On the Anti-Ramsey Number of Spanning Linear Forests with Paths of Lengths 2 and 3

Ali Ghalavand[a,*] Xueliang Li[a]

November 7, 2025

[a] Center for Combinatorics and LPMC, Nankai University, Tianjin 300071, China

Abstract

An edge-coloring of a graph G assigns a color to each edge in the edge set $E(G)$. A graph G is considered to be rainbow under an edge-coloring if all of its edges have different colors. For a positive integer n, the anti-Ramsey number of a graph G, denoted as $AR(n, G)$, represents the maximum number of colors that can be used in an edge-coloring of the complete graph K_n without containing a rainbow copy of G. This concept was introduced by Erdős et al. in 1975. The anti-Ramsey number for the linear forest $kP_3 \cup tP_2$ has been extensively studied for two positive integers k and t. Formulations exist for specific values of t and k, particularly when $k \geq 2$, $t \geq \frac{k^2-k+4}{2}$, and $n \geq 3k+2t+1$. In this work, we present the anti-Ramsey number of the linear forest $kP_3 \cup tP_2$ for the case where $k \geq 1$, $t \geq 2$, and $n = 3k+2t$. Notably, our proof for this case does not require any specific relationship between k and t.

Keywords: rainbow graph; anti-Ramsey number; spanning linear forest.

1 Introduction

In this study, we examine the anti-Ramsey number of forests composed solely of paths of lengths two and three. We will begin by defining key concepts and discussing their historical context. Let G be a finite and simple graph with a vertex set $V(G)$ and an edge set $E(G)$. An edge-coloring of a graph G is a mapping c that assigns

*Corresponding author, email:alighalavand@nankai.edu.cn

a color to each edge in $E(G)$. The graph G is considered to be rainbow under the edge coloring c if all of its edges have different colors. For an edge e in G, the notation $c(e)$ indicates the color assigned to the edge e by the map c. For a subset E' of $E(G)$, the notation $c(E')$ represents the set of colors assigned to the edges in E', that is, $c(E') = \{c(e) : e \in E'\}$. Similarly, for a subgraph H of G, the notation $c(H)$ represents the set of colors assigned to the edges in H, specifically, $c(H) = \{c(e) : e \in E(H)\}$. Let V' be a subset of the vertex set $V(G)$ of a graph G. The subgraph of G that is induced by V' is denoted by $G[V']$. This subgraph has the vertex set V', and two vertices in it are considered adjacent if and only if they are adjacent in the original graph G. For another graph G', the disjoint union of G and G', denoted as $G \cup G'$, is a new graph that includes both G and G'. The vertex set of this new graph is given by $V(G \cup G') = V(G) \cup V(G')$, while the edge set is defined as $E(G \cup G') = E(G) \cup E(G')$. Additionally, for any positive integer a, the notation aG refers to the disjoint union of a copies of the graph G. Let t be a positive integer. We will use the notation $[t]$ to represent the set of natural numbers $\{1, 2, \ldots, t\}$. For clarity, we define $[0] = \emptyset$. The notations K_t, C_t, and P_t refer to the complete graph, the cycle graph, and the path graph on t vertices, respectively.

Let n be a positive integer. The anti-Ramsey number of a graph G, denoted as $AR(n, G)$, represents the maximum number of colors in an edge-coloring of the complete graph K_n that does not contain a rainbow copy of G. This concept was first introduced by Erdős et al. in 1975 [3], who established its connection to Turán numbers and conjectured the anti-Ramsey numbers for paths and cycles. Inspired by their work, a particularly active area of research has focused on characterizing anti-Ramsey numbers for graphs made up of multiple small, disjoint components. A key example of this is a matching, which consists of disjoint edges; the anti-Ramsey numbers for matchings in complete graphs are now fully understood, as referenced in [2, 5, 7, 9, 14]. In recent years, the focus has shifted from single matchings to more complex structures. Notably, researchers have begun exploring anti-Ramsey numbers for graphs formed by the vertex-disjoint union of small connected graphs and matchings. Simonovits and Sós [13] proved that if $t \geq 2$ and n is sufficiently large, then the following holds:

$$AR(n, P_t) = \binom{\lfloor \frac{t-1}{2} \rfloor}{2} + \left(\left\lfloor \frac{t-1}{2} \right\rfloor - 1 \right) \left(n - \left\lfloor \frac{t-1}{2} \right\rfloor + 1 \right) + 1 + \epsilon,$$

where $\epsilon = 1$ if t is even and $\epsilon = 0$ otherwise. Montellano-Ballesteros and Neumann-Lara [11] proved that for any $n \geq t \geq 3$,

$$AR(n, C_t) = \binom{t-1}{2} \left\lfloor \frac{n}{t-1} \right\rfloor + \left\lceil \frac{n}{t-1} \right\rceil + \binom{n \mod (t-1)}{2}.$$

Schiermeyer [12] showed that for all $t \geq 2$ and $n \geq 3t+3$, the following holds:

$$AR(n, tP_2) = \binom{t-2}{2} + (t-2)(n-t+2) + 1.$$

Then, Chen et al. [2] and Fujita et al. [5] demonstrated that for all $t \geq 2$ and $n \geq 2t+1$:

$$AR(n, tP_2) = \begin{cases} (t-2)(2t-3)+1 & \text{when } n \leq \frac{5t-7}{2}, \\ (t-2)(n-\frac{t-1}{2})+1 & \text{when } n \geq \frac{5t-7}{2}. \end{cases}$$

Finally, the remaining case $n = 2t$ was addressed by Haas and Young [7], who confirmed the conjecture made in [5]. They showed that if $n = 2t$, then:

$$AR(n, tP_2) = \begin{cases} \frac{1}{2}(t-2)(3t+1)+1 & \text{when } 3 \leq t \leq 6, \\ (t-2)(2t-3)+2 & \text{when } t \geq 7. \end{cases}$$

Bialostocki et al. [1] examined the anti-Ramsey number of graphs with at most four edges and proved that $AR(n, P_3 \cup P_2) = 2$ when $n \geq 5$ and $AR(n, P_3 \cup 2P_2) = n$ when $n \geq 7$. For more general graph classes, Gilboa and Roditty [6] provided upper bounds on the anti-Ramsey numbers of $L \cup tP_2$ and $L \cup tP_3$ under specific conditions. They proved that, for sufficiently large n,

- $AR(n, P_{k+1} \cup tP_3) = (t + \lfloor \frac{k}{2} \rfloor - 1)(n - \frac{t + \lfloor \frac{k}{2} \rfloor}{2}) + 1 + (k \mod 2)$ when $k \geq 3$;
- $AR(n, kP_3 \cup tP_2) = (k+t-2)(n - \frac{t+k-1}{2}) + 1$ when $k, t \geq 2$;
- $AR(n, P_2 \cup tP_3) = (t-1)(n - \frac{t}{2}) + 1$ when $t \geq 1$;
- $AR(n, P_3 \cup tP_2) = (t-1)(n - \frac{t}{2}) + 1$ when $t \geq 2$;
- $AR(n, P_4 \cup tP_2) = t(n - \frac{t+1}{2}) + 1$ when $t \geq 1$;
- $AR(n, C_3 \cup tP_2) = t(n - \frac{t+1}{2}) + 1$ when $t \geq 1$;
- $AR(n, tP_3) = (t-1)(n - \frac{t}{2}) + 1$ when $t \geq 1$.

He and Jin [8] established that if $t \geq 2$ and $n \geq 2t+3$, then

$$AR(n, P_3 \cup tP_2) = \begin{cases} t(2t-1)+1 & \text{when } 2t+3 \leq n \leq \left\lfloor \frac{5t+2}{2} + \frac{1}{t-1} \right\rfloor, \\ \frac{1}{2}(t-1)(2n-t)+1 & \text{when } n \geq \left\lceil \frac{5t+2}{2} + \frac{1}{t-1} \right\rceil. \end{cases}$$

They also proved that if $t \geq 2$ and $n \geq 2t + 7$, then

$$AR(n, 2P_3 \cup tP_2) = \begin{cases} (t+1)(2t+3)+1 & \text{when } 2t+7 \leq n \leq \left\lfloor \frac{5t+11}{2} + \frac{3}{t} \right\rfloor, \\ \frac{1}{2}t(2n-t-1)+1 & \text{when } n \geq \left\lceil \frac{5t+11}{2} + \frac{3}{t} \right\rceil. \end{cases}$$

Fang et al. [4] considered F to be a linear forest with components of order $p_1, p_2, \ldots, p_t$, where $t \geq 2$ and $p_i \geq 2$ for $1 \leq i \leq t$, and at least one p_i is even. They proved that for sufficiently large n,

$$AR(n, F) = \left(\sum_{i \in [t]} n \left\lfloor \frac{p_i}{2} \right\rfloor - \epsilon \right) n + O(1),$$

where $\epsilon = 1$ if all p_i are odd, and $\epsilon = 2$ otherwise. Following this, Xie et al. [15] established an exact expression for the anti-Ramsey numbers of linear forests containing even components. They proved that if at least one p_i is even, where $t \geq 2$ and $p_i \geq 2$ for all $1 \leq i \leq t$, then for sufficiently large n,

$$AR(n, F) = \binom{\sum_{i \in [t]} \left\lfloor \frac{p_i}{2} \right\rfloor - 2}{2} + \left(\sum_{i \in [t]} \left\lfloor \frac{p_i}{2} \right\rfloor - 2 \right) \left(n - \sum_{i \in [t]} \left\lfloor \frac{p_i}{2} \right\rfloor + 2 \right) + 1 + \epsilon,$$

where $\epsilon = 1$ if exactly one p_i is even, or $\epsilon = 0$ if at least two p_i are even. Note that this expression does not account for the case when $n = \sum_{i \in [t]} p_i$. Let $\lambda = \frac{9k+5t-7}{2} + \frac{k(k+1)}{2(k+t-2)}$. Jie et al. [10] studied the anti-Ramsey number of $kP_3 \cup tP_2$ and proved that if $k \geq 2$, $t \geq \frac{k^2-k+4}{2}$, and $n \geq 3k + 2t + 1$, then

$$AR(n, kP_3 \cup tP_2) = \begin{cases} \frac{1}{2}(3k+2t-3)(3k+2t-4)+1 & \text{when } n \leq \lfloor \lambda \rfloor, \\ \frac{1}{2}(k+t-2)(2n-k-t+1)+1 & \text{when } n \geq \lceil \lambda \rceil. \end{cases}$$

In this paper, we focus on the anti-Ramsey number of the linear forest $kP_3 \cup tP_2$ under the conditions that $k \geq 1$, $t \geq 2$, and $n = 3k + 2t$. We will prove the following theorem:

Theorem 1.1. *For three positive integers k, t, and n, if $k \geq 1$, $t \geq 2$, and $n = 3k + 2t$, then*

$$AR(n, kP_3 \cup tP_2) = \frac{1}{2}(3k+2t-3)(3k+2t-4)+1.$$

To prove this theorem, we require two crucial tools. The first is a result from He and Jin [8]. We will prove the second one ourselves here.

