Stratifying Discriminant Hypersurface

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Abstract

This paper investigates the stratification of the discriminant hypersurface associated with a univariate polynomial via the number of its distinct complex roots. We introduce two novel approaches different from the one based on subdiscriminants. The first approach stratifies the discriminant hypersurface by recursively removing all the *lowest-order* points, while the second one stratifies the discriminant hypersurface by recursively removing all the *smooth* points. Both approaches rely solely on the discriminant itself instead of using high-order subdiscriminants. These results offer new insights into the intrinsic geometry of the discriminant and its connection to root multiplicity.

1 Introduction

Discriminant of a univariate polynomial is a fundamental object in computational algebraic geometry. It plays an essential role on understanding the root structure of a given polynomial. In the past centuries, people proposed various versions of discriminants, extended to subdiscriminants, and developed rich theories on them (to list a few, [31, 32, 33, 9, 14, 23, 5, 4, 6, 13, 12, 37, 35, 20, 21, 28]). Then they applied the theories to tackle various fundamental problems in computational algebraic geometry (see, e.g., [34, 36, 38, 24, 26]), such as finding condition on the coefficients of the univariate polynomial so that the polynomial has a specified number of distinct complex/real roots, or has a specified multiplicities, and so on (see, e.g., [31, 37, 15, 20, 21]).

We revisit the following classical problem: *Stratify* the discriminant hypersurface associated with a univariate polynomial via the number of its distinct complex roots. It is well known that such stratification can be obtained via subdiscriminants. In this paper, we introduce two novel approaches different from the one based on subdiscriminants.

To illustrate the stratification problem and our contributions, consider the monic quartic polynomial $F = x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$ where a_i's are indeterminates/parameters. Its discriminant D is given by

$$\begin{aligned} D = {} & 256a_0^3 - 27a_1^4 - 128a_0^2a_2^2 - 192a_0^2a_1a_3 + 144a_0a_1^2a_2 \\ & - 4a_1^2a_2^3 + 18a_1^3a_2a_3 - 6a_0a_1^2a_3^2 \end{aligned}$$

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$$
\begin{aligned}
&+16a_0a_2^4+144a_0^2a_2a_3^2-80a_0a_1a_2^2a_3+18a_0a_1a_2a_3^3\\
&-4a_1^3a_3^3-4a_0a_2^3a_3^2-27a_0^2a_3^4+a_1^2a_2^2a_3^2.
\end{aligned}
$$

Define

$$Z_k=\left\{(a_0,a_1,a_2,a_3)\in\mathbb{C}^4 : x^4+a_3x^3+a_2x^2+a_1x+a_0=0 \text{ has at most } k \text{ distinct roots}\right\}. \tag{1}$$

It is obvious that $Z_4=\mathbb{C}^4$ and $Z_0=\varnothing$. Also, it is well-known that Z_k's are algebraic sets defined by the subdiscriminants. Moreover, the zero locus of D is the hypersurface Z_3 in the parameter space. It is easy to construct monic quartics having 1, 2, 3, 4 distinct roots respectively:

$$
\begin{array}{lcll}
x^4-4x^3+6x^2-4x+1 & = & (x-1)^4 & \in Z_1\backslash Z_0\\
x^4-5x^3+9x^2-7x+2 & = & (x-1)^3(x-2) & \in Z_2\backslash Z_1\\
x^4-7x^3+17x^2-17x+6 & = & (x-1)^2(x-2)(x-3) & \in Z_3\backslash Z_2\\
x^4-10x^3+35x^2-50x+24 & = & (x-1)(x-2)(x-3)(x-4) & \in Z_4\backslash Z_3
\end{array}
$$

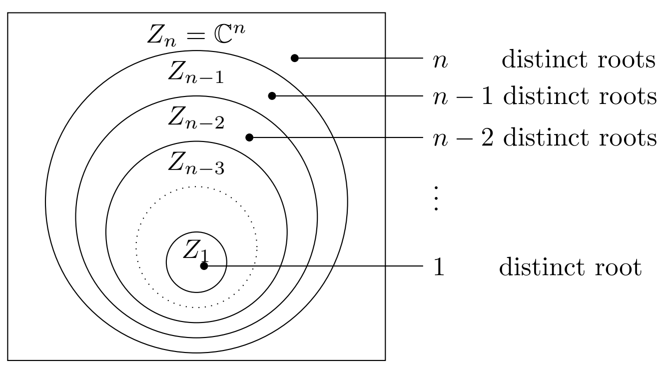
Therefore, we conclude that

$$\varnothing=Z_0\subsetneq Z_1\subsetneq Z_2\subsetneq Z_3=V(D)\subsetneq Z_4=\mathbb{C}^4.$$

In general, let $F=x^n+a_{n-1}x^{n-1}+\cdots+a_0$ be a general monic univariate polynomial of degree n and let D be the discriminant of F. We have

$$\varnothing=Z_0\subsetneq Z_1\subsetneq Z_2\subsetneq\cdots\subsetneq Z_{n-1}=V(D)\subsetneq Z_n=\mathbb{C}^n.$$

which can be illustrated by the following diagram.



Note that $Z_0,\ldots,Z_n$ provides a stratification of $\mathbb{C}^n$ such that, in each strata $S_i=Z_i-Z_{i-1}$, the number of distinct roots of F is invariant, namely i. Naturally we would like to describe the stratification in terms of the coefficients of the polynomial F, that is, to find defining polynomials of Z_i in terms of the coefficients.

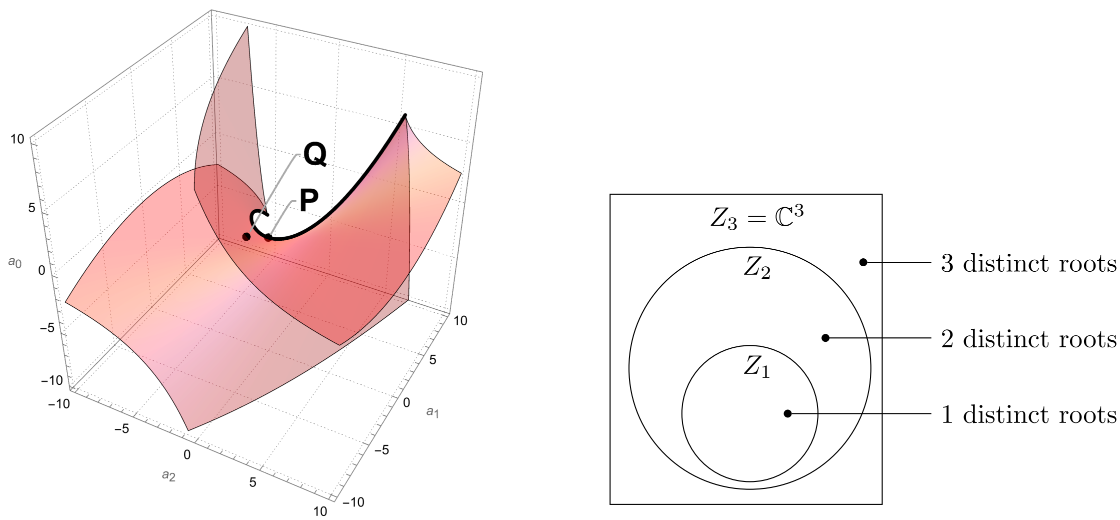
Note that $Z_n=\mathbb{C}^n$ and $Z_0=\varnothing$ are trivial. What is really interesting is the remaining sets $Z_1,\ldots,Z_{n-1}$. It is well-known that Z_i can be described via sub-discriminants (see Section 2).

