Collision types and times in interacting particle systems

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September 29, 2025

Abstract

We consider a system of stochastic interacting particles with general diffusion coefficient and drift functions and we study the types of collisions that arise in them. In particular, interactions between particles are inversely proportional to their separation, and the coupling function of interaction is also considered in great generality. Our main result indicates that under very mild conditions, all collisions are simple almost surely, namely, only one pair of particles collides at any time, while more complicated collisions such as three-body or disjoint two-body collisions occur with zero probability. In order to obtain our results we make use of symmetric polynomials on the square of particle separations; the degree of these polynomials indicates the type of collision, and by a locality argument we show that polynomials indicating a non-simple collision almost surely do not cancel. We make use of our main result to study the Hausdorff dimension of times at which collisions occur, and we show that this dimension is given by the ratio between the interaction coupling and diffusion functions. Our results cover many of the most well-known particle systems, such as the Dyson model and Wishart processes and their extensions to non-constant diffusion coefficients and background drifts.

1 Introduction

Stochastic particle systems have been an important topic of study in both mathematics and physics. In particular, systems like the Dyson model in nuclear physics [5] and Wishart processes in statistics [2] are representative due to their connections to random matrix theory and integrable systems [12, 7]. These particle systems are special cases of the more general particle system $x = (x_1, \ldots, x_N)$ we consider in this paper, which is defined as the solution to the following SDE

$$dx_i(t) = \sigma\big(x_i(t)\big)dB_i(t) + b\big(x_i(t)\big)dt + \sum_{\alpha\in R_+} k_\alpha\big(x(t)\big)\frac{\alpha_i}{\langle x(t), \alpha\rangle}\, dt. \tag{1.1}$$

Details regarding the objects in this SDE are introduced in Section 2.1, but for now we clarify that for a fixed root system R, the collection of selected positive roots is denoted by R_+, and W stands for the corresponding positive Weyl chamber, with $\overline{W}$ its closure, and the process $x \in \overline{W}$. Also, for two vectors $y, z \in \mathbb{R}^N$, we denote the standard (Euclidean) inner product by $\langle y, z\rangle$. Let us denote by $T_\infty := \inf\{t > 0 \,|\, \exists\, k = 1, \ldots, N : |x_k(t)| = \infty\}$ the lifetime of x. For simplicity, we omit the time variable t in parentheses whenever it is not essential for the calculations. We can also write the above system of equations in the form of a single multivariate equation

$$dx = \sigma(x)dB + b(x)dt + \sum_{\alpha\in R_+} k_\alpha(x)\frac{\alpha}{\langle x, \alpha\rangle}\, dt, \tag{1.2}$$

where we slightly disrupt notation by writing $\sigma(x) = (\sigma(x_1), \ldots, \sigma(x_N))$ and $b(x) = (b(x_1), \ldots, b(x_N))$. We will use this notation in the cases where we talk about the diffusion coefficient σ and the drift function b as functions of N variables. In contrast, the coupling interaction function $k_\alpha(x)$ always indicates a multivariate function.

We see from the form of (1.1) that there is a singularity when $\langle x, \alpha \rangle$ approaches zero, which is one of the most interesting features of the class of systems we consider. In fact, this type of singularity corresponds to the situation where two particles collide, and this is equivalent to the situation where x hits the boundary of the Weyl chamber, ∂W. Indeed, it is known that for a subset of the processes we consider, the first time in which x hits ∂W is almost surely finite when the coupling interaction function is sufficiently weak [3]. The objectives of this paper are the following. First, we aim to clarify which types of collision are possible in these processes, more specifically, we show that non-simple collisions, namely collisions that involve more than two particles at any given time, almost surely do not occur. Second, we derive the Hausdorff dimension of the times where collisions occur, providing detailed conditions for their almost-sure occurrence.

Let us introduce the fundamental conditions required for the stating our results. Since the form of the considered SDEs (1.1) is very general, we introduce the following collection of assumptions, which establish the general framework for our considerations. We begin with very general assumptions. The functions σ, b in (1.1) are single-valued and defined on the domain

$$D := \{ y \in \mathbb{R} \,|\, \exists\, z \in \overline{W}, k \in \{1, \dots, N\} : z_k = y \}.$$

(G1) The functions $\sigma, b : D \to \mathbb{R}$ and $k_\alpha : \overline{W} \to \mathbb{R}$ are continuous for every $\alpha \in R$.

(G2) The process $x = (x_1, \dots, x_N)$ is a solution to (1.2) (or equivalently to (1.1)).

(G3) We restrict the root system to $R \in \{A_{N-1}, B_N, D_N\}$ and hence for all $\alpha, \beta \in R$ we have $\langle \alpha, \beta \rangle = \pm 1$, whenever $\langle \alpha, \beta \rangle \neq 0$. Moreover, the SDEs (1.1) and (1.2) are identical for B_N and C_N, so the latter case is covered by the former. Note that these root systems are both reduced and crystallographic.

Here, we point out that our results quite possibly hold for arbitrary root systems, though we leave them out of the scope of this paper as the ones related to particle systems are those included in (G3).

The existence of a solution to (1.2) is ensured by the results given in the upcoming paper [15], where it is shown that continuity of the coefficients together with positivity of k_α, for every $\alpha \in R_+$, are enough to construct a solution. In particular, (G1) together with (A1) presented below are enough to ensure the existence of $x = (x_1, \dots, x_N)$. From now on, all the coefficient functions are continuous as in (G1), whenever we talk about a root system, we work under (G3) and when we consider the process $x = (x_1, \dots, x_N)$, it is always defined as in (G2).

To present the last assumptions let us denote by Δ_+ the family of positive simple roots [11]. Recall that every positive root $\beta \in R_+$ can be written as a sum of some number of positive roots, i.e. $\beta = n_1\alpha_1 + \ldots + n_k\alpha_k$, for some k and $\alpha_1, \dots, \alpha_k \in \Delta_+$ as well as positive natural numbers $n_1, \dots, n_k$. Let us denote this unique set of simple roots by $\Delta_+(\beta)$ and we define the partial root ordering $\beta_1 \leq \beta_2$ to mean that $\Delta_+(\beta_1) \subseteq \Delta_+(\beta_2)$. This allows us to complement the list of general assumptions with the following.

(A1) The functions σ and k_α are strictly positive for every $\alpha \in R_+$.

(A2) The function b satisfies the inequality

$$\sum_{i=1}^{N} \alpha_i b(x_i) \leq 0$$

for every $\alpha \in \Delta_+$.

(A3) For every pair of positive roots $\alpha, \beta \in R_+$ such that $\alpha \leq \beta$ and $\langle \alpha, \beta \rangle \neq 0$ we have

$$\frac{k_\alpha(x)}{\langle x, \alpha \rangle} \geq \frac{k_\beta(x)}{\langle x, \beta \rangle}, \quad x \in \overline{W}.$$

We remark that (A2) takes explicit forms depending on the root system: for A_{N-1}, function b is non-increasing, and for $R = B_N$ and D_N, b is in addition, non-positive. The non-positivity condition may be ignored if all collisions under consideration are between particles and not with spatial boundaries, namely, the origin for B_N and D_N.

Condition (A3) states that the coupling interaction function k_α is such that closer particles always experience a stronger repulsion (for positive roots, the partial ordering $\alpha \le \beta$ implies that $\langle x, \alpha\rangle \le \langle x, \beta\rangle$).

The first result, which enables us to explore more sophisticated properties of the set of collision times, describes the nature of collisions.

Theorem 1. *Let $x = (x_1, \dots, x_N)$ be a solution to* (1.2) *and assume that (G1) and (A1) hold. Then, there are no double collisions after the start, i.e. for every two different $\alpha, \beta \in R_+$ we have*

$$\langle x(t), \alpha\rangle^2 + \langle x(t), \beta\rangle^2 > 0, \quad \textit{for all } t \in (0, T_\infty)$$

almost surely.

We note that this result is in contrast with those in [18], where matrix valued processes with Gaussian entries are considered, but the dynamics considered there are fractional. Our results are applicable to matrix processes with Brownian motions as entries (see Section 5), where, in fact, no collisions occur. However, when matrix entries are fractional Brownian motions, there exist conditions that ensure the existence of multiple collisions with non-zero probability.

The remaining results, which are consequences of Theorem 1 are given in the following two statements.

Theorem 2. *Assume that (G1)-(G3) and (A1) hold, and suppose that the ratios b/σ^2 and k/σ^2 are bounded everywhere in D and $\overline{W}$. Then, the Hausdorff dimension of $x^{-1}(\partial W)$ is almost surely bounded above by*

$$\dim x^{-1}(\partial W) \le \max\left\{0, \frac{1}{2} - \min_{\alpha\in\Delta_+} \inf_{y\in W} \frac{|\alpha|^2 k_\alpha(y)}{\sum_{i=1}^N \alpha_i^2 \sigma^2(y_i)}\right\}.$$

Theorem 3. *Assume that (G1)-(G3) and (A1)-(A3) hold, and suppose that the ratios b/σ^2 and k/σ^2 are bounded everywhere in D and $\overline{W}$. Then, the Hausdorff dimension of $x^{-1}(\partial W)$ is almost surely bounded below by*

$$\dim x^{-1}(\partial W) \ge \max\left\{0, \frac{1}{2} - \min_{\alpha\in\Delta_+} \sup_{y\in W} \frac{|\alpha|^2 k_\alpha(y)}{\sum_{i=1}^N \alpha_i^2 \sigma^2(y_i)}\right\}.$$

Note that Theorem 3 indicates the conditions for almost-sure existence of collisions. A weaker, positive probability form of this last statement can be proved without the assumptions (A2) and (A3).

Lemma 4. *Assume that (G1)-(G3) and (A1) hold, and suppose that the ratios b/σ^2 and k/σ^2 are bounded everywhere in D and $\overline{W}$. Then, the Hausdorff dimension of $x^{-1}(\partial W)$ is bounded below by*

$$\dim x^{-1}(\partial W) \ge \max\left\{0, \frac{1}{2} - \max_{\alpha\in\Delta_+} \sup_{y\in W} \frac{|\alpha|^2 k_\alpha(y)}{\sum_{i=1}^N \alpha_i^2 \sigma^2(y_i)}\right\}$$

with positive probability. Otherwise we have the trivial bound $\dim x^{-1}(\partial W) \ge 0$.

The paper is structured as follows. We begin by recalling several notions and setting notations necessary for our results in Section 2. The proof of Theorem 1 is given in a series of propositions detailed in Section 3. The proofs of Theorems 2 and 3, and Lemma 4 are given in Section 4. We end the paper with several examples of particle systems for which our results are applicable in Section 5.

2 Setting, definitions, and properties

2.1 Reflections, root systems, and Weyl chambers

To understand the general particle system, we first introduce the key concepts that occur in (1.1) and (1.2). For any nonzero vectors y and $z \in \mathbb{R}^N$, we define the *reflection operator* ϱ_y acting on z by the formula

$$\varrho_y z := z - 2\frac{\langle y, z\rangle}{\langle y, y\rangle} y,$$

which yields the reflection of the vector z along the hyperplane of y. We will use the notation $|y| = \sqrt{\langle y, y\rangle}$ for the Euclidean vector norm of $y \in \mathbb{R}^N$, which is naturally reduced to the absolute value function if y is a scalar. A *root system* R is then defined as a finite set of nonzero vectors, called *roots*, satisfying the condition that for any $\alpha, \beta \in R$, the reflection $\varrho_\alpha \beta$ also belongs to R; in other words, the reflection of any root along another root is again a root. Furthermore, we require that the root system be *reduced*, that is, for every $\alpha \in R$ we have $\mathbb{R}\alpha \cap R = \{\pm\alpha\}$. The reflections generated by the roots in R form a *reflection group*, denoted by G.

Some structural properties of reduced root systems will play an important role in the upcoming sections. A *positive subsystem* $R_+ \subset R$ can be selected by choosing an arbitrary vector $u \in \mathbb{R}^N$ such that $\langle \alpha, u\rangle \neq 0$ for all $\alpha \in R$. This choice induces a decomposition of R into positive and negative roots, where

$$R_+ := \{\alpha \in R \mid \langle \alpha, u\rangle > 0\}.$$

The set of negative roots is given by the *negative subsystem* $-R_+$. In the following, we omit the dependence on R. For a fixed positive subsystem R_+, the associated *Weyl chamber* W is

$$W := \left\{y \in \mathbb{R}^N \,\middle|\, \langle \alpha, y\rangle > 0 \text{ for all } \alpha \in R_+\right\}.$$

We denote by Δ_+ the *simple system* corresponding to R_+, consisting of a set of *simple roots* that forms a basis for the positive subsystem. We state a few properties for the simple roots:

(S1) For every root $\alpha \in R_+$ there exists unique $c_\gamma \geq 0$ such that

$$\alpha = \sum_{\gamma \in \Delta_+} c_\gamma \gamma.$$

Every root in $-R_+$ is a linear combination of simple roots with non-positive coefficients.

(S2) For every $\alpha, \beta \in \Delta_+$ with $\alpha \neq \beta$ we have $\langle \alpha, \beta\rangle \leq 0$.

The Properties (S1) and (S2) can be found in [11, Theorem, p. 8; Corollary, p. 9].

2.2 Symmetric polynomials

For a given vector of variables $A = (a_1, \ldots, a_N)$, we denote by $e_n(A)$ the basic symmetric polynomial in A, where $n = 1, \ldots, N$. More precisely, we have

$$e_n(A) := \sum_{i_1 < \ldots < i_n} a_{i_1} \cdot \ldots \cdot a_{i_n}, \quad n = 1, \ldots, N.$$

In particular, $e_1(A) = a_1 + \ldots + a_N$ and $e_N(A) = a_1 \cdot \ldots \cdot a_N$. For completeness, we set $e_0(A) \equiv 1$ and $e_{-1}(A) \equiv 0$. Moreover, for any fixed collection $a_{j_1}, \ldots, a_{j_k}$ of entries of A, we denote by

$$e_n^{\overline{a_{j_1}}, \ldots, \overline{a_{j_k}}}(A) := \sum_{\substack{i_1 < \ldots < i_n \\ i_l \neq j_s}} a_{i_1} \cdot \ldots \cdot a_{i_n},$$

which is a sum of all products of length n in which there are no elements of $\{a_{j_1}, \ldots, a_{j_k}\}$.

The primary tools for studying the collisions of the process x with the boundary of the Weyl chamber are the symmetric polynomials in $\langle x, \alpha\rangle$ for $\alpha \in R_+$. Therefore, we introduce the following notation.

$$\mathcal{A} := \{\langle x, \alpha\rangle^2 : \alpha \in R_+\}.$$

Moreover, for a given set of positive numbers, which we will call weights

$$w := \{w_\alpha > 0 : \alpha \in R_+\}$$

we write α^* for a root normalized by w_α, that is, $\alpha^* = \alpha / w_\alpha$ and consider the corresponding set

$$\mathcal{A}^w := \{\langle x, \alpha^*\rangle^2 : \alpha \in R_+\}.$$

We denote by M the number of elements in $\mathcal{A}^w$, which is obviously independent of weight, and is just the size of R_+. For example, for the A_{N-1} root system we have $M = N(N-1)/2$.

We take into consideration the corresponding symmetric polynomials in $\mathcal{A}^w$. We also simplify the notation and write $e_n^{\overline{\alpha}}(\mathcal{A}^w)$ for the symmetric polynomial of degree n which does not include $\langle x, \alpha^* \rangle^2$. We write similarly $e_n^{\overline{\alpha},\overline{\beta}}(\mathcal{A}^w)$ whenever we want to exclude $\langle x, \alpha^* \rangle^2$ and $\langle x, \beta^* \rangle^2$.

We end this subsection with two technical lemmas that provide algebraic formulas used in what follows.

