Good Ramanujan Expansions: A suitably enhanced decay of coefficients has important consequences

Giovanni Coppola

Abstract. In this self-contained short note, we introduce the new definition of Good Ramanujan Expansion, say G.R.E., for a fixed arithmetic function F, building upon a good decay of its coefficients G; this, gains $\log -$powers w.r.t. the trivial bound for G and precisely $\log^{1+\eta}$, where the present parameter $\eta > 0$ is real. This property alone has important consequences for all the F having a G.R.E. : mainly, 1) the Eratosthenes Transform F' of our F is infinitesimal (see in Theorem 1); 2) when $\eta > 1$ (an enhanced decay) we have uniqueness of G (actually, these are the classic Wintner-Carmichael coefficients, see Th.2); 3) we get a bound for F (in Th.3); 4) an important new class of arithmetic functions F can't have a G.R.E. (see Th.4). These are a generalization of Correlations; which in this way, if are, say, a kind of "far from constants", may not have a G.R.E., whence, a fortiori, can't have the R.E.E.F. This is the Ramanujan Exact Explicit Formula, that we introduced with Prof. Ram Murty.

1. Introduction. Notations and New definitions

In 1918 Ramanujan [R] published a paper that is both a milestone in number theory and the first page of mathematical literature in the subject of Ramanujan Expansions for Arithmetic Functions. We say that, once fixed $F : \mathbb{N} \to \mathbb{C}$, we have a Ramanujan Expansions for F, with coefficient $G : \mathbb{N} \to \mathbb{C}$, IFF (if and only if) by definition:

$$\forall a \in \mathbb{N}, \quad F(a) = \lim_x \sum_{q \le x} G(q) c_q(a) \overset{def}{=} \sum_{q=1}^{\infty} G(q) c_q(a),$$

where it's implicit: $\lim_x$ is over $x \to \infty$ and it exists in complex numbers.

Here we abbreviate with $c_q(a) \overset{def}{=} \sum_{\substack{j \le q \\ (j,q)=1}} \cos \frac{2\pi j a}{q}$ the Ramanujan Sum of modulus $q \in \mathbb{N}$ and argument $a \in \mathbb{N}$. (In this paper, finite & infinite sums are over naturals : within $\mathbb{N}$.)

We'll not give all the details about notations, since we'll follow the standard ones; like for all implicit conventions & standard facts, we follow §1 of [C1].

The idea of Ramanujan was (and still is!) simple and far-reaching : any $F : \mathbb{N} \to \mathbb{C}$ has a Ramanujan Expansion (see Remarks in §4); He soon discovered that <u>it's not unique</u> , whatever is our F. This follows from expanding the null-function, write $\mathbf{0}$, which sends $\mathbb{N}$ in the set $\{0\}$: compare §1 of [C1]. Ramanujan gave [R] explicit non-zero Coefficients for it, see $R_0(q)$, after Corollary 2.

In this short note, we prove how assuming a suitable decay in the, say, $q-$th coefficients $G(q)$ in expansion above entails important consequences:

1) the Eratosthenes Transform $F' \overset{def}{=} F * \mu$, see [C1], of our F is infinitesimal; actually, we'll derive a bound for F' strictly depending on $G(q)$ decay : see Theorem 1; whence Corollary 1;
2) when the decay is "enhanced", say, w.r.t.(with respect to) our standard decay assumption, then we get the <u>uniqueness of coefficients</u> : details in Theorem 2; whence, Corollary 2;
3) a Good Ramanujan Expansion, abbrev. (G.R.E.) , see our new definition, upcoming next, entails also a bound for our F that we expand : precise bound's in Theorem 3;
4) a particular sub-class of periodic arithmetic functions, that we introduce in next definition, has members having Eratosthenes Transform, say F', with a particular & characteristic behavior, ensuring F' is not infinitesimal, excluding for them any (G.R.E.), by Theorem 1; and this is our last main result in this paper : Theorem 4.