For a long time, researchers have believed that the discriminant itself should carry the same amount of information as subdiscriminants. For example, Collins used all subdiscriminants as part of projection polynomials in cylindrical algebraic decomposition but conjectured that, in most cases, only the discriminant is necessary [10]. This insight motivated McCallum to refine projection techniques, successfully designing a much smaller set of projection polynomials that rely on discriminants rather than all subdiscriminants [27]. Yet, despite these advances, it remains unclear how to determine the number of distinct roots using only the discriminant.

This naturally leads to the core question underlying our work: is it possible to describe Z_i in terms of the coefficients without using sub-discriminants and instead using only *discriminant itself*? If so, how? The main contribution of this paper is to provide two different approaches to solving this problem.

Contribution 1. We partition the space of the coefficients into subsets according to the order of discriminant on the points, so that the order of the discriminant is invariant over each subset. All the elements in each subset have the same order and generate polynomials with the same number of distinct roots. In summary, this approach stratifies the discriminant hypersurface by recursively removing all the *lowest-order* points. See Theorem 8.

Contribution 2. We view the hypersurface defined by the discriminant as a variety in $\mathbb{C}^n$ and analyze the singular locus of this variety, the singular locus of the singular variety obtained in the previous step, ..., and so on. This yields a stratification of the discriminant hypersurface, whose strata are smooth varieties. We prove that each stratum exactly corresponds to the set of polynomials having a given number of distinct roots. In summary, this approach stratifies the discriminant hypersurface by recursively removing all the *smooth* points. See Theorem 12.



(a) The discriminant hypersurface Z_2

(b) A diagram showing the case $n = 3$

Figure 1: Stratification for $n = 3$

Example 1. To better illustrate our contribution, we will begin with a simple example, where

$$F = x^3 + a_2x^2 + a_1x + a_0.$$

The discriminant is

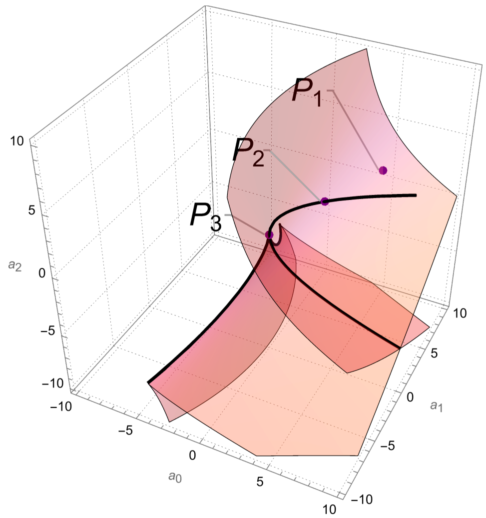
$$D = -4a_2^3a_0 + a_2^2a_1^2 + 18a_2a_1a_0 - 4a_1^3 - 27a_0^2.$$

We plot its zero locus in Figure 1(a) in pink. As one may tell from the figure, the zero locus of D is a surface in the three-dimensional space, and it is singular along the black curve. We can arbitrarily pick one singular point and one smooth point from $V(D)$, say $P = (0, 0, 0)$ and $Q = (0, -3, 2)$, corresponding to x^3 and $x^3 - 3x + 2 = (x - 1)^2(x + 2)$ respectively.

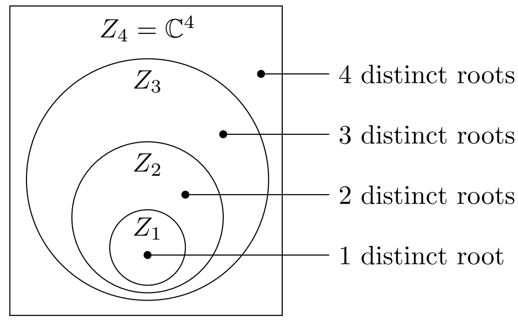
It is easy to verify that D is of order 2 at P, and of order 1 at Q. In fact,

$$\begin{aligned} D &= -27a_0^2 + \left(-4a_1^3 + 18a_2a_1a_0 + a_2^2a_1^2 - 4a_2^3a_0\right) \\ &= -108a + (\text{terms of total degree} \geq 2 \text{ in } a, b+3, c-2). \end{aligned}$$

This agrees with what is predicted in our results. On the one hand, the polynomial x^3 has only one root counted without multiplicity, so the order of D is 2 at P and P is a singularity of $V(D)$. On the other hand, the polynomial $x^3 - 3x + 2$ has two distinct roots, so the order of D is 1 at Q and Q is a smooth point of $V(D)$.



(a) A cross section of the discriminant hypersurface Z_3



(b) A diagram showing the case $n = 4$

Figure 2: Stratification for $n = 4$

Example 2. Let us continue with the monic quartic example. Since it is impossible to directly visualize the 4-dimensional space, we will plot a section of the discriminant hypersurface Z_3, cut by the hyperplane $a_3 = 0$. This is still quite informative, because one can always apply the Tschirnhausen transformation to eliminate the a_3x^3 term. See Figure 2a.

The points corresponding to

$$\begin{array}{lllll} P_1 & = & x^4 + 2x^2 + 8x + 5 & = & (x+1)^2(x-1+2\mathrm{i})(x-1-2\mathrm{i}), \\ P_2 & = & x^4 + 4x^2 + 4 & = & (x-\sqrt{2}\mathrm{i})^2(x+\sqrt{2}\mathrm{i})^2, \\ P_3 & = & x^4 & & \end{array}$$

are also plotted in Figure 2a. One can see that while P_1 is lying on the pink surface, one of the black curves, which are "self-intersections" of the pink surface, passes through P_2. Also, P_3 is the intersection of the two black curves. Moreover, it can be easily checked that the discriminant D is of order 1 at P_1, order 2 at P_2 and order 3 at P_3.

In fact, this is not just a coincidence. Since P_1 has three distinct roots, our Theorem 8 shows that D is of order 1 at P_1, and Theorem 12 concludes that P_1 is a smooth point of Z_3. Similarly, P_2 has only two distinct roots, so D is of order 2 at it and it must be a singular point of Z_3, lying on the black curves. As for the last point, D is of order 3 at P_3 and it is a singularity of the singular locus $\operatorname{Sing} Z_3$, with $\operatorname{Sing} V$ represents the singular locus of the variety V hereinafter.

Remark 3.

- The combination of our two results together shows that the smooth stratification of the discriminant hypersurface can be directly read off from the information of order.
- While the smooth stratification of the discriminant hypersurface is an intrinsic object, the order of the discriminant polynomial depends on the embedding in the ambient space. One may wonder whether our first contribution can be formulated intrinsically. The answer is affirmative because the notion of order can be reinterpreted using commutative algebra. More explicitly, the order of a polynomial F at a point $p \in V(F)$ coincides with the Hilbert-Samuel multiplicity of the local ring $\mathcal{O}_{V(F),p}$. See, for example, [16, Corollary 5.5.9].

In this paper, we will however stick to the classical definition of order because it is conceptually simpler, hence more accessible to the general readers.

