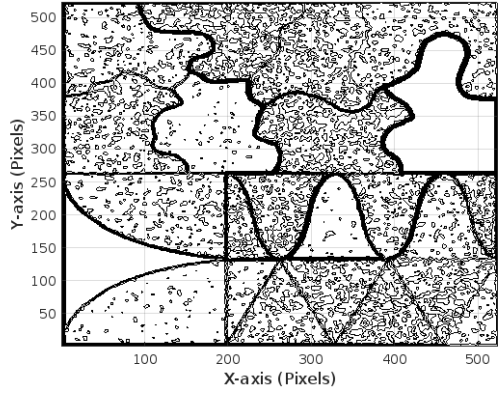
(a) Noisy Image

Black & White 2D Contour Plot of Grayscale Image



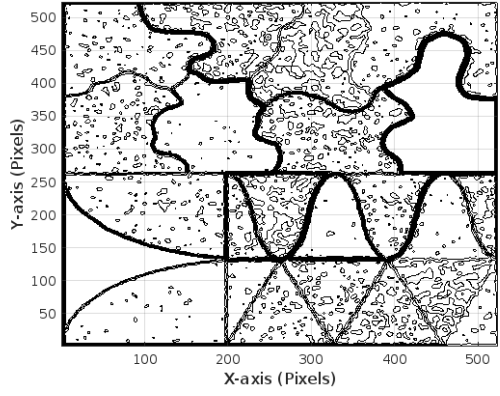
(b) Shan Restored

Black & White 2D Contour Plot of Grayscale Image



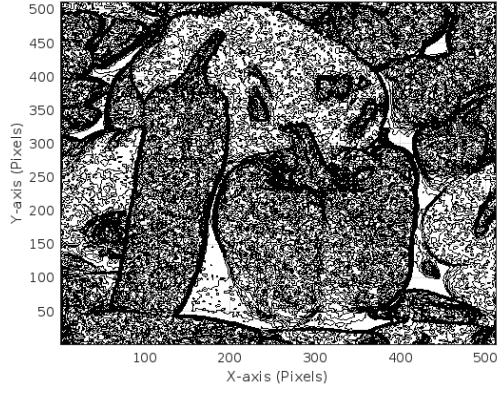
(c) TDM Restored

Black & White 2D Contour Plot of Grayscale Image

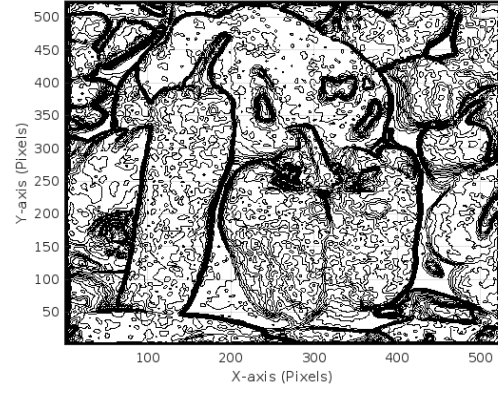


(d) Proposed Restored

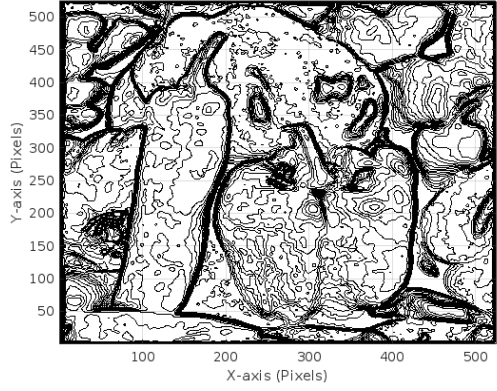
Figure 13: 2D plot of texture image (Look=3) with Restored images



