Generalized Wigner theorem for non-invertible symmetries

Gerardo Ortiz,[1,2] Chinmay Giridhar,[3] Philipp Vojta,[4] Andriy H. Nevidomskyy,[3,5,6] and Zohar Nussinov[4,7,8]

[1]*Department of Physics, Indiana University, Bloomington, IN 47405, USA*
[2]*Institute for Advanced Study, Princeton, NJ 08540, USA*
[3]*Department of Physics and Astronomy, Rice University, Houston, TX 77005, USA*
[4]*Department of Physics, Washington University in St. Louis, MO 63130 USA*
[5]*Rice Center for Quantum Materials and Advanced Materials Institute, Rice University, Houston, TX 77005, USA*
[6]*Division of Condensed Matter Physics and Materials Science, Brookhaven National Laboratory, Upton, NY 11973-5000, USA*
[7]*Institut fur Physik, Technische Universität Chemnitz, 09107 Chemnitz, Germany*
[8]*Department of Physics and Quantum Centre of Excellence for Diamond and Emergent Materials (QuCenDiEM), Indian Institute of Technology Madras, Chennai 600036, India*

(Dated: October 7, 2025)

We establish the conditions under which a conservation law associated with a non-invertible operator may be realized as a symmetry in quantum mechanics. As established by Wigner, all quantum symmetries must be represented by either unitary or antiunitary transformations. Relinquishing an implicit assumption of invertibility, we demonstrate that the fundamental invariance of quantum transition probabilities under the application of symmetries mandates that all non-invertible symmetries may only correspond to *projective* unitary or antiunitary transformations, i.e., *partial isometries*. This extends the notion of physical states beyond conventional rays in Hilbert space to equivalence classes in an *extended, gauged Hilbert space*, thereby broadening the traditional understanding of symmetry transformations in quantum theory. We discuss consequences of this result and explicitly illustrate how, in simple model systems, whether symmetries be invertible or non-invertible may be inextricably related to the particular boundary conditions that are being used.

Introduction. Symmetry is a fundamental organizing principle underlying the laws of nature. At its core, symmetry refers to invariance, the idea that certain properties of a physical system remain unchanged under specific transformations, such as translations or rotations. This concept is not merely aesthetic; it has profound implications for the formulation of physical laws and the conservation principles they entail. For example, the invariance of physical laws under time translation leads to the conservation of energy, while rotational symmetry leads to conservation of angular momentum. As such, symmetry serves not only as a tool for understanding existing phenomena but also as a guiding principle in the pursuit of new physical theories.

In the context of quantum mechanics, one of the most profound consequences of symmetry is encapsulated in Wigner's theorem [1], including non-bijective (isometry) variants [2], which imposes strict constraints on the types of transformations that can be applied to quantum states. Specifically, the theorem asserts that *any symmetry transformation preserving transition probabilities between quantum states must be induced by either a unitary or an antiunitary isometry on the system's Hilbert space*. This result not only ensures the mathematical consistency of quantum theory under symmetry operations but also reveals how deeply symmetries are woven into the fabric of quantum mechanics, shaping everything from conservation laws to the subtleties of quantum measurement and the computation of event probabilities.

Recent developments in quantum many-body and field theories suggest the existence of symmetries that lie beyond the conventional framework of group theory, where each symmetry operation has an inverse. These *non-invertible symmetries*— also known as generalized or categorical symmetries— lack unique inverses and are better described using topological or higher-categorical structures [3]. They arise naturally in two-dimensional conformal field theories, topological quantum field theories, and certain lattice models, and are closely linked to phenomena such as duality transformations [4, 5]. This broader notion of symmetry hints at a deeper and more intricate structure underlying quantum systems. At first glance, however, non-invertible symmetries appear to stand in tension with Wigner's theorem. The aim of this paper is to resolve this apparent contradiction and propose a generalized extension of Wigner's theorem that accommodates such symmetries.

It is all about Boundary Conditions. We begin by illustrating the issue through well-known examples. A paradigmatic case is the transverse-field Ising chain (TFIC) [6–8], which encapsulates the essential features necessary to understand the apparent contradiction. At the outset, it is important to emphasize that the origin of such paradoxes arises from the specific boundary conditions to which quantum systems are subjected [5].

Consider the TFIC with L sites at the self-dual point associated with two different boundary conditions [5]:

$$\begin{aligned} H_1 &= -\sum_{j=1}^{L-1} \sigma^z_j \sigma^z_{j+1} - \sum_{j=1}^{L-1} \sigma^x_j, \\ H_2 &= H_1 - \sigma^x_L - \hat{\eta}\, \sigma^z_L \sigma^z_1, \end{aligned} \tag{1}$$

where $\boldsymbol{\sigma}_j = (\sigma^x_j, \sigma^y_j, \sigma^z_j)$ denotes Pauli matrices acting on site j, and

$$\hat{\eta} = \prod_{j=1}^{L} \sigma_j^x \tag{2}$$

is a $\mathbb{Z}_2$ symmetry common to both Hamiltonians. The first model (H_1) describes a TFIC with open boundary conditions while the closed TFIC (H_2) has a boundary term whose form depends on the eigenvalue of the symmetry operator $\hat{\eta}$.

What does it mean for $\hat{\eta}$ to be a symmetry? At the risk of sounding overly pedantic, this means that (a) $\hat{\eta}$ is a conserved quantity, i.e., commutes with the Hamiltonian, *and* (b) the action of $\hat{\eta}$ must preserve the transition probability between arbitrary quantum states. The latter requirement, which will become essential in what follows, is automatically satisfied if the operator is unitary or antiunitary – the contents of Wigner's original theorem [1].