Lemma 5. *For every $n = 1, \ldots, M$ and every $\alpha, \beta \in R_+$ such that $\alpha \neq \beta$ we have*

$$e_{n-2}^{\overline{\alpha},\overline{\beta}}(\mathcal{A}^w) e_n(\mathcal{A}^w) = e_{n-1}^{\overline{\alpha}}(\mathcal{A}^w) e_{n-1}^{\overline{\beta}}(\mathcal{A}^w) + e_n^{\overline{\alpha},\overline{\beta}}(\mathcal{A}^w) e_{n-2}^{\overline{\alpha},\overline{\beta}}(\mathcal{A}^w) - \left(e_{n-1}^{\overline{\alpha},\overline{\beta}}(\mathcal{A}^w)\right)^2. \tag{2.1}$$

Proof. We have for every $\alpha \neq \beta$ that

$$e_n(\mathcal{A}^w) = \langle x, \alpha^* \rangle^2 e_{n-1}^{\overline{\alpha}}(\mathcal{A}^w) + e_n^{\overline{\alpha}}(\mathcal{A}^w), \quad e_{n-1}^{\overline{\beta}}(\mathcal{A}^w) = \langle x, \alpha^* \rangle^2 e_{n-2}^{\overline{\alpha},\overline{\beta}}(\mathcal{A}^w) + e_{n-1}^{\overline{\alpha},\overline{\beta}}(\mathcal{A}^w),$$

which gives

$$\begin{aligned}
e_{n-2}^{\overline{\alpha},\overline{\beta}}(\mathcal{A}^w) e_n(\mathcal{A}^w) &= e_{n-2}^{\overline{\alpha},\overline{\beta}}(\mathcal{A}^w) \left(\langle x, \alpha^* \rangle^2 e_{n-1}^{\overline{\alpha}}(\mathcal{A}^w) + e_n^{\overline{\alpha}}(\mathcal{A}^w) \right) \\
&= \langle x, \alpha^* \rangle^2 e_{n-2}^{\overline{\alpha},\overline{\beta}}(\mathcal{A}^w) e_{n-1}^{\overline{\alpha}}(\mathcal{A}^w) + e_{n-2}^{\overline{\alpha},\overline{\beta}}(\mathcal{A}^w) e_n^{\overline{\alpha}}(\mathcal{A}^w) \\
&= \left(e_{n-1}^{\overline{\beta}}(\mathcal{A}^w) - e_{n-1}^{\overline{\alpha},\overline{\beta}}(\mathcal{A}^w) \right) e_{n-1}^{\overline{\alpha}}(\mathcal{A}^w) + e_{n-2}^{\overline{\alpha},\overline{\beta}}(\mathcal{A}^w) e_n^{\overline{\alpha}}(\mathcal{A}^w) \\
&= e_{n-1}^{\overline{\alpha}}(\mathcal{A}^w) e_{n-1}^{\overline{\beta}}(\mathcal{A}^w) + e_{n-2}^{\overline{\alpha},\overline{\beta}}(\mathcal{A}^w) e_n^{\overline{\alpha}}(\mathcal{A}^w) - e_{n-1}^{\overline{\alpha},\overline{\beta}}(\mathcal{A}^w) e_{n-1}^{\overline{\alpha}}(\mathcal{A}^w).
\end{aligned}$$

Since $e_n^{\overline{\alpha}}(\mathcal{A}^w) = \langle x, \beta^* \rangle^2 e_{n-1}^{\overline{\alpha},\overline{\beta}}(\mathcal{A}^w) + e_n^{\overline{\alpha},\overline{\beta}}(\mathcal{A}^w)$ and $e_{n-1}^{\overline{\alpha}}(\mathcal{A}^w) = \langle x, \beta^* \rangle^2 e_{n-2}^{\overline{\alpha},\overline{\beta}}(\mathcal{A}^w) + e_{n-1}^{\overline{\alpha},\overline{\beta}}(\mathcal{A}^w)$ we finally get

$$\begin{aligned}
e_{n-2}^{\overline{\alpha},\overline{\beta}}(\mathcal{A}^w) e_n^{\overline{\alpha}}(\mathcal{A}^w) - e_{n-1}^{\overline{\alpha},\overline{\beta}}(\mathcal{A}^w) e_{n-1}^{\overline{\alpha}}(\mathcal{A}^w) &= e_{n-2}^{\overline{\alpha},\overline{\beta}}(\mathcal{A}^w) \left(\langle x, \beta^* \rangle^2 e_{n-1}^{\overline{\alpha},\overline{\beta}}(\mathcal{A}^w) + e_n^{\overline{\alpha},\overline{\beta}}(\mathcal{A}^w) \right) - e_{n-1}^{\overline{\alpha},\overline{\beta}}(\mathcal{A}^w) e_{n-1}^{\overline{\alpha}}(\mathcal{A}^w) \\
&= e_{n-2}^{\overline{\alpha},\overline{\beta}}(\mathcal{A}^w) e_n^{\overline{\alpha},\overline{\beta}}(\mathcal{A}^w) - e_{n-1}^{\overline{\alpha},\overline{\beta}}(\mathcal{A}^w) \left(e_{n-1}^{\overline{\alpha}}(\mathcal{A}^w) - \langle x, \beta^* \rangle^2 e_{n-2}^{\overline{\alpha},\overline{\beta}}(\mathcal{A}^w) \right) \\
&= e_{n-2}^{\overline{\alpha},\overline{\beta}}(\mathcal{A}^w) e_n^{\overline{\alpha},\overline{\beta}}(\mathcal{A}^w) - \left(e_{n-1}^{\overline{\alpha},\overline{\beta}}(\mathcal{A}^w) \right)^2.
\end{aligned}$$

□

Lemma 6. *If $\langle \alpha, \beta \rangle \neq 0$ and $\alpha \neq \beta$, denote by γ one of $\pm \varrho_\beta \alpha = \pm(\alpha - 2\beta \langle \alpha, \beta \rangle / |\beta|^2)$, which is in R_+. Then, we have*

$$\langle \alpha, \beta \rangle \langle x, \alpha \rangle + \langle \beta, \gamma \rangle \langle x, \gamma \rangle = \frac{2 \langle x, \beta \rangle}{|\beta|^2} \tag{2.2}$$

and

$$\langle \alpha, \beta \rangle \langle x, \gamma \rangle + \langle \beta, \gamma \rangle \langle x, \alpha \rangle = \frac{2 \langle \alpha, \beta \rangle \langle \beta, \gamma \rangle \langle x, \beta \rangle}{|\beta|^2} \tag{2.3}$$

Proof. Since the expression $\langle \beta, \gamma \rangle \langle x, \gamma \rangle$ is the same for γ and $-\gamma$ and the equality 2.3 is invariant under the sign change of γ, we can assume that

$$\gamma = \varrho_\beta \alpha = \alpha - \frac{2\beta \langle \alpha, \beta \rangle}{|\beta|^2},$$

which leads to $\langle \beta, \gamma \rangle = \langle \alpha, \beta \rangle - 2 \langle \alpha, \beta \rangle = - \langle \alpha, \beta \rangle$ and hence

$$\begin{aligned}
\langle \beta, \gamma \rangle \langle x, \gamma \rangle &= - \langle \alpha, \beta \rangle \left(\langle x, \alpha \rangle - \frac{2 \langle x, \beta \rangle \langle \alpha, \beta \rangle}{|\beta|^2} \right) = - \langle \alpha, \beta \rangle \langle x, \alpha \rangle + \frac{2 \langle \alpha, \beta \rangle^2 \langle x, \beta \rangle}{|\beta|^2}, \\
\langle \alpha, \beta \rangle \langle x, \gamma \rangle &= \langle \alpha, \beta \rangle \left(\langle x, \alpha \rangle - \frac{2 \langle x, \beta \rangle \langle \alpha, \beta \rangle}{|\beta|^2} \right) = - \langle \beta, \gamma \rangle \langle x, \alpha \rangle + \frac{2 \langle \alpha, \beta \rangle \langle \beta, \gamma \rangle \langle x, \beta \rangle}{|\beta|^2}
\end{aligned}$$

and the results follow from $\langle \alpha, \beta \rangle^2 = 1$, see (G3). □

2.3 Hausdorff dimension

The Hausdorff dimension generalizes the familiar notion of dimension. This means that well-known geometric objects like straight lines, hyperplanes and others with intuitive dimensionality keep the same dimension. The Hausdorff dimension offers a finer distinction, since it admits positive real numbers.

We denote by $B(y,r) := \{z \in \mathbb{R}^N : |y-z| \le r\}$ the closed N-dimensional ball centered at y with radius r. For the monotonically increasing monomial (on the positive halfline) of power $\varkappa \ge 0$ the Hausdorff measure is specified by

$$m_\varkappa(E) := \lim_{\varepsilon\to 0} \inf \Big\{ \sum_{i=1}^{\infty} (2r_i)^\varkappa \,\Big|\, \exists\, 0 \le r_i < \varepsilon,\, y_i \in \mathbb{R}^n : E \subset \bigcup_{i=1}^{\infty} B(y_i, r_i) \Big\}.$$

The Hausdorff dimension is defined by the following lemma, see [1, 8.1 Hausdorff dimension].

Lemma 7. *For any set $E \subset \mathbb{R}^n$ there exists a unique number $d \in [0,n]$, called the Hausdorff dimension of E, for which*

$$\varkappa < d \Rightarrow m_\varkappa(E) = \infty, \quad \varkappa > d \Rightarrow m_\varkappa(E) = 0.$$

This number is denoted by $\dim(E)$ *and satisfies*

$$d = \dim(E) = \sup\{\varkappa > 0 : m_\varkappa(E) = \infty\} = \inf\{\varkappa > 0 : m_\varkappa(E) = 0\}.$$

The Hausdorff dimension satisfies the following properties.

(H1) *Countable stability:* If $F_1, F_2, \ldots$ is a (countable) sequence of sets then

$$\dim\Big(\bigcup_{i=1}^{\infty} F_i\Big) = \sup_{i\in\mathbb{N}} \dim(F_i).$$

(H2) *Monotonicity:* If $E \subset F$ then $\dim(E) \le \dim(F)$.

(H3) *Invariance under bi-Lipschitz mapping:* If $f : E \to \mathbb{R}^N$ is a bi-Lipschitz mapping, that is, if there exist constants $c_2 \ge c_1 > 0$ such that for every $y, z \in E$

$$c_1|y-z| \le |f(y) - f(z)| \le c_2|y-z|,$$

then $\dim(E) = \dim(f(E))$.

Properties (H1) and (H2) can be found in [6, Section 2.2, Hausdorff Dimension], while (H3) is stated in [6, Corollary 2.4].

2.4 Additional notation

We will write

$$\sum_{\substack{\alpha\in R_+}} \sum_{\substack{\beta\in R_+ \\ \alpha\neq\beta}} \qquad \text{and} \qquad \sum_{\alpha\in R_+} \sum_{\beta\in R_+}$$

for the off-diagonal sum and the double sum, respectively. For given $\beta \in R_+$ we introduce the following notation

$$R_+(\beta) = \big\{(\alpha,\gamma) \,|\, \alpha,\gamma \in R_+, \alpha \neq \beta, \langle\alpha,\beta\rangle \neq 0, \gamma = \pm\varrho_\beta\alpha\big\}. \tag{2.4}$$

For example, for $i < j < k$ and $\beta = e_i - e_j$ the pair $(e_i - e_k, e_j - e_k)$ is an element of $R_+(\beta)$.

3 Simple and multiple collisions

In this section we study the nature of collisions in more detail presenting a series of propositions leading to the proof of Theorem 1.

For fixed $y \in \overline{W}$, we introduce two sets indicating the roots that make $\langle y, \alpha \rangle$ equal to zero and those for which this expression is positive, respectively,

$$\mathcal{P}^0(y) = \{\alpha \in R_+ : \langle \alpha, y \rangle = 0\}, \quad \mathcal{P}^+(y) = \{\alpha \in R_+ : \langle \alpha, y \rangle > 0\}.$$

Note that $\mathcal{P}^0(y) \cup \mathcal{P}^+(y) = R_+$ and $\mathcal{P}^0(y) \cap \mathcal{P}^+(y) = \emptyset$ for every $y \in \overline{W}$. Obviously $\mathcal{P}^0(y) = \emptyset$ and $\mathcal{P}^+(y) = R_+$ for every $y \in W$ and $\mathcal{P}^0(y)$ becomes nonempty on the boundary ∂W. We split the boundary ∂W of the Weyl chamber W into parts depending on the number of elements in $\mathcal{P}^0(y)$. More precisely, we introduce the following.

Definition 1. *We say that $y \in \partial W$ is a collision point of order m, where $m \in \{1, \ldots, M\}$, if $|\mathcal{P}^0(y)| = m$. We denote the set of all collision points of order m as $\partial W^{(m)}$.*

Definition 2. *We say that for the general particle system $x = (x_1, \ldots, x_N)$ a collision of order m occurs at time t, if $x_t \in \partial W^{(m)}$. If $m = 1$ we call it a single collision and a multiple collision is a collision of order $m \geq 2$.*

Note that the boundary of the Weyl chamber ∂W decomposes into a disjoint union of $\partial W^{(m)}$ over $m \in \{1, \ldots, M\}$. Moreover, collisions of $x = (x_1, \ldots, x_N)$ and their orders are controlled by the symmetric polynomials $e_n(\mathcal{A}^w)$. In fact, there is a collision of order m at time t if and only if $e_n(\mathcal{A}^w)_t = 0$ and $e_{n-1}(\mathcal{A}^w)_t > 0$, where $n = M - m + 1$. Let us denote the corresponding first hitting times of zero for $e_n(\mathcal{A}^w)$ by

$$\tau_n := \inf\left\{t \in (0, T_\infty] : e_n(\mathcal{A}^w)_t = 0\right\}, \quad n = 0, \ldots, M.$$

Note that the hitting times introduced above are the same for every choice of the weights w due to their positivity. Consequently, we omit w in the notation here. Obviously, the hitting times are ordered as follows

$$\tau_M \leq \tau_{M-1} \leq \ldots \leq \tau_1 \leq \tau_0 = T_\infty.$$

Here, we took advantage of the convention that $e_0(\mathcal{A}^w) \equiv 1$, which immediately gives $\tau_0 = T_\infty$. Our starting point is the stochastic description of polynomials $e_n(\mathcal{A}^w)$, which is provided in the following proposition.

Proposition 1. *For every fixed set of weights $w = \{w_\alpha > 0 : \alpha \in R_+\}$ and $n = 1, \ldots, M$ we have*

$$\begin{aligned} de_n(\mathcal{A}^w) &= 2\sum_{k=1}^N \Bigg(\sum_{\alpha \in R_+} \alpha_k^* \langle x, \alpha^* \rangle \, e_{n-1}^{\overline{\alpha}}(\mathcal{A}^w) \Bigg) \sigma(x_k) dB_k + 2\sum_{k=1}^N b(x_k) \sum_{\alpha \in R_+} \alpha_k^* \langle x, \alpha^* \rangle \, e_{n-1}^{\overline{\alpha}}(\mathcal{A}^w) dt \\ &+2 \sum_{\alpha \in R_+} \sum_{\beta \in R_+} \frac{\langle \alpha^*, \beta^* \rangle \langle x, \alpha^* \rangle}{\langle x, \beta^* \rangle} e_{n-1}^{\overline{\alpha}}(\mathcal{A}^w) k_\beta(x) dt + \sum_{k=1}^N \sigma^2(x_k) \sum_{\alpha \in R_+} (\alpha_k^*)^2 e_{n-1}^{\overline{\alpha}}(\mathcal{A}^w) \, dt \\ &+2 \sum_{k=1}^N \sigma^2(x_k) \sum_{\substack{\alpha \in R_+ \\ \alpha \neq \beta}} \sum_{\beta \in R_+} \alpha_k^* \beta_k^* \langle x, \alpha^* \rangle \langle x, \beta^* \rangle \, e_{n-2}^{\overline{\alpha}, \overline{\beta}}(\mathcal{A}^w) dt \end{aligned}$$

up to the lifetime T_∞.