Theorem 1.2. [8] *For any positive integer t, if $t \geq 2$ and $\mu = \frac{5t+2}{2} + \frac{1}{t-1}$, then*

$$AR(n, P_3 \cup tP_2) = \begin{cases} t(2t-1)+1 & \text{when } 2t+3 \leq n \leq \lfloor \mu \rfloor, \\ (t-1)(n-\frac{t}{2})+1 & \text{when } n \geq \lceil \mu \rceil. \end{cases}$$

Lemma 1.3. *For three positive integers k, t, and n, where $k \geq 1$, $t \geq 2$, and $n = 3k+2t$, let c be an edge-coloring of K_n such that*

$$|c(K_n)| \geq \frac{1}{2}(3k+2t-3)(3k+2t-4)+2.$$

If K_{n-3} is a subgraph of K_n such that $|c(K_{n-3})|$ is maximized, then it holds that

$$|c(K_{n-3})| \geq \frac{1}{2}(3k+2t-6)(3k+2t-7)+2.$$

Proof. For three positive integers k, t, and n where $k \geq 1$, $t \geq 2$, and $n = 3k+2t$, let c be an edge-coloring of K_n such that

$$|c(K_n)| \geq \frac{1}{2}(3k+2t-3)(3k+2t-4)+2.$$

Assume that G is a rainbow spanning subgraph of size $|c(K_n)|$ of K_n, and let K_{n-3} be a subgraph of K_n such that $|E(K_{n-3}) \cap E(G)|$ is maximized. To prove our lemma, it suffices to demonstrate that

$$|E(K_{n-3}) \cap E(G)| \geq \frac{1}{2}(3k+2t-6)(3k+2t-7)+2.$$

To establish this, we will analyze the situation in two distinct cases as follows:
Case I: For every vertex v of K_{n-3}, it holds that

$$|\{uv : u \in V(K_{n-3})\} \cap E(G)| \geq 3k+2t-6.$$

In this case, since $k \geq 1$ and $t \geq 2$, it follows that

$$|E(K_{n-3}) \cap E(G)| \geq \frac{1}{2}(3k+2t-3)(3k+2t-6) \geq \frac{1}{2}(3k+2t-6)(3k+2t-7)+2,$$

which is as desired.
Case II: For a vertex v of K_{n-3}, it holds that

$$|\{uv : u \in V(K_{n-3})\} \cap E(G)| \leq 3k+2t-7,$$

and it also holds that

$$|E(K_{n-3}) \cap E(G)| \leq \frac{1}{2}(3k+2t-6)(3k+2t-7)+1.$$

In this case, since $|E(G)| = |c(K_n)| \geq \frac{1}{2}(3k+2t-3)(3k+2t-4)+2$, it follows that

$$|E(G) - E(K_{n-3})| \geq 9k + 6t - 14.$$

Thus, there exists a vertex x in $V(K_n) - V(K_{n-3})$ such that

$$|\{ux : u \in V(K_{n-3})\} \cap E(G)| \geq 3k + 2t - 5.$$

However, this leads to a contradiction regarding the maximality, because using the vertices v and x, we can construct a graph K_{n-3} such that $|E(K_{n-3}) \cap E(G)|$ is larger.

Now, based on these two cases, we can conclude that

$$|E(K_{n-3}) \cap E(G)| \geq \frac{1}{2}(3k + 2t - 6)(3k + 2t - 7) + 2,$$

therefore completing the proof. □

2 Proof of Theorem 1.1

For three positive integers k, t, and n, assume $k \geq 1$, $t \geq 2$ and $n = 3k + 2t$. To prove our result, we need to confirm two cases:
1. $AR(n, kP_3 \cup tP_2) \geq \frac{1}{2}(3k + 2t - 3)(3k + 2t - 4) + 1$,
2. $AR(n, kP_3 \cup tP_2) \leq \frac{1}{2}(3k + 2t - 3)(3k + 2t - 4) + 1$.

We will address these cases as follows:

Case 1: To establish this case, suppose c is an edge-coloring of K_n such that for a subgraph K_{n-3} of K_n, K_{n-3} is rainbow, and all other edges of K_n that are not in K_{n-3} are colored with a new color. It is evident that $|c(K_n)| = \frac{1}{2}(3k+2t-3)(3k+2t-4)+1$ and that there is no rainbow subgraph of K_n isomorphic to $kP_3 \cup tP_2$. Thus, we conclude that $AR(n, kP_3 \cup tP_2) \geq \frac{1}{2}(3k + 2t - 3)(3k + 2t - 4) + 1$.

Case 2: To prove this case, we need to show that if c is an arbitrary edge-coloring of K_n such that $|c(K_n)| = \frac{1}{2}(3k + 2t - 3)(3k + 2t - 4) + 2$, then K_n has a rainbow subgraph isomorphic to $kP_3 \cup tP_2$. We will use induction on k. If $k = 1$, Theorem 1.2 provides the result. Now, let $k \geq 2$, and consider K_{n-3}, a subgraph of K_n, that has a maximized $|c(K_{n-3})|$. Assume $S = V(K_n) - V(K_{n-3}) = \{s_1, s_2, s_3\}$. By applying Lemma 1.3, we find that $|c(K_{n-3})| \geq \frac{1}{2}(3(k-1)+2t-3)(3(k-1)+2t-4)+2$. Thus, by the inductive hypothesis, the graph K_{n-3} contains a rainbow subgraph H such that $H \cong (k-1)P_3 \cup tP_2$. Now, assume H_1 and H_2 are two distinct subgraphs of H that $H_1 = (k-1)P_3$ and $H_2 = tP_2$. In addition, suppose $H_1 = \{P_3^i : i \in [k-1]\}$, $H_2 = \{P_2^j : j \in [t]\}$, $E(P_3^i) = \{x_1^i x_2^i, x_2^i x_3^i\}$ for $i \in [k-1]$, and $E(P_2^j) = \{y_1^j y_2^j\}$ for

$j \in [t]$. Also, suppose G is a rainbow spanning subgraph of size $|c(K_n)|$ from K_n that satisfies $E(H) \subseteq E(G)$. To complete the proof in this case, we will examine sixteen scenarios as outlined below. Please note that the notation $\mathcal{S}_{2.x}(l_1, \ldots, l_h)$, which we may use in Scenario 2.x, represents the largest subset of the edge set of K_n that we can derive from the conditions l_1 to l_h, such that $\mathcal{S}_{2.x}(l_1, \ldots, l_h) \cap E(G) = \emptyset$.
Scenario 2.1: $|c(K_n[S]) - c(H)| \geq 2$. In this scenario, the graph $K_n[S]$ contains a rainbow subgraph F_1 that $F_1 \cong P_3$ and $c(F_1) \cap c(H) = \emptyset$. Therefore, $H \cup F_1$ forms a rainbow subgraph of K_n that is isomorphic to $kP_3 \cup tP_2$.
Scenario 2.2: $|c(K_n[S])| = 3$ and $|c(K_n[S]) - c(H)| = 1$. For $l \in [k-1]$, it holds that $|c(K_n[S]) \cap c(P_3^l)| = 2$. We also assume that $c(s_1s_2) \notin c(H)$. If one of the following conditions is satisfied, then it is possible to construct a rainbow subgraph of K_n isomorphic to $kP_3 \cup tP_2$.

1. There are $i \in [t]$, $i_1 \in [2]$, and $z \in \{s_3, x_1^l, x_3^l\}$ such that $zy_{i_1}^i \in E(G)$ and $c(zy_{i_1}^i) \neq c(s_1s_2)$.

2. There are $i \in ([k-1]-\{l\})$, $j \in [t]$, $j_1 \in [2]$, $j_2 \in \{1,3\}$, $j_3 \in (\{1,3\}-\{j_2\})$ and $z \in \{s_3, x_1^l, x_3^l\}$ such that $\{zx_{j_2}^i, x_{j_3}^i y_{j_1}^j\} \subseteq E(G)$ and $c(s_1s_2) \notin c(\{zx_{j_2}^i, x_{j_3}^i y_{j_1}^j\})$.

3. There are $i \in ([k-1]-\{l\})$, $j \in [t]$, $j_1 \in [2]$, $j_2 \in \{1,3\}$ and $z \in \{s_3, x_1^l, x_3^l\}$ such that $\{zx_2^i, x_{j_2}^i y_{j_1}^j\} \subseteq E(G)$ and $c(s_1s_2) \notin c(\{zx_2^i, x_{j_2}^i y_{j_1}^j\})$.

4. There are $i \in ([k-1]-\{l\})$, $j \in [t]$, $j_1 \in [2]$, and $z \in \{s_3, x_1^l, x_3^l\}$ such that $\{zx_1^i, zx_3^i, x_2^i y_{j_1}^j\} \subseteq E(G)$ and $c(s_1s_2) \notin c(\{zx_1^i, zx_3^i, x_2^i y_{j_1}^j\})$.

5. There are $i \in [t]$, $i_1 \in [2]$, $i_2 \in ([2]-\{i_1\})$, and $i_3 \in \{1,3\}$ such that either $\{x_{i_3}^l s_1, s_1 y_{i_1}^i, x_2^l y_{i_2}^i\}$, $\{x_{i_3}^l s_1, s_2 y_{i_1}^i, x_2^l y_{i_2}^i\}$, or $\{x_{i_3}^l s_1, s_1 y_{i_1}^i, s_2 y_{i_2}^i\}$ is a subset of $E(G)$.