Do these Hamiltonians exhibit additional symmetries? As demonstrated in Ref. [5], both H_1 and H_2 possess a self-duality transformation (a linear automorphism of bond algebras [4, 5]) implemented by unitary maps whose action on an operator $\hat{O}$ is defined by $\Phi(\hat{O}) = U\hat{O}U^\dagger$, with U unitary. Indeed, the unitary (and Hermitian) operator

$$U^{(1)} = \left(\prod_{j=1}^{\lfloor \frac{L}{2} \rfloor} \mathsf{S}_{j,L-j+1}\right)\left(\prod_{j=1}^{L-1} \mathsf{C}^x_{j+1,j}\right)\mathsf{H}^{\otimes L}, \tag{3}$$

constructed entirely from elements of the Clifford group, including $\mathsf{Controlled}\text{-}\sigma^x$

$$\mathrm{C}^x_{i,j} = e^{i\frac{\pi}{4}\left(\sigma^z_i\sigma^x_j - \sigma^z_i - \sigma^x_j + 1\right)}, \tag{4}$$

Swap, $\mathsf{S}_{i,j} = \mathrm{C}^x_{i,j}\mathrm{C}^x_{j,i}\mathrm{C}^x_{i,j} = \frac{\mathbb{1}+\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j}{2}$ (which interchanges the spin at site i with the one at site j), and Hadamard

$$\mathsf{H}_j = i\, e^{-i\frac{\pi}{2\sqrt{2}}\left(\sigma^z_j + \sigma^x_j\right)}, \tag{5}$$

realizes the self-duality map $\Phi_1(\sigma^x_j) = \sigma^z_{r_j}\sigma^z_{r_j+1}$, $\Phi_1(\sigma^z_j\sigma^z_{j+1}) = \sigma^x_{r_j}$, $\Phi_1(\hat{\eta}) = \sigma^z_L$, with $r_j = L - j$ and $j = 1, \cdots, L-1$, and commutes with H_1. Similarly, the unitary operator

$$\begin{aligned} U^{(2)} &= \Big(\prod_{j=1}^{L-1} e^{-i\frac{\pi}{4}\sigma^x_j} e^{-i\frac{\pi}{4}\sigma^z_j\sigma^z_{j+1}}\Big) e^{-i\frac{\pi}{4}\sigma^x_L} \\ &= e^{-i\frac{\pi}{4}\sigma^x_1} e^{-i\frac{\pi}{4}\sigma^z_1\sigma^z_2} \cdots e^{-i\frac{\pi}{4}\sigma^x_L}, \end{aligned} \tag{6}$$

generates the map $\Phi_2(\sigma^x_j) = \sigma^z_j\sigma^z_{j+1}$, $\Phi_2(\sigma^z_j\sigma^z_{j+1}) = \sigma^x_{j+1}$, with $j = 1, \cdots, L-1$ and $\Phi_2(\sigma^x_L) = \hat{\eta}\,\sigma^z_L\sigma^z_1$, $\Phi_2(\hat{\eta}\,\sigma^z_L\sigma^z_1) = \sigma^x_1$, $\Phi_2(\hat{\eta}) = \hat{\eta}$, and commutes with the Hamiltonian H_2. Consequently, as argued in Refs. [4, 5], these two unitary transformations qualify as respective *symmetries* of the system when the corresponding boundary conditions are implemented. We note that deriving explicit expressions for the unitary operators implementing the dualities is a notably challenging task. For example, even given $U^{(2)}$, deducing the form of $U^{(1)}$ is nontrivial and, to the best of our knowledge, has not appeared in the literature. The operator encodes the automorphism's entanglement characteristics and depends sensitively on the boundary conditions.

Consider now the same TFIC, but with explicit periodic or antiperiodic boundary condition:

$$H^\pm = H_1 - \sigma^x_L \mp \sigma^z_L\sigma^z_1. \tag{7}$$

The operators $U^{(1,2)}$ are no longer conserved quantities of $H^\pm$. In this case, no automorphism exists [5]; instead one may attempt to define *non-invertible operators*

$$D_\pm = U^{(2)}P_\pm, \tag{8}$$

that, respectively, commute with the Hamiltonians $H^\pm$ of Eq. (7), i.e., $[H^\pm, D_\pm] = 0$. Here, $P_\pm = (\mathbb{1} \pm \hat{\eta})/2$ denote projection operators ($P_\pm^2 = P_\pm$) onto the positive/negative symmetry eigenvalue $\eta = \pm 1$ sectors. Importantly, however, the transition probability between two arbitrary quantum states $|\alpha\rangle$, $|\beta\rangle$ in the Hilbert space $\mathcal{H}$ of dimension 2^L is no longer conserved after the action by $D_\pm$. Indeed, explicitly implementing such a transformation,

$$\begin{aligned} \big|\langle D_\pm\beta|D_\pm\alpha\rangle\big|^2 &= \big|\langle\beta|P_\pm U^{(2)\dagger}U^{(2)}P_\pm|\alpha\rangle\big|^2 \\ &= \big|\langle\beta|P_\pm|\alpha\rangle\big|^2 \neq \big|\langle\beta|\alpha\rangle\big|^2. \end{aligned} \tag{9}$$

In other words, despite being a conserved quantity, $D_\pm$ does *not* qualify as a symmetry according to Wigner, since its probability altering action can be detected by measurements– i.e., the application of $D_\pm$ does not lead to an invariance of the system. As we will detail towards the end of this paper, *bona fide* symmetry operators may, nonetheless, be explicitly written down for an amended (gauged) TFIC.

Generalized Symmetry Extension of Wigner's Theorem. A key assumption in Wigner's theorem is that symmetry operators act on the same Hilbert space in which quantum states are defined; consequently, all such symmetries are inherently *invertible*. Since the preservation of transition probabilities is the central physical principle, we propose an extension of Wigner's theorem to include *non-invertible* (symmetry) operators.

Our extension of Wigner's theorem to non-invertible symmetries asserts that: *Any non-invertible symmetry transformation that preserves transition probabilities between quantum states of a Hilbert space, can only be induced by the composition of either a unitary or an antiunitary operator and a projector onto a symmetry sector of an enlarged Hilbert space – such that this projection operator acts as the identity on the original Hilbert space.* Our proof applies to Hilbert spaces of finite or countably infinite (denumerably infinite) dimension.