Proof. Using the fact that $dx_k dx_j = 0$, whenever $k \neq j$, the Itô formula leads to

$$de_n(\mathcal{A}^w) = \sum_{k=1}^N \frac{\partial}{\partial x_k} e_n(\mathcal{A}^w) dx_k + \frac{1}{2} \sum_{k=1}^N \frac{\partial^2}{\partial x_k^2} e_n(\mathcal{A}^w) dx_k dx_k.$$

Since we simply have the following representations of the derivatives

$$\frac{\partial}{\partial x_k} e_n(\mathcal{A}^w) = 2 \sum_{\alpha \in R_+} \alpha_k^* \langle x, \alpha^* \rangle \, e_{n-1}^{\overline{\alpha}}(\mathcal{A}^w),$$

$$\frac{\partial^2}{\partial x_k^2}\, e_n(\mathcal{A}^w) \quad = \quad 2 \sum_{\alpha\in R_+} (\alpha_k^*)^2 e_{n-1}^{\overline{\alpha}}(\mathcal{A}^w) + 4 \sum_{\substack{\alpha\in R_+\\ \alpha\neq\beta}} \sum_{\beta\in R_+} \alpha_k^*\beta_k^* \langle x,\alpha^*\rangle \langle x,\beta^*\rangle\, e_{n-2}^{\overline{\alpha},\overline{\beta}}(\mathcal{A}^w),$$

the result follows now directly from (1.1). □

Let us assume that $e_M(\mathcal{A}^w)_0 > 0$, that is, we start from the interior of the Weyl chamber. We consider a collection of semi-martingales

$$S_n^w(t) := -\ln e_n(\mathcal{A}^w)_t, \quad n = 2,\dots,M$$

for $t < \tau_n$. These processes control the first hitting times τ_n in the way that S_n^w explodes to $+\infty$ before the life-time T_∞ if and only if $e_n(\mathcal{A}^w)$ hits 0. To explore these relations, we have to start with the following stochastic description of S_n^w.

Proposition 2. *Assume that $e_M(\mathcal{A}^w)_0 > 0$. For every fixed set of weights $w = \{w_\alpha > 0 : \alpha \in R_+\}$ we have the semi-martingale decomposition $dS_n^w = dM_n^w + dA_n^w$, where the local martingale part is*

$$dM_n^w = -\frac{2}{e_n(\mathcal{A}^w)} \sum_{k=1}^N \left(\sum_{\alpha\in R_+} \alpha_k^* \langle x,\alpha^*\rangle\, e_{n-1}^{\overline{\alpha}}(\mathcal{A}^w) \right) \sigma(x_k) dB_k$$

and the drift part dA_n^w is given as the sum $(A_1 + A_2 + A_3 + A_4 + A_5 + A_6)\,dt$ of the following six components

$$A_1 \quad = \quad \frac{1}{(e_n(\mathcal{A}^w))^2} \sum_{k=1}^N \sigma^2(x_k) \sum_{\alpha\in R_+} (\alpha_k^*)^2 \left(\langle x,\alpha^*\rangle^2 e_{n-1}^{\overline{\alpha}}(\mathcal{A}^w) - e_n^{\overline{\alpha}}(\mathcal{A}^w) \right) e_{n-1}^{\overline{\alpha}}(\mathcal{A}^w), \tag{3.1}$$

$$A_2 \quad = \quad \frac{2}{(e_n(\mathcal{A}^w))^2} \sum_{k=1}^N \sigma^2(x_k) \sum_{\substack{\alpha\in R_+\\ \alpha\neq\beta}} \sum_{\beta\in R_+} \alpha_k^*\beta_k^* \langle x,\alpha^*\rangle \langle x,\beta^*\rangle \left(e_{n-1}^{\overline{\alpha},\overline{\beta}}(\mathcal{A}^w)\right)^2, \tag{3.2}$$

$$A_3 \quad = \quad -\frac{2}{(e_n(\mathcal{A}^w))^2} \sum_{k=1}^N \sigma^2(x_k) \sum_{\substack{\alpha\in R_+\\ \alpha\neq\beta}} \sum_{\beta\in R_+} \alpha_k^*\beta_k^* \langle x,\alpha^*\rangle \langle x,\beta^*\rangle\, e_n^{\overline{\alpha},\overline{\beta}}(\mathcal{A}^w) e_{n-2}^{\overline{\alpha},\overline{\beta}}(\mathcal{A}^w), \tag{3.3}$$

and

$$A_4 \quad = \quad -\frac{2}{e_n(\mathcal{A}^w)} \sum_{k=1}^N b(x_k) \sum_{\alpha\in R_+} \alpha_k^* \langle x,\alpha^*\rangle\, e_{n-1}^{\overline{\alpha}}(\mathcal{A}^w), \tag{3.4}$$

$$A_5 \quad = \quad -\frac{2}{e_n(\mathcal{A}^w)} \sum_{\alpha\in R_+} |\alpha^*|^2 e_{n-1}^{\overline{\alpha}}(\mathcal{A}^w) k_\alpha(x), \tag{3.5}$$

$$A_6 \quad = \quad -\frac{2}{e_n(\mathcal{A}^w)} \sum_{\substack{\alpha\in R_+\\ \alpha\neq\beta}} \sum_{\beta\in R_+} \frac{\langle\alpha^*,\beta^*\rangle \langle x,\alpha^*\rangle}{\langle x,\beta^*\rangle} e_{n-1}^{\overline{\alpha}}(\mathcal{A}^w) k_\beta(x), \tag{3.6}$$

for $t < \tau_n \wedge T_\infty$.

Proof. Applying the Itô formula for $t < \tau_n \wedge T_\infty$ we get

$$dS_n^w = -\frac{de_n(\mathcal{A}^w)}{e_n(\mathcal{A}^w)} + \frac{de_n(\mathcal{A}^w) de_n(\mathcal{A}^w)}{2(e_n(\mathcal{A}^w))^2}.$$

The stochastic description of $e_n(\mathcal{A}^w)$ given in Proposition 1 directly gives the form of the martingale part dM_n^w, but we can also use it to calculate

$$\frac{1}{2} de_n(\mathcal{A}^w) de_n(\mathcal{A}^w) \quad = \quad 2\sum_{k=1}^N \left(\sum_{\alpha\in R_+} \alpha_k^* \langle x,\alpha^*\rangle\, e_{n-1}^{\overline{\alpha}}(\mathcal{A}^w) \right)^2 \sigma^2(x_k)\, dt$$

$$
\begin{aligned}
&= \quad 2\sum_{k=1}^{N}\sigma^2(x_k)\sum_{\alpha\in R_+}(\alpha_k^*)^2\langle x,\alpha^*\rangle^2\left(e_{n-1}^{\overline{\alpha}}(\mathcal{A}^w)\right)^2dt\\
&\quad +2\sum_{k=1}^{N}\sigma^2(x_k)\sum_{\substack{\alpha\in R_+\\ \alpha\neq\beta}}\sum_{\beta\in R_+}\alpha_k^*\beta_k^*\langle x,\alpha^*\rangle\langle x,\beta^*\rangle e_{n-1}^{\overline{\alpha}}(\mathcal{A}^w)e_{n-1}^{\overline{\beta}}(\mathcal{A}^w)dt.
\end{aligned}
$$

Finally, we can write the drift part of dS_n^w as the sum $(A_1+\widetilde{A_2}+A_4+A_5+A_6)dt$, where

$$
\begin{aligned}
A_1 &= \frac{1}{(e_n(\mathcal{A}^w))^2}\sum_{k=1}^{N}\sigma^2(x_k)\sum_{\alpha\in R_+}(\alpha_k^*)^2\Big(2\langle x,\alpha^*\rangle^2 e_{n-1}^{\overline{\alpha}}(\mathcal{A}^w)-e_n(\mathcal{A}^w)\Big)e_{n-1}^{\overline{\alpha}}(\mathcal{A}^w),\\
\widetilde{A_2} &= \frac{2}{(e_n(\mathcal{A}^w))^2}\sum_{k=1}^{N}\sigma^2(x_k)\sum_{\substack{\alpha\in R_+\\ \alpha\neq\beta}}\sum_{\beta\in R_+}\alpha_k^*\beta_k^*\langle x,\alpha^*\rangle\langle x,\beta^*\rangle\Big(e_{n-1}^{\overline{\alpha}}(\mathcal{A}^w)e_{n-1}^{\overline{\beta}}(\mathcal{A}^w)-e_{n-2}^{\overline{\alpha},\overline{\beta}}(\mathcal{A}^w)e_n(\mathcal{A}^w)\Big)
\end{aligned}
$$

and A_4, A_5, A_6 are given in (3.4), (3.5), (3.6), respectively. Note that A_5 and A_6 are obtained by splitting the sum

$$
\sum_{\alpha\in R_+}\sum_{\beta\in R_+}\frac{\langle\alpha^*,\beta^*\rangle\langle x,\alpha^*\rangle}{\langle x,\beta^*\rangle}e_{n-1}^{\overline{\alpha}}(\mathcal{A}^w)k_\beta(x)
$$

into on and off diagonal sums. The formula (3.1) for A_1 is obtained from the formula given above by $e_n(\mathcal{A}^w)=\langle x,\alpha^*\rangle^2 e_{n-1}^{\overline{\alpha}}(\mathcal{A}^w)+e_n^{\overline{\alpha}}(\mathcal{A}^w)$, which leads to

$$
2\langle x,\alpha^*\rangle^2 e_{n-1}^{\overline{\alpha}}(\mathcal{A}^w)-e_n(\mathcal{A}^w)=\langle x,\alpha^*\rangle^2 e_{n-1}^{\overline{\alpha}}(\mathcal{A}^w)-e_n^{\overline{\alpha}}(\mathcal{A}^w).
$$

From the other side, using (2.1), we get

$$
e_{n-1}^{\overline{\alpha}}(\mathcal{A}^w)e_{n-1}^{\overline{\beta}}(\mathcal{A}^w)-e_{n-2}^{\overline{\alpha},\overline{\beta}}(\mathcal{A}^w)e_n(\mathcal{A}^w)=\left(e_{n-1}^{\overline{\alpha},\overline{\beta}}(\mathcal{A}^w)\right)^2-e_n^{\overline{\alpha},\overline{\beta}}(\mathcal{A}^w)e_{n-2}^{\overline{\alpha},\overline{\beta}}(\mathcal{A}^w)
$$

and we can rewrite $\widetilde{A_2}$ as the sum of A_2 and A_3 given in (3.2) and (3.3). □

In the next proposition, we show that double collisions do not occur up to the life time T_∞, whenever the system starts from the interior of the Weyl chamber. This is the most crucial and difficult part of the proof of Theorem 1. In the base case, where σ is constant and equal to 1, the proof is reduced to showing that all components of the drift of process S are bounded or finite for every finite t, which prevents the process from exploding to infinity in finite time. When the drift parts are modified by a positive σ, the matter becomes more subtle and requires, on one hand, localization — an analysis of the drift of S in the vicinity of a fixed boundary point — and, on the other hand, the proper selection of weights to achieve a similar effect as in the $\sigma=1$ case. The key here is that the explosion of S^w for fixed weight w is equivalent to an explosion for any other positive weight, and thus also for the process without weights.

Proposition 3. *Assume that $e_M(\mathcal{A}^w)_0>0$, that is, the particle system x starts from the interior of the Weyl chamber. Then, under the assumptions of Theorem 1, we have*

$$
\tau_{M-1}=\tau_{M-2}=\ldots=\tau_1=\tau_0=T_\infty
$$

almost surely.

Proof. We fix $n\in\{1,\ldots,M-1\}$ and work on the set $\{\tau_n<\tau_{n-1}\le T_\infty\}$ with the intention of showing that its probability is zero. Note that on $\{\tau_n<\tau_{n-1}\}$ we have $e_n(\mathcal{A}^w)(\tau_n)=0$ and $e_{n-1}(\mathcal{A}^w)(\tau_n)>0$, which is equivalent to saying $x_{\tau_n}\in\partial W^{(m)}$, where $m=M-n+1$.

We begin by constructing the following open cover of $\partial W^{(m)}$. Let us fix $y\in\partial W^{(m)}$. The continuity argument enables us to find an open and bounded set (a ball) $E=E(y)=B(y,r)\subset\mathbb{R}^N$ such that the following statements hold.

(i) For every $x \in E$ we have

$$\mathcal{P}^0(x) \subseteq \mathcal{P}^0(y).$$

(ii) For every $x \in E$ we have

$$\prod_{\alpha \in \mathcal{P}^+(y)} \langle x, \alpha \rangle^2 > 0.$$

(iii) For every $x \in E$ and every $\alpha \in R_+$ we have

$$k_\alpha(x) \geq \varepsilon$$

where $\varepsilon = \varepsilon(y) = \frac{1}{2} \inf_{\alpha \in R_+} k_\alpha(y) > 0$.

(iv) For all $x, z \in E$ and $k = 1, \ldots, N$ we have

$$|\sigma^2(x_k) - \sigma^2(z_k)| \leq \frac{\varepsilon}{8M}.$$

(v) For all $x \in E$ and $\alpha \in \mathcal{P}^0(y)$ we have

$$1 - \frac{\varepsilon_0}{8M} \leq \frac{\sum_{k=1}^N \sigma^2(x_k)\alpha_k^2}{\sum_{k=1}^N \sigma^2(y_k)\alpha_k^2} \leq 1 + \frac{\varepsilon_0}{8M},$$

where $\varepsilon_0 = \varepsilon \min_{\alpha \in \mathcal{P}^0(y)} |\alpha^*|^2$

Existence and positivity of ε follows from (G1) and (A1). Note also that for fixed y the sum $\sum_{k=1}^N \sigma^2(y_k)\alpha_k^2$ appearing in (v) is a fixed positive number. From the above given cover $\left\{E(y) : y \in \partial W^{(m)}\right\}$ we can select a countable sub-cover since we are working on a separable metric space. Let us denote this countable sub-cover of balls by $\{E_i : i \in \mathbb{N}\}$, where $E_i = E(y^i)$ for some $y^i \in \partial W^{(m)}$. Consequently we get

$$\{\tau_n < \tau_{n-1}\} = \bigcup_{i \in \mathbb{N}} \{\tau_n < \tau_{n-1}, x_{\tau_n} \in E_i\}$$

and we can restrict our consideration to $\{\tau_n < \tau_{n-1}, x_{\tau_n} \in E_i\}$ for fixed i and show that this set has probability zero, as then we deal with a countable collection of sets of measure zero.

Our main tool here is the process S_n^w with a suitable chosen set of weights and the fact that it is enough to find one S_n^w which cannot explode on E_i under the additional condition $\tau_n < \tau_{n-1}$ to conclude that $\mathbb{P}(\tau_n < \tau_{n-1}, x_{\tau_n} \in E_i) = 0$. Consequently, the crucial issue is the choice of a suitable set of weights. It turns out that the desired weights are determined by the martingale coefficient σ^2 in the following way

$$w_\alpha = \left(\sum_{k=1}^N \alpha_k^2 \sigma^2(y_k^i) \right)^{1/2}, \quad \alpha \in R_+. \tag{3.7}$$

Positivity of the weights follows from (A1). From now on, we consider $w = \{w_\alpha : \alpha \in R_+\}$, with w_α defined above. Conditions (i) and (ii) ensure that for every collision point y of order $m = M - n + 1$ from E_i we have $\mathcal{P}^0(y) = P^0(y^i)$. Thus, for every $y \in \partial W^{(m)} \cap E_i$ we define the function

$$H_+^w(x) = \prod_{\alpha \in \mathcal{P}^+(y)} \langle x, \alpha^* \rangle^2, \quad x \in W,$$

which does not depend on y as long as $y \in \partial W^{(m)} \cap E_i$ and is strictly positive on E_i since

$$H_+^w(x) = \prod_{\alpha \in \mathcal{P}^+(y)} \frac{\langle x, \alpha \rangle^2}{w_\alpha^2} = \left(\prod_{\alpha \in \mathcal{P}^+(y)} \frac{1}{w_\alpha^2} \right) \prod_{\alpha \in \mathcal{P}^+(y)} \langle x, \alpha \rangle^2 \overset{\text{(ii)}}{>} 0.$$

We will often use the positivity of $H_+^w(x)$ together with the following simple bound

$$e_n(\mathcal{A}^w) \geq H_+^w(x) \sum_{\alpha \in \mathcal{P}^0(y)} \langle x, \alpha^* \rangle^2 . \tag{3.8}$$

For the purposes of this proof, as we are now restricted to the set E_i, we introduce the following auxiliary notation

$$c_1 = \sup_{x \in E_i, \alpha \in R_+} \sum_{k=1}^N |\alpha_k^* \, b(x_k)|, \quad c_2 = \sup_{x \in E_i, \alpha \in \mathcal{P}^+(y)} \frac{1}{\langle x, \alpha^* \rangle}, \qquad c_3 = \sup_{x \in E_i} \frac{1}{H_+^w(x)}.$$

Note that the continuity condition (G1) and the fact that E_i is bounded, with $E_i \subset \overline{E_i}$ where $\overline{E_i}$ is compact, ensure that all the quantities given above are finite.