Related Work:

1. Let $F(a,x)$ and $G(b,x)$ be two univariate polynomials in x with indeterminate coefficients a and b respectively. Let $R(a,b)$ be the resultant of $F(a,x)$ and $G(b,x)$ with respect to x. Let α,β be numeric coefficient vectors such that $\deg_x F(\alpha,x)=\deg_x F(a,x)$ and $\deg_x G(\beta,x)=\deg_x G(b,x)$. Recall the following well-known facts:

   (a) The number of distinct roots of $F(\alpha,x)$ is $n-\deg\gcd(F(\alpha,x),F'(\alpha,x))$.

   (b) The order of R at (α,β) is equal to $\deg\gcd(F(\alpha,x),G(\beta,x))$. In [19, Appendix], Busé provided a short and elegant proof.

   (c) The discriminant of $F(\alpha,x)$ is the resultant of $F(\alpha,x)$ and $F'(\alpha,x)$ up to a non-zero constant.

   Thus one might suspect that our Contribution 1 could be easily obtained from specializing the work of Busé. However, it is not clear whether this is possible because

   - order is not preserved under specialization, and
   - the partial derivatives (with respect to coefficients) of resultant are different from the partial derivatives of discriminant.

2. Let $Y_r(m,n)$ be the determinantal variety consisting of all $m\times n$ matrices of rank $\le r$. Recall the following classical results:

   (a) For every $1\le r\le l=\min(m,n)-1$, the variety $Y_{r-1}(m,n)$ coincides with the singular locus of $Y_r(m,n)$ [14, Proposition 3.2, Chapter 4]. Hence,

   $$Y_l(m,n)\supsetneq\cdots\supsetneq Y_1(m,n)\supsetneq Y_0(m,n)$$

   yields a smooth stratification of $Y_l(m,n)$.

   (b) The number of distinct roots of F is uniquely determined by the rank of the Sylvester matrix of F and F'.

   Thus one may wonder whether our Contribution 2 could be easily obtained from specializing the smooth stratification of $Y_l(m,n)$. However, it is not clear whether this is possible because singularity is not preserved under specialization.

3. The notion of iterated discriminants was considered by several authors [12, 17, 5, 23, 18]. It provides some information about the singular locus of the discriminant hypersurface. For example, [23, Proposition 9] characterizes $\operatorname{Sing} Z_{n-1}$. However it is not clear how to obtain our Contribution 2 because their result do not show how to characterize the other Z_i's.

4. Let C_μ be the family of polynomials of type $(x-x_1)^{\mu_1}\cdots(x-x_k)^{\mu_k}$ where $\mu=(\mu_1,\ldots,\mu_k)$ is a partition of n where again x_i's are not necessarily distinct. It is a folklore that the irreducible components of Z_k consist of C_μ's where $\#\mu=k$ (see our Corollary 17). Therefore researches on Z_k or C_μ are closely intertwined.

   The object C_μ have been studied under several distinct terminologies: coincident root locus [8, 22], multiple root locus [25], λ-Chow variety [29], etc. In [20, 21], Hong and Yang discovered "simple" defining polynomials for C_μ's under the name "multiplicity discriminants". In [8], Chipalkatti fully determined the singular locus of C_μ. Later, the object $\operatorname{Sing} C_\mu$ was given another equivalent characterization by Kurmann in [22].

   Based on Chipalkatti's results and a careful analysis on all the partitions μ by their largest number, we managed to prove our Contribution 2.

The current paper is structured as follows. In Section 2, we give a brief review on the stratification of discriminant hypersurface based on subdiscriminants. In Section 3, we prove that the discriminant hypersurface can be stratified based on the order of the discriminant at the points so that the order of the discriminant remains constant within each stratum. In Section 4, we present another stratification of the discriminant hypersurface based on smoothness where each stratum exactly corresponds to the set of polynomials having a given number of distinct roots. The paper is concluded in Section 5 with some further remarks.

We assume that the reader is familiar with the following notations: order of polynomial at a point and singular locus of a variety.

2 Subdiscriminant-based Stratification

We briefly review the classical notion of subdiscriminants and their use in stratification [31, 32].

Definition 4 (Subdiscrimiant)**.** Let $F(x) = \sum_i a_i x^i$ be a monic polynomial of degree n, that is $a_n = 1$. Let $F'(x) = \sum_i b_i x^i$ be the derivative of F, that is $b_i = (i+1)\, a_{i+1}$. Then the k-th subdiscriminant of F, written as D_k, is defined by

$$D_k = (-1)^{\binom{n-k}{2}} \det \begin{bmatrix} a_n & a_{n-1} & \cdots & \cdots & \cdots & \cdots \\ & \ddots & \ddots & & & \\ & & a_n & a_{n-1} & \cdots & \cdots \\ b_{n-1} & b_{n-2} & \cdots & \cdots & \cdots & \cdots \\ & \ddots & \ddots & & & \\ & & \ddots & \ddots & & \\ & & & b_{n-1} & b_{n-2} & \cdots \end{bmatrix}$$

where # of the top rows (involving a's) is $n-1-k$ and the # of the bottom rows (involving b's) is $n-k$.

Example 5. Let $n = 4$. Recall that $a_4 = 1$. We have

$$D_0 = (-1)^{\frac{4\cdot 3}{2}} \begin{vmatrix} a_4 & a_3 & a_2 & a_1 & a_0 & & \\ & a_4 & a_3 & a_2 & a_1 & a_0 & \\ & & a_4 & a_3 & a_2 & a_1 & a_0 \\ 4a_4 & 3a_3 & 2a_2 & 1a_1 & & & \\ & 4a_4 & 3a_3 & 2a_2 & 1a_1 & & \\ & & 4a_4 & 3a_3 & 2a_2 & 1a_1 & \\ & & & 4a_4 & 3a_3 & 2a_2 & 1a_1 \end{vmatrix}$$

$$D_1 = (-1)^{\frac{3\cdot 2}{2}} \begin{vmatrix} a_4 & a_3 & a_2 & a_1 & a_0 \\ & a_4 & a_3 & a_2 & a_1 \\ 4a_4 & 3a_3 & 2a_2 & 1a_1 & \\ & 4a_4 & 3a_3 & 2a_2 & 1a_1 \\ & & 4a_4 & 3a_3 & 2a_2 \end{vmatrix}$$

$$D_2 = (-1)^{\frac{2\cdot 1}{2}} \begin{vmatrix} a_4 & a_3 & a_2 \\ 4a_4 & 3a_3 & 2a_2 \\ & 4a_4 & 3a_3 \end{vmatrix}$$

$$D_3 = (-1)^{\frac{1\cdot 0}{2}} \begin{vmatrix} 4a_4 \end{vmatrix}$$

Theorem 6. *For $k = 1, \ldots, n-1$, we have*

$$Z_k = V(\{D_0, \ldots, D_{n-k-1}\}).$$

Hence $Z_{n-1} \supsetneq Z_{n-2} \supsetneq \cdots \supsetneq Z_1$ is the subdiscriminant-*based stratification of the discriminant hypersurface Z_{n-1}.*

3 Order-based Stratification

3.1 Main result

For the sake of simplicity, we use the following order-related notations in this section.