Preliminaries:

Consider a quantum system with a Hermitian Hamiltonian H having its support in a Hilbert space $\mathcal{H}$ defined over the field of the complex numbers $\mathbb{C}$. The dimension of $\mathcal{H}$, denoted $\dim \mathcal{H}$, is assumed to be either finite or denumerably infinite. Any pair of arbitrary normalized states $|\alpha\rangle$, $|\beta\rangle \in \mathcal{H}$ can be expanded in the orthonormal eigenbasis of H, i.e., $H|a_\mu\rangle = \epsilon_\mu |a_\mu\rangle$ ($\epsilon_\mu \in \mathbb{R}$ and $\langle a_\mu | a_\nu\rangle = \delta_{\mu\nu}$),

$$|\alpha\rangle = \sum_{\mu=1}^{\dim \mathcal{H}} \mathsf{a}_\mu |a_\mu\rangle \ \text{ and } |\beta\rangle = \sum_{\mu=1}^{\dim \mathcal{H}} \mathsf{b}_\mu |a_\mu\rangle, \tag{10}$$

with $\mathsf{a}_\mu \equiv \langle a_\mu|\alpha\rangle, \mathsf{b}_\mu \equiv \langle a_\mu|\beta\rangle \in \mathbb{C}$. The above system can, in general, be extended to a larger (finite or denumerably infinite of dimension $\dim \tilde{\mathcal{H}} > \dim \mathcal{H}$) space $\tilde{\mathcal{H}} = \mathcal{H}_{\mathsf{e}} \otimes \mathcal{H} = \mathcal{H} \oplus \mathcal{H}_\perp$, with normalized states $|\alpha_\perp\rangle \in \mathcal{H}_\perp$, a subspace orthogonal to $\mathcal{H}$. The subsystem $\mathcal{H}_{\mathsf{e}}$ in the tensor product extension is spanned by orthonormal states $\{|g_\mu\rangle\}$. On this larger space $\tilde{\mathcal{H}}$, the above Hamiltonian H is replaced by a "gauge-enlarged" Hamiltonian H_G; the original Hamiltonian H may be understood as $H = P_{\mathcal{H}} H_G P_{\mathcal{H}}$, where $P_{\mathcal{H}}$ is a projection operator onto $\mathcal{H} \subseteq \tilde{\mathcal{H}}$.

We now return to our central focus: the investigation of non-invertible symmetries. Suppose there exists a conserved quantity D, defined on $\mathcal{H}$, i.e.,

$$[H, D] = 0, \tag{11}$$

and D is represented by a bounded but non-invertible operator. The corresponding operator $\widehat{D}$, defined on the extended space $\tilde{\mathcal{H}}$, satisfies

$$D = P_{\mathcal{H}} \widehat{D} P_{\mathcal{H}}, \tag{12}$$

meaning that the action of the $\mathcal{H}$-component of $\widehat{D}$ on states in $\mathcal{H}$ coincides exactly with the action of D on those states. The action of $\widehat{D}$ on the pair of states $|\alpha\rangle$ and $|\beta\rangle \in \mathcal{H}$ is denoted by

$$|\widetilde{\alpha}\rangle = \widehat{D}|\alpha\rangle \ , \quad |\widetilde{\beta}\rangle = \widehat{D}|\beta\rangle. \tag{13}$$

We stress that $\widehat{D}$ need not leave $\mathcal{H}$ invariant and thus $|\widetilde{\alpha}\rangle$ and $|\widetilde{\beta}\rangle$ may, generally, lie outside $\mathcal{H}$. Recall that automorphisms may be linear or antilinear; antilinear maps can be written as the composition of a linear map with complex conjugation K.

Thus, for the associated transition probability – defined via the inner product structure of the Hilbert space – to remain invariant, it must hold that *for all* such arbitrary states $|\alpha\rangle, |\beta\rangle \in \mathcal{H}$, the conditions

$$\big|\langle\widetilde{\beta}|\widetilde{\alpha}\rangle\big|^2 = \big|\langle \widehat{D}\beta|\widehat{D}\alpha\rangle\big|^2 = \big|\langle\beta|\widehat{D}^\dagger\widehat{D}|\alpha\rangle\big|^2 = \big|\langle\beta|\alpha\rangle\big|^2, \tag{14}$$

and, *for all* $|\alpha_\perp\rangle \in \mathcal{H}_\perp$,

$$\big|\langle\widetilde{\alpha}_\perp|\widetilde{\alpha}\rangle\big|^2 = \big|\langle\widehat{D}\alpha_\perp|\widehat{D}\alpha\rangle\big|^2 = \big|\langle\alpha_\perp|\alpha\rangle\big|^2 = 0, \tag{15}$$

must be satisfied.

Theorem:

Consider a general non-invertible quantum symmetry $\widehat{D}$ that when restricted to the physical Hilbert space $\mathcal{H}$ (following Eq. (12)) satisfies the commutation relation Eq. (11). Such a symmetry may preserve the transition probability conditions of Eqs. (14) and (15) if and only if it is of the form

$$\widehat{D} = \mathcal{U}\,\tilde{P}, \tag{16}$$

where $\mathcal{U}$ is either unitary or antiunitary and $\tilde{P}$ is a positive semi-definite – non-invertible – operator. Both, $\mathcal{U}$ and $\tilde{P}$, are defined on the extended space $\tilde{\mathcal{H}}$, such that $\tilde{P}$ acts as the identity when restricted to $\mathcal{H} \subseteq \tilde{\mathcal{H}}$. Specifically, $\tilde{P} = P_{\mathcal{H}} + \tilde{P}_{\mathcal{H}_\perp}$, where $\tilde{P}_{\mathcal{H}_\perp}$ is a non-invertible operator that lies exclusively in $\mathcal{H}_\perp$. Note that from the respective definitions of $\tilde{P}$ and the projection operator $P_{\mathcal{H}}$, it follows that $P_{\mathcal{H}}\tilde{P} = \tilde{P}P_{\mathcal{H}} = P_{\mathcal{H}}$.

Proof:

In what follows, as befits the "if and only if" nature of the Theorem, we explicitly break the proof into two parts.