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Notations: recalling standard & non-standard

For standard notations, like numbers sets (by the way, $\mathbb{P}$ is the primes' set), we follow [C1] : there we find Möbius Function μ & Euler Totient Function φ (with basic properties). Here, for the non-standard ones, namely those that may differ from one Author to the other, we recall log is in natural base e; we'll call $q-$periodic a function with minimal *period* $\mathtt{T}$ dividing q, but may be $\mathtt{T}< q$ (see [C1]); notation $\mathbb{Z}_q^*$ is standard for the *reduced residue classes* modulo $q \in \mathbb{N}$, but we abuse it writing, for $q-$periodic functions' argument a, OFTEN $a \in \mathbb{Z}_q^*$ TO MEAN: $a \in \mathbb{N}$, $a \le q$, $(a,q)=1$. We write $p \equiv r(q)$ for $p \equiv r(\bmod q)$. For fixed $F : \mathbb{N} \to \mathbb{C}$

$$\mathrm{Win}_q F \stackrel{def}{=} \sum_{d \equiv 0 (\bmod q)} \frac{F'(d)}{d},$$

whenever the series converges pointwise in $\mathbb{C}$, is the $q-$TH WINTNER COEFFICIENT for F; varying $q \in \mathbb{N}$, we get WINTNER'S TRANSFORM Win F: $q \in \mathbb{N} \mapsto \mathrm{Win}_q F \in \mathbb{C}$, a new arithmetic function that exists provided all $\mathrm{Win}_q F$ exist. Analogously,

$$\mathrm{Car}_q F \stackrel{def}{=} \frac{1}{\varphi(q)} \lim_x \frac{1}{x} \sum_{a \le x} F(a) c_q(a),$$

whenever the limit exists in $\mathbb{C}$, is the $q-$TH CARMICHAEL COEFFICIENT for F; varying $q \in \mathbb{N}$, we get CARMICHAEL'S TRANSFORM Car F: $q \in \mathbb{N} \mapsto \mathrm{Car}_q F \in \mathbb{C}$, a new arithmetic function that exists provided all $\mathrm{Car}_q F$ exist. Just to abbreviate a limit over a subsequence of primes, that we need in the following,

$$\lim_{p \equiv r (\bmod q)} H(p)$$

indicates the limit of $H(p)$, for given $H : \mathbb{N} \to \mathbb{C}$, over primes $p \equiv r(q)$, for fixed $q \in \mathbb{N}$ and (trivially) $r \in \mathbb{Z}_q^*$. Like in [C1], $\stackrel{\bullet}{=}$ means "equals, from the definition"; QED is the "end of a part" of a Proof; whose end's $\square$

New Definitions

A GOOD RAMANUJAN EXPANSION, abbrev. (G.R.E.), for a fixed $F : \mathbb{N} \to \mathbb{C}$, means by definition that we have fulfilled the three requirements:

(RAMANUJAN EXPANSION) $$\forall a \in \mathbb{N}, \quad F(a) = \lim_x \sum_{q \le x} G(q) c_q(a) \stackrel{def}{=} \sum_{q=1}^{\infty} G(q) c_q(a)$$

where the RAMANUJAN COEFFICIENT $G : \mathbb{N} \to \mathbb{C}$ in this expansion is the same as in the

(LUCHT EXPANSION) $$\forall d \in \mathbb{N}, \quad F'(d) = d \lim_x \sum_{K \le x} \mu(K) G(dK) \stackrel{def}{=} d \sum_{K=1}^{\infty} \mu(K) G(dK)$$

and it is infinitesimal, with the existence of a real PARAMETER $\eta > 0$, in the following

($\eta-$ DECAY) $$G(q) \ll_\eta \frac{1}{q (\log q)^{1+\eta}}$$

A G.R.E., in general, has only the requirement $\eta > 0$ and may have *no Coefficients uniqueness*; however, compare Theorem 2. The Lucht Expansion is a vital hypothesis, here (see §4 Remarks).

A $\mathcal{Q}-$periodic function $F : \mathbb{N} \to \mathbb{C}$, with $\mathcal{Q} > 2$, by definition, DIVERTS VALUES IN $\mathbb{Z}_{\mathcal{Q}}^*$, when we have: $\exists a \in \mathbb{Z}_{\mathcal{Q}}^* \backslash \{1\}$ with $F(a) \neq F(1)$. The other possibility, i.e. $F(\mathbb{Z}_{\mathcal{Q}}^*) = \{F(1)\}$, defines the $\mathcal{Q}-$periodic functions F that are $\mathcal{Q}-$MONOCHROMATIC. We may abbreviate, like in abstract, "DIVERTS VALUES" with "FAR FROM CONSTANTS", since $\mathcal{Q}-$MONOCHROMATIC sounds like "close to constant".

