ON MINIMIZING SURFACES OF THE CR INVARIANT ENERGY E_1

JIH-HSIN CHENG, HUNG-LIN CHIU, PAUL YANG, AND YONGBING ZHANG

Abstract. We study a CR-invariant equation for vanishing E_1 surfaces in the 3-dimensional Heisenberg group. This is shown to be a hyperbolic equation. We prove the local uniqueness theorem for an initial value problem and classify all such global surfaces with rotational symmetry. We also show that the Clifford torus in the CR 3-sphere is not a local minimizer of E_1 by computing the second variation.

1. Introduction and statement of the results

Let (M,ξ) be a contact 3-manifold M with contact structure ξ. A CR structure is an endomorphism $J:\xi\to\xi$ such that $J^2=-I$. Consider a (smooth: meaning "C^∞-smooth" unless specified otherwise) surface Σ in the CR manifold (M,ξ,J). A CR invariant surface element is a 2-form dA_Σ defined on Σ such that

$$dA_{\Sigma_1}=\varphi^*(dA_{\Sigma_2})$$

for any (local) CR diffeomorphism φ from a neighborhood U_1 of Σ_1 to a neighborhood U_2 of Σ_2 such that $\varphi(\Sigma_1)=\Sigma_2$. In the paper [Che95] of early 90's, one of the authors discovered two CR invariant surface elements dA_1 and dA_2, via Cartan-Chern's method of admissible frames. In a subsequent paper [CYZ18], three of the authors show that for a nonsingular ($T\Sigma\neq\xi$) surface $\Sigma\subset M$, they can express dA_1 and dA_2 by geometric quantities e_1 (characteristic unit tangent), α (the deviation function), H (the p-mean curvature) obtained from the ambient pseudohermitian structure (J,Θ) (see a short review for related quantities in Section 2). In particular, we have (see [CYZ18, Theorem 2] or Proposition 2.2 in Section 2)

$$dA_1=|e_1\alpha+\frac{1}{2}\alpha^2+\frac{1}{6}H^2+\frac{1}{4}W-\operatorname{Im}A_{11}|^{3/2}\Theta\wedge e^1. \tag{1.1}$$

Therefore we can define the CR invariant energy functionals: (assuming the set of singular points has measure 0 in Σ)

$$E_1(\Sigma)=\int_\Sigma dA_1.$$

First note that $E_1(\Sigma)=0$ for a (smooth) surface $\Sigma\subset M$=the Heisenberg group H_1 if and only if the following equation holds true (on the nonsingular domain where e_1, α, H are defined; for H_1, the Webster (scalar) curvature $W=0$ and the torsion $A_{11}=0$):

$$e_1\alpha+\frac{1}{2}\alpha^2+\frac{1}{6}H^2=0. \tag{1.2}$$

We often call Σ or its graph (smooth) function u an absolute E_1 energy minimizer or absolute E_1-minimizer (with vanishing energy).

2020 *Mathematics Subject Classification.* 32V05, 53C45, 53C17.

Key words and phrases. Pseudohermitian structure, Heisenberg group, CR-invariant surface, E_1-energy, hyperbolic equation, initial value problem, rotational symmetry.

On the region where $\alpha \neq 0$, a surface can be defined by the graph of a (smooth) function $t = u(x,y)$ on a (nonsingular: $D \neq 0$) domain of the xy-plane. In terms of $u(x,y)$, equation (1.2) reads as

$$\begin{aligned} 0 &= F(x,y,u,u_x,u_y,u_{xx},u_{xy},u_{yy}) \\ &= \left(\frac{u_y+x}{D}\right) D_x - \left(\frac{u_x-y}{D}\right) D_y - \frac{1}{6}\left(\Delta u - \left(\frac{u_x-y}{D}\right) D_x - \left(\frac{u_y+x}{D}\right) D_y\right)^2 - \frac{1}{2} \end{aligned} \tag{1.3}$$

where $D := [(u_x-y)^2+(u_y+x)^2]^{1/2}$ (see Section 2 for the deduction). By Proposition 2.1 in Section 2, we have

$$F_{u_{xx}}F_{u_{yy}} - \frac{1}{4}F_{u_{xy}}^2 = -\frac{1}{4} < 0,$$

meaning that (1.3) is a second-order hyperbolic equation in the sense that the linearized operator is hyperbolic (see, for instance, [Eva98, (89) on p.398]). Let (r,ϕ) denote the polar coordinates of the xy-plane. Write

$$\cos\theta = \frac{u_x-y}{D}, \quad \sin\theta = \frac{u_y+x}{D}. \tag{1.4}$$

For a curve to be characteristic, see (3.21). We have the uniqueness of absolute E_1-minimizers for the following initial-value problem.

Theorem 1.1. *Suppose that on a nonsingular domain $u = u(r,\phi)$ and $v = v(r,\phi)$ are two (smooth) absolute E_1-minimizers having the same initial values H, α and θ on a non-characteristic (smooth) curve $r = \mathbf{c} > 0$ with respect to u (or U, see (3.15)), lying on the region where $\det A(r,\phi,U) \neq 0$ (see the notation $\det A(r,\phi,U)$ in (3.17)). Then $u = v$ on a neighborhood of the initial curve $r = \mathbf{c}$.*

Note that θ, α involve first derivatives of u while H involves u-derivatives of second order. The $E_1 = 0$ equation (1.2) and the integrability condition (3.3) form the first two equations of (3.13) via (3.4). Notice that H is involved as a coefficient in this two-equations system, so it is not a quasi-linear system. Therefore we need to enlarge the set of variables including H $(= -N^{\perp}\theta)$, but then $N^{\perp}\alpha$ should also be included to make the system "perfect" (so that the Coddazi-like equation (3.1) also plays its role). We finally need to consider a system of four equations, which is quasi-linear; see (3.13). This special structure for a second order nonlinear hyperbolic equation arising from CR geometry is intriguing and worth further study in the future.

Next we classify all complete (say, with respect to the metric induced from Euclidean $\mathbb{R}^3$), rotationally symmetric E_1-minimizer with vanishing energy, i.e. satisfying (1.2).

Theorem 1.2. *With the notation above, we conclude that the complete (smooth), rotationally symmetric absolute E_1-minimizers are divided into four classes up to CR transformations on the Heisenberg group H_1:*

(1) $u(r) = \frac{\sqrt{3}}{2}r^2$,
(2) $u(r) = -\frac{\sqrt{3}}{2}r^2$,
(3) Type I shifted Heisenberg spheres: $\left(r^2 + \frac{\sqrt{3}}{2}\rho_0^2\right)^2 + 4u(r)^2 = \rho_0^4$ *for* $\rho_0 > 0$,
(4) Type II shifted Heisenberg spheres: $\left(r^2 - \frac{\sqrt{3}}{2}\rho_0^2\right)^2 + 4u(r)^2 = \rho_0^4$ *for* $\rho_0 > 0$.

We can obtain the solutions $u(r) = \pm\frac{\sqrt{3}}{2}r^2$ in (1)/(2) of Theorem 1.2 as a limit of a family of dilations applied to type I/II shifted Heisenberg spheres. See Remark 4.8 for a detailed explanation. One may ask whether type I and type II shifted Heisenberg spheres are the only closed (compact with no boundary), (smooth) absolute E_1-minimizers up to CR transformations on H_1 with no assumption on rotational symmetry.

The Clifford torus in the standard CR 3-sphere is a critical point of E_1 and we had conjectured that it is a global minimizer for E_1 among all surfaces of torus type [CYZ18]. However it turns out that the Clifford torus is actually not a local minimizer for E_1. We show this by computing the second variation of E_1 for the Clifford torus. Let $F_t : \Sigma \to S^3$ be a family of immersions such that $F_{t=0}$ =inclusion and

$$\frac{d}{dt}F_t = X = fe_2 + gT,$$

where T is the Reeb vector field, $e_2 = Je_1$ for e_1 being a unit vector in $T\Sigma_t \cap \xi$, $\Sigma_t := F_t(\Sigma)$. We assume f and g are supported in a domain of Σ away from the singular set of Σ. We have the following second variation formula:

Theorem 1.3. *Let Σ be the Clifford torus $\Sigma_{[1/\sqrt{2}]}$ in the CR 3-sphere S^3. Then it holds that*

$$\frac{d^2}{dt^2}|_{t=0}E_1(\Sigma_t) = \frac{\sqrt{2}}{4}\int_{\Sigma_{[1/\sqrt{2}]}} [e_1e_1(f)^2 + 3e_1T(f)^2 - 7e_1(f)^2 + 9fTT(f) + 12f^2]\Theta \wedge e^1. \tag{1.5}$$

Recall (see Subsection 5.3) that for the Clifford torus $\Sigma_{[1/\sqrt{2}]}$, $e_1 = -\frac{\partial}{\partial\varphi_1} + \frac{\partial}{\partial\varphi_2}$, $V = T = \partial_{\varphi_1} + \partial_{\varphi_2}$, $z = (z_1, z_2) = \frac{1}{\sqrt{2}}(e^{i\varphi_1}, e^{i\varphi_2})$. We may take $f(z) = v_l(z) = \cos l(\varphi_1 + \varphi_2)$, $\quad l = 1, 2, \cdots$. Then $TT(v_l) = -4l^2 v_l$, $e_1(v_l) = 0$ and for

$$\frac{d}{dt}\Sigma_t = v_l e_2,$$

we have $f = v_l$ and

$$\begin{aligned}\frac{d^2}{dt^2}|_{t=0}E_1(\Sigma_t) &= \frac{\sqrt{2}}{4}\int_{\Sigma_{[1/\sqrt{2}]}} [9v_lTT(v_l) + 12v_l^2]\Theta \wedge e^1 \\ &= 3\sqrt{2}\int_{\Sigma_{[1/\sqrt{2}]}} (1 - 3l^2)v_l^2\Theta \wedge e^1 \\ &< 0.\end{aligned}$$

Corollary 1.4. *The Clifford torus $\Sigma_{[1/\sqrt{2}]}$ is not an E_1-minimizer among all surfaces of torus type in the CR 3-sphere S^3.*

We wonder what E_1-minimizers are if exist in the above corollary.

The paper is organized as follows. In Section 2, we show the hyperbolicity of equation (1.3) and the CR invariance of dA_1 (see (1.1)). In Section 3, we prove Theorem 1.1. In Section 4, we show the classification theorem Theorem 1.2. Finally Theorem 1.3 is proved in Section 5.

Acknowledgement 1.5. J.-H. Cheng (resp. H.-L. Chiu) would like to thank the National Science and Technology Council of Taiwan for the support: grant no. 112-2115-M-001-012 (resp. 112-2115-M-007-009-MY3). J.-H. Cheng would also like to thank the National Center for Theoretical Sciences for the constant support. P. Yang acknowledges support from the

NSF for the grant DMS 1509505. Y. Zhang acknowledges support from National Key Research and Development Project SQ2020YFA070080 and NSFC 12071450.

2. Hyperbolicity and CR invariance

We refer basic notations such as contact form Θ, characteristic unit tangent e_1, p-mean curvature H, Legendrian normal e_2, Reeb vector field T, "the deviation function" α, etc. to [CHMY05, Section 2]. For a (nonsingular: $D \neq 0$) graph $t = u(x,y)$ in the Heisenberg group, We can express e_1, α, H as follows:

$$\begin{aligned} e_1 &= -\frac{u_y + x}{D}(\frac{\partial}{\partial x} + y\frac{\partial}{\partial t}) + \frac{u_x - y}{D}(\frac{\partial}{\partial y} - x\frac{\partial}{\partial t}), \\ \alpha &= \mp 1/D,\ D := [(u_x - y)^2 + (u_y + x)^2]^{1/2}, \\ H &= D^{-3}\{(u_y + x)^2 u_{xx} - 2(u_y + x)(u_x - y)u_{xy} + (u_x - y)^2 u_{yy}\} \end{aligned} \tag{2.1}$$

([CHMY05, p.137 and (2.10)]). Substituting (2.1) into (1.2) yields (1.3). We then observe that (1.3) is a hyperbolic equation in the sense that the linearized operator is hyperbolic [Eva98, (89) on p.398].