1. We first illustrate the "if" part of the theorem. Towards that end, we assume that Eq. (16) is satisfied with $\tilde{P}$ a non-invertible operator in an enlarged Hilbert space $\tilde{\mathcal{H}}$ that acts as the identity within $\mathcal{H}$. It is then readily verified that Eq. (14) is obeyed when $\widehat{D}$ is of the form of Eq. (16). Indeed, when Eq. (16) is satisfied for $\tilde{P}$ that acts as the identity on arbitrary states $|\alpha\rangle, |\beta\rangle \in \mathcal{H}$ then, from Eq. (13), the inner product

$$\begin{aligned}\langle\widetilde{\beta}|\widetilde{\alpha}\rangle &= \langle \mathcal{U}\tilde{P}\,\beta|\mathcal{U}\tilde{P}\,\alpha\rangle = \langle\beta|\tilde{P}^\dagger\tilde{P}|\alpha\rangle = \langle\beta|P_{\mathcal{H}}|\alpha\rangle \\ &= \sum_{\mu=1}^{\dim\mathcal{H}} \mathsf{b}_\mu^* \mathsf{a}_\mu = \langle\beta|\alpha\rangle \ ,\end{aligned} \tag{17}$$

if $\mathcal{U}$ is unitary, where * denotes complex conjugation. Similarly, when $\mathcal{U}$ is antiunitary, we have

$$\begin{aligned}\langle\widetilde{\beta}|\widetilde{\alpha}\rangle &= \langle \mathcal{U}\tilde{P}\,\beta|\mathcal{U}\tilde{P}\,\alpha\rangle = \langle\beta|\tilde{P}^\dagger\tilde{P}|\alpha\rangle^* = \langle\beta|P_{\mathcal{H}}|\alpha\rangle^* \\ &= \sum_{\mu=1}^{\dim\mathcal{H}} \mathsf{b}_\mu \mathsf{a}_\mu^* = \langle\alpha|\beta\rangle,\end{aligned} \tag{18}$$

i.e., probabilities are, in fact, conserved. Similar arguments readily establish that Eq. (15) is satisfied.

2. We next turn to the "only if" portion of our theorem and demonstrate that if probabilities remain invariant under the action of a non-invertible transformation (a basic defining requirement of symmetries, following Wigner) then the corresponding operator *must be* of the form of Eq. (16). In what follows, we first discuss (i) the case of linear $\widehat{D}$ and then turn to (ii) antilinear $\widehat{D}$. To illustrate this for *linear* $\widehat{D}$, we recall that any non-invertible linear operator admits a (non-unique, left) polar decomposition (see, e.g., [9] and also Appendix A):

$$\widehat{D} = \widehat{\mathcal{U}}\,\widehat{P} \ , \tag{19}$$

where $\widehat{\mathcal{U}}$ is unitary and $\widehat{P}$ represents a *positive semi-definite* (and thus Hermitian) operator with a non-trivial kernel. This implies that $\widehat{D}^\dagger\widehat{D} = \widehat{P}^2$. Equation (15) states that $\forall\,|\alpha\rangle \in \mathcal{H}$ and $|\alpha_\perp\rangle \in \mathcal{H}_\perp$, the matrix elements $\langle\alpha_\perp|\widehat{P}^2|\alpha\rangle = 0$, that is, $\widehat{P}^2$ may be expressed as a direct sum of an operator lying solely in $\mathcal{H}$ and another exclusively in $\mathcal{H}_\perp$, $\widehat{P}^2 = (\widehat{P}^2)_{\mathcal{H}} + (\widehat{P}^2)_{\mathcal{H}_\perp}$. Equation (14) reads

$$\left|\langle\beta|\widehat{P}^2|\alpha\rangle\right|^2 = \left|\langle\beta|\alpha\rangle\right|^2 . \tag{20}$$

Setting $|\beta\rangle = |\alpha\rangle$, we observe that $\langle\alpha|\widehat{P}^2|\alpha\rangle = 1$, $\forall\,|\alpha\rangle$, implying, given that $\langle\alpha_\perp|\widehat{P}^2|\alpha\rangle = 0$, that $\widehat{P}^2|\alpha\rangle = |\alpha\rangle$. In other words, within $\mathcal{H}$, the operator $\widehat{P}^2$ is the identity, $P_{\mathcal{H}}\widehat{P}^2 P_{\mathcal{H}} = P_{\mathcal{H}}$. Thus the above direct sum for $\widehat{P}^2$ simplifies to $\widehat{P}^2 = P_{\mathcal{H}} + (\widehat{P}^2)_{\mathcal{H}_\perp}$. Given that $\widehat{P}$ is positive semidefinite, it must also enjoy the same block diagonal structure, $\widehat{P} = \tilde{P} = P_{\mathcal{H}} + \tilde{P}_{\mathcal{H}_\perp}$ with a non-invertible $\tilde{P}_{\mathcal{H}_\perp}$. Indeed, if $\tilde{P}_{\mathcal{H}_\perp}$ had, hypothetically, an inverse, $(\tilde{P}_{\mathcal{H}_\perp})^{-1}$, then so would the symmetry $\widehat{D}$, i.e., $\widehat{D}^{-1} = (P_{\mathcal{H}} + (\tilde{P}_{\mathcal{H}_\perp})^{-1})\,\mathcal{U}^\dagger$. Thus, for $\widehat{D}$ to be non-invertible, $\tilde{P}_{\mathcal{H}_\perp}$ cannot have an inverse.

We next address the case of *antilinear* non-invertible symmetries. From the linear operator polar decomposition of Eq. (19), all bounded antilinear symmetries $\widehat{D}$ may be written as

$$\widehat{D} = \widehat{\mathcal{U}}\,\widehat{P} K. \tag{21}$$

Replicating *mutatis mutandis* the above linear case proof yet now with the substitution of Eq. (21) (instead of Eq. (19)) leads, once again, to the conclusion that $\widehat{P} = \tilde{P}$ must be a real operator that acts, for general states within $\mathcal{H}$, as the identity. Specifically, Eq. (14) implies that, within $\mathcal{H}$ the operator $K\widehat{P}^2K$ acts as the identity, $\langle\alpha|K\widehat{P}^2K|\alpha\rangle = 1$, for all $|\alpha\rangle \in \mathcal{H}$. Therefore, in $\mathcal{H}$, the positive semi-definite $\widehat{P}$ itself must also act as the identity (trivially commuting with K). Hence, we may express antilinear symmetries $\widehat{D}$ in the form of Eq. (16) where $\mathcal{U} = \widehat{\mathcal{U}}\,K$ is now antiunitary and $\widehat{P} = \tilde{P}$.