6. There are $i \in [t]$ and $i_1 \in [2]$ such that $\{x_1^l s_2, s_1 y_{i_1}^i\}$ is a subset of $E(G)$.

7. There are $i \in [t]$, $i_1 \in [2]$, and $i_2 \in ([2]-\{i_1\})$ such that either $\{x_3^l s_2, s_1 y_{i_1}^i, x_2^l y_{i_2}^i\}$, $\{x_3^l s_2, s_2 y_{i_1}^i, x_2^l y_{i_2}^i\}$, or $\{x_3^l s_2, s_1 y_{i_1}^i, s_2 y_{i_2}^i\}$ is a subset of $E(G)$.

8. There are $i \in [t]$ and $i_1 \in [2]$ such that $\{x_1^l s_3, s_1 y_{i_1}^i\}$ is a subset of $E(G)$.

9. There are $i \in [t]$ and $i_1 \in [2]$ such that $\{x_3^l s_3, s_2 y_{i_1}^i\}$ is a subset of $E(G)$.

10. Both $x_1^l x_3^l \in E(G)$ and $c(x_1^l x_3^l) \neq c(s_1s_2)$ hold.

11. There are $i \in [t]$ and $i_1 \in [2]$ such that $c(x_1^l x_3^l) \in c(K_n[S])$, $x_2^l y_{i_1}^i \in E(G)$, and $c(x_2^l y_{i_1}^i) \neq c(s_1s_2)$.

12. There are $i \in [t]$, $j \in ([t] - \{i\})$, $j_1 \in [2]$, and $j_2 \in ([2] - \{j_1\})$ such that $c(x_1^l x_3^l) = c(y_1^i y_2^i)$, $y_1^i y_{j_1}^j, y_2^i y_{j_2}^j \in E(G)$, and $c(s_1 s_2) \notin c(\{y_1^i y_{j_1}^j, y_2^i y_{j_2}^j\})$.

13. There is $i \in [t]$ such that $c(x_1^l x_3^l) = c(y_1^i y_2^i)$, $y_1^i x_2^l, y_2^i x_2^l \in E(G)$, and $c(s_1 s_2) \notin c(\{y_1^i x_2^l, y_2^i x_2^l\})$.

14. There are $i \in [t]$, $j \in ([k-1] - \{l\})$, $i_1 \in [2]$, $j_1 \in ([3] - \{2\})$ such that $c(x_1^l x_3^l) = c(x_{j_1}^j x_2^j)$, $x_{j_1}^j y_{i_1}^i \in E(G)$, and $c(s_1 s_2) \neq c(x_{j_1}^j y_{i_1}^i)$.

15. There are $i \in ([k-1] - \{l\})$ and $i_1 \in ([3] - \{2\})$ such that $c(x_1^l x_3^l) = c(x_{i_1}^i x_2^i)$, $x_1^i x_3^i \in E(G)$, and $c(s_1 s_2) \neq c(x_1^i x_3^i)$.

If none of the above conditions hold, we can observe that $|\mathcal{S}_{2.2}(1)| \geq 6t$, $|\mathcal{S}_{2.2}(2,3)| \geq 9(k-2)$, and $|\mathcal{S}_{2.2}(4,\ldots,15) \cup \{s_1 s_3, s_2 s_3\}| \geq 11$. Thus, we find that

$$\begin{aligned}|c(K_n)| &\leq \frac{1}{2}(3k+2t)(3k+2t-1) - \big(6t + 9(k-2) + 11\big) \\ &= \frac{1}{2}(3k+2t-3)(3k+2t-4) + 1, \text{a contradiction regarding } |c(K_n)|.\end{aligned}$$

Scenario 2.3: $|c(K_n[S])| = 3$, $|c(K_n[S]) - c(H)| = 1$, and $|c(K_n[S]) \cap c(P_2^a \cup P_2^b)| = 2$, where $a, b \in [t]$. Assume $c(s_1 s_2) \notin c(H)$, $c(s_1 s_3) = c(y_1^a y_2^a)$, and $c(s_2 s_3) = c(y_1^b y_2^b)$. In this scenario, if any of the following conditions hold, we can construct a rainbow subgraph of K_n that is isomorphic to $kP_3 \cup tP_2$.

1. There are $i \in [t]$ and $i_1 \in [2]$ such that $s_3 y_{i_1}^i \in E(G)$ and $c(s_1 s_2) \neq c(s_3 y_{i_1}^i)$.

2. There are $i \in ([t] - \{a, b\})$, $i_1 \in [2]$, $i_2 \in ([2] - \{i_1\})$, and $j \in \{a, b\}$ such that $\{y_{i_1}^i y_1^j, y_{i_2}^i y_2^j\} \subseteq E(G)$ and $c(s_1 s_2) \notin c(\{y_{i_1}^i y_1^j, y_{i_2}^i y_2^j\})$.

3. There are $i \in [k-1]$ and $j \in \{a, b\}$ such that for $A_1 = \{y_1^j x_2^i, y_2^j x_2^i, x_1^i x_3^i\}$, it holds that $A_1 \subseteq E(G)$ and $c(s_1 s_2) \notin c(A_1)$.

4. There are $i \in [k-1]$, $j \in \{a, b\}$, $j_1 \in [2]$, $j_2 \in \{1, 3\}$, and $j_3 \in ([3] - \{j_2\})$ such that for $A_2 = \{y_{j_1}^j x_{j_2}^i, s_3 x_{j_3}^i\}$, it holds that $A_2 \subseteq E(G)$ and $c(s_1 s_2) \notin c(A_2)$.

5. There are $i \in [k-1]$ and $j \in [3]$ such that for $A_3 = \{y_1^a x_2^i, y_2^a x_2^i, y_1^b x_2^i, y_2^b x_2^i, s_j x_1^i, s_j x_2^i\}$, it holds that $A_3 \subseteq E(G)$ and $c(s_1 s_2) \notin c(A_3)$.

6. There are $i \in [k-1]$, $j \in \{a, b\}$, $j_1 \in [3]$, $j_2 \in ([3] - \{j_1\})$ such that for $A_4 = \{y_1^j x_{j_1}^i, y_2^j x_{j_2}^i\}$, it holds that $A_4 \subseteq E(G)$ and $c(s_1 s_2) \notin c(A_4)$.

7. It holds that either $\{y_1^a s_2, y_2^a s_2\}$ or $\{y_1^b s_1, y_2^b s_1\}$ is a subset of $E(G)$.

8. There are $j \in \{a,b\}$, $i_1 \in [2]$, and $i_2 \in ([2]-\{i_1\})$ such that $\{y^j_{i_1}s_1, y^j_{i_2}s_2\} \subseteq E(G)$.

9. There are $i_1 \in [2]$ and $i_2 \in ([2]-\{i_1\})$ such that $\{y^a_1 y^b_{i_1}, y^a_2 y^b_{i_2}\} \subseteq E(G)$.

Assuming that none of the previous statements is true, we obtain that $|\mathcal{S}_{2.3}(1,2)| \geq 2t + 4(t-2)$, $|\mathcal{S}_{2.3}(3,\dots,6)| \geq 11(k-1)$, and $|\mathcal{S}_{2.3}(7,8,9) \cup \{s_1s_3, s_2s_3\}| \geq 8$. From this and $k \geq 2$, we can observe that

$$\begin{aligned}
|c(K_n)| \leq& \frac{1}{2}(3k+2t)(3k+2t-1) - \big(2t + 4(t-2) + 11(k-1) + 8\big)\\
=& \frac{1}{2}(3k+2t-3)(3k+2t-4) + 2 - (2k-3)\\
\leq& \frac{1}{2}(3k+2t-3)(3k+2t-4) + 1, \text{a contradiction.}
\end{aligned}$$

Scenario 2.4: $|c(K_n[S])| = 3$ and $|c(K_n[S]) - c(H)| = 1$. Additionally, for $a \in [t]$ and $b \in [k-1]$, $c(P^a_2) \in c(K_n[S])$ and $|c(K_n[S]) \cap c(P^b_3)| = 1$. We suppose that $c(s_1s_2) \notin c(H)$, $c(y^a_1 y^a_2) = c(s_1s_3)$, and $c(x^b_1 x^b_2) = c(s_2s_3)$. Under this scenario, if one of the following holds, then we can build a rainbow subgraph of K_n that is isomorphic to $kP_3 \cup tP_2$.

1. There are $i \in [t]$, $i_1 \in [2]$, and $z \in \{s_3, x^b_1\}$ such that $zy^i_{i_1} \in E(G)$ and $c(s_1s_2) \neq c(zy^i_{i_1})$.

2. There are $i \in ([t]-\{a\})$, $i_1 \in [2]$, and $i_2 \in ([2]-\{i_1\})$ such that $\{y^i_{i_1}y^a_1, y^i_{i_2}y^a_2\} \subseteq E(G)$ and $c(s_1s_2) \notin c(\{y^i_{i_1}y^a_1, y^i_{i_2}y^a_2\})$.

3. For $i \in ([k-1]-\{b\})$, there is a subset B_1 of $\{y^a_{a_1}x^i_{i_1} : a_1 \in [2], i_1 \in [3]\} \cup \{x^i_1 x^i_3\}$ such that $|B_1| = 3$, $B_1 \subseteq E(G)$, and $c(s_1s_2) \notin c(B_1)$.

4. For $i \in ([k-1]-\{b\})$, there is a subset B_2 of $\{s_{i_1}x^i_{i_2} : i_1, i_2 \in [3]\}$ such that $|B_2| = 7$, $B_2 \subseteq E(G)$, and $c(s_1s_2) \notin c(B_2)$.

5. For $i \in ([k-1]-\{b\})$, there is a subset B_3 of $\{x^b_{i_1}x^i_{i_2} : i_1, i_2 \in [3]\}$ such that $|B_3| = 7$, $B_3 \subseteq E(G)$, and $c(s_1s_2) \notin c(B_3)$.

6. For $a_1 \in [2]$ and $a_2 \in ([2]-\{a_1\})$, either $\{y^a_{a_1}s_1, y^a_{a_2}s_2\} \subseteq E(G)$ or $\{y^a_1 s_2, y^a_2 s_2\} \subseteq E(G)$ holds.

7. For $a_1 \in [2]$ and $a_2 \in ([2]-\{a_1\})$, either $\big(\{y^a_{a_1}x^b_2, y^a_{a_2}x^b_3\} \subseteq E(G)$ and $c(s_1s_2) \notin c(\{y^a_{a_1}x^b_2, y^a_{a_2}x^b_3\})\big)$ or $\big(\{y^a_1 x^b_3, y^a_2 x^b_3\} \subseteq E(G)$ and $c(s_1s_2) \notin c(\{y^a_1 x^b_3, y^a_2 x^b_3\})\big)$ holds.