Notation 7. *For $P \in \mathbb{C}[u_1, \ldots, u_p]$ and $\gamma \in \mathbb{C}^p$,*

- $\partial_{(k_1,\ldots,k_p)} P := \dfrac{\partial^k P}{\partial u_1^{k_1} \cdots \partial u_p^{k_p}}$ *where $k_i \in \mathbb{N}$ and $k = k_1 + \cdots + k_p$;*
- $\operatorname{ord} P(\gamma) :=$ *the order of P at γ.*

Theorem 8. *For $k = 1, \ldots, n-1$, we have $Z_k = O_k$ where*

$$O_k = \{\gamma \in \mathbb{C}^n : \operatorname{ord} D(\gamma) \geq n-k\}.$$

Equivalently,

$$Z_k \;=\; V(\{\partial_\delta D \;:\; |\delta| \;\leq\; n-k-1\}).$$

Hence $Z_{n-1} \supsetneq Z_{n-2} \supsetneq \cdots \supsetneq Z_1$ is the order-*based stratification of the discriminant hypersurface Z_{n-1}.*

Example 9. Let $n = 4$ and $k = 2$. Then $F = x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$. For the polynomial P_2 in Example 2, i.e., $\gamma = (4, 0, 4, 0)$, it is obvious that P_2 has two distinct roots and thus $\gamma \in Z_2$. By Theorem 8, we have $\gamma \in O_2$. In other words, $\operatorname{ord} D(\gamma) \geq 4 - 2 = 2$.

In fact, we can verify $\gamma \in O_2$ by calculation. Recall that

$$\begin{aligned}D =\,& 256a_0^3 - 27a_1^4 - 128a_0^2a_2^2 - 192a_0^2a_1a_3 + 144a_0a_1^2a_2 - 4a_1^2a_2^3 + 18a_1^3a_2a_3 - 6a_0a_1^2a_3^2\\ &+ 16a_0a_2^4 + 144a_0^2a_2a_3^2 - 80a_0a_1a_2^2a_3 + 18a_0a_1a_2a_3^3 - 4a_1^3a_3^3 - 4a_0a_2^3a_3^2 - 27a_0^2a_3^4 + a_1^2a_2^2a_3^2.\end{aligned}$$

Taking the first partial derivative of D with respect to a_i for $i = 0, 1, 2, 3$, we obtain

$$\begin{aligned}\frac{\partial D}{\partial a_0} =& - 54\,a_0a_3^4 + 18\,a_1a_2a_3^3 - 4\,a_2^3a_3^2 + 288\,a_0a_2a_3^2 - 6\,a_1^2a_3^2 - 80\,a_1a_2^2a_3 + 16\,a_2^4 - 384\,a_0a_1a_3\\ &- 256\,a_0a_2^2 + 144\,a_1^2a_2 + 768\,a_0^2,\\ \frac{\partial D}{\partial a_1} =& \,18\,a_0a_2a_3^3 - 12\,a_1^2a_3^3 + 2\,a\;{}_1a_2^2a_3^2 - 12\,a_0a_1a_3^2 - 80\,a_0a_2^2a_3 + 54\,a_1^2a_2a_3 - 8\,a_1a^3{}_2 - 192\,a_0^2a_3\\ &+ 288\,a_0a_1a_2 - 108\,a_1^3,\\ \frac{\partial D}{\partial a_2} =& \,18\,a_0a_1a_3^3 - 12\,a_0a_2^2a_3^2 + 2\,a_1^2a_2a_3^2 + 144\,a_0^2a_3^2 - 160\,a_0a_1a_2a_3 + 64\,a_0a_2^3 + 18\,a_1^3a_3 - 12\,a_1^2a_2^2\\ &- 256\,a_0^2a_2 + 144\,a_0a_1^2,\\ \frac{\partial D}{\partial a_3} =& \,18\,a_0a_1a_3^3 - 12\,a_0a_2^2a_3^2 + 2\,a_1^2a_2a_3^2 + 144\,a_0^2a_3^2 - 160\,a_0a_1a_2a_3 + 64\,a_0a_2^3 + 18\,a_1^3a_3 - 12\,a_1^2a_2^2\\ &- 256\,a_0^2a_2 + 144\,a_0a_1^2.\end{aligned}$$

It can be verified that

$$D(\gamma) = \frac{\partial D}{\partial a_0}(\gamma) = \frac{\partial D}{\partial a_1}(\gamma) = \frac{\partial D}{\partial a_2}(\gamma) = \frac{\partial D}{\partial a_3}(\gamma) = 0$$

and

$$\frac{\partial^2 D}{\partial a_0 \partial a_2}(\gamma) = \left(18\,a_1a_3^3 - 12\,a_2^2a_3^2 + 288\,a_0a_3^2 - 160\,a_1a_2a_3 + 64\,a_2^3 - 512\,a_0a_2 + 144\,a_1^2\right)\big|_{a=\gamma} = -4096 \neq 0.$$

Hence $\operatorname{ord} D(\gamma) = 2$, indicating that $\gamma \in O_2$.

3.2 Proof of Theorem 8

Now we will prove Theorem 8. For this we need the following lemma.

Lemma 10. *The polynomial F_γ has at most m distinct roots if and only if*

$$\frac{\partial^0 D}{\partial a_0^0}(\gamma) \;=\; \frac{\partial^1 D}{\partial a_0^1}(\gamma) \;=\; \cdots \;=\; \frac{\partial^{n-m-1} D}{\partial a_0^{n-m-1}}(\gamma) \;=\; 0. \tag{2}$$

Proof. First, we derive an equivalent condition for Equation (2). By Taylor expansion,

$$\operatorname{res}_x(F+t, F') = D(a_0+t, a_1, \ldots, a_{n-1}) = \sum_{i\geq 0} t^i \cdot \frac{1}{i!}\frac{\partial^i}{\partial a_0^i} D.$$

Therefore, Equation (2) is equivalent to $t^{n-m} \mid \operatorname{res}_x(F_\gamma + t, F'_\gamma)$.

Let $G_\gamma = \gcd\left(F_\gamma, F'_\gamma\right)$ and $d = \deg G_\gamma$. Assume $F'_\gamma = G_\gamma \overline{F'_\gamma}$. It is easy to see that

$$\operatorname{res}_x\left(F_\gamma + t, F'_\gamma\right) = \operatorname{res}_x\left(F_\gamma + t, G_\gamma \overline{F'_\gamma}\right) = \operatorname{res}_x\left(F_\gamma + t, G_\gamma\right)\ \operatorname{res}_x\left(F_\gamma + t, \overline{F'_\gamma}\right). \tag{3}$$

Let $\alpha_1, \ldots, \alpha_d$ be the complex roots of G_γ. Obviously, $F_\gamma(\alpha_i) = 0$. Hence

$$\operatorname{res}_x\left(F_\gamma + t, G_\gamma\right) = (-1)^{nd} \cdot 1^n \cdot \prod_{i=1}^{d}(F_\gamma + t)(\alpha_i) = (-1)^{nd}\prod_{i=1}^{d} t = (-1)^{nd} t^d. \tag{4}$$

Now we are going to show that

$$F_\gamma \text{ has at most } m \text{ distinct roots} \Longleftrightarrow t^{n-m} \mid \operatorname{res}_x(F_\gamma + t, F'_\gamma).$$

$\Longrightarrow$: Combining Equations (3) and (4), we have

$$\operatorname{res}_x\left(F_\gamma + t, F'_\gamma\right) = (-1)^{nd} t^d\ \operatorname{res}_x\left(F_\gamma + t, \overline{F'_\gamma}\right). \tag{5}$$

Therefore, if F_γ has at most m distinct roots, then $n - m \leq d = \deg G_\gamma$. By Equation (5), we immediately have $t^{n-m} \mid \operatorname{res}_x(F_\gamma + t, F'_\gamma)$.