Proposition 2.1. *Let $F = F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy})$ be as in (1.3). Then we have*

$$F_{u_{xx}} F_{u_{yy}} - \frac{1}{4}F^2_{u_{xy}} = -\frac{1}{4} < 0 \tag{2.2}$$

Proof. From (1.4) we recall that θ satisfies

$$\cos\theta = \frac{u_x - y}{D}, \sin\theta = \frac{u_y + x}{D}. \tag{2.3}$$

In terms of θ, D and H, we write (see (1.3) for F)

$$F = (\sin\theta)D_x - (\cos\theta)D_y - \frac{1}{6}D^2H^2 - \frac{1}{2} \tag{2.4}$$

where we have used

$$\begin{aligned} A &:= \Delta u - (\cos\theta)D_x - (\sin\theta)D_y \\ &= (\sin^2\theta)u_{xx} - (2\sin\theta\cos\theta)u_{xy} + (\cos^2\theta)u_{yy} \overset{(2.1)}{=} DH. \end{aligned} \tag{2.5}$$

By the chain rule we easily get

$$\begin{aligned} D_x &= (\cos\theta)u_{xx} + (\sin\theta)(u_{yx} + 1), \\ D_y &= (\cos\theta)(u_{xy} - 1) + (\sin\theta)u_{yy}. \end{aligned} \tag{2.6}$$

From (2.4), (2.6) and (2.5), we can then compute

$$F_{u_{xx}} = \sin\theta\cos\theta - \frac{1}{6}\cdot 2AA_{u_{xx}} = \sin\theta\cos\theta - \frac{1}{3}A\sin^2\theta. \tag{2.7}$$

Similarly, we also obtain

$$\begin{aligned} F_{u_{yy}} &= -\cos\theta\sin\theta - \frac{1}{3}A\cos^2\theta, \\ F_{u_{xy}} &= F_{u_{yx}} = \sin^2\theta - \cos^2\theta + \frac{2}{3}A\sin\theta\cos\theta. \end{aligned} \tag{2.8}$$

Finally, a direct computation using (2.7) and (2.8) yields (2.2). $\square$

We recall the CR invariance of the surface area element dA_1.

Proposition 2.2. *(*[CYZ18, Theorem 2]*) On a domain of nonsingular points of a surface, the surface area element*

$$dA_1 = |e_1\alpha + \frac{1}{2}\alpha^2 + \frac{1}{6}H^2 + \frac{1}{4}W - \operatorname{Im} A_{11}|^{3/2}\Theta \wedge e^1$$

is a (pointwise) CR invariant, i.e. under the conformal change of contact form: $\tilde{\Theta} = \lambda^2\Theta$, $\lambda > 0$, *we have* $d\tilde{A}_1 = dA_1$.

Proof. (direct verification which is different from the original proof via the Cartan method) The transformation laws of e_1, e_2, T, α, H, W and A_{11} under the conformal change of contact form: $\tilde{\Theta} = \lambda^2\Theta$, $\lambda > 0$, read as follows (on a domain of nonsingular points): *i)* $\tilde{e}_1 = \lambda^{-1}e_1$, *ii)* $\tilde{e}_2 = \lambda^{-1}e_2$, *iii)* $\tilde{T} = \lambda^{-2}T + \lambda^{-3}(e_2(\lambda)e_1 - e_1(\lambda)e_2)$, *iv)* $\tilde{\alpha} = \lambda^{-1}\alpha + \lambda^{-2}e_1(\lambda)$, *v)* $\tilde{H} = \lambda^{-1}H - 3\lambda^{-2}e_2(\lambda)$, *vi)* $\tilde{W} = \lambda^{-2}(W - 4\lambda^{-1}\Delta_b\lambda)$, *vii)* $\tilde{A}_{11} = \lambda^{-2}(A_{11} + 2i\lambda^{-1}\lambda_{11} - 6i\lambda^{-2}\lambda_1\lambda_1)$. A lengthy but straightforward computation yields

$$|\tilde{e}_1\tilde{\alpha} + \frac{1}{2}\tilde{\alpha}^2 + \frac{1}{6}\tilde{H}^2 + \frac{1}{4}\tilde{W} - \operatorname{Im}\tilde{A}_{11}| = \lambda^{-2}|e_1\alpha + \frac{1}{2}\alpha^2 + \frac{1}{6}H^2 + \frac{1}{4}W - \operatorname{Im} A_{11}|$$

while $\tilde{\Theta} \wedge \tilde{e}^1 = \lambda^3\Theta \wedge e^1$. That $d\tilde{A}_1 = dA_1$ follows. $\square$

3. Proof of Theorem 1.1: the local uniqueness

For a nonsingular ($T\Sigma \neq \xi$) surface Σ in a contact 3-manifold (M, ξ). We choose a characteristic tangent $e_1 \in T\Sigma \cap \xi$, unit with respect to the Levi-metric $\frac{1}{2}d\Theta(\cdot, J\cdot)$ where Θ is the stardard contact form in $M = H_1$. For basic material in the theory of surfaces in the Heisenberg group, we refer the reader to [CHMY05] and [CHMY12]. We denote by $N^\perp$ (resp. N) the projection of $-e_1$ (resp. $-(e_2 + \alpha^{-1}T)$) onto the domain of u in the xy-plane. From the theory of hyperbolic equations, we find out two "characteristic" directions to be $N^\perp$ and $N + \frac{1}{3}\alpha^{-1}HN^\perp$: (see the formulas for $F_{u_{xx}}$, $F_{u_{xy}}$, $F_{u_{yy}}$ in (2.7) and (2.8))

$$\begin{aligned}&(N + \frac{1}{3}\alpha^{-1}HN^\perp) \circ N^\perp\\ =&F_{u_{xx}}\partial_x^2 + F_{u_{xy}}\partial_x\partial_y + F_{u_{yy}}\partial_y^2 \text{ mod lower order terms.}\end{aligned}$$

The Codazzi-like equation (see, for instance, [CYZ18, (3.24) with $W = 0$]) reads as

$$-\alpha N(H) = N^\perp N^\perp\alpha - 6\alpha N^\perp(\alpha) + 4\alpha^3 + \alpha H^2. \tag{3.1}$$

We have the following identities (see (1.4) for the notation θ)

$$N^\perp\theta = -H \tag{3.2}$$

(see, for instance, [CHMY12, (1.14) with $V = N^\perp$], not to be confused with the notation V in [CYZ18]) and

$$N^\perp(\alpha) - \alpha N(\theta) - 2\alpha^2 = 0. \tag{3.3}$$

(see, for instance, [CHMY12, (1.15) with the notation D replaced by $-\alpha^{-1}$ and $V = N^\perp$]).

Proposition 3.1. *For an absolute E_1-minimizer, we have*

$$\alpha(N + \frac{1}{3}\alpha^{-1}HN^\perp)\theta + N^\perp(\alpha) = -\alpha^2. \tag{3.4}$$

Proof. First of all, we notice that, writing down in the domain of u, (1.2) reads as

$$2N^{\perp}(\alpha) = \alpha^2 + \frac{1}{3}H^2. \tag{3.5}$$

On the other hand,

$$\begin{aligned} \alpha(N + \frac{1}{3}\alpha^{-1}HN^{\perp})\theta &= \alpha N\theta - \frac{1}{3}H^2 \quad \text{(by (3.2))} \\ &= N^{\perp}\alpha - 2\alpha^2 - \frac{1}{3}H^2 \quad \text{(by (3.3))} \\ &= -N^{\perp}(\alpha) - \alpha^2 \quad \text{(by (3.5))}. \end{aligned} \tag{3.6}$$

This completes the proof. □

Later we will see that, in order to get a quasi-linear system, we need an extra equation including the term $N^{\perp}H$, or $N^{\perp}N^{\perp}\theta$.

Proposition 3.2. *For an absolute E_1-minimizer, we have*

$$-\alpha(NH) - \alpha H^2 + 2\alpha^2(N\theta) + \frac{2}{3}HN^{\perp}N^{\perp}\theta + N^{\perp}N^{\perp}\alpha = -2\alpha(N^{\perp}\alpha). \tag{3.7}$$

Proof. We take $N^{\perp}$-derivative of (3.4) to get

$$\left((N^{\perp}\alpha)N + \alpha N^{\perp}N + \frac{1}{3}(N^{\perp}H)N^{\perp} + \frac{1}{3}HN^{\perp}N^{\perp}\right)\theta + N^{\perp}N^{\perp}\alpha = -2\alpha(N^{\perp}\alpha), \tag{3.8}$$

where

$$\frac{1}{3}(N^{\perp}H)N^{\perp}\theta = \frac{1}{3}(-N^{\perp}N^{\perp}\theta)(-H) = \frac{1}{3}HN^{\perp}N^{\perp}\theta. \tag{3.9}$$

Substituting (3.9) into (3.8), we get

$$(N^{\perp}\alpha)N\theta + \alpha N^{\perp}N\theta + \frac{2}{3}HN^{\perp}N^{\perp}\theta + N^{\perp}N^{\perp}\alpha = -2\alpha(N^{\perp}\alpha), \tag{3.10}$$

where

$$N^{\perp}N\theta = NN^{\perp}\theta + [N^{\perp}, N]\theta = -NH + [N^{\perp}, N]\theta. \tag{3.11}$$

On the other hand, writing down the formula (see [CYZ18, (3.42)])

$$[e_1, V] = -\alpha He_1 - 2\alpha V, \quad \text{where } V = \alpha e_2 + T,$$

in the domain of u, we get

$$[-N^{\perp}, -\alpha N] = \alpha HN^{\perp} + 2\alpha^2 N,$$

that is,

$$\alpha[N^{\perp}, N] = \alpha HN^{\perp} + (2\alpha^2 - N^{\perp}\alpha)N. \tag{3.12}$$

Substituting (3.12) into (3.11) and (3.11) into (3.10) respectively, we obtain (3.7). This completes the proof. □

We take all equations (3.1), (3.3), (3.4) and (3.7) together to obtain a system as follows

(3.13)
$$\begin{pmatrix} \alpha(N+\frac{1}{3}\alpha^{-1}HN^{\perp}) & N^{\perp} & 0 & 0 \\ -\alpha N & N^{\perp} & 0 & 0 \\ 0 & 6\alpha N^{\perp} & -\alpha N & N^{\perp} \\ 2\alpha^2 N & 2\alpha N^{\perp} & -\alpha N - \frac{2}{3}HN^{\perp} & -N^{\perp} \end{pmatrix} \begin{pmatrix} \theta \\ \alpha \\ H \\ -N^{\perp}\alpha \end{pmatrix} = \begin{pmatrix} -\alpha^2 \\ 2\alpha^2 \\ 4\alpha^3 + \alpha H^2 \\ \alpha H^2 \end{pmatrix}$$

Notice that (3.13) only holds on the region where $\alpha \neq 0$. In terms of polar coordinates (r, ϕ), we have (see also (4.19))

(3.14)
$$\begin{aligned} N^{\perp} &= \sin(\theta-\phi)\partial_r - \frac{\cos(\theta-\phi)}{r}\partial_\phi \\ N &= \cos(\theta-\phi)\partial_r + \frac{\sin(\theta-\phi)}{r}\partial_\phi. \end{aligned}$$

Writing U as the column vector

(3.15)
$$U = \begin{pmatrix} \theta \\ \alpha \\ H \\ -N^{\perp}\alpha \end{pmatrix},$$

and $s = \sin(\theta-\phi)$, $c = \cos(\theta-\phi)$, then (3.13) reads as

(3.16)
$$A(r,\phi,U)\frac{\partial U}{\partial r} + B(r,\phi,U)\frac{\partial U}{\partial \phi} + C(r,\phi,U)U = 0,$$

where

(3.17)
$$A(r,\phi,U) = \begin{pmatrix} \frac{1}{3}sH + c\alpha & s & 0 & 0 \\ -c\alpha & s & 0 & 0 \\ 0 & 6s\alpha & -c\alpha & s \\ 2c\alpha^2 & 2s\alpha & -\frac{2}{3}sH - c\alpha & -s \end{pmatrix}$$