8. Both $x_1^b x_3^b \in E(G)$ and $c(x_1^b x_3^b) \neq c(s_1 s_2)$ hold.

9. For $i \in \{2,3\}$, it holds that $c(x_1^b x_3^b) = c(s_1 s_i)$, $\{y_1^a x_2^b, y_2^a x_2^b\} \subseteq E(G)$, and $c(s_1 s_2) \notin c(\{y_1^a x_2^b,\ y_2^a x_2^b\})$.

10. It holds that $c(s_1 s_3) = c(x_2^b x_3^b)$, $\{y_1^a x_2^b, y_2^a x_2^b\} \subseteq E(G)$, and $c(s_1 s_2) \notin c(\{y_1^a x_2^b,\ y_2^a x_2^b\})$.

11. There are $i \in ([t] - \{a\})$, $i_1 \in [2]$ and $i_2 \in \{2,3\}$ such that for $B_4 = \{y_{i_1}^a x_{i_2}^b, y_1^a s_1, y_2^a s_1, y_1^i s_2, y_2^i s_2\}$, it holds that $B_4 \subseteq E(G)$ and $c(x_1^b x_3^b) = c(s_2 s_2)$.

12. There are $i \in ([t] - \{a\})$ and $z \in \{s_1, x_2^b\}$ such that $c(x_1^b x_3^b) = c(y_1^i y_2^i)$, $\{z y_1^i, z y_2^i\} \subseteq E(G)$, and $c(s_1 s_2) \notin c(\{z y_1^i, z y_2^i\})$.

13. There are $i \in ([t] - \{a\})$, $j \in ([k-1] - \{b\})$, $i_1 \in [2]$, and $j_1 \in \{1,3\}$ such that $c(x_1^b x_3^b) = c(x_{j_1}^j x_2^j)$, $y_{i_1}^i x_{j_1}^j \in E(G)$, and $c(s_1 s_2) \neq c(y_{i_1}^i x_{j_1}^j)$.

14. There is a subset B_5 of $\{s_i x_j^b : i, j \in [3]\}$ such that $|B_5| = 6$, $B_5 \subseteq E(G)$, and $c(s_1 s_2) \notin c(B_5)$.

Let's consider a situation where none of the above conditions are true. In this case, we can observe that $|\mathcal{S}_{2.4}(1,2)| \geq 4t + 2(t-1)$, $|\mathcal{S}_{2.4}(3,4,5)| \geq 11(k-2)$, and $|\mathcal{S}_{2.4}(6,\ldots,14) \cup \{s_1 s_3, s_2 s_3\}| \geq 13$. From this, and since $k \geq 2$, we can conclude that

$$\begin{aligned}
|c(K_n)| \leq& \frac{1}{2}(3k+2t)(3k+2t-1) - \big(4t + 2(t-1) + 11(k-1) + 13\big) \\
=& \frac{1}{2}(3k+2t-3)(3k+2t-4) + 2 - (2k-3) \\
\leq& \frac{1}{2}(3k+2t-3)(3k+2t-4) + 1, \text{a contradiction.}
\end{aligned}$$

Scenario 2.5: Let $|c(K_n[S])| = 3$ and $|c(K_n[S]) - c(H)| = 1$. Additionally, for two distinct elements a and b in the set $[k-1]$, we have $c(P_3^a) \cap c(K_n[S]) \neq \emptyset$ and $c(P_3^b) \cap c(K_n[S]) \neq \emptyset$. We also assume that $c(s_1 s_2) \notin c(H)$, $c(x_1^a x_2^a) = c(s_1 s_3)$, and $c(x_1^b x_2^b) = c(s_2 s_3)$. Let $Z = \{s_3, x_1^a, x_1^b\}$. Holding one of the following statements true will enable us, either directly or by employing one of the Scenarios 2.3 and 2.4, to create a rainbow subgraph of K_n that is isomorphic to $kP_3 \cup tP_2$.

1. There are $i \in [t]$, $i_1 \in [2]$ and $z \in Z$ such that $y_{i_1}^i z \in E(G)$ and $c(s_1 s_2) \neq c(y_{i_1}^i z)$.

2. There is $i \in ([k-1] - \{a,b\})$ such that $|\{x_{i_1}^i s_j : i_1, j \in [3]\} \cap E(G)| \geq 7$.

3. There are $i \in ([k-1] - \{a,b\})$ and $j \in \{a,b\}$ such that $|\{x^i_{i_1}x^j_{j_1} : i_1, j_1 \in [3]\} \cap E(G)| \geq 7$.

4. For $j \in \{a,b\}$, there is a subset D_1 of $\{s_i x^j_{j_1} : i, j_1 \in [3]\}$ such that $|D_1| = 6$, $D_1 \subseteq E(G)$, and $c(s_1s_2) \notin c(D_1)$.

5. There is a subset D_2 of $\{x^a_{a_1}x^b_{b_1} : a_1, b_1 \in [3]\}$ such that $|D_2| = 6$, $D_2 \subseteq E(G)$, and $c(s_1s_2) \notin c(D_2)$.

6. For $j \in \{a,b\}$, both $x^j_1x^j_3 \in E(G)$ and $c(s_1s_2) \neq c(x^j_1x^j_3)$ occur.

7. There are $i \in [t]$ and $i_1 \in [2]$ such that either $\big(c(x^a_1x^a_3) = c(s_1s_3)$, $y^i_{i_1}x^a_2 \in E(G)$ and $c(y^i_{i_1}x^a_2) \neq c(s_1s_2)\big)$ or $\big(c(x^b_1x^b_3) = c(s_2s_3)$, $y^i_{i_1}x^b_2 \in E(G)$ and $c(y^i_{i_1}x^b_2) \neq c(s_1s_2)\big)$ holds.

8. For $i \in \{a,b\}$, there are $j \in [t]$ and $j_1 \in [2]$ such that $c(x^j_1x^j_3) = c(s_1s_2)$ and $y^i_{i_1}x^j_2 \in E(G)$.

9. There are $i \in [t]$, $i_1 \in [2]$, and $i_2 \in ([2] - \{i_1\})$ such that $c(x^a_1x^a_3) = c(s_2s_3)$, $c(x^b_1x^b_3) = c(s_1s_3)$, $\{y^i_{i_1}x^a_2, y^i_{i_2}x^b_2\} \subseteq E(G)$, and $c(s_1s_2) \notin c(\{y^i_{i_1}x^a_2, y^i_{i_2}x^b_2\})$.

10. For $i \in \{a,b\}$, there are $j \in [t]$, $h \in ([t] - \{j\})$, $h_1 \in [2]$, and $h_2 \in ([2] - \{i_1\})$ such that either $\big(c(x^i_1x^i_3) = c(y^j_1y^j_2)$, $\{y^j_1x^i_2, y^j_2x^i_2\} \subseteq E(G)$, and $c(s_1s_2) \notin c(\{y^j_1x^i_2, y^j_2x^i_2\})\big)$ or $\big(c(x^i_1x^i_3) = c(y^j_1y^j_2)$, $\{y^j_1y^h_{h_1}, y^j_2y^h_{h_2}\} \subseteq E(G)$, and $c(s_1s_2) \notin c(\{y^j_1y^h_{h_1}, y^j_2y^h_{h_2}\})\big)$ holds.

11. For $i \in \{a,b\}$, there are $j \in ([k-1] - \{a,b\})$, $j_1 \in \{1,3\}$, $h \in [t]$, and $h_1 \in [2]$ such that $c(x^i_1x^i_3) = c(x^j_{j_1}x^j_2)$, $y^h_{h_1}x^j_{j_1} \in E(G)$, and $c(s_1s_2) \neq c(y^h_{h_1}x^j_{j_1})$.

Assuming that none of the above qualifications is true, we find that $|\mathcal{S}_{2.5}(1)| \geq 6t$, $|\mathcal{S}_{2.5}(2,3)| \geq 9(k-3)$, and $|\mathcal{S}_{2.5}(4,\ldots,11) \cup \{s_1s_3, s_2s_3\}| \geq 20$. From this, we can deduce that

$$\begin{aligned}|c(K_n)| \leq& \frac{1}{2}(3k+2t)(3k+2t-1) - \big(6t + 9(k-3) + 20\big)\\ =& \frac{1}{2}(3k+2t-3)(3k+2t-4) + 1, \text{a contradiction.}\end{aligned}$$

Scenario 2.6: Let $c(s_1s_2) \notin c(H)$, and $c(s_1s_3) = c(s_2s_3) = c(x^a_1x^a_2)$, where $a \in [k-1]$. Additionally, we assume that $|c(K_n[V(P^a_3)])| = 3$. In this case, we first redefine the set S and the path P^a_3 as follows: let $S = \{x^a_i : i \in [3]\}$, $V(P^a_3) = \{s_i : i \in [3]\}$, and $E(P^a_3) = \{s_1s_2, s_2s_3\}$. After this redefinition, we can apply one of the scenarios from 2.1 to 2.5 to obtain the desired result.

Scenario 2.7: Let $c(s_1s_2) \notin c(H)$, and $c(s_1s_3) = c(s_2s_3) = c(x_1^a x_2^a)$, where $a \in [k-1]$. Plus, $c(s_3x_1^a) \notin (c(H) - \{c(x_1^a x_2^a)\})$ and $|c(K_n[V(P_3^a)])| = 2$. In this case, if one of the following holds, then we can construct a rainbow subgraph of K_n that is isomorphic to $kP_3 \cup tP_2$.