Paper's Plan

- ⋆ in next section we state and prove our results;
- ⋆ in §3 we apply some of our results to the Correlations of two arithmetic functions;
- ⋆ in §4 we offer some more Remarks giving also a glance to future applications for our results.

2. Statements and Proofs of our Results

A Good Ramanujan Expansion for F entails an infinitesimal F' ; more precisely, we have the following.

Theorem 1. *Let* $F : \mathbb{N} \to \mathbb{C}$ *have a* GOOD RAMANUJAN EXPANSION, *of* PARAMETER $\eta > 0$. *Then*

$$F'(d) \ll_\eta (\log d)^{-\eta}.$$

In particular, the ERATOSTHENES TRANSFORM F' IS INFINITESIMAL.

Proof. Above (LUCHT EXPANSION) with $G : \mathbb{N} \to \mathbb{C}$ of (η − DECAY) gives, together with the EASY INEQUALITY : $\log(dK) \ge \max(\log d, \log K)$, $\forall d, K \in \mathbb{N}$, with "LARGE" $d \in \mathbb{N}$ (say, $d \ge 2$ suffices)

$$F'(d) \ll_\eta \sum_{K \le d} \frac{1}{K(\log d)^{1+\eta}} + \sum_{K > d} \frac{1}{K(\log K)^{1+\eta}} \ll_\eta \frac{1}{\log^\eta d},$$

from the two easy bounds:

$$\sum_{K \le d} \frac{1}{K} \ll \log d \qquad \text{AND} \qquad \sum_{K > d} \frac{1}{K(\log K)^{1+\eta}} \le \int_d^\infty \frac{1}{t(\log t)^{1+\eta}} dt \ll_\eta (\log d)^{-\eta},$$

compare [T] for the former. □

The DIVISOR FUNCTION: $a \in \mathbb{N} \mapsto d(a) \stackrel{def}{=} \sum_{d|a} 1 \in \mathbb{C}$ CAN'T HAVE a G.R.E., HAVING ERATOSTHENES TRANSFORM $\mathbf{1}(d) \stackrel{def}{=} 1, \forall d \in \mathbb{N}$ NOT INFINITESIMAL. As a kind of CURIOSITY, RAMANUJAN [R] GAVE

$$d(a) = \sum_{q=1}^{\infty} \left(-\frac{\log q}{q} \right) c_q(a), \quad \forall a \in \mathbb{N}$$

and, AS YOU SEE, these COEFFICIENTS "LOOSE A log" INSTEAD OF GAINING $\log^{1+\eta}$.

As a straight consequence of Theorem 1, we get

Corollary 1. *Let* $F : \mathbb{N} \to \mathbb{C}$ *have a* GOOD RAMANUJAN EXPANSION, *of* PARAMETER $\eta > 0$. *Then*

as $p \to \infty$ (*in prime numbers*), $F(2p) = F(2) + O_\eta\left((\log p)^{-\eta}\right)$

and, IN PARTICULAR, $\lim_p F(2p) = F(2)$. *Furthermore, in the same way*, FIXING $a_0 \in \mathbb{N}$,

as $p \to \infty$, $F(a_0 p) = F(a_0) + O_\eta\left((\log p)^{-\eta}\right)$

whence, IN PARTICULAR, $\lim_p F(a_0 p) = F(a_0)$.

Remark 1. The result may be GENERALIZED: take A FINITE NUMBER OF PRIMES $p_1 < p_2 < \cdots < p_r$ and then send $p_1 \to \infty$, all staying in prime numbers, TO MULTIPLY A FIXED $a_0 \in \mathbb{N}$. ◇

Proof. Writing $F(a) = \sum_{d|a} F'(d)$, we get $F(2p) - F(2) = F'(p) + F'(2p)$, whence Theorem 1 entails BOTH $F'(p) \ll_\eta (\log p)^{-\eta}$ AND $F'(2p) \ll_\eta (\log(2p))^{-\eta} \ll_\eta (\log p)^{-\eta}$. QED