(3.18)
$$B(r,\phi,U) = \begin{pmatrix} -\frac{1}{3}\frac{c}{r}H + \frac{s}{r}\alpha & -\frac{c}{r} & 0 & 0 \\ -\frac{s}{r}\alpha & -\frac{c}{r} & 0 & 0 \\ 0 & -6\frac{c}{r}\alpha & -\frac{s}{r}\alpha & -\frac{c}{r} \\ 2\frac{s}{r}\alpha^2 & -2\frac{c}{r}\alpha & \frac{2}{3}\frac{c}{r}H - \frac{s}{r}\alpha & \frac{c}{r} \end{pmatrix},$$

(3.19)
$$C(r,\phi,U) = (-1)\begin{pmatrix} 0 & -\alpha & 0 & 0 \\ 0 & 2\alpha & 0 & 0 \\ 0 & 4\alpha^2 & \alpha H & 0 \\ 0 & 0 & \alpha H & 0 \end{pmatrix}.$$

Equation (3.16) is a quasi-linear equation of first order. The Cauchy problem for (3.16) prescribes the value of U on a curve $r = r(\phi)$ in the ϕr-plane

(3.20)
$$U = U(r(\phi), \phi) = f(\phi).$$

The curve is characteristic [John82, p.47] if

(3.21)
$$\det(Ad\phi - Bdr) = 0$$

for each point $(r(\phi), \phi)$ on the curve. It depends on the solution U. On the region where $\det A(r, \phi, U) \neq 0$, the quasi-linear equation (3.16) becomes

$$U_r + a(r, \phi, U)U_\phi + b(r, \phi, U) = 0, \tag{3.22}$$

where $a = A^{-1}B$ and $b = A^{-1}CU$. On this region, we obtain immediately from the characteristic differential equation (3.21) that the curve $r = \mathbf{c} > 0$, where $\frac{dr}{d\phi} = 0$, is always non-characteristic. The initial-value problem to be studied here is to prescribe the values of U on the curve $r = \mathbf{c} > 0$, which is assumed to be non-characteristic: $U(\mathbf{c}, \phi) = f(\phi)$.

3.1. **The smooth category.** In this subsection, we show the uniqueness theorem for the initial-value problem.

Theorem 3.3. *Let U be a solutions to (3.16) or to (3.22) and the curve $r = \mathbf{c}$ is lying in the region where $\det A(r, \phi, U) \neq 0$. If V is another solution to (3.16) or to (3.22) having the same initial values as U, on the curve $r = \mathbf{c} > 0$. Then $U = V$ on a neighborhood of the initial curve $r = \mathbf{c}$.*

Proof. Notice that

$$\det A(r, \phi, U) = s^2(\frac{1}{3}sH + 2c\alpha)(\frac{2}{3}sH + 2c\alpha). \tag{3.23}$$

Since $\det A(r, \phi, U) \neq 0$, and V and U have the same initial values on the curve $r = \mathbf{c}$, we conclude that $\det A(r, \phi, V) \neq 0$ on a neighborhood of the initial curve $r = \mathbf{c}$. We consider the vector function

$$Z(r, \phi) := V - U.$$

Then Z has the initial values zero on the curve $r = \mathbf{c}$.

Subtracting the differential equation (3.22) for U from that for V we have, omitting explicit reference to r and ϕ,

$$Z_r + a(U)Z_\phi + [a(V) - a(U)]V_\phi + b(V) - b(U) = 0. \tag{3.24}$$

On the other hand, we may apply the mean value theorem and have

$$a(V) - a(U) = h(V, U)Z; \quad b(V) - b(U) = k(V, U)Z,$$

for some h and k which are both smooth. We now consider U, V, V_ϕ as known expressions in r, ϕ and substitute these expressions in h and k as well as in $a(U)$; thereby (3.24) becomes a linear differential equation for Z of the form

$$Z_r + a(r, \phi)Z_\phi + b(r, \phi)Z = 0, \tag{3.25}$$

with initial values zero. Here $a(r, \phi) = a(U) = a(r, \phi, U)$. In the next subsection, we will show that the system (3.25) is hyperbolic (see [John82, p.48]) in the sense that there exists a complete set of real eigenvectors $\xi^1, \xi^2, \xi^3, \xi^4$ of $a(r, \phi)$ by Proposition 3.5, such that

$$a(r, \phi)\xi^k = \lambda_k \xi^k,$$

where the ξ^k are linearly independent and depend regularly on r and ϕ. The uniqueness theorem for a first-order system of linear equations ([John82, Section 5 in Chapter 2]) asserts now that Z is identically zero, and thus $U = V$ around the curve $r = \mathbf{c}$. □

Proof. **(of Theorem 1.1)** From (4.19), we see that the same value for θ via Theorem 3.3 implies that the two graphs $u = u(r,\phi)$ and $v = v(r,\phi)$ have the same first fundamental form, as well as the characteristic vectors $N^{\perp}$, (or e_1). Together with the same value for α and H by Theorem 3.3, we conclude, by the fundamental theorem for surfaces in H_1 (for the details, see **Theorem 1.3** in [CL15], also cf. [CHMY12, Theorem H]), that $u = v$ on a neighborhood of the initial curve $r = \mathbf{c}$. ☐

3.2. **Hyperbolic system.** In this subsection we are going to show that the system (3.25) is hyperbolic in the sense of [John82, p.48] so that Theorem 3.3, and thus Theorem 1.1, hold. Let

$$\sigma = \frac{1}{3}sH + 2c\alpha, \quad \eta = \frac{2}{3}sH + 2c\alpha, \tag{3.26}$$

that is, from (3.23), $\det A(r,\phi,U) = s^2\sigma\eta$. Then, applying a sequence of elementary row operations, we obtain that the inverse of $A(r,\phi,U)$ is

$$A^{-1}(r,\phi,U) = \begin{pmatrix} \frac{1}{\sigma} & -\frac{1}{\sigma} & 0 & 0 \\ \frac{c\alpha}{s\sigma} & \frac{\sigma - c\alpha}{s\sigma} & 0 & 0 \\ \frac{10c\alpha^2}{\sigma\eta} & \frac{8\sigma\alpha - 10c\alpha^2}{\sigma\eta} & -\frac{1}{\eta} & -\frac{1}{\eta} \\ \frac{-6\eta c\alpha^2 + 10c^2\alpha^3}{s\sigma\eta} & \frac{8\sigma c\alpha^2 - 10c^2\alpha^3 - 6\sigma\eta\alpha + 6\eta c\alpha^2}{s\sigma\eta} & \frac{\eta - c\alpha}{s\eta} & -\frac{c\alpha}{s\eta} \end{pmatrix}, \tag{3.27}$$

and thus

$$a(r,\phi) = A^{-1}B(r,\phi,U) = \begin{pmatrix} \frac{-\frac{1}{3}cH + 2s\alpha}{r\sigma} & 0 & 0 & 0 \\ \frac{-\frac{1}{3}\alpha H}{sr\sigma} & \frac{-c}{sr} & 0 & 0 \\ \frac{-\frac{10}{3}H\alpha^2}{r\sigma\eta} & 0 & \frac{-\frac{2}{3}cH + 2s\alpha}{r\eta} & 0 \\ \frac{\frac{4}{3}s\alpha^2H^2 + \frac{2}{3}c\alpha^3 H}{sr\sigma\eta} & 0 & \frac{-\frac{2}{3}\alpha H}{sr\eta} & \frac{-c}{sr} \end{pmatrix}, \tag{3.28}$$

with eigenvalues

$$\lambda_1 = \frac{-\frac{1}{3}cH + 2s\alpha}{r\sigma}, \quad \lambda_2 = \lambda_4 = \frac{-c}{sr}, \quad \lambda_3 = \frac{-\frac{2}{3}cH + 2s\alpha}{r\eta}. \tag{3.29}$$

Proposition 3.4. *We have that* $\lambda_1 \neq 0 \quad \lambda_3 \neq 0$. *And*

(1) *If* $\alpha H \neq 0$, *then* $\lambda_1 \neq \lambda_2,\ \lambda_2 \neq \lambda_3,\ \lambda_1 \neq \lambda_3$.
(2) *If* $\alpha \neq 0,\ H = 0$, *then* $\lambda_1 \neq \lambda_2,\ \lambda_2 \neq \lambda_3,\ \lambda_1 = \lambda_3$.
(3) *If* $\alpha = 0,\ H \neq 0$, *then* $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4$.

Proof. Let

$$\tilde{\sigma} = -\frac{1}{3}cH + 2s\alpha, \quad \tilde{\eta} = -\frac{2}{3}cH + 2s\alpha, \tag{3.30}$$

it is easy to see that

$$\tilde{\sigma}(r,\phi + \frac{\pi}{2}, U) = -\sigma(r,\phi,U), \quad \tilde{\eta}(r,\phi + \frac{\pi}{2}, U) = -\eta(r,\phi,U), \tag{3.31}$$

thus, on the region where $\det A(r,\phi,U) \neq 0$, we have that $\lambda_1 \neq 0$ and $\lambda_3 \neq 0$. On the other hand, a straightforward computation shows that

$$\lambda_1 - \lambda_2 = \frac{2\alpha}{sr\sigma}, \quad \lambda_3 - \lambda_2 = \frac{2\alpha}{sr\eta}, \quad \lambda_1 - \lambda_3 = \frac{\frac{2}{3}\alpha H}{r\sigma\eta}. \tag{3.32}$$

This completes the proof of the proposition. ☐

We have

Proposition 3.5. *The eigenspaces with respect to the eigenvalues $\lambda_1, \lambda_2(=\lambda_4), \lambda_3$ respectively are spanned by the vectors*

$$\xi^1 = \begin{pmatrix} 1 \\ -\frac{1}{6}H \\ -5\alpha \\ \frac{11}{6}\alpha H \end{pmatrix}, \quad \xi^2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \xi^4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \xi^3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -\frac{1}{3}H \end{pmatrix}.$$

It is clear that the $\xi^k, k = 1 \cdots 4$ are linearly independent and depend regularly on r and ϕ. This means that the system (3.25) is hyperbolic.

Proof. We in addition put

$$\lambda_{21} = \frac{-\frac{1}{3}\alpha H}{sr\sigma}, \quad \lambda_{31} = \frac{-\frac{10}{3}H\alpha^2}{r\sigma\eta}, \quad \lambda_{41} = \frac{\frac{4}{3}s\alpha^2H^2 + \frac{2}{3}c\alpha^3H}{sr\sigma\eta}, \quad \lambda_{43} = \frac{-\frac{2}{3}\alpha H}{sr\eta}, \tag{3.33}$$

which are the other nonzero entries in $a(r,\phi)$ in (3.28). First, we suppose that $\alpha H \neq 0$. Assume that $\xi = (a_1, a_2, a_3, a_4)^t$ is an eigenvector of $a(r,\phi)$ with respect to λ_1, then $a(r,\phi)\xi = \lambda_1\xi$ means that

$$\begin{aligned} \lambda_1 a_1 &= \lambda_1 a_1 \\ \lambda_{21}a_1 + \lambda_2 a_2 &= \lambda_1 a_2 \\ \lambda_{31}a_1 + \lambda_3 a_3 &= \lambda_1 a_3 \\ \lambda_{41}a_1 + \lambda_{43}a_3 + \lambda_4 a_4 &= \lambda_1 a_4, \end{aligned}$$

which implies that

$$\xi = a_1 \begin{pmatrix} 1 \\ \frac{\lambda_{21}}{\lambda_1-\lambda_2} \\ \frac{\lambda_{31}}{\lambda_1-\lambda_3} \\ \frac{\lambda_{41}}{\lambda_1-\lambda_4} + \frac{\lambda_{31}\lambda_{43}}{(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)} \end{pmatrix},$$

for a_1 arbitrary. After a straightforward computation, and noticing that $\alpha H \neq 0$, we have

$$\xi^1 = \begin{pmatrix} 1 \\ \frac{\lambda_{21}}{\lambda_1-\lambda_2} \\ \frac{\lambda_{31}}{\lambda_1-\lambda_3} \\ \frac{\lambda_{41}}{\lambda_1-\lambda_4} + \frac{\lambda_{31}\lambda_{43}}{(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)} \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{1}{6}H \\ -5\alpha \\ \frac{11}{6}\alpha H \end{pmatrix}.$$