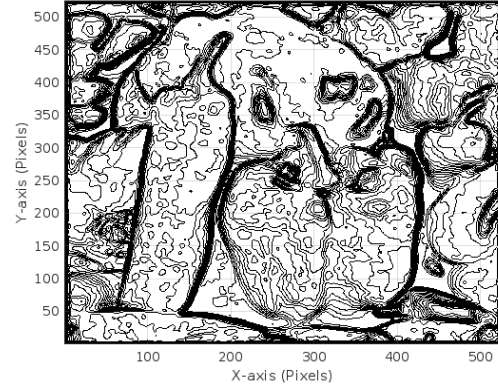
(a) Noisy Image



(b) Shan Restored



(c) TDM Restored



(d) Proposed Restored

Figure 14: 2D contour plot of pepper image (Look=5) with Restored images

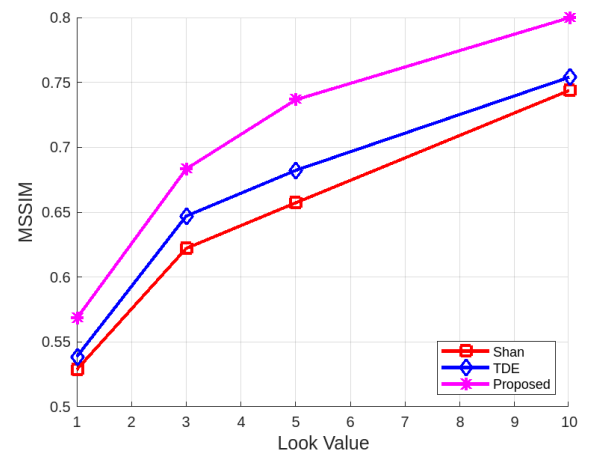
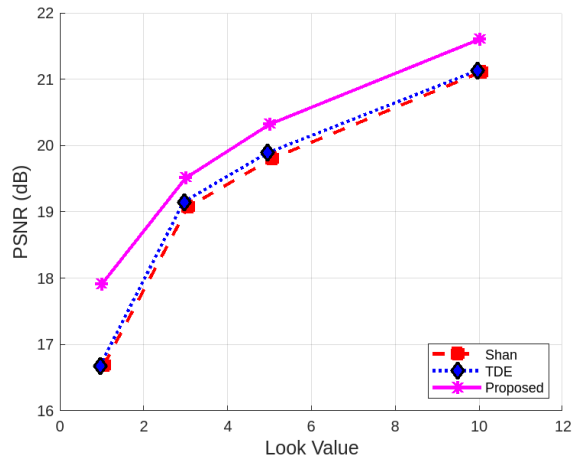


Figure 15: Camparsion of PSNR and MSSIM Proposed Model with State-of-art Models for the Color baboon image

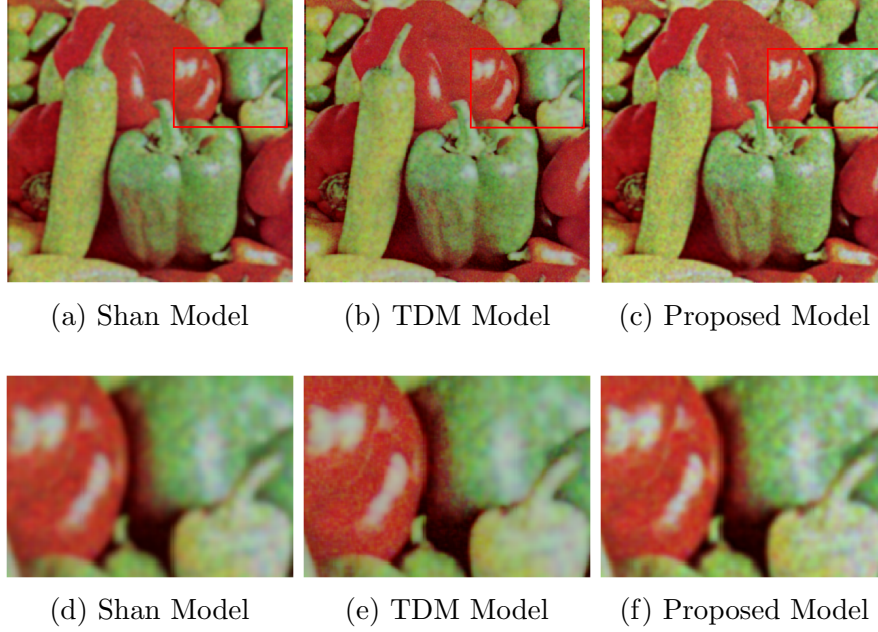


Figure 16: Comparison of Denoising Performance: The top row displays the full Peppers images ($L = 3$) with red boxes marking key despeckled areas, while the bottom row shows zoomed ROIs for a side-by-side evaluation of each model’s noise removal effectiveness.

4 Conclusion

This work introduces a novel fourth-order telegraph diffusion model designed to address the challenges of multiplicative speckle noise in image denoising. The proposed model effectively balances noise reduction and feature preservation, mitigating common issues such as staircase artifacts that are prevalent in second-order models. Through extensive numerical simulations, the model demonstrates superior performance in maintaining intricate details like contours and textures while efficiently suppressing noise. The results highlight the model’s ability to outperform existing state-of-the-art second order pde based image despeckling, particularly in applications involving noisy SAR images. Future research could focus on developing more efficient numerical methods to reduce computational time without compromising the model’s performance. Overall, the fourth-order PDE model represents a significant advancement in image restoration, offering a robust solution for handling noisy images in various real-world applications.

Author Declaration

The authors declare that there are no conflicts of interest.

Data Availability

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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