We want to show that on $\{\tau_n < \tau_{n-1}\}$ the process S_n^w cannot explode when $x_{\tau_n} \in E_i$. Since the local martingale part of S_n^w, by the McKean argument [16, Problem 7, p. 47] (as a Brownian motion with changed time), cannot explode to ∞, it remains to show that the drift part dA_n^w given in Proposition 2 cannot explode. Formally, for every $\omega \in \{\tau_n < \tau_{n-1}, x_{\tau_n} \in E_i\}$ there exists $\delta = \delta(\omega) > 0$ such that $x_t \in E_i$ for every $t \in (\tau_n - \delta, \tau_n)$, as E_i is open. Thus, our aim is to show that

$$\int_{\tau_n - \delta}^{\tau_n} A_n^w(s) ds < \infty$$

because it is equivalent to

$$\int_0^{\tau_n} A_n^w(s) ds < \infty$$

as the function $s \mapsto A_n^w(s)$ is integrable over $[0, \tau_n - \delta]$ as a continuous function on a compact interval. As a consequence, we find that $S_n^w(\tau_n)$ is bounded from above, which is a contradiction that implies that $\mathbb{P}(\tau_n < \tau_{n-1}, x_{\tau_n} \in E_i) = 0$. Although we omit the time variable in the following calculations, we assume that our considerations apply to times in the interval $[\tau_n - \delta, \tau_n]$. Finally, in the following arguments, we denote $y = x_{\tau_n}$, which is a point in $E_i \cap \partial W^{(m)}$. We stress that for all elements of $E_i \cap \partial W^{(m)}$ the set $\mathcal{P}^0(y)$ is the same.

Step 1. We begin with the first two parts A_1 and A_2 given in (3.1) and (3.2), respectively. Note that if in any component of the sum in (3.1) we find two factors that go to zero, that is, the product $\langle x, \alpha^* \rangle^2 \langle x, \beta^* \rangle^2$ with $\alpha, \beta \in \mathcal{P}^0(y)$ (α could be equal to β), then this component divided by $(e_n(\mathcal{A}^w))^2$ is generally bounded by $[H_+^w(x)]^{-2}$ and then by c_3^2 using (3.8). This allows us to focus only on the components where elements $\langle x, \alpha^* \rangle^2$ with $\alpha \in \mathcal{P}^0(y)$ occur individually. Indeed, for every fixed $\alpha \in \mathcal{P}^+(y)$, in every component of $e_{n-1}^{\overline{\alpha}}(\mathcal{A}^w)$ and $e_n^{\overline{\alpha}}(\mathcal{A}^w)$, we can always find $\langle x, \beta^* \rangle^2$ such that $\beta \in \mathcal{P}^0(y)$. Therefore, in products of the form $e_{n-1}^{\overline{\alpha}}(\mathcal{A}^w) e_{n-1}^{\overline{\alpha}}(\mathcal{A}^w)$ and $e_{n-1}^{\overline{\alpha}}(\mathcal{A}^w) e_n^{\overline{\alpha}}(\mathcal{A}^w)$, we will always find a product of two such vanishing expressions, which, in light of the previous observation, makes the sum over $\mathcal{P}^+(y)$ in (3.1) bounded from above. Thus, we can just focus on

$$\overline{A_1} = \left(\frac{H_+^w(x)}{e_n(\mathcal{A}^w)} \right)^2 \sum_{k=1}^N \sigma^2(x_k) \sum_{\alpha \in \mathcal{P}^0(y)} (\alpha_k^*)^2 \Bigg(\langle x, \alpha^* \rangle^2 - \sum_{\beta \in \mathcal{P}^0(y) \setminus \{\alpha\}} \langle x, \beta^* \rangle^2 \Bigg).$$

In a similar way, we have to only deal with

$$\overline{A_2} \quad = \quad 2 \left(\frac{H_+^w(x)}{e_n(\mathcal{A}^w)} \right)^2 \sum_{k=1}^N \sigma^2(x_k) \sum_{\substack{\alpha \in \mathcal{P}^0(y) \\ \alpha \neq \beta}} \sum_{\beta \in \mathcal{P}^0(y)} \alpha_k^* \beta_k^* \langle x, \alpha^* \rangle \langle x, \beta^* \rangle .$$

The first thing to do is replace $\sigma^2(x_k)$ with $\sigma^2(y_k)$ in the sums given above using (iv). Since we have exactly $M - n + 1$ elements in $\mathcal{P}^0(y)$, by the Cauchy-Schwarz inequality, we get the bounds

$$\Bigg| \sum_{\substack{\alpha \in \mathcal{P}^0(y) \\ \alpha \neq \beta}} \sum_{\beta \in \mathcal{P}^0(y)} \alpha_k^* \beta_k^* \langle x, \alpha^* \rangle \langle x, \beta^* \rangle \Bigg| \quad \leq \quad \Bigg(\sum_{\alpha \in \mathcal{P}^0(y)} |\alpha_k^*| \langle x, \alpha^* \rangle \Bigg)^2 \leq (M - n + 1) \sum_{\alpha \in \mathcal{P}^0(y)} |\alpha_k^*|^2 \langle x, \alpha^* \rangle^2 ,$$

which together with (iv) gives

$$\begin{aligned}
\overline{A_2} &\le 2\left(\frac{H_+^w(x)}{e_n(\mathcal{A}^w)}\right)^2 \sum_{k=1}^N \left(\sigma^2(y_k) + |\sigma^2(x_k) - \sigma^2(y_k)|\right) \sum_{\substack{\alpha\in\mathcal{P}^0(y)\\ \alpha\neq\beta}} \sum_{\beta\in\mathcal{P}^0(y)} \alpha_k^*\beta_k^* \langle x,\alpha^*\rangle \langle x,\beta^*\rangle \\
&\le \left(\frac{H_+^w(x)}{e_n(\mathcal{A}^w)}\right)^2 \left[\sum_{k=1}^N \sigma^2(y_k) \sum_{\substack{\alpha\in\mathcal{P}^0(y)\\ \alpha\neq\beta}} \sum_{\beta\in\mathcal{P}^0(y)} 2\alpha_k^*\beta_k^* \langle x,\alpha^*\rangle \langle x,\beta^*\rangle + \frac{\varepsilon}{4} \sum_{\alpha\in\mathcal{P}^0(y)} |\alpha^*|^2 \langle x,\alpha^*\rangle^2\right].
\end{aligned}$$

Note that for $\alpha \in \mathcal{P}^0(y)$ we have $\langle x,\alpha^*\rangle^2 H_+^w(x) \le e_n(\mathcal{A}^w)$ (see (3.8)) and $H_+^w(x) \le e_{n-1}^{\overline{\alpha}}(\mathcal{A}^w)$, which together with the definition of ε given in (iii) gives that

$$\frac{\varepsilon}{4}\left(\frac{H_+^w(x)}{e_n(\mathcal{A}^w)}\right)^2 \sum_{\alpha\in\mathcal{P}^0(y)} |\alpha^*|^2 \langle x,\alpha^*\rangle^2 \le -\frac{1}{8}A_5.$$

Exactly in the same way we get

$$\begin{aligned}
\overline{A_1} + \frac{1}{4}A_5 &\le \left(\frac{H_+^w(x)}{e_n(\mathcal{A}^w)}\right)^2 \sum_{k=1}^N \sigma^2(y_k) \sum_{\alpha\in\mathcal{P}^0(y)} (\alpha_k^*)^2 \Bigg(\langle x,\alpha^*\rangle^2 - \sum_{\beta\in\mathcal{P}^0(y)\setminus\{\alpha\}} \langle x,\beta^*\rangle^2 \Bigg) \\
&= \left(\frac{H_+^w(x)}{e_n(\mathcal{A}^w)}\right)^2 \sum_{k=1}^N \sigma^2(y_k) \Bigg(\sum_{\alpha\in\mathcal{P}^0(y)} 2(\alpha_k^*)^2 \langle x,\alpha^*\rangle^2 - \sum_{\alpha,\beta\in\mathcal{P}^0(y)} (\alpha_k^*)^2 \langle x,\beta^*\rangle^2 \Bigg) \\
&= \left(\frac{H_+^w(x)}{e_n(\mathcal{A}^w)}\right)^2 \sum_{\alpha\in\mathcal{P}^0(y)} \langle x,\alpha^*\rangle^2 \sum_{k=1}^N \sigma^2(y_k) \Bigg(2(\alpha_k^*)^2 - \sum_{\beta\in\mathcal{P}^0(y)} (\beta_k^*)^2 \Bigg).
\end{aligned}$$

To get the last expression note that we just collect coefficients appearing with fixed $\langle x,\alpha^*\rangle^2$ and k. Every $\langle x,\alpha^*\rangle^2$ appears once with positive $(\alpha_k^*)^2$, but also with $-(\beta_k^*)^2$ as many times as many $\beta \in \mathcal{P}^0(y)$ different from α we can find. Let us now focus on the double sum of $2\alpha_k^*\beta_k^* \langle x,\alpha^*\rangle \langle x,\beta^*\rangle$ in the upper bound for $\overline{A_2}$ and consider $\alpha,\beta \in \mathcal{P}^0(y)$ such that $\alpha \neq \beta$. These two roots may contribute to the sum if and only if $\alpha_k\beta_k \neq 0$ for some k. In this case it may still happen that $\langle \alpha,\beta\rangle = 0$, but this is only if $\alpha = e_i \pm e_j$, $\beta = e_i \mp e_j$ and then we simply have $\alpha_i^*\beta_i^* = -\alpha_j^*\beta_j^*$, $y_i = y_j$, $w_\alpha = w_\beta$ and finally

$$\sigma^2(y_i)\alpha_i^*\beta_i^* \langle x,\alpha^*\rangle \langle x,\beta^*\rangle + \sigma^2(y_j)\alpha_j^*\beta_j^* \langle x,\alpha^*\rangle \langle x,\beta^*\rangle = 0,$$

which means that this kind of pairs $\{\alpha,\beta\}$ does not contribute to the considered sum and we can focus on $\alpha,\beta \in \mathcal{P}^0(y)$, $\alpha \neq \beta$ such that $\langle \alpha,\beta\rangle \neq 0$. Fix $\alpha \in \mathcal{P}^0(y)$. For every $\beta \in \mathcal{P}^0(y)$ such that $\beta \neq \alpha$ and $\langle \alpha,\beta\rangle \neq 0$ there exists

$$\gamma = \gamma_{\alpha,\beta} = \pm\varrho_\alpha\beta \in R_+,$$

i.e., γ (up to the sign) is a reflection of β in α. It is easy to see that $|\beta| = |\gamma|$, $\langle \alpha,\gamma\rangle = \mp\langle \alpha,\beta\rangle \neq 0$ and recall (2.2), which gives

$$2\langle \alpha,\beta\rangle \langle x,\alpha\rangle \langle x,\beta\rangle + 2\langle \alpha,\gamma\rangle \langle x,\alpha\rangle \langle x,\gamma\rangle = \frac{4\langle x,\alpha\rangle^2}{|\alpha|^2}. \tag{3.9}$$

Note that condition $\alpha \in \mathcal{P}^0(y)$ implies $\langle y,\alpha\rangle = 0$ and consequently $\sigma^2(y_k) = \sigma^2$ for every k such that $\alpha_k^2 > 0$. It is obvious for $\alpha = e_i$ and $\alpha = e_i - e_j$ and for $\alpha = e_i + e_j$ we simply have $y_i = -y_j = 0$. Clearly, the same holds for y^i, which is the center of the ball E^i and being used to establish the weights. Thus, we get

$$w_\gamma^2 = \sigma^2|\gamma|^2, \quad w_\beta^2 = \sigma^2|\beta|^2, \quad w_\alpha^2 = \sigma^2|\alpha|^2$$

and since $|\beta| = |\gamma|$ we get $w_\gamma = w_\beta = |\beta|w_\alpha/|\alpha|$.

Taking into consideration only selected β, γ from the sum over all roots from $\mathcal{P}^0(y)$ not equal to α, but having nonzero inner product with α

$$\sum_{k=1}^{N} \sigma^2(y_k)\big(2\alpha_k^*\beta_k^* \langle x, \alpha^*\rangle \langle x, \beta^*\rangle + 2\alpha_k^*\gamma_k^* \langle x, \alpha^*\rangle \langle x, \gamma^*\rangle\big)$$

we can use (3.9) to write it as

$$\frac{2\sigma^2|\alpha|^2\big(\langle \alpha, \beta\rangle \langle x, \alpha\rangle \langle x, \beta\rangle + \langle \alpha, \gamma\rangle \langle x, \alpha\rangle \langle x, \gamma\rangle\big)}{|\beta|^2 w_\alpha^4} = \frac{4\sigma^2 \langle x, \alpha\rangle^2}{|\beta|^2 w_\alpha^4} = \frac{4}{|\alpha|^2|\beta|^2}\sum_{k=1}^{N} \sigma^2(y_k)^2(\alpha_k^*)^2 \langle x, \alpha^*\rangle^2 .$$

In summary, we see that the expression $\overline{A_2}$ will produce additional positive elements $\sigma^2(y_k)(\alpha_k^*)^2 \langle x, \alpha^*\rangle^2$ with coefficient $4/(|\alpha||\beta|)^2$, which appears every time we can find a pair $\{\beta, \gamma\}$ as above. Note that $4/(|\alpha||\beta|)^2 = 1$ if $|\alpha| = |\beta| = \sqrt{2}$ and $4/(|\alpha||\beta|)^2 = 2$ if either $|\alpha| = 1$ and $|\beta| = \sqrt{2}$ or the opposite; note that $\langle \alpha, \beta\rangle = 0$ if $|\alpha| = |\beta| = 1$ and $\alpha \neq \beta$.

Recall that the collision order $m = M - n + 1$ indicates the number of elements in $\mathcal{P}^0(y)$, and consider n_α as the number of appearances of $\langle x, \alpha^*\rangle^2 \sigma^2(y_k)(\alpha_k^*)^2$ from the procedure described above. For $\alpha = e_i \pm e_j$ there is at most one pair $\{\beta, \gamma\}$ such that $|\beta| = |\gamma| = 1$, which is $\{e_i, e_j\}$ and that will produce the considered expression with coefficient 2. The rest of the pairs will produce the expression with coefficients equal to 1. We estimate the number of such pairs roughly by excluding four roots $\alpha = e_i \pm e_j$, e_i, e_j and $e_i \mp e_j$, then counting the remaining roots in $\mathcal{P}^0(y)$ and dividing this number by 2, since we do not want to count pairs twice. This leads to the upper bound $(m-4)/2$ for the number of such pairs. In summary, we get $n_\alpha \leq 2 + (m-4)/2 \leq m - 2$ as $m \geq 4$ in this case since we have $e_i \pm e_j, e_i, e_j, e_i \mp e_j \in \mathcal{P}^0(y)$. If there are no pairs of roots with length 1, but there is at least one pair of length $\sqrt{2}$, then we have the upper bound $(m-1)/2$ for possible pairs and get $n_\alpha \leq (m-1)/2 \leq m-2$, where the last inequality follows as $m \geq 3$ in this case. Finally, if $\alpha = e_i$, then every possible pair will produce the expression with coefficient 2, but there are at most $(m-2)/2$ such pairs, because we have to exclude all other roots of length 1 as they are orthogonal to α. This also gives $n_\alpha \leq m - 2$.