Similarly, since $\lambda_1 \neq \lambda_2,\ \lambda_3 \neq \lambda_2$ and $\lambda_1 \neq \lambda_3$, it is easy to get that

$$\xi^2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \xi^4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \xi^3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -\frac{1}{3}H \end{pmatrix}.$$

Next we assume that $\alpha \neq 0,\ H = 0$, then $\lambda_1 = \lambda_3$, and a similar computation shows that the eigensapce with respect to λ_1(or λ_3) is spanned by the two vectors

$$\xi^1 = \begin{pmatrix} 1 \\ 0 \\ -5\alpha \\ 0 \end{pmatrix}, \quad \xi^3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

And the eigensapce with respect to λ_2(or λ_4) is spanned by the two vectors

$$\xi^2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \xi^4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Finally, we assume that $\alpha = 0,\ H \neq 0$, then $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4$, that is, $a(r, \phi)$ is a multiple of the identity matrix of order 4. This implies that each nonzero vector is an eigenvector of $a(r, \phi)$. This completes the proof. $\square$

3.3. **The real analytic category.** In the real analytic category, $r \in \mathbb{R},\ \phi \in \mathbb{R}$ and $U \in \mathbb{R}^4$, both a and b in (3.22) can be regarded as functions defined on $\mathbb{R}^1 \times \mathbb{R}^1 \times \mathbb{R}^4$, mapping into $R^{4\times 4}$ and R^4, respectively. Looking at formulae (3.27), (3.17), (3.18) and (3.19), we see that both a and b are clearly analytic. If we in addition assume that the initial value function $f(\phi) = U(\mathbf{c}, \phi)$ is analytic, then by Cauchy-Kovalevskaya theorem, (3.16) or (3.22) has a unique solution on a neighborhood of $r = \mathbf{c}$ with initial value $f(\phi)$.

4. Proof of Theorem 1.2: the classification

4.1. **Rotationally symmetric surfaces.** Suppose in addition that it is rotationally symmetric, that is, $t = u(x, y) = u(r)$, which only depends on $r, r = (x^2 + y^2)^{1/2}$. Then (1.3) is reduced to a non-linear ODE of second order

$$\frac{1}{3} r^4 u_{rr}^2 - (\frac{4}{3} r u_r^3 + 2r^3 u_r) u_{rr} + \frac{1}{3}\frac{u_r^6}{r^2} + u_r^4 - r^4 = 0. \tag{4.1}$$

We observe that u is a solution to (4.1) if and only if $-u$ is a solution to (4.1).

Proposition 4.1. *The function $u = u(r)$ satisfies equation (4.1) if and only if it satisfies either*

$$u_{rr} = \frac{2u_r^3 + 3r^2 u_r + \sqrt{3}(u_r^2 + r^2)^{3/2}}{r^3} \tag{4.2}$$

or

$$u_{rr} = \frac{2u_r^3 + 3r^2 u_r - \sqrt{3}(u_r^2 + r^2)^{3/2}}{r^3} \tag{4.3}$$

Proof. Let

$$A(r) = \frac{1}{3} r^4, \quad B(r, u_r) = -(\frac{4}{3} r u_r^3 + 2r^3 u_r), \quad C(r, u_r) = \frac{1}{3}\frac{u_r^6}{r^2} + u_r^4 - r^4.$$

Then (4.1) can be written as

$$A(r) u_{rr}^2 + B(r, u_r) u_{rr} + C(r, u_r) = 0.$$

Thus we immediately have

$$u_{rr} = \frac{-B(r, u_r) \pm \sqrt{B^2(r, u_r) - 4A(r)C(r, u_r)}}{2A(r)}, \tag{4.4}$$

where

$$\begin{aligned} B^2(r,u_r) - 4A(r)C(r,u_r) &= \frac{4}{3}r^2u_r^6 + 4r^4u_r^4 + 4r^6u_r^2 + \frac{4}{3}r^8 \\ &= \frac{4}{3}r^2(u_r^2+r^2)^3 \geq 0. \end{aligned}$$

This completes the proof. □

Proposition 4.2. *The function $u = u(r)(or - u(r))$ satisfies equation (4.2) if and only if $-u(or\ u(r))$ satisfies equation (4.3).*

Proof. First notice that $D^2 = u_r^2 + r^2$. To reach a contradiction, we assume that both u and $-u$ satisfy (4.2). Then

$$\begin{aligned} 0 &= u_{rr} + (-u)_{rr} \\ &= \frac{2u_r^3 + 3r^2u_r + \sqrt{3}(u_r^2+r^2)^{3/2}}{r^3} + \frac{-2u_r^3 - 3r^2u_r + \sqrt{3}(u_r^2+r^2)^{3/2}}{r^3} \\ &= \frac{2\sqrt{3}(u_r^2+r^2)^{3/2}}{r^3}, \end{aligned}$$

which is a contradiction on the regular part in which $D > 0$. Similarly, if we assume that both u and $-u$ satisfy (4.3), we will also get a contradiction. This completes the proof. □

In view of Proposition 4.1 and Proposition 4.2, we see that, to study equation (4.1), it is sufficient to study equation (4.3). Since (4.3) is independent of u, essentially it is an ODE of first order for u_r. That is, if we let $v = u_r$ then (4.3) is just the first order equation for v

$$v_r = \frac{2v^3 + 3r^2v - \sqrt{3}(v^2+r^2)^{3/2}}{r^3}. \tag{4.5}$$

Moreover if we put $w = \frac{v}{r}$ then (4.5) can be expressed as a little more simple form

$$w_r = \frac{2w^3 + 2w - \sqrt{3}(1+w^2)^{3/2}}{r}, \quad r > 0. \tag{4.6}$$

which can be solved in a closed form. We observe the numerator of the right hand side of (4.6) and notice that $w_r \geq 0$ if and only if $2w^3 + 2w - \sqrt{3}(1+w^2)^{3/2} \geq 0$, which is equivalent to that $w \geq \sqrt{3}$. And it is easy to see that $w = \sqrt{3}$ is a solution to (4.6). Actually we have Theorem 4.3 for the general solutions as follows. See Figure 1 for the graph of $w = w(r)$.

Theorem 4.3. *The general solutions to (4.6) is as follows:*

(1) The constant function $w = \sqrt{3}$ is a solution.
(2) For each $\rho_0 > 0$, the function

$$w(r) = \frac{r^2 + \frac{\sqrt{3}}{2}\rho_0^2}{\sqrt{\rho_0^4 - (r^2 + \frac{\sqrt{3}}{2}\rho_0^2)^2}}, \quad r > 0, \tag{4.7}$$

is a solution. We have that $w(r) > \sqrt{3}$ for all $r > 0$. And

$$w(r) \to \begin{cases} \sqrt{3}, & \text{as } r \to 0, \\ \infty, & \text{as } r \to r_0 = \sqrt{\frac{2-\sqrt{3}}{2}}\rho_0. \end{cases}$$

Moreover, fixing a number r, we have $w(r) \to \sqrt{3}$ as $\rho_0 \to \infty$.

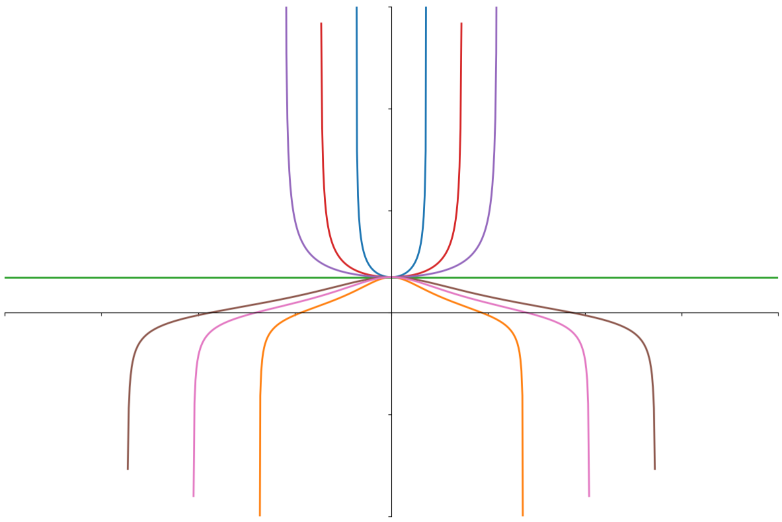

FIGURE 1. (4.7) and (4.8)

(3) For each $a > 0$ *(or* $\rho_0 > 0, a^2 = \frac{\sqrt{3}}{2}\rho_0^2$*), the function*

$$w(r) = \frac{-\sqrt{3}(r^2 - a^2)}{\sqrt{4a^4 - 3(r^2 - a^2)^2}} = \frac{-(r^2 - \frac{\sqrt{3}}{2}\rho_0^2)}{\sqrt{\rho_0^4 - (r^2 - \frac{\sqrt{3}}{2}\rho_0^2)^2}}, \quad r > 0, \tag{4.8}$$

is a solution with $w(a) = 0$*. We have that* $w(r) < \sqrt{3}$ *for all* $r > 0$*, and*

$$w(r) \to \begin{cases} \sqrt{3}, & \text{as } r \to 0, \\ -\infty, & \text{as } r \to b = \sqrt{\frac{2+\sqrt{3}}{\sqrt{3}}}a = \sqrt{\frac{2+\sqrt{3}}{2}}\rho_0. \end{cases}$$

Moreover, fixing a number r*, we have* $w(r) \to \sqrt{3}$ *as* $a \to \infty$*.*

Proof. We will only show formulae (4.7) and (4.8) for $w(r)$ while the others follow easily. As equation (4.6) is separable, taking the substitution $w = \tan\theta$, $\theta \in (\frac{-\pi}{2}, \frac{\pi}{3})$, we have

$w \in (-\infty, \sqrt{3})$ and

$$\begin{aligned}
\int \frac{dr}{r} &= \int \frac{dw}{2w^3 + 2w - \sqrt{3}(1+w^2)^{3/2}} \\
&= \int \frac{\sec^2\theta \; d\theta}{\sec^2\theta(2\tan\theta - \sqrt{3}(\sec^2\theta)^{1/2})} \\
&= \int \frac{d\theta}{2\tan\theta - \sqrt{3}\sec\theta}, \quad \sec\theta > 0 \\
&= \int \frac{\cos\theta \; d\theta}{2\sin\theta - \sqrt{3}}, \quad \cos\theta > 0 \\
&= \int \frac{d\sin\theta}{2\sin\theta - \sqrt{3}} \\
&= \frac{1}{2}\ln(\sqrt{3} - 2\sin\theta), \quad \text{where } \sqrt{3} - 2\sin\theta > 0 \\
&= \ln\left(\sqrt{3} - \frac{2w}{\sqrt{1+w^2}}\right)^{1/2}.
\end{aligned}$$

Therefore

$$\left(\sqrt{3} - \frac{2w}{\sqrt{1+w^2}}\right)^{1/2} = Cr, \tag{4.9}$$

for some constant $C > 0$. If we take $C = \left(\frac{\sqrt{3}}{a^2}\right)^{1/2}$, $a > 0$, the solution (4.9) reads as

$$\frac{4w^2}{1+w^2} = \frac{3}{a^4}(a^2 - r^2)^2$$

or

$$w = \frac{-\sqrt{3}(r^2 - a^2)}{\sqrt{4a^4 - 3(r^2 - a^2)^2}} = \frac{-(r^2 - \frac{\sqrt{3}}{2}\rho_0^2)}{\sqrt{\rho_0^4 - (r^2 - \frac{\sqrt{3}}{2}\rho_0^2)^2}}.$$

This is the formula (4.8). Similarly, for $w > \sqrt{3}$, we can show that

$$\int \frac{dr}{r} = \ln\left(\frac{2w}{\sqrt{1+w^2}} - \sqrt{3}\right)^{1/2}$$

Therefore

$$\left(\frac{2w}{\sqrt{1+w^2}} - \sqrt{3}\right)^{1/2} = Cr, \tag{4.10}$$

for some constant $C > 0$. If we take $C = \left(\frac{2}{\rho_0^2}\right)^{1/2}$, the solution (4.10) reads as

$$w = \frac{r^2 + \frac{\sqrt{3}}{2}\rho_0^2}{\sqrt{\rho_0^4 - (r^2 + \frac{\sqrt{3}}{2}\rho_0^2)^2}}.$$