The union of the two above "if" and "only if" portions of our proof concludes the demonstration of our theorem. We emphasize that $\widehat{D}$ is non-invertible if and only if the positive semi-definite Hermitian operator $\tilde{P}$ has a nontrivial kernel on the extended Hilbert space $\tilde{\mathcal{H}}$, i.e., at least one of the eigenvalues of $\tilde{P}$ must vanish. ■

Corollary: If $[H_G, P_{\mathcal{H}}] = 0$, the (anti)unitary operator $\mathcal{U}$ commutes with H_G within $\mathcal{H}$.

Proof: Employing Eq. (11), writing $\widehat{D} = \mathcal{U}\tilde{P}$, and noting that H_G and $P_{\mathcal{H}}$ commute, leads to the conclusion that $P_{\mathcal{H}}[H_G,\mathcal{U}\tilde{P}]P_{\mathcal{H}} = 0$. Substituting $P_{\mathcal{H}}\tilde{P} = \tilde{P}P_{\mathcal{H}} = P_{\mathcal{H}}$ and, once again, invoking the commutativity $[H_G, P_{\mathcal{H}}] = 0$ immediately implies that $P_{\mathcal{H}}[H_G,\mathcal{U}]P_{\mathcal{H}} = 0$. ■

Our theorem and its corollary underscore the utility of the bond algebraic approach [4, 5] in constructing explicit realizations of the linear (or antilinear) map in Eq. (16) across different settings. In particular, defining the automorphism in the enlarged Hilbert space $\tilde{\mathcal{H}}$ guarantees that $[H_G,\mathcal{U}] = 0$ in $\tilde{\mathcal{H}}$.

Realizing the theorem in the TFIC. To close our circle of ideas and see how Eq. (16) appears in a simple system, we return to the (anti)periodic TFIC which is now *minimally gauged* with a single $\mathbb{Z}_2$ gauge field Z_{L+1} on the $(L,1)$ link ($\mathcal{H}_{\mathsf{e}} = \mathbb{C}^2$),

$$H_G = H_1 - \sigma^x_L - \sigma^z_L Z_{L+1}\sigma^z_1. \tag{22}$$

This enlarges the Hilbert space $\mathcal{H}$ by a factor of two, yielding $\tilde{\mathcal{H}} = \mathbb{C}^2 \otimes \mathcal{H}$, with dimension $\dim\tilde{\mathcal{H}} = 2^{L+1}$. Projecting onto the gauge sector $Z_{L+1} = 1(-1)$ gives us $H^{+(-)}$ which acts on $\mathcal{H}\,(\mathcal{H}_\perp)$. These TFICs are known to possess the non-invertible conserved quantity $D_\pm$ (Eq. (8)), implementing the self-duality. The motivation behind considering the minimally gauged model H_G is the existence of a bond-algebraic automorphism [4, 5] Φ_G implementing the self-duality on the enlarged Hilbert space, such that $[H_G,\mathcal{U}] = 0$ with unitary $\mathcal{U}$. Precisely, it acts as: $\Phi_G(\sigma^z_j\sigma^z_{j+1}) = \sigma^x_{j+1}$, $\Phi_G(\sigma^x_j) = \sigma^z_j\sigma^z_{j+1}$, $j = 1, \ldots, L-1$, and the boundary bonds $\Phi_G(\sigma^z_L Z_{L+1}\sigma^z_1) = \sigma^x_1$, and $\Phi_G(\sigma^x_L) = \sigma^z_L Z_{L+1}\sigma^z_1$. Following Ref. [10], that unitary operator is given by

$$\mathcal{U} = \mathsf{C}^x_{1,L+1}\left(\prod_{j=1}^{L}\mathsf{H}_j\mathsf{C}^z_{j+1,j}\right)\mathsf{H}_{L+1}. \tag{23}$$

To explicitly connect this unitary operator to $U^{(2)}$ in Eq. (6), we express the various factors in Eq. (23) in terms of the local Pauli operators. From Eq. (4) we have

$$\mathsf{C}^x_{1,L+1} = e^{i\frac{\pi}{4}\,(\sigma^z_1 X_{L+1} - \sigma^z_1 - X_{L+1} + 1)},$$

the local Controlled-σ^z Clifford operations are given by

$$\mathsf{C}^z_{j+1,j} = e^{-i\frac{\pi}{4}\left(\sigma^z_j\sigma^z_{j+1} - \sigma^z_j - \sigma^z_{j+1} + 1\right)}, \tag{24}$$

while the Hadamard operator is defined in Eq. (5), with the understanding that when acting on the last link ($j = L+1$) one should interpret σ^x (σ^z) operators as the gauge fields X_{L+1} (Z_{L+1}) respectively.

The non-invertibility of the operator $\widehat{D} = \mathcal{U}\tilde{P}$ stems from the projection operator $\tilde{P}$, which is defined on the extended Hilbert space by its action on the gauge degree of freedom: $\tilde{P}_\pm = (\mathbb{1} \pm Z_{L+1})/2$, while acting trivially on the original Hilbert space, as required by the theorem.

We note that if one attempted instead to define the operator $\widehat{D} = \mathcal{U}P$ yet with a projector P that *acts non-trivially* on the original Hilbert space $\mathcal{H}$, such a non-invertible operator will, generally, *not* preserve transition probabilities. An explicit example is that of Eq. (8) for the TFIC. That is why one must go to the trouble of defining a gauged Hamiltonian in Eq. (22) and the associated enlarged Hilbert space $\tilde{\mathcal{H}}$.