Analogously, $F(a_0 p) - F(a_0) = \sum_{d|a_0} F'(dp) \ll_\eta d(a_0) \cdot (\log p)^{-\eta}$ and $a_0 \in \mathbb{N}$ FIXED $\Rightarrow$ $d(a_0)$ FIXED. □

Remark 2. The behavior of our F in Corollary 1 resembles, also for the "general" case $a_0 \in \mathbb{N}$ and we assume it now TO BE EVEN, say $a_0 = 2k$, that of THE EXPECTED HEURISTIC FOR $F(2k) =$"$2k-$TWIN PRIMES COUNT", as given in HARDY-LITTLEWOOD CONJECTURE (see [HL] Conjecture B). FOCUS ON THE "SINGULAR SERIES" : IT HAS FACTORS AS A KIND OF FINITE PRODUCTS $\prod_{p|(2k)} (1 + O(1/p))$ WHICH, JOINING LARGE FACTORS p, SAY, "DON'T CHANGE MUCH". ◇

Gaining a little bit more in the decay of coefficients entails their uniqueness.

Theorem 2. *Let* $F : \mathbb{N} \to \mathbb{C}$ *have a* G.R.E. *with* Ramanujan Coefficient $G : \mathbb{N} \to \mathbb{C}$ and parameter $\eta > 1$. *Then*

a) the Transforms Win F and Car F exist ;

b) Car F = Win F ;

c) G = Car F = Win F.

Remark 3. In other words, point *c*) says that any $F : \mathbb{N} \to \mathbb{C}$ with a G.R.E. featuring $\eta > 1$ can only have as Ramanujan Coefficients the so-called Carmichael-Wintner Coefficients. $\diamond$

Proof. From "Wintner's Criterion", compare [C0] Proposition 2, point 1, above *a*) and *b*) follow from "Wintner Assumption"(Criterion's hypothesis):

$$\text{(WA)} \qquad \sum_{d=1}^{\infty} \frac{|F'(d)|}{d} < \infty.$$

We apply, in order to prove (WA), the $\eta-$decay with $\eta > 1$ to bound $F'(d)/d$, like in previous Proof, getting

$$\sum_{d=1}^{\infty} \frac{|F'(d)|}{d} \ll_\eta |F'(1)| + \sum_{d \ge 2} \frac{1}{d(\log d)^\eta} < \infty,$$

from $\eta > 1$. QED

Using point *b*), in order to get *c*) we need to show G = Win F : it comes from (7) in Theorem 1 [C0], i.e.

$$\sum_{q=1}^{\infty} 2^{\omega(q)} |G(q)| < \infty;$$

recall $\omega(q) \stackrel{def}{=} |\{p \in \mathbb{P} : p|q\}|$ is the number of prime factors of q, so this convergence may be proved as follows from the trivial $2^{\omega(q)} \le d(q)$, $\forall q \in \mathbb{N}$, whence

$$(*) \qquad \sum_{q=1}^{\infty} 2^{\omega(q)} |G(q)| \ll_\eta |G(1)| + \sum_{q=2}^{\infty} \frac{d(q)}{q(\log q)^{1+\eta}} \ll_\eta 1;$$

in fact, Partial Summation entails

$$\sum_{1<q\le x} \frac{d(q)}{q(\log q)^{1+\eta}} = \frac{1}{x(\log x)^{1+\eta}} \sum_{1<q\le x} d(q) + \int_2^x \sum_{2 \le q \le t} d(q) \frac{(t(\log t)^{1+\eta})'}{t^2(\log t)^{2+2\eta}} dt$$

and with classic bound [T] : $\sum_{q \le x} d(q) \ll x \log x$, for all large $x \ge 2$, we get

$$\sum_{q=2}^{\infty} \frac{d(q)}{q(\log q)^{1+\eta}} \stackrel{\bullet}{=} \lim_x \sum_{1<q\le x} \frac{d(q)}{q(\log q)^{1+\eta}} = \int_2^\infty \sum_{1<q\le t} d(q) \frac{(\log t) + (1+\eta)}{t^2 (\log t)^{2+\eta}} dt \ll_\eta \int_2^\infty \frac{1}{t(\log t)^\eta} dt \ll_\eta 1,$$

whence $(*)$ above follows again from $\eta > 1$. $\square$

Theorem 2 entails : no (G.R.E.) with $\eta > 1$ for the null-function, if we want a non-trivial Coefficient G.