This is the formula (4.7). We thus complete the proof. $\square$

Proof. **(of Theorem 1.2)** For solutions $w(r)$ to (4.6), the corresponding solutions $u(r)$ to (4.3) are, respectively, computed as follows. If $w(r) = \sqrt{3}$, then

$$\begin{aligned} u(r) - u(0) &= \int_0^r u_s \, ds = \int_0^r sw(s) \, ds \\ &= \int_0^r \sqrt{3}s \, ds = \frac{\sqrt{3}}{2} r^2. \end{aligned} \tag{4.11}$$

So if $u(0) = 0$ then we have $u(r) = \frac{\sqrt{3}}{2}r^2$. By Proposition 4.2, $u(r) = -\frac{\sqrt{3}}{2}r^2$ is another solution. If $w(r)$ is one of (4.7), then

$$\begin{aligned} u(r) - u(0) &= \int_0^r u_s \, ds = \int_0^r sw(s) \, ds \\ &= \int_0^r s \frac{s^2 + \frac{\sqrt{3}}{2}\rho_0^2}{\sqrt{\rho_0^4 - (s^2 + \frac{\sqrt{3}}{2}\rho_0^2)^2}} \, ds \\ &= \frac{1}{2} \int_{\frac{\sqrt{3}}{2}\rho_0^2}^{r^2 + \frac{\sqrt{3}}{2}\rho_0^2} \frac{x}{\sqrt{\rho_0^4 - x^2}} \, dx, \quad x = s^2 + \frac{\sqrt{3}}{2}\rho_0^2 \\ &= \frac{1}{4}\rho_0^2 - \frac{1}{2}\sqrt{\rho_0^4 - \left(r^2 + \frac{\sqrt{3}}{2}\rho_0^2\right)^2}. \end{aligned} \tag{4.12}$$

Therefore if $u(0) = -\frac{1}{4}\rho_0^2$ then we have

$$u(r) = -\frac{1}{2}\sqrt{\rho_0^4 - \left(r^2 + \frac{\sqrt{3}}{2}\rho_0^2\right)^2}, \quad 0 < r < \sqrt{\frac{2 - \sqrt{3}}{2}}\rho_0, \tag{4.13}$$

which is part of a type I shifted Heisenberg sphere. If $w(r)$ is one of (4.8), then

$$\begin{aligned} u(r) - u(0) &= \int_0^r u_s \, ds = \int_0^r sw(s) \, ds \\ &= -\int_0^r s \frac{s^2 - \frac{\sqrt{3}}{2}\rho_0^2}{\sqrt{\rho_0^4 - (s^2 - \frac{\sqrt{3}}{2}\rho_0^2)^2}} \, ds \\ &= -\frac{1}{2} \int_{-\frac{\sqrt{3}}{2}\rho_0^2}^{r^2 - \frac{\sqrt{3}}{2}\rho_0^2} \frac{x}{\sqrt{\rho_0^4 - x^2}} \, dx, \quad x = s^2 - \frac{\sqrt{3}}{2}\rho_0^2 \\ &= -\frac{1}{4}\rho_0^2 + \frac{1}{2}\sqrt{\rho_0^4 - \left(r^2 - \frac{\sqrt{3}}{2}\rho_0^2\right)^2}. \end{aligned} \tag{4.14}$$

Therefore if $u(0) = \frac{1}{4}\rho_0^2$, then we have

$$u(r) = \frac{1}{2}\sqrt{\rho_0^4 - \left(r^2 - \frac{\sqrt{3}}{2}\rho_0^2\right)^2}, \quad 0 < r < \sqrt{\frac{2 + \sqrt{3}}{2}}\rho_0. \tag{4.15}$$

This is part/a graph of a type II shifted Heisenberg sphere. It is not possible to glue parts of type I and type II shifted Heisenberg spheres to form a new surface of C^2. See Remark 4.4 below. $\square$

Remark 4.4. *It is easy to see that for a type I shifted Heisenberg sphere $u(\sqrt{\frac{2-\sqrt{3}}{2}}\rho_0) = 0$ and after a straightforward computation for r_{uu} at $u = 0$, we have*

$$r_{uu}(0) = \frac{-2\sqrt{2}}{(2-\sqrt{3})^{\frac{1}{2}}\rho_0^3} \tag{4.16}$$

From (4.15) we see that $u(b) = 0$ for some $b > 0$ and after a straightforward computation for r_{uu} at $u = 0$, we have

$$r_{uu}(0) = \frac{-(2+\sqrt{3})}{b^3}, \tag{4.17}$$

where the number b is the radius in some sense. If we take the radius $b = r_0 = \sqrt{\frac{2-\sqrt{3}}{2}}\rho_0$, then

$$r_{uu}(0) = \frac{-(2+\sqrt{3})}{b^3} = -\left(\frac{2+\sqrt{3}}{2-\sqrt{3}}\right)\frac{2\sqrt{2}}{(2-\sqrt{3})^{\frac{1}{2}}\rho_0^3}. \tag{4.18}$$

From (4.16) and (4.18), we conclude that the two graphs (4.13) and (4.15) can be glued together at $u = 0$ to become a C^1 smooth curve, but not of C^2.

In terms of polar coordinates (r, ϕ), we have

$$\begin{aligned} N^\perp &= \sin\theta\ \partial_x - \cos\theta\ \partial_y \\ &= \sin\theta(\cos\phi\ \partial_r - \frac{\sin\phi}{r}\ \partial_\phi) - \cos\theta(\sin\phi\ \partial_r - \frac{\cos\phi}{r}\ \partial_\phi) \\ &= \sin(\theta-\phi)\partial_r - \frac{\cos(\theta-\phi)}{r}\partial_\phi \\ N &= \cos\theta\ \partial_x + \sin\theta\ \partial_y \\ &= \cos\theta(\cos\phi\ \partial_r - \frac{\sin\phi}{r}\ \partial_\phi) + \sin\theta(\sin\phi\ \partial_r - \frac{\cos\phi}{r}\ \partial_\phi) \\ &= \cos(\theta-\phi)\partial_r + \frac{\sin(\theta-\phi)}{r}\partial_\phi. \end{aligned} \tag{4.19}$$

From (4.19), we have

$$\begin{aligned} N^\perp &= \sin(\theta-\phi)\partial_r - \frac{\cos(\theta-\phi)}{r}\partial_\phi, \\ N &= \cos(\theta-\phi)\partial_r + \frac{\sin(\theta-\phi)}{r}\partial_\phi, \end{aligned}$$

where

$$\begin{aligned}\sin(\theta-\phi) &= \sin\theta\cos\phi - \cos\theta\sin\phi\\ &= \frac{(u_y+x)}{D}\frac{x}{r} - \frac{(u_x-y)}{D}\frac{y}{r}\\ &= \frac{(xu_y-yu_x)+r^2}{rD}\\ &= \frac{r^2+u_\phi}{rD},\end{aligned}$$

and

$$\begin{aligned}\cos(\theta-\phi) &= \cos\theta\cos\phi - \sin\theta\sin\phi\\ &= \frac{(u_x-y)}{D}\frac{x}{r} - \frac{(u_y+x)}{D}\frac{y}{r}\\ &= \frac{(xu_x+yu_y)}{rD}\\ &= \frac{ru_r}{rD} = \frac{u_r}{D}.\end{aligned}$$

Therefore

$$\begin{aligned}N^\perp &= \left(\frac{r^2+u_\phi}{rD}\right)\partial_r - \left(\frac{u_r}{rD}\right)\partial_\phi,\\ N &= \left(\frac{u_r}{D}\right)\partial_r + \left(\frac{r^2+u_\phi}{r^2D}\right)\partial_\phi.\end{aligned} \tag{4.20}$$

Example 4.5. Let $u = \pm\frac{\sqrt{3}}{2}r^2$. We have

$$u_x - y = \pm\sqrt{3}x - y, \quad u_y + x = x \pm \sqrt{3}y,$$

thus

$$D = \sqrt{(u_x-y)^2+(u_y+x)^2} = \sqrt{4(x^2+y^2)} = 2r,$$

and hence

$$\alpha = -\frac{1}{D} = -\frac{1}{2r},$$

and

$$\begin{aligned}N^\perp &= \left(\frac{r^2+u_\phi}{rD}\right)\partial_r - \left(\frac{u_r}{rD}\right)\partial_\phi,\\ &= \frac{1}{2}\partial_r \mp \frac{\sqrt{3}}{2r}\partial_\phi.\end{aligned}$$

From (4.19), we have

$$s = \sin(\theta-\phi) = \frac{1}{2},\ c = \cos(\theta-\phi) = \pm\frac{\sqrt{3}}{2}.$$

This implies that

$$\theta = \begin{cases}\phi+\frac{\pi}{6}, & \text{for } u = \frac{\sqrt{3}}{2}r^2\\ \phi+\frac{5\pi}{6}, & \text{for } u = -\frac{\sqrt{3}}{2}r^2\end{cases}$$

and hence

$$\begin{aligned} H &= -N^{\perp}\theta = \pm\frac{\sqrt{3}}{2r}, \\ -N^{\perp}\alpha &= -\frac{1}{4r^2}, \end{aligned}$$

Thus

$$e_1(\alpha) + \frac{1}{2}\alpha^2 + \frac{1}{6}H^2 = 0, \quad \text{where } e_1 = -N^{\perp}.$$

On the other hand, we compute

$$\det A(r, \phi, U) = s^2(\frac{1}{3}sH + 2c\alpha)(\frac{2}{3}sH + 2c\alpha) = \frac{5}{48r^2} \neq 0, \quad \text{for} \quad r > 0.$$

Therefore each curve defined by $r = \mathbf{c} > 0$ is a non-characteristic curve.

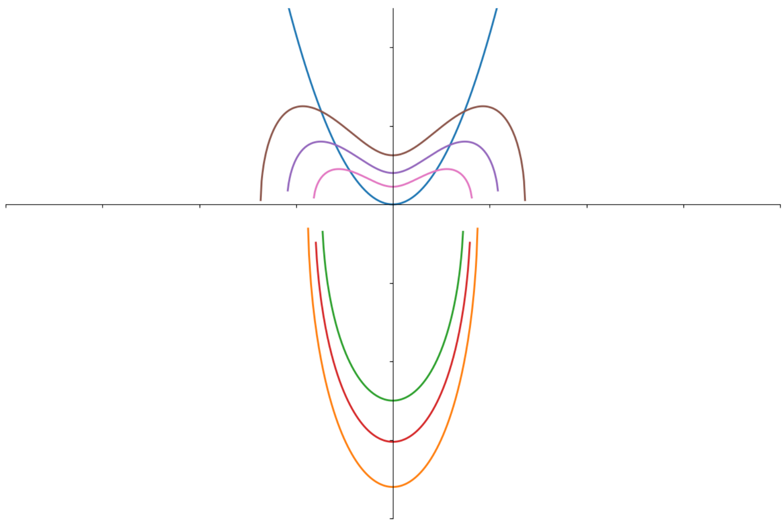

FIGURE 2. (4.21)

Example 4.6. For a type I shifted Heisenberg sphere and a type II shifted Heisenberg sphere, we let

$$u(r) = \mp\frac{1}{2}\sqrt{\rho_0^4 - \left(r^2 \pm \frac{\sqrt{3}}{2}\rho_0^2\right)^2}, \quad 0 < r < \sqrt{\frac{2 \mp \sqrt{3}}{2}}\rho_0, \tag{4.21}$$

which, respectively, describes the lower semi-sphere of type I and the upper semi-sphere of type II. We have

$$u_r = \frac{\pm\left(r^2 \pm \frac{\sqrt{3}}{2}\rho_0^2\right)}{\sqrt{\rho_0^4 - \left(r^2 \pm \frac{\sqrt{3}}{2}\rho_0^2\right)^2}},$$

thus

$$D = \sqrt{r^2 + u_r^2} = \frac{r\rho_0^2}{\sqrt{\rho_0^4 - \left(r^2 \pm \frac{\sqrt{3}}{2}\rho_0^2\right)^2}},$$

and hence

$$\alpha = -\frac{1}{D},$$

and

$$\begin{aligned}
N^{\perp} &= \left(\frac{r^2+u_\phi}{rD}\right)\partial_r - \left(\frac{u_r}{rD}\right)\partial_\phi, \\
&= \left(\frac{r}{D}\right)\partial_r - \left(\frac{u_r}{rD}\right)\partial_\phi, \quad u_\phi = 0 \\
&= \frac{\sqrt{\rho_0^4 - \left(r^2 \pm \frac{\sqrt{3}}{2}\rho_0^2\right)^2}}{\rho_0^2}\partial_r \mp \frac{\left(r^2 \pm \frac{\sqrt{3}}{2}\rho_0^2\right)}{r\rho_0^2}\partial_\phi.
\end{aligned}$$