We can now go back to the bounds of $\overline{A_1} + \overline{A_2} + \frac{1}{2}A_5$ and get

$$\begin{aligned}
\overline{A_1} + \overline{A_2} + \frac{3}{8}A_5 \quad &\leq \quad \left(\frac{H_+^w(x)}{e_n(\mathcal{A}^w)}\right)^2 \sum_{\alpha\in\mathcal{P}^0(y)} \langle x, \alpha^*\rangle^2 \sum_{k=1}^{N}\sigma^2(y_k)\left((\alpha_k^*)^2(2+n_\alpha) - \sum_{\beta\in\mathcal{P}^0(y)}(\beta_k^*)^2\right) \\
&\overset{(3.7)}{\leq} \quad \left(\frac{H_+^w(x)}{e_n(\mathcal{A}^w)}\right)^2 \sum_{\alpha\in\mathcal{P}^0(y)} \langle x, \alpha^*\rangle^2 \left[m\frac{\sum_{k=1}^N \sigma^2(y_k)\alpha_k^2}{\sum_{k=1}^N \sigma^2(y_k^i)\alpha_k^2} - \sum_{\beta\in\mathcal{P}^0(y)}\frac{\sum_{k=1}^N \sigma^2(y_k)\beta_k^2}{\sum_{k=1}^N \sigma^2(y_k^i)\beta_k^2}\right] \\
&\overset{(v)}{\leq} \quad \left(\frac{H_+^w(x)}{e_n(\mathcal{A}^w)}\right)^2 \sum_{\alpha\in\mathcal{P}^0(y)} \langle x, \alpha^*\rangle^2 \left[m\left(1+\frac{\varepsilon_0}{8M}\right) - m\left(1-\frac{\varepsilon_0}{8M}\right)\right] \\
&= \quad \frac{m}{M}\cdot\frac{\varepsilon}{4}\left(\frac{H_+^w(x)}{e_n(\mathcal{A}^w)}\right)^2 \sum_{\alpha\in\mathcal{P}^0(y)} \min_{\beta\in\mathcal{P}^0(y)} |\beta^*|^2 \langle x, \alpha^*\rangle^2 \\
&\leq \quad \frac{\varepsilon}{4}\left(\frac{H_+^w(x)}{e_n(\mathcal{A}^w)}\right)^2 \sum_{\alpha\in\mathcal{P}^0(y)} |\alpha^*|^2 \langle x, \alpha^*\rangle^2 \leq \frac{\varepsilon}{4}\frac{1}{e_n(\mathcal{A}^w)}\sum_{\alpha\in\mathcal{P}^0(y)} |\alpha^*|^2 e_{n-1}^{\overline{\alpha}}(\mathcal{A}^w),
\end{aligned}$$

which directly gives, by using (iii) and $A_5 \leq 0$, that

$$\overline{A_1} + \overline{A_2} + \frac{3}{4}A_5 \quad \leq \quad \frac{1}{e_n(\mathcal{A}^w)}\sum_{\alpha\in\mathcal{P}^0(y)} |\alpha^*|^2 e_{n-1}^{\overline{\alpha}}(\mathcal{A}^w)\left(\frac{\varepsilon - 2k_\alpha(x)}{4}\right) \leq 0.$$

Step 2. In the next step, we show that the drift coefficients $b(x_k)dt$ appearing in (1.1) cannot cause multiple collisions. More precisely, the drift part $A_5 dt$ in the stochastic description of dS_n^w, which comes from the repulsive part prevents collisions that could be caused by the drift part A_4dt described by the function b in the following way

$$\begin{aligned}
A_4+\frac{1}{4}A_5 &\le \frac{2}{e_n(\mathcal{A}^w)}\sum_{k=1}^N|b(x_k)|\sum_{\alpha\in R_+}|\alpha_k^*|\left\langle x,\alpha^*\right\rangle e_{n-1}^{\overline{\alpha}}(\mathcal{A}^w)-\frac{1}{2e_n(\mathcal{A}^w)}\sum_{\alpha\in R_+}|\alpha^*|^2e_{n-1}^{\overline{\alpha}}(\mathcal{A}^w)k_\alpha(x)\\
&\le \frac{2c_1}{e_n(\mathcal{A}^w)}\left(\sum_{\alpha\in\mathcal{P}^+(y)}+\sum_{\alpha\in\mathcal{P}^0(y)}\right)\left\langle x,\alpha^*\right\rangle e_{n-1}^{\overline{\alpha}}(\mathcal{A}^w)-\frac{1}{2e_n(\mathcal{A}^w)}\sum_{\alpha\in R_+}|\alpha^*|^2e_{n-1}^{\overline{\alpha}}(\mathcal{A}^w)k_\alpha(x)\\
&\le 2c_1c_2\sum_{\alpha\in\mathcal{P}^+(y)}\frac{\left\langle x,\alpha^*\right\rangle^2e_{n-1}^{\overline{\alpha}}(\mathcal{A}^w)}{e_n(\mathcal{A}^w)}+\sum_{\alpha\in\mathcal{P}^0(y)}\frac{[4c_1\left\langle x,\alpha^*\right\rangle-|\alpha^*|^2k_\alpha(x)]e_{n-1}^{\overline{\alpha}}(\mathcal{A}^w)}{2e_n(\mathcal{A}^w)}.
\end{aligned}$$

As we have previously observed, every component of the first sum is simply bounded by 1. To deal with the other sum, it is enough to see that $4c_1\left\langle x,\alpha^*\right\rangle-|\alpha^*|^2k_\alpha(x)$ becomes negative near y, because $k_\alpha(y)$ is strictly positive by (A1) and $\left\langle x,\alpha^*\right\rangle$ goes to 0 as $\left\langle y,\alpha^*\right\rangle=0$ for every $\alpha\in\mathcal{P}^0(y)$.

Step 3. In the last step, we have to show that the two remaining parts of the drift A_3 and A_6 do not cause an explosion. Here, we do not need to compensate for those parts with the help of A_5. Indeed, taking into account every single component of the inner sum defining A_3 in (3.3), which is

$$\alpha_k^*\beta_k^*\frac{2\left\langle x,\alpha^*\right\rangle\left\langle x,\beta^*\right\rangle e_n^{\overline{\alpha},\overline{\beta}}(\mathcal{A}^w)e_{n-2}^{\overline{\alpha},\overline{\beta}}(\mathcal{A}^w)}{(e_n(\mathcal{A}^w))^2},\qquad \alpha,\beta\in R_+,\alpha\neq\beta, \tag{3.10}$$

we can consider all possible scenarios for α and β. If both of them are in $\mathcal{P}^0(y)$, then by (3.8) we get

$$\frac{2\left\langle x,\alpha^*\right\rangle\left\langle x,\beta^*\right\rangle}{e_n(\mathcal{A}^w)}\le\frac{\left\langle x,\alpha^*\right\rangle^2+\left\langle x,\beta^*\right\rangle^2}{e_n(\mathcal{A}^w)}\le c_3\frac{(\left\langle x,\alpha^*\right\rangle^2+\left\langle x,\beta^*\right\rangle^2)H_+^w(x)}{e_n(\mathcal{A}^w)}\le c_3,$$

which together with the obvious bound $e_n^{\overline{\alpha},\overline{\beta}}(\mathcal{A}^w)\le e_n(\mathcal{A}^w)$ gives finiteness of (3.10). If at least one of the roots α or β is in $\mathcal{P}^+(y)$, then all components of $e_n^{\overline{\alpha},\overline{\beta}}(\mathcal{A}^w)$ must contain a product of the form $\left\langle x,\gamma^*\right\rangle^2\left\langle x,\delta^*\right\rangle^2$ for two different roots $\gamma,\delta\in\mathcal{P}^0(y)$. Since, as we have seen just before, for every $\gamma\in\mathcal{P}^0(y)$ we have $\left\langle x,\gamma^*\right\rangle^2\le c_3e_n(\mathcal{A}^w)$, this implies the finiteness of $e_n^{\overline{\alpha},\overline{\beta}}(\mathcal{A}^w)/(e_n(\mathcal{A}^w))^2$ and consequently (3.10) as well.

The final part relates to A_6, which is given by (3.6) and can be slightly rewritten as follows

$$A_6=-2\sum_{\beta\in R_+}k_\beta(x)\sum_{\alpha\in R_+\setminus\{\beta\}}\frac{\left\langle\alpha^*,\beta^*\right\rangle\left\langle x,\alpha^*\right\rangle}{\left\langle x,\beta^*\right\rangle}\frac{e_{n-1}^{\overline{\alpha}}(\mathcal{A}^w)}{e_n(\mathcal{A}^w)}.$$

First observe that if $\beta\in\mathcal{P}^+(y)$ then we might, at most, get a singularity of the form $1/\left\langle x,\beta^*\right\rangle$, which is integrable and can not cause an explosion. Thus, from now on, we fix $\beta\in\mathcal{P}^0(y)$ and focus on the inner sum and take $\alpha\in R_+$ different from β, and we only consider those roots for which $\left\langle\alpha,\beta\right\rangle\neq0$. Then, recall that there exists $\gamma=\gamma_{\beta,\alpha}\in R_+$ such that $\gamma=\pm\varrho_\beta\alpha$. Moreover, we can write $e_{n-1}^{\overline{\alpha}}(\mathcal{A}^w)=\left\langle x,\beta^*\right\rangle^2e_{n-2}^{\overline{\alpha},\overline{\beta}}(\mathcal{A}^w)+e_{n-1}^{\overline{\alpha},\overline{\beta}}(\mathcal{A}^w)$, and then we symmetrize over α and γ in $e_{n-1}^{\overline{\alpha},\overline{\beta}}(\mathcal{A}^w)=\left\langle x,\gamma^*\right\rangle^2e_{n-2}^{\overline{\alpha},\overline{\beta},\overline{\gamma}}(\mathcal{A}^w)+e_{n-1}^{\overline{\alpha},\overline{\beta},\overline{\gamma}}(\mathcal{A}^w)$. Collecting the components for α and γ together and applying the facts that the reflection of γ in β is again α, and $\beta\in\mathcal{P}^0(y)$ implies $w_\alpha=w_\gamma$, we can reduce our consideration into three parts

$$\begin{aligned}
A_6^{(1)} &= -2\sum_{\beta\in\mathcal{P}^0(y)}\frac{k_\beta(x)}{\left\langle x,\beta^*\right\rangle}\sum_{\alpha\in R_+\setminus\{\beta\}}\left\langle\alpha^*,\beta^*\right\rangle\left\langle x,\alpha^*\right\rangle\left\langle x,\beta^*\right\rangle^2\frac{e_{n-2}^{\overline{\alpha},\overline{\beta}}(\mathcal{A}^w)}{e_n(\mathcal{A}^w)},\\
A_6^{(2)} &= -2\sum_{\beta\in\mathcal{P}^0(y)}k_\beta(x)\sum_{(\alpha,\gamma)\in R_+(\beta)}\left\langle x,\alpha^*\right\rangle\left\langle x,\gamma^*\right\rangle\frac{\left\langle\alpha^*,\beta^*\right\rangle\left\langle x,\gamma^*\right\rangle+\left\langle\beta^*,\gamma^*\right\rangle\left\langle x,\alpha^*\right\rangle}{\left\langle x,\beta^*\right\rangle}\frac{e_{n-2}^{\overline{\alpha},\overline{\beta},\overline{\gamma}}(\mathcal{A}^w)}{e_n(\mathcal{A}^w)}\\
&= -2\sum_{\beta\in\mathcal{P}^0(y)}k_\beta(x)\sum_{(\alpha,\gamma)\in R_+(\beta)}\frac{\left\langle x,\alpha^*\right\rangle\left\langle x,\gamma^*\right\rangle}{w_\alpha w_\gamma}\frac{\left\langle\alpha,\beta\right\rangle\left\langle x,\gamma\right\rangle+\left\langle\beta,\gamma\right\rangle\left\langle x,\alpha\right\rangle}{\left\langle x,\beta\right\rangle}\frac{e_{n-2}^{\overline{\alpha},\overline{\beta},\overline{\gamma}}(\mathcal{A}^w)}{e_n(\mathcal{A}^w)}
\end{aligned}$$

$$\overset{(2.3)}{=} \quad -4\sum_{\beta\in\mathcal{P}^0(y)} k_\beta(x) \sum_{(\alpha,\gamma)\in R_+(\beta)} \frac{\langle x,\alpha^*\rangle\langle x,\gamma^*\rangle}{w_\alpha w_\gamma}\frac{\langle\alpha,\beta\rangle\langle\beta,\gamma\rangle}{|\beta|^2}\frac{e_{n-2}^{\overline{\alpha},\overline{\beta},\overline{\gamma}}(\mathcal{A}^w)}{e_n(\mathcal{A}^w)},$$

$$A_6^{(3)} = -2\sum_{\beta\in\mathcal{P}^0(y)} k_\beta(x) \sum_{(\alpha,\gamma)\in R_+(\beta)} \frac{\langle\alpha^*,\beta^*\rangle\langle x,\alpha^*\rangle+\langle\beta^*,\gamma^*\rangle\langle x,\gamma^*\rangle}{\langle x,\beta^*\rangle}\frac{e_{n-1}^{\overline{\alpha},\overline{\beta},\overline{\gamma}}(\mathcal{A}^w)}{e_n(\mathcal{A}^w)}$$

$$= -2\sum_{\beta\in\mathcal{P}^0(y)} k_\beta(x) \sum_{(\alpha,\gamma)\in R_+(\beta)} \frac{1}{w_\alpha w_\gamma}\frac{\langle\alpha,\beta\rangle\langle x,\alpha\rangle+\langle\beta,\gamma\rangle\langle x,\gamma\rangle}{\langle x,\beta\rangle}\frac{e_{n-1}^{\overline{\alpha},\overline{\beta},\overline{\gamma}}(\mathcal{A}^w)}{e_n(\mathcal{A}^w)}$$

$$\overset{(2.2)}{=} \quad -4\sum_{\beta\in\mathcal{P}^0(y)} k_\beta(x) \sum_{(\alpha,\gamma)\in R_+(\beta)} \frac{1}{w_\alpha w_\gamma|\beta|^2}\frac{e_{n-1}^{\overline{\alpha},\overline{\beta},\overline{\gamma}}(\mathcal{A}^w)}{e_n(\mathcal{A}^w)},$$

where the notation $R_+(\beta)$ is introduced in (2.4).