$\Longleftarrow$: The combination of the assumption $t^{n-m} \mid \operatorname{res}_x(F_\gamma + t, F'_\gamma)$ and Equation (5) yields

$$t^{n-m} \mid t^d\ \operatorname{res}_x\left(F_\gamma + t, \overline{F'_\gamma}\right).$$

So in order to prove $d \geq n - m$, it suffices to prove that $t \nmid \operatorname{res}_x\left(F_\gamma + t, \overline{F'_\gamma}\right)$, or equivalently, $\operatorname{res}_x\left(F_\gamma, \overline{F'_\gamma}\right) \neq 0$.

But this is almost trivial, as

$$\operatorname{res}_x\left(F_\gamma, \overline{F'_\gamma}\right) \neq 0 \iff \gcd(F_\gamma, \overline{F'_\gamma}) = 1 \iff \gcd(G_\gamma, \overline{F'_\gamma}) = 1,$$

and $G_\gamma, \overline{F'_\gamma}$ are clearly coprime by counting the multiplicity of common roots of F_γ and F'_γ.

□

Remark 11. In Lemma 10, we obtain the following nice result:

$$Z_m \;=\; V\left(\left\{D, \frac{\partial D}{\partial a_0}, \ldots, \frac{\partial^{n-m-1} D}{\partial a_0^{n-m-1}}\right\}\right).$$

This characterizes Z_m in terms of the partial derivatives of D with respect to a_0. A curious reader might wonder what if a_1, a_2 or other coefficients were used instead of a_0.

In fact, similar relations hold, but slight adjustments have to be made in both the statement and the proof. We prepare a satisfactory answer in the Appendix for those with the greatest curiosity.

Now we are ready to prove Theorem 8.

Proof of Theorem 8. The proof will be divided into several steps for easier reading.

1. It is easy to see that

$$\text{the order of } D \text{ at } \gamma \text{ with respect to } a \geq n-k \iff \gamma \in V(\{\partial_\delta (D) : 0 \leq |\delta| \leq n-k-1\})$$

where $|\delta|$ is the shorthand notation of the sum of elements in δ hereinafter. Equivalently,

$$O_k = V(\{\partial_\delta (D) \,|\, 0 \leq |\delta| \leq n-k-1\}).$$

Next we will prove $Z_k = O_k$ by showing $Z_k \subseteq O_k$ and $Z_k \supseteq O_k$.

2. $Z_k \subseteq O_k$.

(a) Let $\gamma \in Z_k$. Then F_γ has at most k distinct roots. Hence

$$\deg \gcd(F_\gamma, F'_\gamma) \geq n-k.$$

(b) Let M_γ denote the Sylvester matrix of F_γ and F'_γ with respect to x. By [2, Chapter 1, Proposition 1],

$$\operatorname{rank} M_\gamma = 2n-1-\deg\gcd(F_\gamma, F'_\gamma) \leq 2n-1-(n-k) = n+k-1.$$

(c) Assume $\delta = (\delta_0, \ldots, \delta_{n-1})$ satisfies $|\delta| \leq n-k-1$. Let e_i denote the i-th unit vector of length n. One can write δ into the sum of $|\delta|$ unit vectors, i.e., $\delta = \sum_{j=1}^{|\delta|} e_{s_j}$. Thus

$$\partial_\delta D = \partial_{e_{s_{|\delta|}}} \left(\cdots \partial_{e_{s_2}} \left(\partial_{e_{s_1}} D\right)\right).$$

Let M be the Sylvester matrix of F and F' with respect to x. By [1, Chapter 11, Section 1, Remark on Theorem 1],

$$\partial_{e_{s_1}} D = \sum_{i=1}^{n} \det[M_1, \ldots, M_{i-1}, \hat{M}_i, M_{i+1}, \ldots, M_n],$$

where M_j is the j-th column of M, and $\hat{M}_i = \dfrac{\partial M_i}{\partial a_{s_1-1}}$ with the partial derivative taken entrywise.
Note that the entries of M are linear with respect to a_i's. Thus each $\hat{M}_i$ is a constant vector. Taking the Laplace expansion along the column $\hat{M}_i$ for each of the above summands, we can write $\partial_{e_{s_1}} D$ as a sum of minors of M with size $2n-2$.
Now apply the $\partial_{e_{s_2}}$ operator on each minor. Noting that the entries of each minor are again linear with respect to a_i's, we conclude that $\partial_{e_{s_2}} \left(\partial_{e_{s_1}} D\right)$ is a sum of minors of M with size $2n-3$.
Repeating the process, we can eventually write $\partial_\delta D$ as a sum of minors of M with size $2n-1-|\delta|$.

(d) From (b), we deduce that

$$2n-1-|\delta| \geq 2n-1-(n-k-1) = n+k > n+k-1 \geq \operatorname{rank} M_\gamma.$$

Thus the partial derivative $(\partial_\delta D)(\gamma)$ is a sum of minors of M_γ with size greater than the rank. Hence each of the summands is zero, which implies that $(\partial_\delta D)(\gamma) = 0$. Hence $\gamma \in O_k$.

3. $Z_k \supseteq O_k$.
From Lemma 10, we have

$$Z_k \;=\; V\left(\left\{D, \frac{\partial D}{\partial a_0}, \ldots, \frac{\partial^{n-k-1} D}{\partial a_0^{n-k-1}}\right\}\right).$$

It immediately follows that $Z_k \supseteq O_k$.

∎

4 Smoothness-based Stratification

Theorem 12. *For $k = n-1, n-2, \dots, 2$, we have*

$$\operatorname{Sing} Z_k = Z_{k-1}.$$

Equivalently, for $k = 1, \dots, n-1$, we have

$$Z_k = \underbrace{\operatorname{Sing} \cdots \operatorname{Sing}}_{n-1-k \text{ iterations}} V(D).$$

Hence $Z_{n-1} \supsetneq Z_{n-2} \supsetneq \cdots \supsetneq Z_1$ is the smoothness-*based stratification of the discriminant hypersurface Z_{n-1}.*

Now we will prove Theorem 12. For this, we need to investigate the irreducible decomposition of Z_k. Since a monic polynomial of degree n is determined by its non-leading coefficients, the affine space $\mathbb{C}^n$ parameterizes the set of monic polynomials of degree n.

Definition 13. Let n be a positive integer and let $\mu = (\mu_1, \dots, \mu_m)$ be a partition of n. We say a monic polynomial P of degree n lies in the **coincidence root locus** C_μ, if $P(x)$ splits into m (not necessarily distinct) linear factors raised to powers $\mu_1, \dots, \mu_m$ over $\mathbb{C}$:

$$P(x) = \prod_{i=1}^{m} (x - x_i)^{\mu_i}.$$

In other words,

$$C_\mu = \left\{ P \in \mathbb{C}^n \,\middle|\, P(x) = \prod_{i=1}^{m} (x - x_i)^{\mu_i} \right\}.$$

Example 14. For example, let $P = x^4 - 2x^2 + 1$. Then

$$P = (x-1)^2(x+1)^2 = (x-1)(x-1)(x+1)^2 = (x-1)(x-1)(x+1)(x+1).$$

Therefore P lies in $C_{(1,1,1,1)}, C_{(1,1,2)}$ and $C_{(2,2)}$.