1. For $z \in \{s_3, x_1^a\}$, there are $i \in [t]$ and $j \in [2]$ such that $zy_j^i \in E(G)$ and $c(s_1s_2) \neq c(zy_j^i)$.
2. There are $i \in [t]$, $i_1 \in [2]$, $i_2 \in ([2] - \{i_1\})$, $z_1 \in \{s_1, s_2\}$, and $z_2 \in \{x_2^a, x_3^a\}$ such that $\{y_{i_1}^i z_1, y_{i_2}^i z_2\} \subseteq E(G)$ and $c(s_1s_2), c(s_3x_1^a) \notin c(\{y_{i_1}^i z_1, y_{i_2}^i z_2\})$.
3. There are $i \in [t]$, $i_1 \in [2]$, $j \in ([k-1] - \{a\})$, $j_1 \in \{1, 3\}$, $z \in \{s_3, x_1^a\}$, and $j_2 \in (\{1, 3\} - \{j_1\})$ such that $\{zx_{j_1}^j, x_{j_2}^j y_{i_1}^i\} \subseteq E(G)$ and $c(s_1s_2) \notin c(\{zx_{j_1}^j, x_{j_2}^j y_{i_1}^i\})$.
4. There are $i \in [t]$, $i_1 \in [2]$, $j \in ([k-1] - \{a\})$, $j_1 \in \{1, 3\}$, and $z \in \{s_3, x_1^a\}$ such that $\{zx_2^j, x_{j_1}^j y_{i_1}^i\} \subseteq E(G)$ and $c(s_1s_2) \notin c(\{zx_2^j, x_{j_1}^j y_{i_1}^i\})$.
5. There are $i \in [t]$, $i_1 \in [2]$, $j \in ([k-1] - \{a\})$, and $z \in \{s_3, x_1^a\}$ such that $\{zx_1^j, zx_3^j, x_2^j y_{i_1}^i\} \subseteq E(G)$ and $c(s_1s_2) \notin c(\{zx_1^j, zx_3^j, x_2^j y_{i_1}^i\})$.
6. For two distinct elements $j_1, j_2 \in [3]$, there are $i \in [t]$, $i_1 \in [2]$, $j \in ([k-1] - \{a\})$, $z_1 \in \{s_1, s_2\}$, $z_2 \in \{x_2^a, x_3^a\}$, and $j_3 \in ([3] - \{j_1, j_2\})$ such that $\{z_1x_{j_1}^j, z_2x_{j_2}^j, x_{j_3}^j y_{i_1}^i\} \subseteq E(G)$ and $c(s_1s_2) \notin c(\{z_1x_{j_1}^j, z_2x_{j_2}^j, x_{j_3}^j y_{i_1}^i\})$.
7. Both $x_1^a x_3^a \in E(G)$ and $c(s_1s_2) \neq c(x_1^a x_3^a)$ hold.
8. There is a subset F of $\{s_i x_{a_1}^a : i, a_1 \in [3]\}$ such that $|F| = 6$, $F \subseteq E(G)$, and $c(s_1s_2) \notin c(F)$.

Assuming none of the previously mentioned conditions is true, we conclude that $|\mathcal{S}_{2.7}(1,2)| \geq 8t$, $|\mathcal{S}_{2.7}(3, \ldots, 6)| \geq 12(k-2)$, and $|\mathcal{S}_{2.7}(7,8) \cup \{s_1s_3, s_2s_3\}| \geq 7$. From this, and $k, t \geq 2$, we can find that

$$
\begin{aligned}
|c(K_n)| \leq & \frac{1}{2}(3k+2t)(3k+2t-1) - (8t + 12(k-2) + 7) \\
= & \frac{1}{2}(3k+2t-3)(3k+2t-4) + 2 - (3k+2t-9) \\
\leq & \frac{1}{2}(3k+2t-3)(3k+2t-4) + 1, \text{a contradiction.}
\end{aligned}
$$

Scenario 2.8: Let $c(s_1s_2) \notin c(H)$, and $c(s_1s_3) = c(s_2s_3) = c(x_1^a x_2^a)$, where $a \in [k-1]$. Plus, $c(s_3x_1^a) = c(s_1s_2)$ and $|c(K_n[V(P_3^a)])| = 2$. In this case, if one of the following holds, then we can construct a rainbow subgraph of K_n that is isomorphic to $kP_3 \cup tP_2$.

1. There are $i \in [t]$, $j \in [2]$, and $z \in \{s_1, s_2, s_3, x_1^a\}$ such that $zy_j^i \in E(G)$ and $c(s_1s_2) \neq c(zy_j^i)$.

2. There are $i \in [t]$, $i_1 \in [2]$, $j \in ([k-1]-\{a\})$, $j_1 \in \{1,3\}$, $z \in \{s_3, x_1^a\}$, and $j_2 \in (\{1,3\}-\{j_1\})$ such that $\{zx_{j_1}^j, x_{j_2}^j y_{i_1}^i\} \subseteq E(G)$ and $c(s_1s_2) \notin c(\{zx_{j_1}^j, x_{j_2}^j y_{i_1}^i\})$.

3. There are $i \in [t]$, $i_1 \in [2]$, $j \in ([k-1]-\{a\})$, $j_1 \in \{1,3\}$, and $z \in \{s_3, x_1^a\}$ such that $\{zx_2^j, x_{j_1}^j y_{i_1}^i\} \subseteq E(G)$ and $c(s_1s_2) \notin c(\{zx_2^j, x_{j_1}^j y_{i_1}^i\})$.

4. There are $i \in [t]$, $i_1 \in [2]$, $j \in ([k-1]-\{a\})$, and $z \in \{s_3, x_1^a\}$ such that $\{zx_1^j, zx_3^j, x_2^j y_{i_1}^i\} \subseteq E(G)$ and $c(s_1s_2) \notin c(\{zx_1^j, zx_3^j, x_2^j y_{i_1}^i\})$.

5. There are $j \in ([k-1]-\{a\})$, $j_1 \in \{1,3\}$, and $j_2 \in ([3]-\{j_1\})$ such that $\{s_1x_{j_1}^j, s_2x_{j_1}^j, x_3^a x_{j_2}^j\} \subseteq E(G)$ and $c(s_1s_2) \notin c(\{s_1x_{j_1}^j, s_2x_{j_1}^j, x_3^a x_{j_2}^j\})$.

6. There are $j \in ([k-1]-\{a\})$ and $j_1 \in \{1,3\}$ such that $\{s_1x_2^j, s_2x_2^j, x_3^a x_{j_1}^j, x_1^j x_3^j\} \subseteq E(G)$ and $c(s_1s_2) \notin c(\{s_1x_2^j, s_2x_2^j, x_3^a x_{j_1}^j, x_1^j x_3^j\})$.

7. Both $x_1^a x_3^a \in E(G)$ and $c(s_1s_2) \neq c(x_1^a x_3^a)$ hold.

8. There are a subset F_1 of $\{s_i x_j^a : i \in [2], j \in [3]\} \cup \{s_3 x_j^a : j \in \{2,3\}\}$ such that $F_1 \subseteq E(G)$, $|F_1| = 6$, and $c(s_1s_2) \notin c(F_1)$.

Let's assume that none of the statements mentioned above are true. We can observe that $|\mathcal{S}_{2.8}(1)| \geq 8t$, $|\mathcal{S}_{2.8}(2,\ldots,6)| \geq 9(k-2)$, and $|\mathcal{S}_{2.8}(7,8) \cup \{s_1s_3, s_2s_3\}| \geq 7$. Given that $t \geq 2$, we can derive that

$$\begin{aligned}|c(K_n)| \leq& \frac{1}{2}(3k+2t)(3k+2t-1) - (8t + 9(k-2) + 7)\\ =& \frac{1}{2}(3k+2t-3)(3k+2t-4) + 2 - (2t-3)\\ \leq& \frac{1}{2}(3k+2t-3)(3k+2t-4) + 1, \text{a contradiction.}\end{aligned}$$

Scenario 2.9: Let $c(s_1s_2) \notin c(H)$, and $c(s_1s_3) = c(s_2s_3) = c(x_1^a x_2^a)$, where $a \in [k-1]$. Plus, $c(s_3x_1^a) = c(x_2^a x_3^a)$ and $|c(K_n[V(P_3^a)])| = 2$. In this scenario, possessing one of the following will enable us to construct a rainbow subgraph of K_n that is isomorphic to $kP_3 \cup tP_2$.

1. There are $i \in [t]$, $j \in [2]$, and $z \in \{s_3, x_1^a, x_2^a, x_3^a\}$ such that $zy_j^i \in E(G)$ and $c(s_1s_2) \neq c(zy_j^i)$.

2. There are $i \in [t]$, $i_1 \in [2]$, $j \in ([k-1]-\{a\})$, $j_1 \in \{1,3\}$, $z \in \{s_3, x_1^a\}$, and $j_2 \in (\{1,3\}-\{j_1\})$ such that $\{zx_{j_1}^j, x_{j_2}^j y_{i_1}^i\} \subseteq E(G)$ and $c(s_1s_2) \notin c(\{zx_{j_1}^j, x_{j_2}^j y_{i_1}^i\})$.

3. There are $i \in [t]$, $i_1 \in [2]$, $j \in ([k-1]-\{a\})$, $j_1 \in \{1,3\}$, and $z \in \{s_3, x_1^a\}$ such that $\{zx_2^j, x_{j_1}^j y_{i_1}^i\} \subseteq E(G)$ and $c(s_1s_2) \notin c(\{zx_2^j, x_{j_1}^j y_{i_1}^i\})$.

4. There are $i \in [t]$, $i_1 \in [2]$, $j \in ([k-1]-\{a\})$, and $z \in \{s_3, x_1^a\}$ such that $\{zx_1^j, zx_3^j, x_2^j y_{i_1}^i\} \subseteq E(G)$ and $c(s_1s_2) \notin c(\{zx_1^j, zx_3^j, x_2^j y_{i_1}^i\})$.

5. There are $j \in ([k-1]-\{a\})$, $j_1 \in \{1,3\}$, and $j_2 \in ([3]-\{j_1\})$ such that $\{x_2^a x_{j_1}^j, x_3^a x_{j_1}^j, s_1 x_{j_2}^j\} \subseteq E(G)$.

6. There are $j \in ([k-1]-\{a\})$ and $j_1 \in \{1,3\}$ such that $\{x_2^a x_2^j, x_3^a x_2^j, s_1 x_{j_1}^j, x_1^j x_3^j\} \subseteq E(G)$.

7. Both $x_1^a x_3^a \in E(G)$ and $c(s_1s_2) \neq c(x_1^a x_3^a)$ hold.

8. There are a subset F_2 of $\{s_i x_j^a : i \in [2], j \in [3]\} \cup \{s_3 x_j^a : j \in \{2,3\}\}$ such that $F_2 \subseteq E(G)$, $|F_2| = 6$, and $c(s_1s_2) \notin c(F_1)$.