Here, as indicated, we have used formulas from Proposition 6. Starting from the simplest one, observe that the last expression is just non-positive. To deal with the final form of $A_6^{(2)}$, observe that since $\beta\in\mathcal{P}^0(y)$, then for $(\alpha,\gamma)\in R_+(\beta)$ either both are in $\mathcal{P}^0(y)$ or both are in $\mathcal{P}^+(y)$. In the first case, we simply have

$$\frac{2\langle x,\alpha^*\rangle\langle x,\gamma^*\rangle}{e_n(\mathcal{A}^w)} \le \frac{\langle x,\alpha^*\rangle^2+\langle x,\gamma^*\rangle^2}{e_n(\mathcal{A}^w)}$$

and the last ratio is bounded. If $\alpha,\gamma\in\mathcal{P}^+(y)$, then there are only $(n-3)$ roots left in $\mathcal{P}^+(y)$ and consequently, every component of $e_{n-2}^{\overline{\alpha},\overline{\beta},\overline{\gamma}}(\mathcal{A}^w)$ consists of at least one factor $\langle x,\delta^*\rangle^2$, where $\delta\in\mathcal{P}^0(y)$, which, similarly to before, makes the ratio $e_{n-2}^{\overline{\alpha},\overline{\beta},\overline{\gamma}}(\mathcal{A}^w)/e_n(\mathcal{A}^w)$ bounded. Finally, since $\langle x,\beta^*\rangle^2/e_n(\mathcal{A}^w)$ is bounded and $k_\beta(x)/\langle x,\beta^*\rangle$ is integrable due to the fact that (1.1) has a solution and therefore all of its terms are integrable, all components of $A_6^{(1)}$ are integrable and consequently cannot explode.

In summary of all the considerations conducted, we have just shown that from $n=M-1$ to $n=1$, the event $\{\tau_n<\tau_{n-1}\}$ has probability 0. Recall that $e_0\equiv 1$, which finishes the proof as then $\tau_0=T_\infty$. □

The remainder, which completes the proof of Theorem 1, concerns showing that the process enters the interior of the Weyl chamber immediately after starting.

Proposition 4. *Assume that (G1) and (A1) hold. For $x=(x_1,\dots,x_N)$ being a solution to* (1.1) *define*

$$\tau_W=\inf\{t>0: x(t)\in W\}.$$

Then for every $x(0)\in\overline{W}$ we have $\tau_W=0$ a.s.

Proof. Let us denote by M the number of roots in R_+ and consider the symmetric polynomials $e_n(\mathcal{A})$ for $n=1,\dots,M$. We have

$$\begin{aligned} de_n(\mathcal{A}) &= 2\sum_{k=1}^N\sum_{\alpha\in R_+}\alpha_k\langle x,\alpha\rangle e_{n-1}^{\overline{\alpha}}(\mathcal{A})\sigma(x_k)dB_k + 2\sum_{k=1}^N\sum_{\alpha\in R_+}\alpha_k\langle x,\alpha\rangle e_{n-1}^{\overline{\alpha}}(\mathcal{A})b(x_k)dt \\ &\quad +2\sum_{\alpha\in R_+} e_{n-1}^{\overline{\alpha}}(\mathcal{A})|\alpha|^2k_\alpha(x)\,dt + 2\sum_{\beta\in R_+}\frac{k_\beta(x)}{\langle x,\beta\rangle}\sum_{\substack{\alpha\in R_+\\ \alpha\neq\beta,\langle\alpha,\beta\rangle\neq 0}}\langle\alpha,\beta\rangle\langle x,\alpha\rangle e_{n-1}^{\overline{\alpha}}(\mathcal{A})dt \\ &\quad +2\sum_{k=1}^N\sum_{\alpha\in R_+}\sum_{\substack{\beta\in R_+\\ \beta\neq\alpha}}\alpha_k\beta_k\langle x,\alpha\rangle\langle x,\beta\rangle e_{n-2}^{\overline{\alpha},\overline{\beta}}(\mathcal{A})\sigma^2(x_k)dt \\ &\quad +\sum_{k=1}^N\sum_{\alpha\in R_+}\alpha_k^2 e_{n-1}^{\overline{\alpha}}(\mathcal{A})\sigma^2(x_k)dt. \end{aligned}$$

As previously, to remove the singularity in $k_\beta / \langle x, \beta \rangle$ we use (2.2) and (2.3) to get for fixed $\beta \in R_+$ that

$$\begin{aligned} 2 \sum_{\substack{\alpha \in R_+ \\ \alpha \neq \beta, \langle \alpha, \beta \rangle \neq 0}} \langle \alpha, \beta \rangle \langle x, \alpha \rangle e_{n-1}^{\overline{\alpha}}(\mathcal{A}) &= \sum_{(\alpha,\gamma) \in R_+(\beta)} (\langle \alpha, \beta \rangle \langle x, \alpha \rangle + \langle \gamma, \beta \rangle \langle x, \gamma \rangle) e_{n-1}^{\overline{\alpha},\overline{\gamma}}(\mathcal{A}) \\ &\quad + \sum_{(\alpha,\gamma) \in R_+(\beta)} \langle x, \alpha \rangle \langle x, \gamma \rangle (\langle \alpha, \beta \rangle \langle x, \gamma \rangle + \langle \gamma, \beta \rangle \langle x, \alpha \rangle) e_{n-2}^{\overline{\alpha},\overline{\gamma}}(\mathcal{A}) \\ &\overset{(2.2)}{\underset{(2.3)}{=}} \frac{2 \langle x, \beta \rangle}{|\beta|^2} \sum_{(\alpha,\gamma) \in R_+(\beta)} (e_{n-1}^{\overline{\alpha},\overline{\gamma}}(\mathcal{A}) + \langle \alpha, \beta \rangle \langle \beta, \gamma \rangle \langle x, \alpha \rangle \langle x, \gamma \rangle e_{n-2}^{\overline{\alpha},\overline{\gamma}}(\mathcal{A})). \end{aligned}$$

Assume now that one of the processes $e_n(\mathcal{A}^w)$, with $n = 1, \ldots, M$, stays at zero for some positive time interval with positive probability. This means that the drift part of $e_n(\mathcal{A}^w)$ cancels on this time interval. On the other hand, all the products of $\langle x, \alpha \rangle$ of length m, where each of them appears only once becomes zero as well. In particular, we have $\langle x, \alpha \rangle e_{n-1}^{\overline{\alpha}}(\mathcal{A}) = 0$ and $\langle x, \alpha \rangle \langle x, \beta \rangle e_{n-2}^{\overline{\alpha},\overline{\beta}}(\mathcal{A}) = 0$ for every $\alpha, \beta \in R_+$ and $\alpha \neq \beta$. Consequently, positivity of k_α and σ^2 imply that for every $\alpha \in R_+$ we have

$$\sum_{\alpha \in R_+} e_{n-1}^{\overline{\alpha}}(\mathcal{A}^w) |\alpha|^2 = 0$$

on the positive time interval with positive probability, i.e., $e_{n-1}(\mathcal{A}^w) = 0$ and inductively

$$e_M(\mathcal{A}^w) = \ldots = e_1(\mathcal{A}^w) = e_0(\mathcal{A}^w) = 0$$

Since $e_0(\mathcal{A}^w) \equiv 1$ we get a contradiction. This means that the process enters immediately the interior of the Weyl chamber. □

4 Hausdorff dimension bounds

In order to find bounds on the Hausdorff dimension of collision times, we adopt the following general strategy. First, we choose a functional of our general particle system, the squared projection of x onto a simple root β, that cancels whenever there is a collision. This functional has upper and lower bounds given by a time-transformed Bessel process. Then, by verifying that the transformation in each case is bi-Lipschitz, we can relate the Hausdorff dimensions of the functional and the time-transformed Bessel process; the bi-Lipschitz property is a direct consequence of Theorem 1. Finally, we use the Hausdorff dimension of times where Bessel processes hit zero to collect our results. We note here that the Hausdorff dimension bounds given here are valid for every starting point in $\overline{W}$. Indeed, the only problematic starting points are those in ∂W, namely initial configurations where two or more particles start from the same position, but by Proposition 4 we know that particles separate immediately, as the process leaves the boundary of W immediately after starting, and the contribution of such a starting point does not change the dimensionality of the collision time set.

Let us emphasize that for the calculations that follow, we reset the weights w as

$$w_\alpha := |\alpha|$$

so that $\alpha^* = \alpha / |\alpha|$ and $|\alpha^*| = 1$. A critical fact that we use in our derivations follows immediately from Theorem 1 and is given below.

Corollary 8. *Consider the projections $\langle x, \alpha^* \rangle \geq 0$ for $\alpha \in R_+$ after the process x has started. There exists a sufficiently small number $\varepsilon > 0$ for which, whenever the smallest projection $\langle x, \zeta^* \rangle$ satisfies $0 \leq \langle x, \zeta^* \rangle < \varepsilon$, then it is the unique projection that satisfies this inequality, and a second-smallest projection $\langle x, \beta^* \rangle$ satisfies $\langle x, \beta^* \rangle \geq \langle x, \zeta^* \rangle + \delta(\epsilon)$ almost surely, where $\delta(\varepsilon) > 0$ has a positive limit as $\varepsilon \to 0$.*

Proof. Because $\varepsilon > 0$ is arbitrarily small, we only need to think of the situation where a collision occurs. Theorem 1 states that all collisions of x with ∂W are simple. Therefore, at any collision time, say $t > 0$, there exists exactly one root $\zeta \in R_+$ such that $\langle x(t), \zeta^* \rangle = 0$, and $\langle x(t), \alpha^* \rangle > 0$ for $\alpha \in R_+ \backslash \{\zeta\}$ almost surely. Because x is a continuous Markov process, all polynomials of x, and in particular all projections $\langle x, \alpha^* \rangle$ are continuous Markov processes. Only for this proof, define

$$d := \min_{\alpha \in R_+ \backslash \{\zeta\}} \langle x(t), \alpha^* \rangle$$

and the times

$$t_{\varepsilon+} := \inf\{s > t : \langle x(s), \zeta^* \rangle \geq \varepsilon\}, \qquad t_{\varepsilon-} := \sup\{s < t : \langle x(s), \zeta^* \rangle \geq \varepsilon\}.$$

Denote a second-smallest projection by $\langle x(t), \beta^* \rangle$. When $s \in [t_{\varepsilon-}, t_{\varepsilon+}]$, we see that $0 \leq \langle x(s), \zeta^* \rangle \leq \varepsilon$ and $d - r(\varepsilon) \leq \langle x(s), \beta^* \rangle \leq d + r(\varepsilon)$, with $r(\varepsilon) \to 0$ as $\varepsilon \to 0$ because $\langle x, \beta^* \rangle$ is continuous. Therefore, we have $\langle x(s), \beta^* \rangle - \langle x(s), \zeta^* \rangle \geq d - r(\varepsilon) - \varepsilon =: \delta(\varepsilon)$. As $\varepsilon \to 0$, $\delta(\varepsilon)$ becomes positive and tends to $d > 0$. ☐

4.1 Upper bound

Proof of Theorem 2. Let us recall that a collision occurs if and only if there exists a root $\beta \in R_+$ such that $\langle x, \beta \rangle = 0$. Let us also recall that $\beta \leq \alpha$ implies $\langle x, \beta \rangle \leq \langle x, \alpha \rangle$, and the minimal roots are all simple roots by (S1). So we can simply focus on the *squared* projections onto a simple root β,

$$y_\beta := \langle x, \beta^* \rangle^2$$

to study the collision times. The SDE of y_β reads

$$\begin{aligned} dy_\beta &= 2 \langle x, \beta^* \rangle \langle \beta^*, dx \rangle + \langle \beta^*, dx \rangle^2 \\ &= 2 \langle x, \beta^* \rangle \sum_{i=1}^N \beta_i^* \big(\sigma(x_i) \, dB_i + b(x_i) \, dt \big) + 2 \langle x, \beta^* \rangle \sum_{\alpha \in R_+} k_\alpha(x) \frac{\langle \alpha^*, \beta^* \rangle}{\langle x, \alpha^* \rangle} \, dt + \sum_{i=1}^N (\beta_i^*)^2 \sigma^2(x_i) \, dt. \end{aligned}$$

The quadratic variation is

$$4 \langle x, \beta^* \rangle^2 \sum_{i=1}^N (\beta_i^*)^2 \sigma^2(x_i) \, dt = 4 y_\beta \sum_{i=1}^N (\beta_i^*)^2 \sigma^2(x_i) \, dt,$$

so by the Lévy characterization theorem, and introducing the new Wiener process W, we can rewrite the martingale part as

$$2 \sqrt{y_\beta} \sqrt{\sum_{i=1}^N (\beta_i^*)^2 \sigma^2(x_i)} \, dW.$$

In addition, we have

$$2 \langle x, \beta^* \rangle \sum_{\alpha \in R_+} k_\alpha(x) \frac{\langle \alpha^*, \beta^* \rangle}{\langle x, \alpha^* \rangle} = 2 k_\beta(x) + 2 \sqrt{y_\beta} \sum_{\alpha \in R_+ \backslash \{\beta\}} k_\alpha(x) \frac{\langle \alpha^*, \beta^* \rangle}{\sqrt{y_\alpha}},$$

so the SDE becomes

$$\begin{aligned} dy_\beta = 2 \sqrt{y_\beta} \sqrt{\sum_{i=1}^N (\beta_i^*)^2 \sigma^2(x_i)} \, dW + 2 \sqrt{y_\beta} \sum_{i=1}^N \beta_i^* b(x_i) \, dt + 2 k_\beta(x) \, dt \\ + 2 \sqrt{y_\beta} \sum_{\alpha \in R_+ \backslash \{\beta\}} k_\alpha(x) \langle \alpha^*, \beta^* \rangle \frac{dt}{\sqrt{y_\alpha}} + \sum_{i=1}^N (\beta_i^*)^2 \sigma^2(x_i) \, dt. \end{aligned}$$

Now, we perform the following time change,

$$s = \Theta(t) := \int_0^t C_\beta(\tau) \, d\tau, \quad C_\beta(t) := \sum_{i=1}^N (\beta_i^*)^2 \sigma^2(x_i(t)) > 0,$$

which is bi-Lipschitz thanks to (A1). The SDE in s-time reads

$$\begin{aligned} dy_\beta &= 2\sqrt{y_\beta}\, dW + 2\frac{\sqrt{y_\beta}}{C_\beta}\sum_{i=1}^N \beta_i^* b(x_i)\, ds + 2\frac{k_\beta(x)}{C_\beta}\, ds + 2\sqrt{y_\beta}\sum_{\alpha\in R_+\setminus\{\beta\}} \frac{k_\alpha(x)}{C_\beta}\left\langle \alpha^*, \beta^*\right\rangle \frac{ds}{\sqrt{y_\alpha}} + ds \\ &= 2\sqrt{y_\beta}\, dW + 2\frac{\sqrt{y_\beta}}{C_\beta}\left(\sum_{i=1}^N \beta_i^* b(x_i) + \sum_{\alpha\in R_+\setminus\{\beta\}} k_\alpha(x)\left\langle \alpha^*, \beta^*\right\rangle \frac{1}{\sqrt{y_\alpha}}\right) ds + \left(2\frac{k_\beta(x)}{C_\beta} + 1\right) ds. \end{aligned} \tag{4.1}$$

Our task now is to find an appropriate lower bound for this process. Let us define

$$\check{\eta}_\beta := \inf_{y\in W} \frac{k_\beta(y)}{\sum_{i=1}^N (\beta_i^*)^2 \sigma^2(y_i)}$$

in order to bound the second drift term as follows,

$$\left(2\frac{k_\beta(x)}{C_\beta} + 1\right) \geq (2\check{\eta}_\beta + 1).$$

Next, we introduce the constants

$$\hat{b} := \sup_{y\in D}\left|\frac{b(y)}{\sigma^2(y)}\right|, \tag{4.2}$$

and

$$c_R := \max_{\substack{\alpha\in R_+ \\ i\in\{1,\dots,N\}: \\ \alpha_i\neq 0}} \left|\frac{1}{\alpha_i^*}\right|. \tag{4.3}$$

which allow us to write

$$\frac{1}{C_\beta}\sum_{i=1}^N \beta_i^* b(x_i) = \frac{1}{C_\beta}\sum_{i=1}^N (\beta_i^*)^2\sigma^2(x_i)\frac{1}{\beta_i^*}\frac{b(x_i)}{\sigma^2(x_i)} \geq -\frac{c_R\hat{b}}{C_\beta}\sum_{i=1}^N (\beta_i^*)^2\sigma^2(x_i) = -c_R\hat{b}.$$