From (4.19), we have

$$s = \sin(\theta - \phi) = \frac{\sqrt{\rho_0^4 - \left(r^2 \pm \frac{\sqrt{3}}{2}\rho_0^2\right)^2}}{\rho_0^2},$$

and

$$c = \cos(\theta - \phi) = \pm\frac{\left(r^2 \pm \frac{\sqrt{3}}{2}\rho_0^2\right)}{\rho_0^2}.$$

Therefore

$$\theta - \phi = \sin^{-1}\left(\frac{\sqrt{\rho_0^4 - \left(r^2 \pm \frac{\sqrt{3}}{2}\rho_0^2\right)^2}}{\rho_0^2}\right).$$

We compute the p-mean curvature

$$\begin{aligned}
H &= -N^{\perp}\theta \\
&= \pm\frac{\left(r^2 \pm \frac{\sqrt{3}}{2}\rho_0^2\right)}{r\rho_0^2} - \frac{\sqrt{\rho_0^4 - \left(r^2 \pm \frac{\sqrt{3}}{2}\rho_0^2\right)^2}}{\rho_0^2}\partial_r\left(\sin^{-1}\frac{\sqrt{\rho_0^4 - \left(r^2 \pm \frac{\sqrt{3}}{2}\rho_0^2\right)^2}}{\rho_0^2}\right),
\end{aligned}$$

where, by chain rules,

$$\begin{aligned}
\partial_r\left(\sin^{-1}\frac{\sqrt{\rho_0^4 - \left(r^2 \pm \frac{\sqrt{3}}{2}\rho_0^2\right)^2}}{\rho_0^2}\right) &= \frac{1}{\cos(\theta-\phi)}\partial_r\left(\frac{\sqrt{\rho_0^4 - \left(r^2 \pm \frac{\sqrt{3}}{2}\rho_0^2\right)^2}}{\rho_0^2}\right) \\
&= \frac{1}{\cos(\theta-\phi)}\left(\frac{-2r\left(r^2 \pm \frac{\sqrt{3}}{2}\rho_0^2\right)}{\rho_0^2\sqrt{\rho_0^4 - \left(r^2 \pm \frac{\sqrt{3}}{2}\rho_0^2\right)^2}}\right).
\end{aligned}$$

Therefore

$$\begin{aligned} H &= -N^{\perp}\theta \\ &= \pm\frac{\left(r^2 \pm \frac{\sqrt{3}}{2}\rho_0^2\right)}{r\rho_0^2} - \frac{1}{\cos(\theta-\phi)}\left(\frac{-2r\left(r^2 \pm \frac{\sqrt{3}}{2}\rho_0^2\right)}{\rho_0^4}\right) \\ &= \pm\frac{\left(r^2 \pm \frac{\sqrt{3}}{2}\rho_0^2\right)}{r\rho_0^2} \pm \frac{2r}{\rho_0^2} \\ &= \pm\frac{\left(3r^2 \pm \frac{\sqrt{3}}{2}\rho_0^2\right)}{r\rho_0^2}. \end{aligned}$$

Also, after a straightforwards computation, we have

$$-N^{\perp}\alpha = \frac{4}{\rho_0^4}\left(-\frac{u^2}{r^2} - \frac{r^2 \pm \frac{\sqrt{3}}{2}\rho_0^2}{2}\right).$$

Thus

$$e_1(\alpha) + \frac{1}{2}\alpha^2 + \frac{1}{6}H^2 = 0, \quad \text{where } e_1 = -N^{\perp}.$$

Remark 4.7. *Both the geometric invariants α and H we compute in examples are with respect to the horizontal normal*

$$e_2 = \frac{\nabla_b\psi}{\|\nabla_b\psi\|},$$

where ψ is the defining function defined by

$$\psi = t - u.$$

With respect to such a horizontal normal e_2, we have

(1) The function α is always negative, i.e., $\alpha = -\frac{1}{D} \leq 0$.
(2) $e_1 = -N^{\perp}$, when projected onto the xy-plane.

If we choose the horizontal normal $\tilde{e}_2$ such that

$$\tilde{e}_2 = \frac{\nabla_b\tilde{\psi}}{\|\nabla_b\tilde{\psi}\|}$$

with the defining function

$$\tilde{\psi} = u^2 - t^2.$$

Since

$$\begin{aligned} \nabla_b\tilde{\psi} &= (u+t)\nabla_b(u-t) + (u-t)\nabla_b(u+t) \\ &= 2u\nabla_b(u-t), \quad \text{on } t-u=0, \end{aligned}$$

we see that on the surface Σ defined by the graph of $t = u$, we have

$$\tilde{e}_2 = \begin{cases} -e_2, & \text{when } u > 0 \\ e_2, & \text{when } u < 0 \end{cases}$$

Therefore the invariants α and H with respect to $\tilde{e}_2$ will have the same sign as those with respect to e_2 when $u < 0$; and will differ by a sign when $u > 0$. Notice that, for a Pansu sphere, the horizontal normal $\tilde{e}_2$ is globally defined on the whole regular part and the corresponding p-mean curvature H is a positive constant.

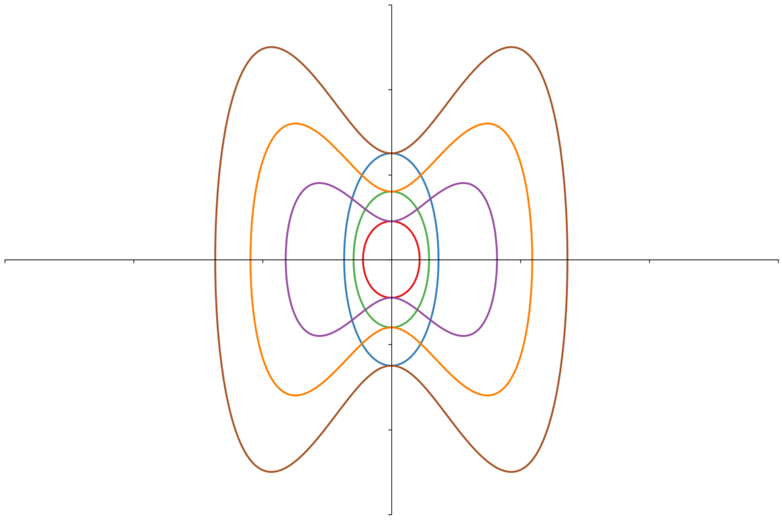

FIGURE 3. closed pictures

Remark 4.8. *The solutions* $u(r) = \pm\frac{\sqrt{3}}{2}r^2$ *in (1)/(2) of Theorem 1.2 can be obtained as a limit of dilations applied to type I/II shifted Heisenberg spheres (see Figure 3 for closed pictures) after translating their "south pole" to the origin. Let* $\tilde{t} = t + \frac{1}{4}\rho_0^2$*. The surfaces defined by* $(r^2 \pm \frac{\sqrt{3}}{2}\rho_0^2)^2 + 4t^2 = \rho_0^4$, $\rho_0 > 0$, *can be translated to the form* $(r^2 \pm \frac{\sqrt{3}}{2}\rho_0^2)^2 + 4(\tilde{t} - \frac{1}{4}\rho_0^2)^2 = \rho_0^4$ *passing through* $(r = 0, \tilde{t} = 0)$*. Now applying the dilations* $(r, \tilde{t}) \to (\lambda r, \lambda^2\tilde{t})$, $\lambda > 0$, *and letting* $\lambda \to 0$ *after dividing by* λ^2 *yield*

$$\pm\sqrt{3}\rho_0^2 r^2 - 2\rho_0^2\tilde{t} = 0$$

which are solutions (1)/(2) of Theorem 1.2.

5. Proof of Theorem 1.3: second variation of E_1 for the Clifford torus

5.1. **First variation of E_1 :review.** We first review some material in [CYZ18], including the first variation of E_1 and lemmas which will be used to compute the second variation of E_1 for the Clifford torus in the standard CR 3-sphere, which is a critical point of E_1.

Let $\Sigma_t = F_t(\Sigma)$ be a family of surfaces in a 3-dimensional pseudohermitian manifold (M, J, Θ) with $\Sigma_0 = \Sigma$, such that

$$\frac{d}{dt}F_t = X = fe_2 + gT. \tag{5.1}$$

Here we let e_1 be the unit vector in $T\Sigma_t \cap \xi$ and $e_2 = Je_1$. We assume f and g are supported in a domain of Σ away from the singular set of Σ. We denote $h := f - \alpha g$ and $V := T + \alpha e_2$. As satisfied by the standard CR 3-sphere, we assume (M, J, Θ) has vanishing torsion and constant Webster scalar curvature in this section.

Theorem 5.1. *(*[CYZ18, Theorem 3]*) Assume* (M, J, Θ) *has vanishing torsion and constant Webster scalar curvature. Let* $F_t(\Sigma)$ *be given by (5.1). We have*

$$\frac{d}{dt}E_1[\Sigma_t] = \frac{d}{dt}\int_{\Sigma_t} dA_1 = \int_{\Sigma_t} \mathcal{E}_1 h\Theta \wedge e^1, \tag{5.2}$$

where

$$dA_1 = |H_{cr}|^{3/2}\Theta \wedge e^1, \quad H_{cr} := e_1(\alpha) + \frac{1}{2}\alpha^2 + \frac{1}{6}H^2 + \frac{1}{4}W,$$

$$\begin{aligned} \mathcal{E}_1 &= \frac{1}{2}e_1(|H_{cr}|^{1/2}\mathfrak{f}) + \frac{3}{2}|H_{cr}|^{1/2}\alpha\mathfrak{f} \\ &\quad + \frac{1}{2}sign(H_{cr})|H_{cr}|^{1/2}\{9V(\alpha) + 6HH_{cr} - \frac{1}{3}H^3\}, \end{aligned} \tag{5.3}$$

and $\mathfrak{f}$ *is the function on* Σ_t *given by (see* [CYZ18, (3.11)]*)*

$$\begin{aligned} \mathfrak{f} &= |H_{cr}|^{-1}\{e_1(H)(e_1(\alpha) + \frac{1}{2}\alpha^2 + \frac{1}{3}H^2 + \frac{1}{4}W) \\ &\quad + HV(H) + \frac{3}{2}V(e_1(\alpha) + \frac{1}{2}\alpha^2) \\ &\quad - \frac{7}{2}\alpha He_1(\alpha) - \frac{5}{2}\alpha^3 H - \frac{2}{3}\alpha H^3 - \frac{5}{4}\alpha HW\}. \end{aligned}$$

We rewrite $|H_{cr}|\mathfrak{f}$ and $\mathcal{E}_1$ in the following form:

Proposition 5.2. *We have*

$$|H_{cr}|\mathfrak{f} = e_1(H)H_{cr} + \frac{3}{2}V(H_{cr}) + \frac{1}{2}He_1(H_{cr}) - \alpha HH_{cr},$$

and

$$\begin{aligned} \mathcal{E}_1 &= |H_{cr}|^{-\frac{1}{2}}\{-\frac{1}{4}sign(H_{cr})\mathfrak{f}e_1(H_{cr}) + \frac{1}{2}e_1(|H_{cr}|\mathfrak{f}) \\ &\quad + \frac{3}{2}|H_{cr}|\mathfrak{f}\alpha + H_{cr}[\frac{9}{2}V(\alpha) + 3HH_{cr} - \frac{1}{6}H^3]\}. \end{aligned} \tag{5.4}$$