Proposition 15 (Folklore)**.** *The set C_μ is an irreducible closed subset of $\mathbb{C}^n$, of dimension $\#\mu$.*

Proof. Consider the Vièta map ν sending roots $(x_1, \dots, x_n)$ to the coefficients $(a_0, \dots, a_{n-1})$ defined by the elementary symmetric polynomials. The map ν is a finite flat surjective morphism from $\mathbb{C}^n$ to $\mathbb{C}^n$, because the polynomial ring $\mathbb{C}[x_1, \dots, x_n]$ is a free module of rank $n!$ over the subring of symmetric polynomials [3, Chapter IV, Section 6, No. 1, Theorem 1(c)]. Then C_μ is the image of a linear subspace of dimension $\#\mu$ under ν, hence irreducible, closed and of dimension $\#\mu$. □

Example 16. For example, the set $C_{(2,2)} \subseteq \mathbb{C}^4$ is the image of $H = \{(x_1, \dots, x_4) \in \mathbb{C}^4 \,|\, x_1 = x_2, x_3 = x_4\}$ under the Vièta map ν. Clearly $H \simeq \mathbb{C}^2$ is a two-dimensional irreducible closed set, so is $C_{(2,2)}$.

Corollary 17. *We have the following irreducible decomposition of Z_k:*

$$Z_k = \bigcup_{\#\mu = k} C_\mu.$$

Therefore, Z_k is equidimensional and each C_μ is an irreducible component of Z_k.

Proof. This immediately follows from Proposition 15 and the fact that a polynomial P has at most k distinct roots if and only if P lies in a coincident root locus C_μ for some $\#\mu = k$. Also notice that $\dim C_\mu = \#\mu$. □

Now we are ready to prove Theorem 12.

Proof of Theorem 12. Let $\bigcup_{\#\mu=k} C_\mu$ be the irreducible decomposition of the variety Z_k. Then we have

$$\operatorname{Sing} Z_k = \bigcup_{\substack{\mu\neq\mu' \\ \#\mu=\#\mu'=k}} (C_\mu \cap C_{\mu'}) \cup \bigcup_{\#\mu=k} \operatorname{Sing} C_\mu \tag{6}$$

by [11, Exercise 9.6.11(c)].

- $\operatorname{Sing} Z_k \subseteq Z_{k-1}$.

  It suffices to show that, for $\mu \neq \mu'$ and $\#\mu = \#\mu' = k$, we have $C_\mu \cap C_{\mu'} \subseteq Z_{k-1}$ and $\operatorname{Sing} C_\mu \subseteq Z_{k-1}$.

  1. For every monic polynomial $P \in C_\mu \cap C_{\mu'}$ with $\#\mu = \#\mu' = k$, we see that the (monic) linear factors of P can be grouped in two different ways, namely μ and μ'. Thus the number of distinct roots of P must be less than k. Therefore,
  $$C_\mu \cap C_{\mu'} \subseteq Z_{k-1}.$$
  2. In [8, Theorem 5.4], Chipalkatti showed that for every μ the set $\operatorname{Sing} C_\mu$ is a union of certain C_λ's where each λ is a coarsening of μ. Note that $\#\lambda < k$ and $C_\lambda \subseteq Z_{\#\lambda} \subseteq Z_{k-1}$. Hence
  $$\operatorname{Sing} C_\mu \subseteq Z_{k-1}.$$

- $\operatorname{Sing} Z_k \supseteq Z_{k-1}$.
  Now we will show that all irreducible components of Z_{k-1} are contained in $\operatorname{Sing} Z_k$. This is done by analyzing the irreducible components of $Z_{k-1} = \bigcup_{\#\lambda=k-1} C_\lambda$ by the largest number in $\lambda = (\lambda_1, \ldots, \lambda_{k-1})$.

  1. Suppose $\max_i \lambda_i \geq 4$. Without loss of generality, we may assume that $\lambda_1 \geq 4$. Then λ can be refined to two different partitions with length k, e.g., $\mu = (1, \lambda_1 - 1, \lambda_2, \ldots, \lambda_{k-1})$ and $\mu' = (2, \lambda_1 - 2, \lambda_2, \ldots, \lambda_{k-1})$. So $C_\lambda \subseteq (C_\mu \cap C_{\mu'}) \subseteq \operatorname{Sing} Z_k$.
  2. Suppose $\max_i \lambda_i = 3$. Let
  $$\lambda = (\underbrace{3, \ldots, 3}_{\ell}, \underbrace{2, \ldots, 2}_{\ell'}, \underbrace{1, \ldots, 1}_{\ell''})$$
  and
  $$\mu = (\underbrace{3, \ldots, 3}_{\ell-1}, \underbrace{2, \ldots, 2}_{\ell'+1}, \underbrace{1, \ldots, 1}_{\ell''+1}).$$
  Note that λ is a coarsening of μ by merging 2 and 1 from μ into 3. From [8, Theorem 5.4], we have $C_\lambda \subseteq \operatorname{Sing} C_\mu$. Recalling (6), we see that $\operatorname{Sing} C_\mu \subseteq \operatorname{Sing} Z_k$. Hence we have
  $$C_\lambda \subseteq \operatorname{Sing} Z_k.$$
  3. Suppose $\max_i \lambda_i = 2$. Let
  $$\lambda = (\underbrace{2, \ldots, 2}_{\ell}, \underbrace{1, \ldots, 1}_{\ell'})$$
  and
  $$\mu = (\underbrace{2, \ldots, 2}_{\ell-1}, \underbrace{1, \ldots, 1}_{\ell'+2}).$$
  Note that λ is a coarsening of μ by merging two copies of 1 from μ into 2, where 2 is already in μ. From [8, Theorem 5.4], we have $C_\lambda \subseteq \operatorname{Sing} C_\mu$. Recalling (6), we see that $\operatorname{Sing} C_\mu \subseteq \operatorname{Sing} Z_k$ Hence we have
  $$C_\lambda \subseteq \operatorname{Sing} Z_k.$$

Therefore $\operatorname{Sing} Z_k \supseteq Z_{k-1}$.

So we have the claimed equality $\operatorname{Sing} Z_k = Z_{k-1}$. Notice that Z_1 is isomorphic to the affine line because it is the image of the map

$$t \mapsto \left((-1)^1\binom{n}{1}t, (-1)^2\binom{n}{2}t^2, \ldots, (-1)^n\binom{n}{n}t^n\right)$$

(recall Vieta's formulas). Thus Z_1 is smooth and $Z_{n-1} \supsetneq Z_{n-2} \supsetneq \cdots \supsetneq Z_1$ is indeed the smooth stratification of Z_{n-1}. □

5 Conclusion

Summary: In this paper, we considered stratification of the discriminant hypersurface:

$$Z_{n-1} \supsetneq Z_{n-2} \supsetneq \cdots \supsetneq Z_1$$

where Z_k is the set of all monic univariate polynomials of degree n with at most k distinct roots. We reviewed a classical approach to describe Z_k and then proposed two new approaches.