In Appendix B, we demonstrate that an alternative global gauging of the original Hamiltonian $H^\pm$ is spectrally equivalent —up to uniform degeneracies— to our minimally gauged Hamiltonian H_G. This equivalence is established via a dual mapping between the two Hamiltonians, indicating that all information about the isometry is encoded in H_G. In this way we can freely enlarge the kernel of $\tilde{P}$ while $\mathcal{U}$ remains fixed.

Conclusions. In earlier investigations on non-invertible symmetries, a defining focus has been the need to satisfy $[H, D] = 0$ to ensure that D is a conserved quantity. Building on the central idea underlying Wigner's original theorem, we further imposed a fundamental physical requirement that any symmetry, including a non-invertible one, must leave all transition probabilities invariant, see Eq. (14). This led us to the rigorous conclusion that all non-invertible symmetries satisfying both of the conditions of Eqs. (11), (14) must take the form of an (anti)unitary operator compounded by a positive semi-definite operator $\tilde{P}$ that acts as the identity on physical states, as expressed in Eq. (16). In other words, they are *partial isometries* that correspond to the gauge-reducing/enlarging dualities introduced in Ref. [5]. Crucially, we show that $\tilde{P}$ cannot be a nontrivial projector on the original Hilbert space $\mathcal{H}$ (i.e., projecting from the Hilbert space $\mathcal{H}$ to a smaller subspace $\mathcal{H}_{\mathsf{sub}} \subset \mathcal{H}$). If $\tilde{P}$ were such a projection operator to a smaller subspace, then transition probabilities would not be preserved, as per Eq. (9); the transformation of Eq. (8) for the periodic TFIC exemplifies the case in point. As we further expounded on the TFIC example, the non-invertibility of the symmetry is sensitive to (and may be triggered by) the choice of the boundary conditions. In particular, a non-invertible symmetry can be rendered to be of a familiar unitary form by a particular choice of boundary conditions, as exemplified by H_1 and H_2 in Eq. (1). Elsewhere [11], building on the bond-algebraic technique, we illustrate this to be the case for other systems. The strength of the bond-algebraic duality technique [4, 5] lies in its ability to explicitly identify and construct automorphisms $\mathcal{U}$.

Following Wigner's foundational insight, we emphasize that physical states are not represented by individual vectors in a Hilbert space, but rather by equivalence classes of vectors. In Wigner's case, vectors differing by an overall phase are equivalent – corresponding to the same *ray* – reflecting the fact that global phase factors are physically unobservable. In the context of non-invertible symmetries, this observation necessitates extending the notion of rays to accommodate gauging. Thus, a proper treatment of generalized symmetries requires a corresponding generalization of the concept of physical states beyond simple Hilbert space vectors, incorporating the additional structure introduced by gauge redundancy:

$$\boxed{\textsf{Physical states} \leftrightarrow \textsf{Equivalence classes}\, |\Psi\rangle \simeq e^{i\gamma}|g\rangle \otimes |\alpha\rangle}$$

where $|g\rangle$ denotes the specific sector in the (enlarged) gauged Hilbert space. The above structure requires a mathematical structure akin to the concept of a vector bundle, and is reminiscent of the fiber bundles (and related field-bundles) used in describing gauge field theories (see Ref. [12] for a pedagogical exposition). Different gauge choices correspond to the sections of the field bundle. We note that there is a rich body of mathematical literature where the notion of a field bundle is further extended in order to accommodate certain key properties, for instance the so-called Batalin–Vilkovisky formalism [13–15] (see also Ref. [16] for a review), which appears most notably in the description of Yang–Mills theories [17] where the physical fields are enumerated by the cohomology of the Becchi–Rouet–Stora–Tyutin (BRST) differential (see, e.g., Ref. [18]).

Acknowledgements. We thank Thomas Bartsch for insightful comments. G.O. gratefully acknowledges support from the Institute for Advanced Study. Z.N. is grateful for support from TU Chemnitz and the IIT Madras. Part of this work was performed at the Aspen Center for Physics, which is supported by National Science Foundation grant PHY-2210452. C.G. and A.H.N. were supported by the Department of Energy under the Basic Energy Sciences award no. DE-SC0025047.

[1] E. Wigner, *Group theory and its application* (Academic Press, New York, 1959).
[2] G. Gehér, An elementary proof for the non-bijective version of wigner's theorem, Physics Letters A **378**, 2054 (2014).
[3] L. Lootens, C. Delcamp, G. Ortiz, and F. Verstraete, Dualities in One-Dimensional Quantum Lattice Models: Symmetric Hamiltonians and Matrix Product Operator Intertwiners, PRX Quantum **4**, 020357 (2023), publisher: American Physical Society.
[4] E. Cobanera, G. Ortiz, and Z. Nussinov, Unified Approach to Quantum and Classical Dualities, Physical Review Letters **104**, 020402 (2010), publisher: American Physical Society.
[5] E. Cobanera, G. Ortiz, and Z. Nussinov, The bond-algebraic approach to dualities, Advances in Physics **60**, 679 (2011), publisher: Taylor & Francis _eprint: https://doi.org/10.1080/00018732.2011.619814.
[6] H. Nishimori and G. Ortiz, *Elements of Phase Transitions and Critical Phenomena* (Oxford University Press, Oxford, 2010).
[7] E. Fradkin and L. Susskind, Order and disorder in gauge systems and magnets, Physical Review D **17**, 2637 (1978).
[8] D. Seo, G. Y. Cho, and R.-J. Slager, Noninvertible duality and symmetry topological order of one-dimensional lattice models with spatially modulated symmetry, Phys. Rev. B **111**, 115112 (2025).
[9] P. Wójcik, A simple proof of the polar decomposition theorem, Annales Mathematicae Silesianae **31**, 165 (2017).
[10] N. Seiberg, S. Seifnashri, and S.-H. Shao, Non-invertible symmetries and LSM-type constraints on a tensor product Hilbert space, SciPost Phys. **16**, 154 (2024).
[11] C. Giridhar, P. Vojta, Z. Nussinov, G. Ortiz, and A. H. Nevidomskyy, Algebraic fusion in a (2+1) lattice model with generalized symmetries, (in preparation).
[12] U. Schreiber, Examples of prequantum field theories I: Gauge fields.
[13] I. A. Batalin and G. A. Vilkovisky, Gauge algebra and quantization, Physics Letters B **102**, 27 (1981).
[14] I. A. Batalin and G. A. Vilkovisky, Quantization

of gauge theories with linearly dependent generators, Phys. Rev. D **28**, 2567 (1983).