Proof. We compute

$$\begin{aligned} |H_{cr}|\mathfrak{f} &= e_1(H)(H_{cr} + \frac{1}{6}H^2) \\ &\quad + HV(H) + \frac{3}{2}V(H_{cr} - \frac{1}{6}H^2) \\ &\quad - \frac{7}{2}\alpha He_1(\alpha) - \frac{5}{2}\alpha^3 H - \frac{2}{3}\alpha H^3 - \frac{5}{4}\alpha HW \\ &= e_1(H)H_{cr} + \frac{3}{2}V(H_{cr}) \\ &\quad + \frac{1}{2}He_1(H_{cr} - e_1(\alpha) - \frac{1}{2}\alpha^2) + \frac{1}{2}HV(H) \\ &\quad - \frac{7}{2}\alpha He_1(\alpha) - \frac{5}{2}\alpha^3 H - \frac{2}{3}\alpha H^3 - \frac{5}{4}\alpha HW. \end{aligned}$$

Then, using the Codazzi-like equation ([CYZ18, (3.24)])

$$e_1e_1(\alpha) = -6\alpha e_1(\alpha) + V(H) - \alpha H^2 - 4\alpha^2 - 2W\alpha,$$

we find

$$\begin{aligned}
|H_{cr}|\mathfrak{f} &= e_1(H)H_{cr}+\frac{3}{2}V(H_{cr})+\frac{1}{2}He_1(H_{cr})\\
&\quad -\frac{1}{2}H[-5\alpha e_1(\alpha)-\alpha H^2-4\alpha^3-2W\alpha]\\
&\quad -\frac{7}{2}\alpha He_1(\alpha)-\frac{5}{2}\alpha^3H-\frac{2}{3}\alpha H^3-\frac{5}{4}\alpha HW\\
&= e_1(H)H_{cr}+\frac{3}{2}V(H_{cr})+\frac{1}{2}He_1(H_{cr})\\
&\quad -H\alpha[e_1(\alpha)+\frac{1}{2}\alpha^2+\frac{1}{6}H^2+\frac{1}{4}W]\\
&= e_1(H)H_{cr}+\frac{3}{2}V(H_{cr})+\frac{1}{2}He_1(H_{cr})-\alpha HH_{cr}.
\end{aligned}$$

We rewrite (5.3) as

$$\begin{aligned}
\mathcal{E}_1 &= \frac{1}{2}e_1(|H_{cr}|^{1/2}\mathfrak{f})+\frac{3}{2}|H_{cr}|^{1/2}\alpha\mathfrak{f}\\
&\quad +\frac{1}{2}sign(H_{cr})|H_{cr}|^{1/2}\{9V(\alpha)+6HH_{cr}-\frac{1}{3}H^3\}\\
&= |H_{cr}|^{-\frac{1}{2}}\{\frac{3}{2}|H_{cr}|\mathfrak{f}\alpha+\frac{1}{2}H_{cr}[9V(\alpha)+6HH_{cr}-\frac{1}{3}H^3]\\
&\quad +\frac{1}{2}e_1(|H_{cr}|\mathfrak{f})-\frac{1}{4}sign(H_{cr})\mathfrak{f}e_1(H_{cr})\}.
\end{aligned}$$

$\square$

5.2. **Second variation of E_1.** We assume that $\Sigma\hookrightarrow(M,J,\Theta)$ satisfies $\mathcal{E}_1(\Sigma)=0$, i.e. a critical point of E_1, and $H_{cr}\neq 0$. Let $\Sigma_t=F_t(\Sigma)$ be a family of surfaces in (M,J,Θ) with $\Sigma_0=\Sigma$, such that (recall (5.1))

$$\frac{d}{dt}F_t=X=fe_2+gT.$$

Here we let e_1 be the unit vector in $T\Sigma_t\cap\xi$ and $e_2=Je_1$. We denote $h:=f-\alpha g$ and $V:=T+\alpha e_2$. Then by (5.2) and (5.4), the second variation of E_1 for Σ is given by

$$\begin{aligned}
\frac{d^2}{dt^2}|_{t=0}E_1(\Sigma_t) &= \int_\Sigma |H_{cr}|^{-\frac{1}{2}}\frac{d}{dt}\{-\frac{1}{4}|H_{cr}|\mathfrak{f}\frac{1}{H_{cr}}e_1(H_{cr})+\frac{1}{2}e_1(|H_{cr}|\mathfrak{f})\\
&\quad +\frac{3}{2}|H_{cr}|\mathfrak{f}\alpha+H_{cr}[\frac{9}{2}V(\alpha)+3HH_{cr}-\frac{1}{6}H^3]\}h\Theta\wedge e^1.
\end{aligned}\tag{5.5}$$

We need some lemmas to compute the above second variation of E_1 for the Clifford torus, some were proved in [CYZ18].

Assume $\{e_1,e_2=Je_1,T\}$ is an orthonormal frame of (M,J,Θ) with respect to the adapted metric $g_\Theta=\frac{1}{2}d\Theta(\cdot,J\cdot)+\Theta\otimes\Theta$ and $\{e^1,e^2,\Theta\}$ is its dual coframe. There is a real 1-form ω such that the following holds (see, for instance, Appendix of [CYZ18]), under the assumption that (M,J,Θ) has vanishing torsion,

$$\begin{aligned}
de^1 &= -e^2\wedge\omega,\ de^2=e^1\wedge\omega,\\
[e_2,e_1] &= \omega(e_2)e_2+\omega(e_1)e_1+2T,\\
[e_1,T] &= -\omega(T)e_2,\ [e_2,T]=\omega(T)e_1.
\end{aligned}$$

Lemma 5.3. *(*[CYZ18, Lemma 4]*) Let $h := f - \alpha g$ and $V := T + \alpha e_2$. Under the torsion free condition, we have*

$$\omega(e_1) = H, \quad \omega(e_2) = h^{-1}e_1(h) + 2\alpha, \quad \omega(T) = e_1(\alpha) - \alpha h^{-1}e_1(h),$$
$$e_2(\alpha) = h^{-1}V(h),$$
$$[e_1, V] = -\alpha H e_1 - 2\alpha V,$$

Lemma 5.4. *(*[CYZ18, Lemma 6]*) Suppose the torsion vanishes, i.e., $A_1^1 = 0$. Then we have*

$$\frac{dH}{dt} = e_1e_1(h) + 2\alpha e_1(h) + 4h(e_1(\alpha) + \alpha^2 + \frac{1}{4}H^2 + \frac{1}{2}W) + gV(H),$$
$$\frac{d\alpha}{dt} = V(h) + gV(\alpha),$$
$$\frac{d}{dt}e_1(\alpha) = e_1V(h) + ge_1V(\alpha) + 2fV(\alpha) + fHe_1(\alpha).$$

In addition, we compute the following

Lemma 5.5. *We have*

$$\frac{d}{dt}V(\alpha) = V(\frac{d\alpha}{dt}) + he_1(\alpha)e_1(\alpha) - \alpha e_1(h)e_1(\alpha) - V(g)V(\alpha),$$

and

$$\frac{d}{dt}[e_1(|H_{cr}|\mathfrak{f})] = e_1[\frac{d}{dt}(|H_{cr}|\mathfrak{f})] + [fHe_1 + 2fV - e_1(g)V](|H_{cr}|\mathfrak{f}),$$

where

$$\begin{aligned}\frac{d}{dt}(|H_{cr}|\mathfrak{f}) \quad &= \quad H_{cr}\frac{d}{dt}[e_1(H)] + e_1(H)\frac{d}{dt}H_{cr} + \frac{3}{2}\frac{d}{dt}V(H_{cr}) \\ &\quad + \frac{1}{2}e_1(H_{cr})\frac{dH}{dt} + \frac{1}{2}H\frac{d}{dt}[e_1(H_{cr})] \\ &\quad - \frac{d\alpha}{dt}HH_{cr} - \alpha\frac{dH}{dt}H_{cr} - \alpha H\frac{dH_{cr}}{dt},\end{aligned}$$

here

$$\begin{aligned}\frac{d}{dt}e_1(H) \quad &= \quad e_1[e_1e_1(h) + 2\alpha e_1(h) + 4he_1(\alpha) + H^2h + 4\alpha^2h + 2Wh] \\ &\quad + ge_1V(H) + fHe_1(H) + 2fV(H),\end{aligned}$$

$$\frac{d}{dt}V(H_{cr}) = V(\frac{dH_{cr}}{dt}) + he_1(\alpha)e_1(H_{cr}) - \alpha e_1(h)e_1(H_{cr}) - V(g)V(H_{cr}),$$

and

$$\begin{aligned}\frac{d}{dt}H_{cr} \quad &= \quad e_1V(h) + \alpha V(h) + \frac{1}{3}H[e_1e_1(h) + 2\alpha e_1(h)] \\ &\quad + \frac{4}{3}H(e_1(\alpha) + \alpha^2 + \frac{1}{4}H^2 + \frac{1}{2}W)h \\ &\quad + (2V(\alpha) + He_1(\alpha))f + (e_1V(\alpha) + \alpha V(\alpha) + \frac{1}{3}HV(H))g.\end{aligned}$$

Proof. It follows from Lemma 5.3 that

$$\begin{aligned}[fe_2+gT,V] &= f[e_2,V]+g[T,V]-V(f)e_2-V(g)T\\ &= f[e_2(\alpha)e_2+e_1(\alpha)e_1-\alpha h^{-1}e_1(h)e_1]\\ &\quad +g[T(\alpha)e_2-\alpha e_1(\alpha)e_1+\alpha^2h^{-1}e_1(h)e_1]\\ &\quad -V(f)e_2-V(g)T\\ &= he_1(\alpha)e_1-\alpha e_1(h)e_1-V(g)V.\end{aligned}$$

Then

$$\frac{d}{dt}V(\alpha)=V(\frac{d\alpha}{dt})+he_1(\alpha)e_1(\alpha)-\alpha e_1(h)e_1(\alpha)-V(g)V(\alpha).$$

In a similar manner,

$$\begin{aligned}\frac{d}{dt}V(H_{cr}) &= V(\frac{d}{dt}H_{cr})+[fe_2+gT,V](H_{cr})\\ &= V(\frac{dH_{cr}}{dt})+he_1(\alpha)e_1(H_{cr})-\alpha e_1(h)e_1(H_{cr})-V(g)V(H_{cr}).\end{aligned}$$

It follows from Lemma 5.3 that

$$\begin{aligned}[fe_2+gT,e_1] &= f[e_2,e_1]-e_1(f)e_2+g[T,e_1]-e_1(g)T\\ &= f[\omega(e_2)e_2+\omega(e_1)e_1+2T]-e_1(f)e_2+g\omega(T)e_2-e_1(g)T\\ &= fHe_1+2fV-e_1(g)V,\end{aligned}$$

then by Lemma 5.4

$$\begin{aligned}\frac{d}{dt}e_1(H) &= e_1(\frac{dH}{dt})+[fe_2+gT,e_1](H)\\ &= e_1[e_1e_1(h)+2\alpha e_1(h)+4h(e_1(\alpha)+\alpha^2+\frac{1}{4}H^2+\frac{1}{2}W)+gV(H)]\\ &\quad +fHe_1(H)+2fV(H)-e_1(g)V(H)\\ &= e_1[e_1e_1(h)+2\alpha e_1(h)+4he_1(\alpha)+H^2h+4\alpha^2h+2Wh]\\ &\quad +ge_1V(H)+fHe_1(H)+2fV(H).\end{aligned}$$

In a similar manner, we have

$$\frac{d}{dt}[e_1(|H_{cr}|\mathfrak{f})]=e_1[\frac{d}{dt}(|H_{cr}|\mathfrak{f})]+[fHe_1+2fV-e_1(g)V](|H_{cr}|\mathfrak{f}).$$

Recall that

$$|H_{cr}|\mathfrak{f}{=}e_1(H)H_{cr}+\frac{3}{2}V(H_{cr})+\frac{1}{2}He_1(H_{cr})-\alpha HH_{cr}.$$