We also define

$$\tilde{\eta}_\alpha := \max_{1\leq i\leq N}\sup_{y\in W}\frac{k_\alpha(y)}{\sigma^2(y_i)} \tag{4.4}$$

in order to obtain

$$\begin{aligned} \frac{1}{C_\beta}\sum_{\alpha\in R_+\setminus\{\beta\}} k_\alpha(x)\left\langle \alpha^*, \beta^*\right\rangle \frac{1}{\sqrt{y_\alpha}} &= \frac{1}{C_\beta}\sum_{\alpha\in R_+\setminus\{\beta\}}\sum_{i=1}^N (\beta_i^*)^2 k_\alpha(x)\left\langle \alpha^*, \beta^*\right\rangle \frac{1}{\sqrt{y_\alpha}} \\ &\geq -\frac{1}{C_\beta}\sum_{\alpha\in R_+\setminus\{\beta\}} \tilde{\eta}_\alpha \sum_{i=1}^N (\beta_i^*)^2\sigma^2(x_i)\frac{1}{\sqrt{y_\alpha}} = -\sum_{\alpha\in R_+\setminus\{\beta\}}\frac{\tilde{\eta}_\alpha}{\sqrt{y_\alpha}}, \end{aligned}$$

while recalling that $|\beta^*|^2 = \sum_{i=1}^N (\beta_i^*)^2 = 1$ and that $|\left\langle \alpha^*, \beta^*\right\rangle| \geq -1$ due to the Cauchy-Schwarz inequality. With these inequalities, we can write the bound

$$dy_\beta \geq 2\sqrt{y_\beta}\, dW - 2\sqrt{y_\beta}\left(c_R\hat{b} + \sum_{\alpha\in R_+\setminus\{\beta\}}\frac{\tilde{\eta}_\alpha}{\sqrt{y_\alpha}}\right) ds + (2\check{\eta}_\beta + 1)\, ds. \tag{4.5}$$

The critical point of the derivation is the following: because the drift term proportional to $\sqrt{y_\beta}$ is negative, the right hand side is a squared Bessel process with an additional attraction term, and it almost surely hits zero if $\check{\eta}_\beta < 1/2$ [9]. By Corollary 8, we know that for every time s at which a collision occurs, x has a distance at most $\varepsilon > 0$ from ∂W, and the smallest projection is unique when ε is sufficiently small. Hence, if the smallest projection is $\sqrt{y_\beta}$, we have $\sqrt{y_\beta} \leq \varepsilon$ and there exists $\delta(\varepsilon) > 0$ such that any other projection $\sqrt{y_\alpha}$ satisfies $\sqrt{y_\alpha} \geq \sqrt{y_\beta} + \delta(\varepsilon)$ within the s-time transformed interval $[s_{\varepsilon-}, s_{\varepsilon+}]$. Thus, we can write

$$\begin{aligned} dy_\beta &\geq 2\sqrt{y_\beta}\, dW - 2\sqrt{y_\beta}\bigg(c_R\hat{b} + \sum_{\alpha \in R_+\setminus\{\beta\}} \frac{\tilde{\eta}_\alpha}{\sqrt{y_\alpha}}\bigg)\, ds + (2\check{\eta}_\beta + 1)\, ds \\ &\geq 2\sqrt{y_\beta}\, dW + \left(2\left[\check{\eta}_\beta - \varepsilon c_R\hat{b} - \varepsilon\frac{\tilde{h}(M-1)}{\delta(\varepsilon)}\right] + 1\right) ds, \end{aligned}$$

where

$$\tilde{h} := \max_{\alpha \in R_+} \tilde{\eta}_\alpha. \tag{4.6}$$

In other words, we see that y_β is bounded below by a squared Bessel process $\check{z}_{\beta,\varepsilon}$ that has a zero set with Hausdorff dimension

$$\dim \check{z}_{\beta,\varepsilon}^{-1}(0) = \max\Big\{0, \frac{1}{2} - \check{\eta}_\beta + \varepsilon c_R\hat{b} + \varepsilon\frac{\tilde{h}(M-1)}{\delta(\varepsilon)}\Big\},$$

and which hits zero almost surely whenever $\check{\eta} < 1/2$ [9, 14]. This implies that by (H2), for every time $t^{(\beta)}$ where $y_\beta(t^{(\beta)}) = 0$ there exists a closed interval $[t_{\varepsilon-}^{(\beta)}, t_{\varepsilon+}^{(\beta)}] \ni t^{(\beta)}$ such that

$$\dim\big(y_\beta^{-1}(0) \cap \big[t_{\varepsilon-}^{(\beta)}, t_{\varepsilon+}^{(\beta)}\big]\big) \leq \dim \check{z}_{\beta,\varepsilon}^{-1}(0) = \max\left\{0, \frac{1}{2} - \check{\eta}_\beta + c_R\hat{b}\varepsilon + \tilde{h}(M-1)\frac{\varepsilon}{\delta(\varepsilon)}\right\}.$$

Now, for an arbitrary positive integer n, we can use (H1) to write

$$\dim y_\beta^{-1}(0) = \dim y_\beta^{-1}(0) \cap \bigcup_{n=1}^{\infty}[0,n] = \sup_{n\in\mathbb{N}} \dim y_\beta^{-1}(0) \cap [0,n],$$

and because the lower bound in (4.5) hits zero almost surely for $\check{\eta}_\beta < 1/2$, it follows that as $n \to \infty$ there exists an interval $[0,n]$ which contains a collision, so we write

$$\begin{aligned} \dim y_\beta^{-1}(0) &= \sup_{n\in\mathbb{N}} \dim y_\beta^{-1}(0) \cap [0,n] \leq \sup_{n\in\mathbb{N}} \max\left\{0, \frac{1}{2} - \check{\eta}_\beta + c_R\hat{b}\varepsilon + \tilde{h}(M-1)\frac{\varepsilon}{\delta(\varepsilon)}\right\} \\ &= \max\left\{0, \frac{1}{2} - \check{\eta}_\beta + c_R\hat{b}\varepsilon + \tilde{h}(M-1)\frac{\varepsilon}{\delta(\varepsilon)}\right\}, \end{aligned}$$

and because we can choose ε arbitrarily small, we see that

$$\dim y_\beta^{-1}(0) \leq \max\left\{0, \frac{1}{2} - \check{\eta}_\beta\right\}.$$

We finish by using (H1) and writing

$$\dim x^{-1}(\partial W) = \dim \bigcup_{\beta\in\Delta_+} y_\beta^{-1}(0) = \max_{\beta\in\Delta_+} \dim y_\beta^{-1}(0) \leq \max_{\beta\in\Delta_+} \max\left\{0, \frac{1}{2} - \check{\eta}_\beta\right\} = \max\left\{0, \frac{1}{2} - \min_{\beta\in\Delta_+} \check{\eta}_\beta\right\}.$$

□

4.2 Lower bound

Proof of Theorem 3. In the same vein as the proof of Theorem 2, we consider the squared projection onto the arbitrary simple root β, and carry on the same calculations up to (4.1). By (A3), we can show that

$$2\sum_{\alpha\in R_+\setminus\{\beta\}} k_\alpha(x)\,\langle\alpha^*, \beta^*\rangle\,\frac{1}{\sqrt{y_\alpha}}$$

is not positive. We begin by expanding the sum as follows,

$$2\sum_{\alpha\in R_+\setminus\{\beta\}} k_\alpha(x)\left\langle \alpha^*,\beta^*\right\rangle \frac{1}{\sqrt{y_\alpha}} = \sum_{\alpha\in R_+\setminus\{\beta\}} \frac{k_\alpha(x)}{\left\langle x,\alpha^*\right\rangle}\left\langle \alpha^*,\beta^*\right\rangle + \sum_{\varrho_\beta\alpha\in R_+\setminus\{\beta\}} \frac{k_{\varrho_\beta\alpha}(x)}{\left\langle x,\varrho_\beta\alpha^*\right\rangle}\left\langle \varrho_\beta\alpha^*,\beta^*\right\rangle.$$

The rhs follows from the fact that for any simple root β, every positive root $\alpha \neq \beta$ is reflected onto another positive root, namely $\varrho_\beta\alpha \in R_+ \setminus \{\beta\}$. We can rewrite this expression using (2.4) and setting $\gamma = \varrho_\beta\alpha$,

$$2\sum_{\alpha\in R_+\setminus\{\beta\}} k_\alpha(x)\left\langle \alpha^*,\beta^*\right\rangle \frac{1}{\sqrt{y_\alpha}} = \sum_{(\alpha,\gamma)\in R_+(\beta)} \Big(\frac{k_\alpha(x)}{\left\langle x,\alpha^*\right\rangle}\left\langle \alpha^*,\beta^*\right\rangle + \frac{k_\gamma(x)}{\left\langle x,\gamma^*\right\rangle}\left\langle \gamma^*,\beta^*\right\rangle\Big),$$

while keeping in mind that $\left\langle \gamma^*,\beta^*\right\rangle = \left\langle \varrho_\beta\alpha^*,\beta^*\right\rangle = \left\langle \alpha^*,\varrho_\beta\beta^*\right\rangle = -\left\langle \alpha^*,\beta^*\right\rangle$. Now, we apply (A3). First, suppose that $\left\langle \alpha^*,\beta^*\right\rangle > 0$; we can then write

$$\gamma^* = \alpha^* - 2\left\langle \alpha^*,\beta^*\right\rangle \beta^*,$$

and because both α^* and γ^* are positive roots, each can be written as a linear combination of simple roots with non-negative coefficients. From this last expression, we see that the expansion of γ^* in terms of simple roots includes a coefficient of β^* that decreases by $-2\left\langle \alpha^*,\beta^*\right\rangle$, so the coefficient of β^* either decreases while staying positive, or cancels, as it cannot be negative, and in the root systems we consider it cancels exactly. Therefore $\gamma^* \leq \alpha^*$, and by (A3) we see that

$$\frac{k_\alpha(x)}{\left\langle x,\alpha^*\right\rangle}\left\langle \alpha^*,\beta^*\right\rangle + \frac{k_\gamma(x)}{\left\langle x,\gamma^*\right\rangle}\left\langle \gamma^*,\beta^*\right\rangle \leq 0. \tag{4.7}$$

Conversely, if we suppose that $\left\langle \alpha^*,\beta^*\right\rangle < 0$, then $-\left\langle \alpha^*,\beta^*\right\rangle = \left\langle \varrho_\beta\alpha^*,\beta^*\right\rangle = \left\langle \gamma^*,\beta^*\right\rangle > 0$, and by a similar argument we can write

$$\alpha^* = \gamma^* + 2\left\langle \alpha^*,\beta^*\right\rangle \beta^* = \gamma^* - 2\left\langle \varrho_\beta\alpha^*,\beta^*\right\rangle \beta^* = \gamma^* - 2\left\langle \gamma^*,\beta^*\right\rangle \beta^*,$$

which allows us to conclude that $\alpha^* \leq \gamma^*$. By (A3), we recover (4.7) for all possible cases, and consequently we can write

$$\sum_{(\alpha,\gamma)\in R_+(\beta)} \Big(\frac{k_\alpha(x)}{\left\langle x,\alpha^*\right\rangle}\left\langle \alpha^*,\beta^*\right\rangle + \frac{k_\gamma(x)}{\left\langle x,\gamma^*\right\rangle}\left\langle \gamma^*,\beta^*\right\rangle\Big) \leq 0.$$

By assumption (A2), we can also see that

$$\sum_{i=1}^N \beta_i^* b(x_i) \leq 0$$

is satisfied as follows. The root systems we consider always include simple roots of the form $\beta^* = (e_{j+1} - e_j)/\sqrt{2}$. For these, we see that

$$\sum_{i=1}^N \frac{e_{j+1}-e_j}{\sqrt{2}} b(x_i) = \frac{1}{\sqrt{2}}(b(x_{j+1}) - b(x_j)) \leq 0$$

because all points $x \in W$ are such that $x_j \leq x_{j+1}$ and b is non-increasing. In addition, for the simple roots e_1 (for B_N) and $e_2 + e_1$ (for D_N) we have

$$\sum_{i=1}^N \frac{e_2+e_1}{\sqrt{2}} b(x_i) = \frac{1}{\sqrt{2}}(b(x_2) + b(x_1)) \leq 0 \text{ and}$$

$$\sum_{i=1}^N e_1 b(x_i) = b(x_1) \leq 0$$

because b is non-positive. Therefore, for all simple roots β the upper bound

$$dy_\beta \le 2\sqrt{y_\beta}\, dW + \left(2\frac{k_\beta(x)}{C_\beta} + 1\right) ds$$

is satisfied for the root systems in consideration.

We now define for every simple root

$$\hat{\eta}_\beta := \sup_{y\in W} \frac{k_\beta(y)}{\sum_{i=1}^N (\beta_i^*)^2\sigma^2(y_i)} = \sup_{y\in W} \frac{k_\beta(y)}{C_\beta(y)} \tag{4.8}$$

in order to write

$$dy_\beta \le 2\sqrt{y_\beta}\, dW + \left(2\hat{\eta}_\beta + 1\right) ds.$$

This inequality implies that y_β is bounded above by a squared Bessel process, say $\hat{z}_{\hat{\eta}_\beta}$, of dimension $2\hat{\eta}_\beta + 1$ which hits zero almost surely when $\hat{\eta}_\beta < 1/2$ and has a Hausdorff dimension of hitting times at zero given almost surely by [9, 14]

$$\dim \hat{z}_{\hat{\eta}_\beta}^{-1}(0) = \max\Big\{0, \frac{1}{2} - \hat{\eta}_\beta\Big\}.$$

Recalling that $\hat{z}_{\hat{\eta}_\beta}^{-1}(0) \subseteq y_\beta^{-1}(0)$, we are now in position to write

$$\begin{aligned}\dim\Big(x^{-1}(\partial W)\Big) &= \dim\Big(\bigcup_{\beta\in\Delta_+} y_\beta^{-1}(0)\Big) = \max_{\beta\in\Delta_+} \dim y_\beta^{-1}(0)\\ &\ge \max_{\beta\in\Delta_+} \dim \hat{z}_{\hat{\eta}_\beta}^{-1}(0) = \max_{\beta\in\Delta_+} \max\Big\{0, \frac{1}{2} - \hat{\eta}_\beta\Big\} = \max\Big\{0, \frac{1}{2} - \min_{\beta\in\Delta_+} \hat{\eta}_\beta\Big\}.\end{aligned}$$

The second equality follows from (H1), and the inequality in the second line follows from (H2). □

Proof of Lemma 4. In the same way as in the proof of Theorem 3, we can start from (4.1), as no assumptions are used up to that point. The next step is to try and bound the SDE of y_β from above, and in order to do this, we impose conditions on k_α, σ, and b. Recall the constants $\hat{b}$, c_R, and $\tilde{\eta}_\alpha$ introduced in (4.2), (4.3), and (4.4). With these, we write

$$\frac{1}{C_\beta}\sum_{i=1}^N \beta_i^* b(x_i) = \frac{1}{C_\beta}\sum_{i=1}^N (\beta_i^*)^2\sigma^2(x_i)\frac{1}{\beta_i^*}\frac{b(x_i)}{\sigma^2(x_i)} \le \frac{c_R\hat{b}}{C_\beta}\sum_{i=1}^N (\beta_i^*)^2\sigma^2(x_i) = c_R\hat{b},$$

and

$$\begin{aligned}\frac{1}{C_\beta}\sum_{\alpha\in R_+\setminus\{\beta\}} k_\alpha(x)\,\langle\alpha^*,\beta^*\rangle\,\frac{1}{\sqrt{y_\alpha}} &= \frac{1}{C_\beta}\sum_{\alpha\in R_+\setminus\{\beta\}}\sum_{i=1}^N (\beta_i^*)^2 k_\alpha(x)\,\langle\alpha^*,\beta^*\rangle\,\frac{1}{\sqrt{y_\alpha}}\\ &\le \frac{1}{C_\beta}\sum_{\alpha\in R_+\setminus\{\beta\}} \tilde{\eta}_\alpha \sum_{i=1}^N (\beta_i^*)^2\sigma^2(x_i)\frac{1}{\sqrt{y_\alpha}} = \sum_{\alpha\in R_+\setminus\{\beta\}} \frac{\tilde{\eta}_\alpha}{\sqrt{y_\alpha}}.\end{aligned}$$