[15] I. A. Batalin and G. A. Vilkovisky, Existence theorem for gauge algebra, Journal of Mathematical Physics **26**, 172 (1985).

[16] A. S. Cattaneo, P. Mnev, and M. Schiavina, BV Quantization - Encyclopedia of Math Phys, in *Encyclopedia of Mathematical Physics (Second Edition)*, edited by R. Szabo and M. Bojowald (Academic Press, Oxford, 2025) second edition ed., pp. 543–555.

[17] To describe the instanton sector of the Yang–Mills theory requires an even more general mathematical structure, extending the notion of fiber bundles to so-called ∞-fiber bundles [19].

[18] G. Barnich, F. Brandt, and M. Henneaux, Local BRST cohomology in gauge theories, Physics Reports **338**, 439 (2000).

[19] T. Nikolaus, U. Schreiber, and D. Stevenson, Principal ∞-bundles: general theory, Journal of Homotopy and Related Structures **10**, 749 (2015).

Appendix A. Polar decomposition of operators

The polar decomposition theorem (Eq. (19)) is well known and may be readily demonstrated by various inter-related approaches. This decomposition is most often discussed and employed for invertible operators. Since the use of this general theorem (applicable to all operators whether they happen to be invertible or not) is germane to our central result, for completeness and clarity (using compact Dirac notation), we pedagogically provide a proof that reviews, in passing, the Singular Value Decomposition for *arbitrary* operators A.

Given such general operators, the two products $(A^\dagger A)$ and $(AA^\dagger)$ are Hermitian. Thus, these two operators trivially enjoy a spectral decomposition into complete basis spanned by sets (respectively, $\{|v_\mu\rangle\}$ and $\{|w_\mu\rangle\}$) of orthonormal eigenvectors,

$$\langle v_\mu|v_\nu\rangle = \langle w_\mu|w_\nu\rangle = \delta_{\mu\nu}, \tag{25}$$

that is,

$$\begin{aligned} A^\dagger A|v_\mu\rangle &= \lambda_\mu|v_\mu\rangle, \\ AA^\dagger|w_\mu\rangle &= \lambda_\mu|w_\mu\rangle, \end{aligned} \tag{26}$$

with $\lambda_\mu \in \mathbb{R}$. In the outer product basis formed by these complete orthonormal states, the general operator A may be consistently expressed as

$$A = \sum_\mu \sigma_\mu |w_\mu\rangle\langle v_\mu|, \tag{27}$$

with $\lambda_\mu = |\sigma_\mu|^2$. Equation (27) implies, for these orthonormal bases vectors, that $A|v_\mu\rangle = \sigma_\mu|w_\mu\rangle$ – a generalization of the eigenvector equation to also include operators A that do not necessarily have eigenvectors. By judiciously multiplying individual eigenvectors $|v_\mu\rangle$ and $|w_\mu\rangle$ by appropriate phase factors in Eqs. (26) (doing so leaves Eqs. (26) invariant), we can set all "singular values" $\{\sigma_\mu\}$ to be non-negative, $\sigma_\mu \geq 0$. Defining operators V (and W) whose columns are given by $\{|v_\mu\rangle\}$ (respectively, $\{|w_\mu\rangle\}$), and a diagonal operator Σ whose eigenvalues are the non-negative $\{\sigma_\mu\}$, we may recast the Singular Value Decomposition of Eq. (27) into the more common (non-Dirac notation) form appearing in linear algebra textbooks, viz.,

$$A = W\Sigma V^\dagger. \tag{28}$$

Here, given the orthonormality of their column vectors (Eq. (25)), the operators W and V are unitary.

We underscore that no assumptions have been made in deriving the standard Singular Value Decomposition form of Eq. (28). As such and as is well appreciated, the Singular Value Decomposition indeed applies to *all operators* A including those operators having a rectangular matrix representation (in which case, Σ is represented by a rectangular matrix and W and V, generally, by matrices of different dimension since in Eqs. (26), $(AA^\dagger)$ and $(A^\dagger A)$ are Hermitian operators of different dimension). Our focus is, however, on physically pertinent square matrix representations A in which W, Σ, and V become square matrices of equal dimension (with the number of columns/rows equal to the dimension of the Hilbert space $\tilde{\mathcal{H}}$ discussed in the main text). Given that W and V are unitary (and thus have trivial inverses $W^{-1} = W^\dagger$ and $V^{-1} = V^\dagger$), the Singular Value Decomposition of Eq. (28) implies that the operator A does not have an inverse if and only if at least one of the singular values vanishes, $\sigma_\mu = 0$. Equivalently, A is non-invertible if and only if the singular matrix Σ has at least one null vector – that is, Σ has a nontrivial kernel. Indeed, if all singular values are positive $\sigma_\mu > 0$ then it follows from Eq. (28) that the operator A is invertible.

As we emphasized earlier, the polar decomposition theorem (employed in Eq. (19) of the main text) applies to all operators (whether invertible or non-invertible). The derivation of this theorem from the Singular Value Decomposition of Eqs. (27) and (28) is immediate. To see this, we may simply set

$$\widehat{\mathcal{U}} \equiv WV^\dagger, \quad \widehat{P} \equiv V\Sigma V^\dagger. \tag{29}$$

The unitarity of the so-defined operator $\widehat{\mathcal{U}}$ is evident given that both W and V are unitary. Furthermore, since Σ is a diagonal real operator with non-negative entries and V is unitary, it readily follows that $\widehat{P}$ is a Hermitian operator. Returning to our earlier review of the Singular Value Decomposition, if A is non-invertible then $\widehat{P}$ has a null space (a nontrivial kernel) and is a positive semi-definite operator. Conversely, if A is invertible then $\widehat{P}$ is a positive definite operator.