1. In Section 2 (Theorem 6), we briefly reviewed a classical description of Z_k via subdiscriminants D_i's. Precisely put,
$$Z_k \;=\; V\left(\{D_0, D_1, \ldots, D_{n-k-1}\}\right).$$
2. In Section 3 (Theorem 8), we gave a new description of Z_k via partial derivatives of the discriminant D (that is D_0). Precisely put,
$$Z_k \;=\; V(\{\partial_\delta D \;:\; |\delta| \;\leq\; n-k-1\}).$$
3. In Section 4 (Theorem 12), we gave another new description of Z_k via repeated singular locus of the discriminant D. Precisely put
$$Z_k = \underbrace{\operatorname{Sing}\cdots\operatorname{Sing}}_{n-1-k \text{ iterations}}\; V(D).$$

Future works:

- Note that the first two descriptions of Z_k imply that
$$V(D_0, D_1, \ldots, D_{n-k-1}) \;=\; V(\langle\partial_\delta D \;:\; |\delta| \;\leq\; n-k-1\rangle),$$
or equivalently,
$$\sqrt{\langle D_0, D_1, \ldots, D_{n-k-1}\rangle} \;=\; \sqrt{\langle\partial_\delta D \;:\; |\delta| \;\leq\; n-k-1\;\rangle}$$
However, numerical evidences suggest that
$$\begin{array}{l} \langle D_0, D_1, \ldots, D_{n-k-1}\rangle \;\neq\; \langle\partial_\delta D \;:\; |\delta| \;\leq\; n-k-1\;\rangle \\ \langle D_0, D_1, \ldots, D_{n-k-1}\rangle \qquad \text{is not radical} \\ \langle\partial_\delta D \;:\; |\delta| \;\leq\; n-k-1\;\rangle \text{ is not radical} \end{array}$$
Thus a question naturally arises:
Is there a "good" set of generators for their radicals?
Knowing those would allow us to apply the Jacobian criterion to write down the defining polynomials of the singular locus of Z_k as we did in the third approach.

- Recall the relation from the above

$$V(\{D_0, D_1, \ldots, D_{n-k-1}\}) \;=\; V(\{\partial_\delta D \;:\; |\delta| \;\le\; n-k-1\})$$

Hilbert's Nullstellensatz implies that

$$\underset{s}{\exists} D_k^s \in \langle \partial_\delta D \;:\; |\delta| \;\le\; n-k-1 \;\rangle$$

Another question naturally arises: *What is the smallest number for s?*

Based on some computations, we conjecture that the smallest exponent is $k+1$. The claim has been verified for small n and k using a Maple program that can be downloaded from the following site:

`https://github.com/JYangMATH/DiscriminantHypersurface`

However, it is still unclear how to prove/disprove the claim for the general case.

- In this paper, we partition the discriminant hypersurface by the number of distinct roots. Recall that the number of distinct roots can be captured via subdiscriminants and their variants [30, 37, 15] and (signed) subdiscriminants can also be used to retrieve the information of real roots. Then a natural question arises: can we partition the discriminant hypersurface by the number of distinct roots in the real space? This question is worthy of further investigation in the future.

- Finally the most significant generalization of the current work is to extend the theory of the discriminant hypersurface from the single univariate polynomial case to the multivariate polynomial system case. For this, one would need to revisit the works on multivariate discriminants [14, 13, 17, 12] and/or on multivariate subresultants [7] in view of stratifying the discriminant hypersurface in terms of number of distinct complex/real solutions.

Appendix

The main result of the paper does *not* depend on the result hereinafter. This appendix is for the readers who want to learn more about details behind Remark 11.

In Lemma 10, we show that Z_k can be characterized as the vanishing set of D and first $n-k-1$ partial derivatives of D with respect to a_0. This is no longer true if we replace a_0 with any other coefficients. However it is not very far from being true if we are willing to make a small adjustment.

Proposition 18. *For $1 \le s \le n-1$ and $1 \le k \le n-1$, we have*

$$Z_k = \overline{V\left(\left\{D, \frac{\partial D}{\partial a_s}, \ldots, \frac{\partial^{n-k-1} D}{\partial a_s^{n-k-1}}\right\}\right) \backslash V(a_0)}.$$

In order to prove Proposition 18, we need the following lemma.

Lemma 19. *Suppose $P = x^n + c_{n-1}x^{n-1} + \cdots + c_0$ is a monic polynomial that splits as $P = \prod_{i=1}^m (x-r_i)^{\mu_i}$, where $r_1, \ldots, r_m$ are distinct complex roots. Let $1 \le s \le n-1$ be a positive integer. Then*

1. $D(c_0, \ldots, c_{n-1}) = \dfrac{\partial D}{\partial a_s}(c_0, \ldots, c_{n-1}) = \cdots = \dfrac{\partial^{n-m-1} D}{\partial a_s^{n-m-1}}(c_0, \ldots, c_{n-1}) = 0$;

2. $\dfrac{\partial^{n-m} D}{\partial a_s^{n-m}}(c_0, \ldots, c_{n-1}) = 0$ *if and only if* 0 *is a multiple root of* P.

Proof. Recall that $F = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ is a general monic univariate polynomial of degree n and D is the discriminant of F. Now we introduce a fresh variable t and consider $F_s = F + tx^s$. The discriminant of F_s will simply be $D_s = D(a_0, \ldots, a_{s-1}, a_s + t, a_{s+1}, \ldots, a_{n-1}) \in \mathbb{C}[a_0, \ldots, a_{n-1}, t]$. By Taylor's formula, we have

$$D_s = \sum_{i=0}^{+\infty} \left(\frac{1}{i!} \frac{\partial^i D}{\partial^i_{a_s}} \right) t^i.$$

Notice that this is actually a finite sum.

To show the lemma, it suffices to show that

1. $t^{n-m} \big| D_s(c_0, \ldots, c_{n-1}; t)$, and
2. $t^{n-m+1} \big| D_s(c_0, \ldots, c_{n-1}; t)$ if and only if 0 is a multiple root of P.

Write $Q = \prod_{i=1}^m (x - r_i)^{\mu_i - 1} = \gcd(P, P')$. Notice that

$$\begin{aligned}
D_s(c_0, \ldots, c_{n-1}; t) &= \operatorname{res}_x \left(P + tx^s, P' + stx^{s-1}\right) \\
&= \operatorname{res}_x \left(P + tx^s - \frac{1}{s} x \left(P' + stx^{s-1}\right), P' + stx^{s-1}\right) \\
&= \operatorname{res}_x \left(P - \frac{1}{s} xP', P' + stx^{s-1}\right) \\
&= \operatorname{res}_x \left(Q \frac{P - \frac{1}{s} xP'}{Q}, P' + stx^{s-1}\right) \\
&= \operatorname{res}_x \left(Q, P' + stx^{s-1}\right) \operatorname{res}_x \left(\frac{P - \frac{1}{s} xP'}{Q}, P' + stx^{s-1}\right) \\
&= \operatorname{res}_x \left(Q, stx^{s-1}\right) \operatorname{res}_x \left(\frac{P - \frac{1}{s} xP'}{Q}, P' + stx^{s-1}\right).
\end{aligned}$$

Now we claim the following for the first and the second factor respectively:

1. $t^{n-m} \big| \operatorname{res}_x \left(Q, stx^{s-1}\right)$, and
2. $t \big| \operatorname{res}_x \left(\frac{P - \frac{1}{s} xP'}{Q}, P' + stx^{s-1}\right)$ if and only if 0 is a multiple root of P.

For the first claim, we have

$$\begin{aligned}
\operatorname{res}_x \left(Q, stx^{s-1}\right) &= \operatorname{res}_x (Q, t) \operatorname{res}_x \left(Q, sx^{s-1}\right) \\
&= \operatorname{res}_x \left(\prod_{1 \le i \le m} (x - r_i)^{\mu_i - 1}, t\right) \operatorname{res}_x \left(\prod_{1 \le i \le m} (x - r_i)^{\mu_i - 1}, sx^{s-1}\right) \\
&= t^{n-m} \cdot s^{n-m} \cdot \prod_{1 \le i \le m} (r_i - 0)^{(\mu_i - 1)(s-1)} \\
&= t^{n-m} \cdot s^{n-m} \cdot \prod_{1 \le i \le m} r_i^{(\mu_i - 1)(s-1)}.
\end{aligned}$$

Therefore the first factor is indeed a multiple of t^{n-m}. The first part of our lemma follows from this claim. Notice also that if 0 is not a multiple root of P, then $\operatorname{res}_x \left(Q, stx^{s-1}\right) = \alpha t^{n-m}$ for some non-zero constant α.