Let's consider that none of the conditions mentioned above are true. We can see that $|\mathcal{S}_{2.9}(1)| \geq 8t$, $|\mathcal{S}_{2.9}(2,\ldots,6)| \geq 9(k-2)$, and $|\mathcal{S}_{2.9}(7,8) \cup \{s_1s_3, s_2s_3\}| \geq 7$. From this, and $t \geq 2$, we can find that

$$\begin{aligned}|c(K_n)| \leq &\frac{1}{2}(3k+2t)(3k+2t-1) - (8t + 9(k-2) + 7)\\ =&\frac{1}{2}(3k+2t-3)(3k+2t-4) + 2 - (2t-3)\\ \leq&\frac{1}{2}(3k+2t-3)(3k+2t-4) + 1, \text{a contradiction.}\end{aligned}$$

Scenario 2.10: Let $c(s_1s_2) \notin c(H)$ and $c(s_1s_3) = c(s_2s_3) = c(x_1^a x_2^a)$, where $a \in [k-1]$. Additionally, $c(s_3x_1^a) = c(y_1^l y_2^l)$, where $l \in [t]$. In this case, if any of the following conditions hold, we can either construct a rainbow subgraph of K_n that is isomorphic to $kP_3 \cup tP_2$ or redefine the set S and the graph H. We can then apply one of the scenarios from sections 2.6 to 2.9 to achieve the desired outcome.

1. There are $i \in [t]$, $j \in [2]$, and $z \in \{s_3, x_1^a\}$ such that $zy_j^i \in E(G)$ and $c(s_1s_2) \neq c(zy_j^i)$.

2. There is $i \in ([t]-\{l\})$, $i_1 \in [2]$, and $i_2 \in ([2]-\{i_1\})$ such that $\{y_{i_1}^i y_1^l, y_{i_2}^i y_2^l\} \subseteq E(G)$ and $c(s_1s_2) \notin c(\{y_{i_1}^i y_1^l, y_{i_2}^i y_2^l\})$.

3. There are $i \in [t]$, $i_1 \in [2]$, $j \in ([k-1]-\{a\})$, $j_1 \in \{1,3\}$, $z \in \{s_3, x_1^a\}$, and $j_2 \in (\{1,3\}-\{j_1\})$ such that $\{zx_{j_1}^j, x_{j_2}^j y_{i_1}^i\} \subseteq E(G)$ and $c(s_1s_2) \notin c(\{zx_{j_1}^j, x_{j_2}^j y_{i_1}^i\})$.

4. There are $i \in [t]$, $i_1 \in [2]$, $j \in ([k-1] - \{a\})$, $j_1 \in \{1,3\}$, and $z \in \{s_3, x_1^a\}$ such that $\{zx_2^j, x_{j_1}^j y_{i_1}^i\} \subseteq E(G)$ and $c(s_1s_2) \notin c(\{zx_2^j, x_{j_1}^j y_{i_1}^i\})$.
5. There are $i \in [t]$, $i_1 \in [2]$, $j \in ([k-1] - \{a\})$, and $z \in \{s_3, x_1^a\}$ such that $\{zx_1^j, zx_3^j, x_2^j y_{i_1}^i\} \subseteq E(G)$ and $c(s_1s_2) \notin c(\{zx_1^j, zx_3^j, x_2^j y_{i_1}^i\})$.
6. There are $j \in ([k-1] - \{a\})$, $j_1 \in [3]$, and $j_2 \in ([3] - \{j_1\})$ such that $\{y_1^l x_{j_1}^j, y_2^l x_{j_2}^j\} \subseteq E(G)$ and $c(s_1s_2) \notin c(\{y_1^l x_{j_1}^j, y_2^l x_{j_2}^j\})$.
7. There are $j \in ([k-1] - \{a\})$ and $j_1 \in \{1,3\}$ such that $\{y_1^l x_{j_1}^j, y_2^l x_{j_1}^j\} \subseteq E(G)$ and $c(s_1s_2) \notin c(\{y_1^l x_{j_1}^j, y_2^l x_{j_1}^j\})$.
8. There are $i \in [t]$ and $j \in [2]$ such that $\{s_3x_2^a, x_3^a y_j^i\} \subseteq E(G)$ and $c(s_1s_2) \notin c(\{s_3x_2^a, x_3^a y_j^i\})$.
9. There are $i \in [2]$ such that $\{s_3x_2^a, s_i x_3^a\} \subseteq E(G)$ and $c(s_1s_2) \notin c(\{s_3x_2^a, s_i x_3^a\})$.
10. There are $i \in [t]$, $i_1, j \in [2]$, and $i_2 \in ([2] - \{i_1\})$ such that either $\big(\{s_3x_3^a, s_j y_{i_1}^i, x_2^a y_{i_2}^i\} \subseteq E(G)$ and $c(s_1s_2) \notin c(\{s_3x_3^a, s_j y_{i_1}^i, x_2^a y_{i_2}^i\})\big)$, $\big(\{s_3x_3^a, s_j y_{i_1}^i, x_3^a y_{i_2}^i\} \subseteq E(G)$ and $c(s_1s_2) \notin c(\{s_3x_3^a, s_j y_{i_1}^i, x_3^a y_{i_2}^i\})\big)$, or $\big(\{s_3x_3^a, x_2^a y_{i_1}^i, x_3^a y_{i_2}^i\} \subseteq E(G)$ and $c(s_1s_2) \notin c(\{s_3x_3^a, x_2^a y_{i_1}^i, x_3^a y_{i_2}^i\})\big)$ holds.
11. There are $i \in [t]$, $i_1, j_1 \in [2]$, $i_2 \in ([2] - \{i_1\})$, and $j_2 \in ([3] - \{1\})$ such that either $\{x_1^a s_{j_1}, y_{i_1}^i s_1, y_{i_2}^i x_{j_2}^a\}$, $\{x_1^a s_{j_1}, y_{i_1}^i s_2, y_{i_2}^i x_{j_2}^a\}$, or $\{x_1^a s_{j_1}, y_{i_1}^i s_1, y_{i_2}^i s_2\}$ holds.
12. There are $i_1 \in [2]$, $i_2 \in ([2] - \{i_1\})$, $j_1 \in [2]$, and $j_2 \in \{2,3\}$ such that $\{y_{i_1}^l s_{j_1}, y_{i_2}^l x_{j_2}^a\} \subseteq E(G)$ and $c(s_1s_2) \notin c(\{y_{i_1}^l s_{j_1}, y_{i_2}^l x_{j_2}^a\})$.
13. There are $i \in ([t] - \{l\})$, $i_1 \in [2]$, $i_2 \in ([2] - \{i_1\})$, and $z \in \{s_1, s_2, x_2^a, x_3^a\}$ such that $\{y_1^l y_{i_1}^i, y_2^l y_{i_1}^i, y_{i_2}^i z\} \subseteq E(G)$ and $c(s_1s_2) \notin c(\{y_1^l y_{i_1}^i, y_2^l y_{i_1}^i, y_{i_2}^i z\})$.
14. There are $i \in ([t] - \{l\})$, $i_1, l_1 \in [2]$, $l_2 \in ([2] - \{l_1\})$, and $z \in \{s_1, s_2, x_2^a, x_3^a\}$ such that $\{y_{l_1}^l y_{i_1}^i, y_{l_2}^l z\} \subseteq E(G)$ and $c(s_1s_2) \notin c(\{y_{l_1}^l y_{i_1}^i, y_{l_2}^l z\})$.

Assuming that none of the above conditions are true, we calculate that $|\mathcal{S}_{2.10}(1,2)| \geq 4t + 2(t-1)$, $|\mathcal{S}_{2.10}(3,\ldots,7)| \geq 9(k-2)$, and $|\mathcal{S}_{2.10}(8,\ldots,14) \cup \{s_1s_3, s_2s_3\}| \geq 13$. From this, we can conclude that

$$\begin{aligned} |c(K_n)| \leq& \frac{1}{2}(3k+2t)(3k+2t-1) - (4t + 2(t-1) + 9(k-2) + 13) \\ =& \frac{1}{2}(3k+2t-3)(3k+2t-4) + 1, \text{a contradiction.} \end{aligned}$$

Scenario 2.11: Let $c(s_1s_2) \notin c(H)$, and let $c(s_1s_3) = c(s_2s_3) = c(x_1^a x_2^a)$, where $a \in [k-1]$. Additionally, we have $c(s_3x_1^a) = c(x_2^r x_3^r)$, where $r \in ([k-1] - \{a\})$. Under these conditions, if we hold one of the following assumptions, we can either construct a rainbow subgraph of K_n that is isomorphic to $kP_3 \cup tP_2$ or redefine the set S and the graph H. We can then apply one of the scenarios from Sections 2.6 to 2.10 to achieve the desired result.