Thus, putting all of the pieces together, it is seen from the Singular Value Decomposition of Eq. (28) – a decomposition that is applicable to all operators – that any operator A can be written as

$$A = \widehat{\mathcal{U}}\widehat{P}, \tag{30}$$

with $\widehat{\mathcal{U}}$ denoting a unitary operator and $\widehat{P}$ a positive semi-definite one. In the main text, we focused on operators $A = \widehat{D}$ (Eq. (19)) with the Hermitian operators of $\widehat{D}^\dagger\widehat{D}$ whose elements determine the transition probabilities (Eq. (14)) corresponding to the general Hermitian operators of Eq. (26). In such instances, $\widehat{P}$ is a positive semi-definite operator, and the discussion after Eq. (19) follows.

Appendix B. From full gauging to minimal gauging via duality

In the main text, we present an example demonstrating the application of the generalized Wigner theorem to the so-called minimally-gauged TFIC, Eq. (22). Here, we show how this model can be reached starting from a fully-gauged TFIC model, where $\mathbb{Z}_2$ gauge degrees of freedom $\tilde{X}_{j+\frac{1}{2}}$ are placed on every link $(j, j+1)$ of the chain,

$$\tilde{H}_G = -\sum_{j=1}^{L-1}(\sigma^z_j\tilde{X}_{j+\frac{1}{2}}\sigma^z_{j+1} + \sigma^x_j) - \sigma^x_L - \sigma^z_L\tilde{X}_{\frac{1}{2}}\sigma^z_1, \tag{31}$$

subject to the Gauss law on every site $G_j = \tilde{Z}_{j-\frac{1}{2}}\sigma_j^x\tilde{Z}_{j+\frac{1}{2}} = 1$. At first glance, it might appear that we have enlarged the Hilbert space to having dimension 2^{2L}. However, the Gauss laws enforce the gauge redundancy, resulting in a smaller physical Hilbert space. To see this explicitly, we introduce the dual variables containing a string of $\tilde{X}$'s in their definition [10]

$$\tilde{\sigma}_j^z = \tilde{X}_{\frac{1}{2}}\tilde{X}_{\frac{3}{2}}\dots\tilde{X}_{j-\frac{1}{2}}\sigma_j^z; \quad \tilde{\sigma}_j^x = \sigma_j^x. \tag{32}$$

Note that such variables have a long history, closely related to the disorder variables in the context of TFIC [5, 7]. In terms of these new matter fields (replacing $\tilde{\sigma} \to \sigma$ for clarity), the gauged Hamiltonian becomes

$$\begin{aligned} H_G &= -\sum_{j=1}^{L-1}(\sigma_j^z\sigma_{j+1}^z + \sigma_j^x) - \sigma_L^x - \sigma_L^z Z_{L+1}\sigma_1^z \\ &= H_1 - \sigma_L^x - \sigma_L^z Z_{L+1}\sigma_1^z, \end{aligned} \tag{33}$$

where

$$Z_{L+1} = \prod_{j=1}^{L}\tilde{X}_{j-\frac{1}{2}}. \tag{34}$$

Crucially, the only remnant of the gauge degrees of freedom enters via the last $(L,1)$ link. This is exactly the minimally-gauged model in Eq. (22). It follows that it is therefore sufficient to enlarge the Hilbert space by one ancilla qubit: $\tilde{\mathcal{H}} = \mathbb{C}^2\otimes\mathcal{H}$, with dimension $\dim\tilde{\mathcal{H}} = 2^{L+1}$, thus justifying the minimal gauging in Eq. (22).

Proceeding as in the main text, we now define the projector operator $\tilde{P}_\pm = (\mathbb{1}\pm Z_{L+1})/2$ which projects onto a definite ancilla eigenvalue $\tilde{\eta} = \pm 1$, while acting trivially on the original Hilbert space: $\tilde{P}|_{\mathcal{H}} = P_{\mathcal{H}}$. As shown in the main text, the operator $\widehat{D}_\pm = \mathcal{U}\tilde{P}_\pm$, with $\mathcal{U}$ defined in Eq. (23) commutes with the Hamiltonian H_G above, thus implementing the non-invertible symmetry as required by the generalized Wigner theorem.

Several remarks are in order: (i) $\tilde{\mathcal{H}}$ contains two copies of the original Hilbert space, distinguished by the eigenvalue $\tilde{\eta} = \pm 1$ of the ancilla: $\tilde{\mathcal{H}} = \mathcal{H}\oplus\mathcal{H}$, and (ii) the Hamiltonian in the "twisted" sector, corresponding to $\tilde{\eta} = -1$, contains an η-defect on the $(L,1)$ link, relative to the sign on the rest of the bonds in H_1. Adopting the language of Ref. [10], the Hamiltonian H_G in the enlarged Hilbert space $\tilde{\mathcal{H}}$ can thus be interpreted as containing an $1\oplus\eta$ defect – this observation is central to the interpretation of $1\oplus\eta = D\otimes D$ as the result of fusing two D-defects in Ref. [10].

Finally, we point out that (iii) the Hamiltonian H_G in Eq. (33) bears, modulo a replacement of the global symmetry operator $\hat{\eta} = \prod_{j=1}^{L}\sigma_j^x$ by the emergent *dual symmetry* operator $\widehat{\tilde{\eta}} = \prod_{j=1}^{L}\tilde{X}_{j-\frac{1}{2}} \equiv Z_{L+1}$ in Eq. (34), an uncanny resemblance to H_2 in Eq. (1). This echoes the observation, made in Ref. [10], that the duality implemented by the operator $\widehat{D}_\pm = U\tilde{P}_\pm$ interchanges $\eta \longleftrightarrow \tilde{\eta}$. Physically, the eigenvalue η marks the (linear superposition of) the symmetry-broken ground states in the ordered phase, while $\tilde{\eta}$ corresponds to the so-called disorder variable. The D-duality thus interchanges the order and disorder variables of the TFIC.