We carried out these calculations in similar way to those in the proof of Theorem 2. We can now make use of $\hat{\eta}_\beta$, as defined in (4.8) to write

$$\begin{aligned}dy_\beta &= 2\sqrt{y_\beta}\, dW + 2\sqrt{y_\beta}\left(\frac{1}{C_\beta}\sum_{i=1}^N \beta_i^* b(x_i) + \sum_{\alpha\in R_+\setminus\{\beta\}} \frac{k_\alpha(x)}{C_\beta}\,\langle\alpha^*,\beta^*\rangle\,\frac{1}{\sqrt{y_\alpha}}\right) ds + \left(2\frac{k_\beta(x)}{C_\beta} + 1\right) ds\\ &\le 2\sqrt{y_\beta}\, dW + 2\sqrt{y_\beta}\left(c_R\hat{b} + \sum_{\alpha\in R_+\setminus\{\beta\}} \frac{\tilde{\eta}_\alpha}{\sqrt{y_\alpha}}\right) ds + \left(2\hat{\eta}_\beta + 1\right) ds.\end{aligned}$$

Let us assume that a collision occurs. By Corollary 8, and a similar argument to that given in the proof of Theorem 2, we know that for every time s at which x is at a distance at most $\varepsilon > 0$ from ∂W, and the smallest unique projection is $\sqrt{y_\beta}$, there exists $\delta(\varepsilon) > 0$ such that any other projection $\sqrt{y_\alpha}$ satisfies $\sqrt{y_\alpha} \geq \sqrt{y_\beta} + \delta(\varepsilon)$ within the s-time transformed interval $[s_{\varepsilon-}, s_{\varepsilon+}]$. Then, we can write

$$
\begin{aligned}
dy_\beta &\leq 2\sqrt{y_\beta}\, dW + 2\sqrt{y_\beta}\left(c_R\hat{b} + \sum_{\alpha\in R_+\setminus\{\beta\}} \frac{\tilde{\eta}_\alpha}{\sqrt{y_\alpha}}\right) ds + (2\hat{\eta}_\beta + 1)\, ds \\
&\leq 2\sqrt{y_\beta}\, dW + \left(2\left[\hat{\eta}_\beta + \varepsilon c_R\hat{b} + \varepsilon\frac{\tilde{h}(M-1)}{\delta(\varepsilon)}\right] + 1\right) ds,
\end{aligned}
$$

where $\tilde{h}$ is given in (4.6). This implies that for every time $t^{(\beta)}$ where $y_\beta(t^{(\beta)}) = 0$ we have a closed interval $[t^{(\beta)}_{\varepsilon-}, t^{(\beta)}_{\varepsilon+}] \ni t^{(\beta)}$ for which

$$
\dim\left(y_\beta^{-1}(0) \cap \left[t^{(\beta)}_{\varepsilon-}, t^{(\beta)}_{\varepsilon+}\right]\right) \geq \max\left\{0, \frac{1}{2} - \hat{\eta}_\beta - c_R\hat{b}\varepsilon - \tilde{h}(M-1)\frac{\varepsilon}{\delta(\varepsilon)}\right\}.
$$

We recall now that on every collision there exists exactly one simple root β for which $y_\beta = 0$. Therefore, we can choose any collision time to write

$$
\begin{aligned}
\dim\left(x^{-1}(\partial W)\right) &= \dim\left(\bigcup_{\beta\in\Delta_+} y_\beta^{-1}(0)\right) \geq \dim\left(y_\beta^{-1}(0)\right) \geq \dim\left(y_\beta^{-1}(0) \cap \left[t^{(\beta)}_{\varepsilon-}, t^{(\beta)}_{\varepsilon+}\right]\right) \\
&\geq \max\left\{0, \frac{1}{2} - \hat{\eta}_\beta + c_R\hat{b}\varepsilon + \hat{h}(M-1)\frac{\varepsilon}{\delta(\varepsilon)}\right\}.
\end{aligned}
$$

In the first line, we used (H2). We can take ε arbitrarily small, so we arrive at

$$
\dim\left(x^{-1}(\partial W)\right) \geq \max\left\{0, \frac{1}{2} - \hat{\eta}_\beta\right\}.
$$

Here, we used $\varepsilon/\delta(\varepsilon) \to 0$ by Corollary 8. We conclude that

$$
\dim\left(x^{-1}(\partial W)\right) \geq \max\left\{0, \frac{1}{2} - \max_{\beta\in\Delta_+} \hat{\eta}_\beta\right\}.
$$

Here we note that, when applying Corollary 8, we have assumed that a collision occurs. This is not always the case, as the drift b may work to push x away from ∂W, and depending on the initial condition imposed on x it may happen that there are no collisions. Indeed, comparing the possibly positive coefficient of $\sqrt{y_\beta}\, ds$ in (4.1) with (9) in [9] (as well as the text under (10) of the same reference), we see that we can only state that the lower bound we have found holds with positive probability, with the trivial bound of zero when no collisions occur. □

5 Particular cases

Our results are readily applied to several well-known cases.

5.1 Multivariate Bessel processes

Also known as radial Dunkl processes [17, 8], these are processes given by the SDE

$$
dx_i = dB_i + \sum_{\alpha\in R_+} k_\alpha \frac{\alpha_i}{\langle x, \alpha\rangle}\, dt,
$$

that is, $\sigma = 1$, $b = 0$, and $k_\alpha(x) = k_\alpha$, with the added constraint that k_α must be invariant under reflections along the roots in R_+, namely $k_{\varrho_\beta\alpha} = k_\alpha$ for every $\alpha, \beta \in R_+$. Assuming that $R = A_{N-1}$, B_N, or D_N, from Theorems 1-3 the following is immediate.

Corollary 9. *The collision times of the Bessel process with its Weyl chamber have the following Hausdorff dimension,*

$$\dim x^{-1}(\partial W) = \max\Big\{0, \frac{1}{2} - \min_{\alpha\in\Delta_+} k_\alpha\Big\}.$$

This can be extended immediately to a process with functions k, σ and b satisfying (A1)-(A3). The requirement that b be a decreasing function has the following physical interpretation: b represents a force due to an external potential, namely $b(x_i) = -V'(x_i)$, and if $V(x_i)$ is a convex function, it forms a potential well where all particles are confined. For instance, our results include the invariant measure case where $\sigma = 1$ and $b(x_i) = -x_i$, or $V(x_i) = \frac{1}{2}x_i^2$, which is the case where the particles are confined to a harmonic potential. These processes are specialized to the several well-known multiple-particle systems, which we summarize as follows.

5.2 Dyson model

The Dyson model [5] is obtained by setting $R = A_{N-1}$, $\sigma(x) = 1$, $b(x) = 0$, and $k_\alpha(x) = k > 0$, with the positive roots being $\{e_j - e_i\}_{1\le i<j\le N}$. The Weyl chamber is described by $W_{A_{N-1}} = \{x \in \mathbb{R}^N : x_i \le x_j, 1 \le i < j \le N\}$, and the process is given by the following SDE,

$$dx_i = dB_i + k\sum_{j=1:j\neq i}^{N} \frac{dt}{x_i - x_j}.$$

By Corollary 9, we recover the following familiar result [10]: the set of times where particle collisions occur in the Dyson model has a Hausdorff dimension given by

$$\dim x^{-1}(\partial W) = \max\Big\{0, \frac{1}{2} - k\Big\}.$$

An important feature of this process is that it can be expressed as the eigenvalue process of a real orthogonal or complex Hermitian random matrix with independent Brownian motions up to symmetry [12]. These matrix-valued processes are known for showing no collisions, which can be observed in the Hausdorff dimension formula above, as they correspond to $k = 1/2$ and 1.

Note in addition that there is a large freedom in this case, as by Theorems 1–3 we can add a non-increasing drift function b without changing the Hausdorff dimension.

5.3 Multivariate Bessel process of type B

Of particular interest is the case where $R = B_N$, in which k_α is reduced to two independent parameters, k_1 for the positive roots $\{e_i\}_{1\le i\le N}$, and k_2 for the positive roots $\{e_j \pm e_i\}_{1\le i<j\le N}$. Its Weyl chamber is given by $W_{B_N} = \{x \in \mathbb{R}^N : 0 \le x_i \le x_j, 1 \le i < j \le N\}$, and its SDE reads

$$\begin{aligned} dx_i &= dB_i + \frac{k_1}{x_i}\,dt + k_2\sum_{j=1:j\neq i}^{N}\Big\{\frac{1}{x_i - x_j} + \frac{1}{x_i + x_j}\Big\}\,dt \\ &= dB_i + \frac{k_1}{x_i}\,dt + k_2\sum_{j=1:j\neq i}^{N}\frac{2x_i}{x_i^2 - x_j^2}\,dt, \end{aligned}$$

and the particles are in the Weyl chamber $W_{B_N} = \{x \in \mathbb{R}^N : 0 \le x_i \le x_j, 1 \le i < j \le N\}$. By Corollary 9, the corresponding Hausdorff dimension of collision times is

$$\dim x^{-1}(\partial W_B) = \max\Big\{0, \frac{1}{2} - \min\{k_1, k_2\}\Big\}.$$

The parameters k_1 and k_2 control the type of collision that occurs. Indeed, by [3, Prop. 1], the first hitting time of zero is finite if $k_1 < 1/2$, and infinite otherwise, while the first collision time between particles is finite if $k_2 < 1/2$ and infinite otherwise. This observation will be useful for the next example.

5.4 Wishart processes

The Wishart process [2, 13] is another important system that can be formulated as the matrix eigenvalue process of the product of a rectangular matrix with independent Brownian motion entries with its transpose. As in the Dyson case, the values $\kappa = 1$ and 2 correspond to real and complex matrix formulations of the system, but here we consider the general case given by the SDE

$$dy_i = 2\sqrt{y_i}\, dB_i + \kappa a\, dt + \kappa \sum_{j=1:j\neq i}^{N} \frac{y_i + y_j}{y_i - y_j}\, dt.$$

Here, κ and a are positive parameters, and particles are in the Weyl chamber $W_{B_N} = \{x \in \mathbb{R}^N : 0 \leq x_i \leq x_j, 1 \leq i < j \leq N\}$, but the repulsive interaction between particles is given by the root system A_{N-1}. The Wishart process can be obtained from the multivariate Bessel process of type B_N by the relationships $y_i = x_i^2$, $\kappa a = 2k_1 + 2k_2(N-1) + 1$, and $\kappa = 2k_2$. This implies that Wishart process particles do not hit zero provided

$$k_1 \geq 1/2, \text{ or } a \geq \frac{2}{\kappa} + N - 1.$$

If this condition is fulfilled, our results can be applied to this process as well: observe that $\sigma(y_i) = 2\sqrt{y_i}$, $b(y_i) = \kappa a$, $k_\alpha(y) = \kappa(y_i + y_j)$, and due to $y_i > 0$ for every $i = 1, \ldots, N$, (A1) is satisfied. Then, we see that

$$\frac{|\alpha|^2 k_\alpha(y)}{\sum_{i=1}^{N} \alpha_i^2 \sigma^2(y_i)} = \frac{2(2k_2(y_i + y_j))}{4(y_i + y_j)} = k_2,$$

but because $\sigma^2(x_i) = 4x_i$, the ratio b/σ^2 is not bounded close to the origin of the positive half line. However, since we can guarantee there are no collisions with the origin, the constant $\hat{b}$ in the proofs of Theorem 2 and Lemma 4 becomes an almost surely bounded positive random variable, and the calculations can be completed without problems. Ultimately, we obtain the following statement.

Corollary 10. *When $a \geq 2/\kappa + N - 1$, the Hausdorff dimension of collision times between particles in a Wishart process is given by*

$$\dim y^{-1}(\partial W_B) = \max\Big\{0, \frac{1-\kappa}{2}\Big\}.$$

5.5 Jacobi processes

Jacobi processes are mutually repelling particle systems in a finite segment of the real line. As introduced in [4], all particles are ordered and lie in the interval $(0, 1)$, but for the sake of symmetry we consider the process on $(-1, 1)$, namely, the N particles $\{\lambda_i\}_{i=1}^N$ in [4] are replaced by $x_i = 2\lambda_i - 1$, and the corresponding SDE reads

$$dx_i = \sqrt{1 - x_i^2}\, dB_i + k\left[\frac{1}{2}(p - q - (p+q)x_i) + \sum_{j=1:j\neq i}^{N} \frac{1 - x_i x_j}{x_i - x_j}\right] dt,$$

where we have replaced the parameter $\beta > 0$ by k for notational convenience, and p and q are integer parameters satisfying $\min\{p, q\} \geq N - 1 + 2/k$ in order to ensure the SDE has a unique strong solution. We note here that the cases $k = 1$ and 2 can be formulated as dynamical extensions of the Jacobi ensembles, very much in the same way as the Wishart and Dyson cases above. From the SDE, it is clear that $k > 0$ governs the repulsion between particles, while p and q represent the repulsion strength of the left and right walls respectively.

Similar to the Wishart process, we see that $\sigma(x_i) = \sqrt{1 - x_i^2}$, $b(x_i) = k[p - q - (p+q)x_i]/2$, and $k_{i,j}(x) = k(1 - x_i x_j)$. Because the condition $\min\{p, q\} \geq N - 1 + 2/k$ ensures that $k \min\{p, q\} \geq 2$, x_1 will not hit -1, and x_N will not hit 1 almost surely, so effectively (A1)-(A3) are met here as well. Indeed, both $\sigma(x_i)$ and $k_{i,j}(x)$ are effectively positive as no particle hits ± 1, $b(x_i)$ is a decreasing function, and for $m < i < j < n$,

$$\frac{1 - x_i x_j}{x_j - x_i} - \frac{1 - x_m x_n}{x_n - x_m} = \frac{(x_n - x_j)(1 - x_i x_m) + (x_i - x_m)(1 - x_j x_n)}{(x_j - x_i)(x_n - x_m)} \geq 0,$$

so (A3) holds as

$$\frac{k_{i,j}(x)}{x_j - x_i} = k\frac{1 - x_i x_j}{x_j - x_i} \geq k\frac{1 - x_m x_n}{x_n - x_m} = \frac{k_{m,n}(x)}{x_n - x_m}.$$

Furthermore,

$$\frac{|\alpha|^2 k_\alpha(x)}{\sum_{i=1}^N \alpha_i^2 \sigma^2(x_i)} = k\frac{2(1 - x_i x_j)}{1 - x_i^2 + 1 - x_j^2} = k\frac{2 - x_i^2 - x_j^2 + (x_j - x_i)^2}{2 - x_i^2 - x_j^2} = k\Big\{1 + \frac{(x_j - x_i)^2}{2 - x_i^2 - x_j^2}\Big\} \geq k, \tag{5.1}$$

which implies that

$$\dim x^{-1}(\partial W) \leq \max\Big\{0, \frac{1}{2} - k\Big\}.$$

Note that we do not have a non-trivial lower bound here. This is because the ratio of k and σ^2 in (5.1) is not bounded above. In a manner similar to the Wishart process case, for this process the constants $\tilde{\eta}_\alpha$ become random variables with almost sure upper bounds, which allows us to complete the calculations in the proof of Theorem 2. However, not having an upper bound for (5.1) implies that near the walls at ± 1 the repulsion interaction between particles may be much larger than the fluctuations induced by the Brownian motion, so we cannot state that particle collisions occur with certainty.

Acknowledgments

SA was supported by the Kakenhi grant number JP19K14617 of the Japanese Society for the Promotion of Science (JSPS).

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