1. There are $i \in [t]$, $j \in [2]$, and $z \in \{s_3, x_1^a, x_1^r, x_3^r\}$ such that $zy_j^i \in E(G)$ and $c(s_1s_2) \neq c(zy_j^i)$.
2. There are $i \in [t]$, $i_1 \in [2]$, $j \in ([k-1]-\{a,r\})$, $j_1 \in \{1,3\}$, $z \in \{s_3, x_1^a, x_1^r, x_3^r\}$, and $j_2 \in (\{1,3\} - \{j_1\})$ such that $\{zx_{j_1}^j, x_{j_2}^j y_{i_1}^i\} \subseteq E(G)$ and $c(s_1s_2) \notin c(\{zx_{j_1}^j, x_{j_2}^j y_{i_1}^i\})$.
3. There are $i \in [t]$, $i_1 \in [2]$, $j \in ([k-1] - \{a,r\})$, $j_1 \in \{1,3\}$, and $z \in \{s_3, x_1^a, x_1^r, x_3^r\}$ such that $\{zx_2^j, x_{j_1}^j y_{i_1}^i\} \subseteq E(G)$ and $c(s_1s_2) \notin c(\{zx_2^j, x_{j_1}^j y_{i_1}^i\})$.
4. There are $i \in [t]$, $i_1 \in [2]$, $j \in ([k-1] - \{a,r\})$, and $z \in \{s_3, x_1^a, x_1^r, x_3^r\}$ such that $\{zx_1^j, zx_3^j, x_2^j y_{i_1}^i\} \subseteq E(G)$ and $c(s_1s_2) \notin c(\{zx_1^j, zx_3^j, x_2^j y_{i_1}^i\})$.
5. There is $i \in \{a,r\}$ such that $|\{x_{i_1}^i s_j : i_1, j \in [3]\} \cap E(G)| \geq 7$.
6. There is a subset A_1 of $\{x_{a_1}^a x_{r_1}^r : a_1, r_1 \in [3]\}$ such that $A_1 \subseteq E(G)$, $|A_1| = 7$, and $c(s_1s_2) \notin c(A_1)$.
7. There are $i \in [t]$ and $j \in [2]$ such that $\{s_3x_2^a, x_3^a y_j^i\} \subseteq E(G)$ and $c(s_1s_2) \notin c(\{s_3x_2^a, x_3^a y_j^i\})$.
8. There is $i \in [2]$ such that $\{s_3x_2^a, s_ix_3^a\} \subseteq E(G)$ and $c(s_1s_2) \notin c(\{s_3x_2^a, s_ix_3^a\})$.
9. There are $i \in [t]$, $i_1 \in [2]$, $i_2 \in ([2] - \{i_1\})$, and $j \in [2]$ such that either $\big(\{s_3x_3^a, s_jy_{i_1}^i, x_2^a y_{i_2}^i\} \subseteq E(G)$ and $c(s_1s_2) \notin c(\{s_3x_3^a, s_jy_{i_1}^i, x_2^a y_{i_2}^i\})\big)$, $\big(\{s_3x_3^a, s_jy_{i_1}^i, x_3^a y_{i_2}^i\} \subseteq E(G)$ and $c(s_1s_2) \notin c(\{s_3x_3^a, s_jy_{i_1}^i, x_3^a y_{i_2}^i\})\big)$, or $\big(\{s_3x_3^a, x_2^a y_{i_1}^i, x_3^a y_{i_2}^i\} \subseteq E(G)$ and $c(s_1s_2) \notin c(\{s_3x_3^a, x_2^a y_{i_1}^i, x_3^a y_{i_2}^i\})\big)$ holds.
10. There are $i \in [t]$, $i_1 \in [2]$, $i_2 \in ([2] - \{i_1\})$, $j_1 \in [2]$, and $j_2 \in ([3] - \{1\})$ such that either $\{x_1^a s_{j_1}, y_{i_1}^i s_1, y_{i_2}^i x_{j_2}^a\}$, $\{x_1^a s_{j_1}, y_{i_1}^i s_2, y_{i_2}^i x_{j_2}^a\}$, or $\{x_1^a s_{j_1}, y_{i_1}^i s_1, y_{i_2}^i s_2\}$ is a subset of $E(G)$.
11. There are $i \in [t]$, $i_1 \in [2]$, $i_2 \in ([2] - \{i_1\})$, $j_1 \in \{1,3\}$, and $j_2 \in [2]$ such that $\{s_3x_{j_1}^r, s_{j_2}y_{i_1}^i, x_2^r y_{i_2}^i\} \subseteq E(G)$ and $c(s_1s_2) \notin c(\{s_3x_{j_1}^r, s_{j_2}y_{i_1}^i, x_2^r y_{i_2}^i\})$.

12. There are $i \in [t]$, $i_1 \in [2]$, and $j \in [3]$ such that $\{s_jx_1^r, s_jx_3^r, y_{i_1}^i x_2^r\} \subseteq E(G)$ and $c(s_1s_2) \notin c(\{s_jx_1^r, s_jx_3^r, y_{i_1}^i x_2^r\})$.

13. There are $i \in [3]$ and $j \in \{2,3\}$ such that $\{s_ix_1^r, s_ix_3^r, x_j^a x_2^r\} \subseteq E(G)$ and $c(s_1s_2) \notin c(\{s_ix_1^r, s_ix_3^r, x_j^a x_2^r\})$.

14. There is $i \in \{a, r\}$ such that $x_1^i x_3^i \in E(G)$ and $c(s_1s_2) \neq c(x_1^i x_3^i)$.

Let's assume that none of the above statements are true. We can observe that $|\mathcal{S}_{2.11}(1)| \geq 8t$, $|\mathcal{S}_{2.11}(2,3,4)| \geq 12(k-3)$, and $|\mathcal{S}_{2.11}(5,\ldots,14) \cup \{s_1s_3, s_2s_3\}| \geq 16$. Since $t \geq 2$ and $k \geq 3$, it follows that

$$\begin{aligned}|c(K_n)| \leq& \frac{1}{2}(3k+2t)(3k+2t-1) - (8t + 9(k-3) + 16)\\ =& \frac{1}{2}(3k+2t-3)(3k+2t-4) + 2 - (3k+2t-12)\\ \leq& \frac{1}{2}(3k+2t-3)(3k+2t-4) + 1, \text{a contradiction.}\end{aligned}$$

Scenario 2.12: Let $c_1 = c(s_1s_2) = c(s_1s_3) \notin c(H)$, and $c(s_2s_3) = c(y_1^l y_2^l)$, where $l \in [t]$. In this case, if one of the following holds, then we can construct a rainbow subgraph of K_n that is isomorphic to $kP_3 \cup tP_2$.

1. There are $i \in [t]$, $i_1 \in [2]$, and $j \in \{2,3\}$ such that $s_jy_{i_1}^i \in E(G)$ and $c(s_jy_{i_1}^i) \neq c_1$.

2. There are $i \in ([t]-\{l\})$, $i_1 \in [2]$, and $i_2 \in ([2]-\{i_1\})$ such that $\{y_{i_1}^i y_1^l, y_{i_2}^i y_2^l\} \subseteq E(G)$ and $c_1 \notin c(\{y_{i_1}^i y_1^l, y_{i_2}^i y_2^l\})$.

3. For $i \in [k-1]$, there is a subset B_1 of $\{s_{i_1}x_{i_2}^i : i_1, i_2 \in [3]\}$ such that $B_1 \subseteq E(G)$, $|B_1| = 4$, and $c_1 \notin c(B_1)$.

4. For $i \in [k-1]$, there is a subset B_2 of $\{y_{l_1}^l x_{i_1}^i : l_1 \in [2], i_1 \in [3]\}$ such that $B_2 \subseteq E(G)$, $|B_2| = 4$, and $c_1 \notin c(B_2)$.

5. There are $i \in ([t]-\{l\})$, $l_1, i_1 \in [2]$, and $l_2 \in ([2]-\{l_1\})$ such that $\{s_1y_{l_1}^l, y_{l_2}^l y_{i_1}^i\} \subseteq E(G)$.

6. There are $i \in ([t]-\{l\})$ and $l_1 \in [2]$ such that $\{s_1y_{l_1}^l, s_1y_1^i, s_1y_2^i, y_{l_1}^l y_1^i, y_{l_1}^l y_2^i\} \subseteq E(G)$.

7. It holds that $\{s_1y_1^l, s_1y_2^l\} \subseteq E(G)$.

If none of the above conditions are true, we can see that $|\mathcal{S}_{2.12}(1,2)| \geq 4t+2(t-1)$, $|\mathcal{S}_{2.12}(3,4)| \geq 9(k-1)$, and $|\mathcal{S}_{2.12}(5,\ldots,14) \cup \{s_2s_3\}| \geq 4$. Thus, we have

$$\begin{aligned}|c(K_n)| \leq & \frac{1}{2}(3k+2t)(3k+2t-1) - (4t+2(t-1)+9(k-1)+4)\\ = & \frac{1}{2}(3k+2t-3)(3k+2t-4)+1, \text{a contradiction.}\end{aligned}$$

Scenario 2.13: Let $c_1 = c(s_1s_2) = c(s_1s_3) \notin c(H)$, and $c(s_2s_3) = c(x_1^r x_2^r)$, where $r \in [k-1]$. In this case, if one of the following holds, then we can either construct a rainbow subgraph of K_n that is isomorphic to $kP_3 \cup tP_2$ or redefine the set S and the graph H. We can then apply Section 2.12 to achieve the desired result.

1. There are $i \in [t]$, $i_1 \in [2]$, and $z \in \{s_2, s_3, x_1^r, x_3^r\}$ such that $zy_{i_1}^i \in E(G)$ and $c_1 \neq c(zy_{i_1}^i)$.
2. For $i \in ([k-1]-\{r\})$, there is a subset F_1 of $\{s_{i_1}x_{i_2}^i : i_1, i_2 \in [3]\}$ such that $F_1 \subseteq E(G)$, $|F_1| = 4$, and $c_1 \notin c(F_1)$.
3. For $i \in ([k-1]-\{r\})$, there is a subset F_2 of $\{x_{r_1}^r x_{i_1}^i : r_1, i_1 \in [3]\}$ such that $F_2 \subseteq E(G)$, $|F_2| = 7$, and $c_1 \notin c(F_2)$.
4. There is a subset F_3 of $\{s_{i_1}x_{r_1}^r : i_1, r_1 \in [3]\}$ such that $F_3 \subseteq E(G)$, $|F_3| = 6$, and $c_1 \notin c(F_3)$.
5. Both $x_1^r x_3^r \in E(G)$ and $c_1 \neq c(x_1^r x_3^r)$ happen.

Assuming that none of the previous statements are true, we calculate that $|\mathcal{S}_{2.13}(1)| \geq 8t$, $|\mathcal{S}_{2.13}(2,3)| \geq 9(k-2)$, and $|\mathcal{S}_{2.13}(4,5) \cup \{s_2s_3\}| \geq 7$. Given that $t \geq 2$, we derive that

$$\begin{aligned}|c(K_n)| \leq & \frac{1}{2}(3k+2t)(3k+2t-1) - (8t+9(k-2)+7)\\ = & \frac{1}{2}(3k+2t-3)(3k+2t-4)+2-(2t-3)\\ \leq & \frac{1}{2}(3k+2t-3)(3k+2t-4)+1, \text{a contradiction.}\end{aligned}$$

Scenario 2.14: Let $c_1 = c(s_1s_2) = c(s_1s_3) = c(s_2s_3) \notin c(H)$. In this case, if any of the following conditions occur, then we can construct a rainbow subgraph of K_n that is isomorphic to $kP_3 \cup tP_2$.

1. There are $i \in [t]$, $i_1 \in [2]$, and $j \in [3]$ such that $y_{i_1}^i s_j \in E(G)$ and $c_1 \neq c(y_{i_1}^i s_j)$.

2. For $i \in [k-1]$, there is a subset A_1 of $\{s_{i_1}x^i_{i_2} : i_1, i_2 \in [3]\} \cup \{x^i_1 x^i_3\}$ such that $A_1 \subseteq E(G)$ and $|A_1| = 2$, and $c_1 \notin c(A_1)$.