For the second claim, we will compute the constant term of $\operatorname{res}_x \left(\frac{P - \frac{1}{s} xP'}{Q}, P' + stx^{s-1}\right)$, which is

$$\operatorname{res}_x \left(\frac{P - \frac{1}{s} xP'}{Q}, P' + stx^{s-1}\right)\bigg|_{t=0} = \operatorname{res}_x \left(\frac{P - \frac{1}{s} xP'}{Q}, P'\right)$$

$$
\begin{aligned}
&= \operatorname{res}_x\left(\frac{P - \frac{1}{s}xP'}{Q}, Q\frac{P'}{Q}\right)\\
&= \operatorname{res}_x\left(\frac{P - \frac{1}{s}xP'}{Q}, Q\right)\operatorname{res}_x\left(\frac{P - \frac{1}{s}xP'}{Q}, \frac{P'}{Q}\right)\\
&= \operatorname{res}_x\left(\frac{P - \frac{1}{s}xP'}{Q}, Q\right)\operatorname{res}_x\left(\frac{P}{Q}, \frac{P'}{Q}\right).
\end{aligned}
$$

Notice that $\operatorname{res}_x\left(\frac{P}{Q}, \frac{P'}{Q}\right)$ is clearly non-zero, as these polynomials are coprime. So it suffices to show that $\operatorname{res}_x\left(\frac{P-\frac{1}{s}xP'}{Q}, Q\right)$ vanishes if and only if 0 is a multiple root of P. A direct computation shows

$$
\begin{aligned}
\operatorname{res}_x\left(\frac{P - \frac{1}{s}xP'}{Q}, Q\right) &= \operatorname{res}_x\left(\frac{\prod_{i=1}^m (x-r_i)^{\mu_i} - \frac{1}{s}x\sum_{i=1}^m \mu_i(x-r_i)^{\mu_i-1}\prod_{j\neq i}(x-r_j)^{\mu_j}}{\prod_{i=1}^m (x-r_i)^{\mu_i-1}}, \prod_{i=1}^m (x-r_i)^{\mu_i-1}\right)\\
&= \operatorname{res}_x\left(\prod_{i=1}^m (x-r_i) - \frac{1}{s}x\sum_{i=1}^m \mu_i\prod_{j\neq i}(x-r_j), \prod_{i=1}^m (x-r_i)^{\mu_i-1}\right)\\
&= (-1)^{m(n-m)}\prod_{k=1}^m\left(\prod_{i=1}^m (r_k-r_i) - \frac{1}{s}r_k\sum_{i=1}^m \mu_i\prod_{j\neq i}(r_k-r_j)\right)^{\mu_k-1}\\
&= (-1)^{m(n-m)}\prod_{k=1}^m\left(-\frac{1}{s}r_k\mu_k\prod_{j\neq k}(r_k-r_j)\right)^{\mu_k-1}.
\end{aligned}
$$

The last line is obtained by observing that any product inside the biggest parenthesis in the previous line would be zero if $i = k$ or $j = k$.

It is now obvious that $\operatorname{res}_x\left(\frac{P-\frac{1}{s}xP'}{Q}, Q\right)$ is zero if and only if $r_k = 0$ and $\mu_k > 1$, i.e. 0 is a multiple root of P. This shows our second claim and the second part of the lemma follows. $\square$

Now we are ready to prove Proposition 18.

Proof of Proposition 18. We divide the proof into the following four steps:

1. Show $Z_k \subseteq V\left(\left\{D, \frac{\partial D}{\partial a_s}, \ldots, \frac{\partial^{n-k-1}D}{\partial a_s^{n-k-1}}\right\}\right)$.

   Suppose $\gamma \in Z_k$ and the number of distinct roots of F_γ is m. Then $m \leq k$. By Lemma 19, we have

   $$(\partial_{a_s}^{\ell} D)(\gamma) = 0 \ \text{ for } s \geq 1, \ell = 0, \ldots, n-m-1.$$

   Since $m \leq k$, we have $n-k-1 \leq n-m-1$ and thus

   $$(\partial_{a_s}^{0} D)(\gamma) \ = \ \cdots \ = \ (\partial_{a_s}^{n-k-1} D)(\gamma) \ = \ 0.$$

   In other words, $\gamma \in V\left(\left\{D, \frac{\partial D}{\partial a_s}, \ldots, \frac{\partial^{n-k-1}D}{\partial a_s^{n-k-1}}\right\}\right)$.

2. Show $Z_k \supseteq V\left(\left\{D, \dfrac{\partial D}{\partial a_s}, \ldots, \dfrac{\partial^{n-k-1} D}{\partial a_s^{n-k-1}}\right\}\right) \backslash V(a_0)$.

   We show by contradiction. Let $\gamma \in V\left(\left\{D, \dfrac{\partial D}{\partial a_s}, \ldots, \dfrac{\partial^{n-k-1} D}{\partial a_s^{n-k-1}}\right\}\right) \backslash V(a_0)$ but suppose that $\gamma \notin Z_k$. Let $m > k$ be the number of distinct roots of F_γ. By Lemma 19, $\dfrac{\partial^{n-m} D}{\partial a_s}$ does not vanish at γ since $\gamma_0 \neq 0$ and 0 is not even a root of F_γ. However, this contradicts with our choice of γ. So we must have $\gamma \in Z_k$.

3. Show $Z_k \backslash V(a_0) = V\left(\left\{D, \dfrac{\partial D}{\partial a_s}, \ldots, \dfrac{\partial^{n-k-1} D}{\partial a_s^{n-k-1}}\right\}\right) \backslash V(a_0)$.

   Combining 1 and 2, we have

$$V\left(\left\{D, \frac{\partial D}{\partial a_s}, \ldots, \frac{\partial^{n-k-1} D}{\partial a_s^{n-k-1}}\right\}\right) \backslash V(a_0) \;\subseteq\; Z_k \;\subseteq\; V\left(\left\{D, \frac{\partial D}{\partial a_s}, \ldots, \frac{\partial^{n-k-1} D}{\partial a_s^{n-k-1}}\right\}\right).$$

   Removing $V(a_0)$ from the above three sets, we obtain that the desired relationship.

4. Show $Z_k = \overline{Z_k \backslash V(a_0)}$.

   Note that none of the irreducible components of Z_k is contained in $V(a_0)$. So if W is an irreducible component of Z_k, $\overline{W \backslash V(a_0)} = W$. It follows that

$$\begin{aligned}
\overline{Z_k \backslash V(a_0)} &= \overline{\left(\bigcup_{W \in \mathrm{Comp}(Z_k)} W\right) \backslash V(a_0)} \\
&= \overline{\bigcup_{W \in \mathrm{Comp}(Z_k)} (W \backslash V(a_0))} \\
&= \bigcup_{W \in \mathrm{Comp}(Z_k)} \overline{W \backslash V(a_0)} \\
&= \bigcup_{W \in \mathrm{Comp}(Z_k)} W \\
&= Z_k.
\end{aligned}$$

Finally, the proof is completed. □

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