So we have

$$\begin{aligned}\frac{d}{dt}(|H_{cr}|\mathfrak{f}) &= H_{cr}\frac{d}{dt}[e_1(H)]+e_1(H)\frac{d}{dt}H_{cr}+\frac{3}{2}\frac{d}{dt}V(H_{cr})\\ &\quad +\frac{1}{2}e_1(H_{cr})\frac{dH}{dt}+\frac{1}{2}H\frac{d}{dt}[e_1(H_{cr})]\\ &\quad -\frac{d\alpha}{dt}HH_{cr}-\alpha\frac{dH}{dt}H_{cr}-\alpha H\frac{dH_{cr}}{dt}.\end{aligned}$$

Recall that

$$H_{cr}=e_1(\alpha)+\frac{1}{2}\alpha^2+\frac{1}{6}H^2+\frac{1}{4}W.$$

Then by Lemma 5.4, we have

$$\begin{aligned}\frac{d}{dt}H_{cr} &= e_1V(h)+\alpha V(h)+\frac{1}{3}H[e_1e_1(h)+2\alpha e_1(h)]\\ &\quad+\frac{4}{3}H(e_1(\alpha)+\alpha^2+\frac{1}{4}H^2+\frac{1}{2}W)h\\ &\quad+(2V(\alpha)+He_1(\alpha))f+(e_1V(\alpha)+\alpha V(\alpha)+\frac{1}{3}HV(H))g.\end{aligned}$$

□

5.3. **Second variation of E_1 for the Clifford torus: proof of Theorem 1.3.** The Clifford torus of the standard CR 3-sphere S^3 have been studied in [CYZ18, Example 4 in Subsection 4.1]. The contact form $\hat{\Theta}$ on the standard CR 3-sphere S^3 reads as $\hat{\Theta}=\sum_{i=1}^{2}(x^idy^i-y^idx^i)$, hence $d\hat{\Theta}=2\sum_{i=1}^{2}dx^i\wedge dy^i$. So the Reeb vector field $T=\sum_{i=1}^{2}(x^i\frac{\partial}{\partial y^i}-y^i\frac{\partial}{\partial x^i})$. The standard CR 3-sphere S^3 has vanishing torsion and Webster scalar curvature $W=2$. A (nonsingular) surface $\Sigma\subset S^3$ has $\alpha=0$ if and only if $T\in T\Sigma$. The following examples have the property that $\alpha=0$:

$$\Sigma_{[\rho_1]}:=\{|z^1|^2=(x^1)^2+(y^1)^2=\rho_1^2,\quad |z^2|^2=\rho_2^2=1-\rho_1^2\}.$$

We introduce the coordinates (φ_1,φ_2) for $\Sigma_{[\rho_1]}$ such that $\Sigma_{[\rho_1]}=\{(\rho_1e^{i\varphi_1},\rho_2e^{i\varphi_2})\}$. For $\Sigma_{[\rho_1]}$, we have

$$e_1=-\frac{\rho_2}{\rho_1}\frac{\partial}{\partial\varphi_1}+\frac{\rho_1}{\rho_2}\frac{\partial}{\partial\varphi_2},$$

where $\frac{\partial}{\partial\varphi_i}=x^i\frac{\partial}{\partial y^i}-y^i\frac{\partial}{\partial x^i}$, Note also that $V=T=\partial_{\varphi_1}+\partial_{\varphi_2}$. The p-mean curvature of $\Sigma_{[\rho_1]}$ is given by

$$H=\frac{\rho_1^2-\rho_2^2}{\rho_1\rho_2}=\frac{\rho_1}{\rho_2}-\frac{\rho_2}{\rho_1}.$$

When $\rho_1=\rho_2=\frac{1}{\sqrt{2}}$, it is the Clifford torus $\Sigma_{1/\sqrt{2}}$ with

$$\alpha=H=0.$$

Besides, for the Clifford torus

$$e_1=-\frac{\partial}{\partial\varphi_1}+\frac{\partial}{\partial\varphi_2},\quad V=T=\partial_{\varphi_1}+\partial_{\varphi_2},\quad [e_1,V]=0.$$

For the Clifford torus, we compute

$$H_{cr}=e_1(\alpha)+\frac{1}{2}\alpha^2+\frac{1}{6}H^2+\frac{1}{4}W=\frac{1}{2}, \tag{5.6}$$

$$|H_{cr}|\mathfrak{f}=e_1(H)H_{cr}+\frac{3}{2}V(H_{cr})+\frac{1}{2}He_1(H_{cr})-\alpha HH_{cr}=0. \tag{5.7}$$

We now compute the second variation of E_1 for the Clifford torus $\Sigma_{[1/\sqrt{2}]}$, i.e. (1.5) of Theorem 1.3.

Theorem 5.6. *Let Σ be the Clifford torus $\Sigma_{[1/\sqrt{2}]}$ in the CR 3-sphere S^3. Then it holds that*

$$\frac{d^2}{dt^2}|_{t=0}E_1(\Sigma_t)=\frac{\sqrt{2}}{4}\int_{\Sigma_{[1/\sqrt{2}]}}[e_1e_1(f)^2+3e_1T(f)^2-7e_1(f)^2+9fTT(f)+12f^2]\Theta\wedge e^1.$$

Proof. By (5.5) (5.6) and (5.7), we have the second variation of E_1 at $\Sigma_0 = \Sigma_{[1/\sqrt{2}]}$: the Clifford torus

$$\frac{d^2}{dt^2}|_{t=0}E_1(\Sigma_t) = \int_{\Sigma_{[1/\sqrt{2}]}} |H_{cr}|^{-\frac{1}{2}}\frac{d}{dt}\{-\frac{1}{4}|H_{cr}|\mathfrak{f}\frac{1}{H_{cr}}e_1(H_{cr}) + \frac{1}{2}e_1(|H_{cr}|\mathfrak{f})$$
$$+\frac{3}{2}|H_{cr}|\mathfrak{f}\alpha + H_{cr}[\frac{9}{2}V(\alpha) + 3HH_{cr} - \frac{1}{6}H^3]\}h\Theta\wedge e^1$$
$$= \sqrt{2}\int_{\Sigma_{[1/\sqrt{2}]}}[\frac{1}{2}\frac{d}{dt}e_1(|H_{cr}|\mathfrak{f}) + \frac{9}{4}\frac{d}{dt}V(\alpha) + \frac{3}{4}\frac{dH}{dt}]h\Theta\wedge e^1.$$

It follows from Lemma 5.4 that

$$\frac{d}{dt}|_{t=0}H = e_1e_1(h) + 4h.$$

If follows from Lemma 5.4 and Lemma 5.5 that

$$\frac{d}{dt}|_{t=0}V(\alpha) = V\frac{d\alpha}{dt} = VV(h).$$

Using Lemma 5.5, we compute

$$\begin{aligned}
\frac{d}{dt}|_{t=0}[e_1(|H_{cr}|\mathfrak{f})] &= e_1[\frac{d}{dt}(|H_{cr}|\mathfrak{f})] \\
&= e_1\{H_{cr}\frac{d}{dt}[e_1(H)] + \frac{3}{2}\frac{d}{dt}V(H_{cr})\} \\
&= e_1\{\frac{1}{2}[e_1e_1e_1(h) + 4e_1(h)] + \frac{3}{2}V[\frac{d}{dt}H_{cr}]\} \\
&= \frac{1}{2}e_1e_1e_1e_1(h) + 2e_1e_1(h) + \frac{3}{2}e_1Ve_1V(h).
\end{aligned}$$

So we have

$$\frac{d^2}{dt^2}|_{t=0}E_1(\Sigma_t) = \sqrt{2}\int_{\Sigma_{[1/\sqrt{2}]}}[\frac{1}{2}\frac{d}{dt}e_1(|H_{cr}|\mathfrak{f}) + \frac{9}{4}\frac{d}{dt}V(\alpha) + \frac{3}{4}\frac{dH}{dt}]h\Theta\wedge e^1$$
$$= \frac{\sqrt{2}}{4}\int_{\Sigma_{[1/\sqrt{2}]}}[e_1e_1e_1e_1(h) + 3e_1Ve_1V(h) + 7e_1e_1(h) + 9VV(h) + 12h]h\Theta\wedge e^1. \tag{5.8}$$

Suppose either f_1 or f_2 has compact support in the nonsingular domain of Σ. Then we have (see [CYZ18, Lemma 7])

$$\int_\Sigma f_1e_1(f_2)\theta\wedge e^1 = \int_\Sigma -[e_1(f_1) + 2\alpha f_1]f_2\Theta\wedge e^1, \tag{5.9}$$

$$\int_\Sigma f_1V(f_2)\theta\wedge e^1 = -\int_\Sigma [V(f_1) - \alpha Hf_1]f_2\Theta\wedge e^1. \tag{5.10}$$

In particular, on the Clifford torus $\Sigma_{[1/\sqrt{2}]}$, $\alpha = 0$ so (5.9) and (5.10) are reduced to

$$\int_{\Sigma_{[1/\sqrt{2}]}} f_1e_1(f_2)\Theta\wedge e^1 = -\int_{\Sigma_{[1/\sqrt{2}]}} e_1(f_1)f_2\Theta\wedge e^1,$$

$$\int_{\Sigma_{[1/\sqrt{2}]}} f_1V(f_2)\Theta\wedge e^1 = -\int_{\Sigma_{[1/\sqrt{2}]}} V(f_1)f_2\Theta\wedge e^1$$

respectively. Note also that on the Clifford torus $h = f - \alpha g = f, [e_1, V] = 0$, and $V = T$. Then for $\Sigma_0 = \Sigma_{[1/\sqrt{2}]}$: the Clifford torus, by applying the above integration by parts formulas, we reduce (5.8) to

$$\frac{d^2}{dt^2}|_{t=0}E_1(\Sigma_t) = \frac{\sqrt{2}}{4}\int_{\Sigma_{[1/\sqrt{2}]}}[e_1e_1(f)^2 + 3e_1T(f)^2 - 7e_1(f)^2 + 9fTT(f) + 12f^2]\Theta \wedge e^1.$$

We have obtained (1.5), completing the proof of Theorem 1.3. □

References

[Che95] Cheng, J.-H., *Some applications of Cartan's theory on three-dimensional Cauchy-Riemann geometry*, Math. Zeit, 218 (1995), 527-548.

[CHMY05] Cheng, J.-H., Hwang, J.-F., Malchiodi, A. and Yang, P., *Minimal surfaces in pseudohermitian geometry*, Annali della Scuola Normale Superiore di Pisa Classe di Scienze V, 4(1) (2005), 129-177.

[CHMY12] Cheng, J.-H., Hwang, J.-F., Malchiodi, A. and Yang, P., *A Codazzi-like equation and the singular set for C^1 smooth surfaces in the Heisenberg group,* Journal fur die reine und angewandte Mathematik 671 (2012), 131-198.

[CL15] Chiu, H.-L. and Lai, S.-H., *The fundamental theorem for hypersurfaces in Heisenberg groups*, Calc. Var. 54 (2015), 1091-1118.

[CYZ18] Cheng, J.-H., Yang, P. and Zhang, Y., *Invariant surface area functionals and singular Yamabe problem in 3-dimensional CR geometry*, Advances in Mathematics 335 (2018), 405-465.

[Eva98] Evans, L. C., *Partial differential equations*, GSM 19, Amer. Math. Soc. 1998.

[John82] John, F., *Partial differential equations*, 4th ed.Springer-Verlag New York Inc. 1982.

Institute of Mathematics, Academia Sinica and National Center for Theoretical Sciences, Taipei, Taiwan, R.O.C.

Email address: cheng@math.sinica.edu.tw

Department of Mathematics, National Tsing-Hua University and National Center for Theoretical Sciences, Hsinchu, Taiwan, R.O.C.

Email address: hlchiu@math.nthu.edu.tw

Department of Mathematics, Princeton University, Princeton, NJ 08544, U.S.A.

Email address: yang@Math.Princeton.EDU

School of Mathematical Sciences and Wu Wen-Tsun Key Laboratory of Mathematics, University of Science and Technology of China, Hefei, Anhui, 230026, P. R. of China.

Email address: ybzhang@amss.ac.cn