If neither of the two conditions above holds, we can see that $|\mathcal{S}_{2.14}(1)| \geq 6t$ and $|\mathcal{S}_{2.14}(2)| \geq 9(k-1)$. Therefore, since $|E(G) \cap \{s_1s_2, s_1s_3, s_2s_3\}| \leq 1$, it follows that

$$\begin{aligned}|c(K_n)| \leq& \frac{1}{2}(3k+2t)(3k+2t-1) - (6t+9(k-1)+2)\\ =& \frac{1}{2}(3k+2t-3)(3k+2t-4)+1, \text{a contradiction.}\end{aligned}$$

Scenario 2.15: Let $c(K_n[S]) \subseteq c(H)$. Moreover, suppose that $c(x^a_1x^a_2) \in c(K_n[S])$, where $a \in [k-1]$. Under these conditions, if any of the following statements are true, we can either construct a rainbow subgraph of K_n that is isomorphic to $kP_3 \cup tP_2$, or we can apply one of the scenarios from 2.1 to 2.14 to achieve the desired result.

1. There are $i \in [t]$, $j_1 \in [3]$, and $j_2 \in ([3]-\{j_1\})$ such that $\{y^i_1 s_{j_1}, y^i_2 s_{j_2}\} \subseteq E(G)$.
2. There are $i \in [t]$ and $i_1 \in [2]$ such that $|\{y^i_{i_1} s_i : i \in [3]\} \cap E(G)| = 3$.
3. There are $i \in [t]$ and $i_1 \in [2]$ such that $x^a_1 y^i_{i_1} \in E(G)$.
4. For each $i \in [k-1]$, there are two distinct elements $i_1, i_2 \in [3]$, as well as two other distinct elements $j_1, j_2 \in [3]$, not necessarily different from i_1 and i_2, such that $\{x^i_{i_1} s_{j_1}, x^i_{i_2} s_{j_2}\} \subseteq E(G)$.
5. There is $i \in ([k-1]-\{a\})$ such that $|\{x^i_{i_1} x^a_{a_1} : i_1, a_1 \in [3]\} \cap E(G)| \geq 7$.
6. There is $i \in [3]$ such that $|\{x^a_i s_j : j \in [3]\} \cap E(G)| = 3$.
7. It holds that $x^a_1 x^a_3 \in E(G)$.

Let's assume that none of the previous statements are true. Thus, we can conclude that $|\mathcal{S}_{2.15}(1,2,3)| \geq 6t$, $|\mathcal{S}_{2.15}(4,5)| \geq 6(k-1)+3(k-2)$, and $|\mathcal{S}_{2.15}(6,7) \cup \{s_1s_2, s_1s_3, s_2s_3\}| \geq 5$. Therefore, we have that

$$\begin{aligned}|c(K_n)| \leq& \frac{1}{2}(3k+2t)(3k+2t-1) - (6t+6(k-1)+3(k-2)+5)\\ =& \frac{1}{2}(3k+2t-3)(3k+2t-4)+1, \text{a contradiction.}\end{aligned}$$

Scenario 2.16: Let $c(K_n[S]) \subseteq c(H)$. Moreover, suppose that $c(y^b_1y^b_2) \in c(K_n[S])$, where $b \in [t]$. Under these conditions, if any of the following statements are true, we can either construct a rainbow subgraph of K_n that is isomorphic to $kP_3 \cup tP_2$, or we can apply one of the scenarios from 2.1 to 2.14 to achieve the desired result.

1. There are $i \in [t]$, $j_1 \in [3]$, and $j_2 \in ([3]-\{j_1\})$ such that $\{y_1^i s_{j_1}, y_2^i s_{j_2}\} \subseteq E(G)$.
2. There are $i \in ([t]-\{b\})$, $i_1 \in [2]$, and $i_2 \in ([2]-\{i_1\})$ such that $\{y_{i_1}^i y_1^b, y_{i_2}^i y_2^b\} \subseteq E(G)$.
3. There are $i \in ([t]-\{b\})$ and $i_1, i_2 \in [2]$ such that $\{y_{i_1}^b y_1^i, y_{i_1}^b y_2^i\} \subseteq E(G)$ and $|\{y_{i_2}^i s_i : i \in [3]\} \cap E(G)| = 3$.
4. There are $i \in ([t]-\{b\})$, $j \in [k-1]$, $i_1 \in [2]$, $i_2 \in ([2]-\{i_1\})$, and $j_1 \in \{1,3\}$ such that $\{y_1^b y_{i_1}^i, y_2^b y_{i_1}^i\} \subseteq E(G)$ and $y_{i_2}^i x_{j_1}^j \in E(G)$.
5. For each $i \in [k-1]$, there are two distinct elements $i_1, i_2 \in [3]$, as well as two other distinct elements $j_1, j_2 \in [3]$, not necessarily different from i_1 and i_2, such that $\{x_{i_1}^i s_{j_1}, x_{i_2}^i s_{j_2}\} \subseteq E(G)$.
6. There are $i \in [k-1]$, $i_1 \in [3]$, and $i_2 \in ([3]-\{i_1\})$ such that $\{y_1^b x_{i_1}^i, y_2^b x_{i_2}^i\} \subseteq E(G)$.
7. There is $i \in [2]$ such that $|\{y_i^b s_j : j \in [3]\} \cap E(G)| = 3$.
8. There are $i \in [k-1]$, $j \in [3]$, and $j_1 \in [2]$ such that $|\{y_{j_1}^b s_h : h \in [3]\} \cap E(G)| = 2$ and $|\{s_j x_{i_1}^i : i_1 \in [3]\} \cap E(G)| = 3$.

Assuming that none of the above conditions are true, we can observe that $|\mathcal{S}_{2.16}(1,2,3,4)| \geq 3t + 3(t-1)$, $|\mathcal{S}_{2.16}(5,6)| \geq 9(k-1)$, and $|\mathcal{S}_{2.16}(7,8) \cup \{s_1s_2, s_1s_3, s_2s_3\}| \geq 5$. Thus, we conclude that

$$\begin{aligned}|c(K_n)| \leq& \frac{1}{2}(3k+2t)(3k+2t-1) - (3t+3(t-1)+9(k-1)+5)\\ =& \frac{1}{2}(3k+2t-3)(3k+2t-4)+1, \text{a contradiction.}\end{aligned}$$

Based on Scenarios 2.1 to 2.16, we can conclude that if $c(K_n) = \frac{1}{2}(3k+2t-3)(3k+2t-4)+2$, then the graph K_n contains a rainbow subgraph that is isomorphic to $kP_3 \cup tP_2$. Therefore, we have $AR(n, kP_3 \cup tP_2) \leq \frac{1}{2}(3k+2t-3)(3k+2t-4)+1$. This conclusion, along with Case 1, completes the proof of Theorem 1.1.

3 Concluding remarks

In this paper, we establish the anti-Ramsey number for the linear forest $kP_3 \cup tP_2$ in the critical case where the order of the host graph n is precisely equal to the size of the forest, that is, $n = 3k + 2t$. For all $k \geq 1$ and $t \geq 2$, we demonstrate that

$$AR(n, kP_3 \cup tP_2) = \frac{1}{2}(3k+2t-3)(3k+2t-4)+1.$$

Our result significantly extends prior work by Jie et al. [10], which required the parameter t to be quadratically large in relation to k, specifically $t \geq \frac{k^2-k+4}{2}$ and $n \geq 3k+2t+1$. By removing this condition for the case $n = 3k+2t$, our proof is more general and completes the characterization for this specific graph size. However, a gap in the literature remains for host graphs larger than the forest itself ($n \geq 3k + 2t + 1$) when t is relatively small. Determining $AR(n, kP_3 \cup tP_2)$ under these constraints continues to be an intriguing open problem.

Acknowledgments

This research was supported by the NSFC under the grant 12131013.

Conflicts of interest

The authors declare no conflict of interest.

Data availability

No data was used in this investigation.

References

[1] A. Bialostocki, S. Gilboa, Y. Roditty, Anti-Ramsey numbers of small graphs, Ars Combin. 123(2015), 41-53.

[2] H. Chen, X.L. Li, J.H. Tu, Complete solution for the rainbow number of matchings, Discrete Math. 309(10)(2009), 3370-3380.

[3] P. Erdős, M. Simonovits, V.T. Sós, Anti-Ramsey theorems, Colloq. Math. Soc. János Bolyai, Vol. 10 North-Holland Publishing Co., Amsterdam-London, 1975, pp. 633- 643.

[4] C.Q. Fang, E. Győri, M. Lu, J.M. Xiao, On the anti-Ramsey number of forests, Discrete Appl. Math. 291(11)(2021), 129-142.

[5] S. Fujita, A. Kaneko, I. Schiermeyer, K. Suzuki, A rainbow k-matching in the complete graph with r colors, Electron. J. Combin. 16(2009), #R51.

[6] S. Gilboa, Y. Roditty, Anti-Ramsey numbers of graphs with small connected components, Graphs Combin. 32(2)(2016), 649-662.

[7] R. Haas, M. Young, The anti-Ramsey number of perfect matching, Discrete Math. 312(2012)933-937.

[8] M.L. He, Z.M. Jin, Rainbow short linear forests in edge-colored complete graph, Discrete Appl. Math. 361(2025), 523-536.

[9] S. Jahanbekam, D.B. West, Anti-Ramsey problems for t edge-disjoint rainbow spanning subgraphs: cycles, matchings, or trees, J. Graph Theory 82(1)(2016), 75-89.

[10] Q. Jie, M.L. He, Z.M. Jin, Rainbow forest consisting of short paths in Kn, Discrete Appl. Math. 376(2025), 260-269.

[11] J.J. Montellano-Ballesteros, V. Neumann-Lara, An anti-Ramsey theorem on cycles, Graphs Combin. 21 (3) (2005), 343-354.

[12] I. Schiermeyer, Rainbow numbers for matchings and complete graphs, Discrete Math. 286 (2004) 157-162.

[13] M. Simonovits, V.T. Sós, On restricted coloring of Kn, Combinatorica 4 (1) (1984) 101–110.

[14] T.Y. Xie, L.T. Yuan, On the anti-Ramsey numbers of linear forests, Discrete Math. 343(12)(2020), 112130.

[15] T.Y. Xie, L.T. Yuan, On the anti-Ramsey numbers of linear forests, Discrete Math. 343(12)(2020), 112130.