Corollary 2. *The null-function, i.e.,* $\mathbf{0} : n \in \mathbb{N} \mapsto 0 \in \mathbb{C}$, may not have *any* (G.R.E.) with *non-trivial coefficient, i.e.* $G \ne \mathbf{0}$, and (η − Decay) with $\eta > 1$.

Proof. From Theorem 2, a (G.R.E.) with $\eta > 1$ for $\mathbf{0}$ entails G = Win F = Car $F = \mathbf{0}$. $\square$

By the way, apart from the RAMANUJAN COEFFICIENT FOR **0**, say

$$R_0(q) \stackrel{def}{=} \frac{1}{q}, \quad \forall q \in \mathbb{N} \qquad \text{GIVEN BY RAMANUJAN IN [R]}$$

which gives a Ramanujan expansion, actually, equivalent to the Prime Number Theorem, see §1 in [C1], we also have

$$H_0(q) \stackrel{def}{=} \frac{1}{\varphi(q)}, \quad \forall q \in \mathbb{N} \qquad \text{GIVEN BY HARDY IN [H]}$$

as coefficient for **0**. Furthermore, we know that there are INFINITELY MANY NON-TRIVIAL Coefficients for the null-function. None of them, however, may come from (G.R.E.) with $\eta > 1$.

The question arises : MAY WE GIVE A (G.R.E.) FOR **0**, WITH $0 < \eta < 1$, OR EVEN WITH $\eta = 1$? (Remember, NON-TRIVIAL Coefficients!)

A Good Ramanujan Expansion for F entails also a non-trivial bound for the function expanded itself.

Theorem 3. *Let* $F:\mathbb{N}\to\mathbb{C}$ *have a* GOOD RAMANUJAN EXPANSION, *of* PARAMETER $\eta>0$. *Then*

$$F(a)=O(|G(1)|)+O_\eta(1)+O_\eta\left(\sum_{\substack{d|a\\ d>1}}\frac{1}{\log^\eta d}\right),\quad \text{UNIFORMLY } \forall a\in\mathbb{N}.$$

Remark 4. Of course, the corresponding error term vanishes when $G(1)=0$, in the above. $\diamond$

Proof. We'll abbreviate present calculations when they have already been performed like in our previous Theorems' Proofs. Actually, we'll be fairly explicit when we have new details: mainly, for divisors of $a\in\mathbb{N}$.

In fact, we start FIXING such an ARGUMENT and USE THE BOUND

$$|c_q(a)|\le(a,q),\qquad \forall a,q\in\mathbb{N},$$

recall: $\forall a,q\in\mathbb{N}$, (a,b) is their Greatest Common Divisor, for which the reader may compare Lemma A.1 in [CM] for an elementary Proof.

Thus

$$F(a)=O(|G(1)|)+O_\eta\left(\sum_{q=2}^{\infty}\frac{(a,q)}{q(\log q)^{1+\eta}}\right),$$

where from

$$\sum_{q=2}^{\infty}\frac{(a,q)}{q(\log q)^{1+\eta}}\le\sum_{d|a}d\sum_{\substack{q=2\\ q\equiv 0(\bmod d)}}^{\infty}\frac{1}{q(\log q)^{1+\eta}}$$

we get

$$\sum_{q=2}^{\infty}\frac{(a,q)}{q(\log q)^{1+\eta}}\ll_\eta 1+\sum_{\substack{d|a\\ d>1}}\left(\sum_{K\le d}\frac{1}{K(\log d)^{1+\eta}}+\sum_{K>d}\frac{1}{K(\log K)^{1+\eta}}\right)\ll_\eta 1+\sum_{\substack{d|a\\ d>1}}\frac{1}{\log^\eta d},$$

entailing above bound. $\square$

Remark 5. Actually, TH.M 3 follows from TH.M 1, writing $\forall a\in\mathbb{N}$, $\sum_{d|a}F'(d)$. However, we preferred to both avoid giving another Corollary and GIVE ALL DETAILS. $\diamond$

The set of ARITHMETIC FUNCTIONS F THAT DIVERT VALUES IN $\mathbb{Z}^*_{\mathcal{Q}}$, *for some fixed* $\mathcal{Q}>2$, GIVES A big SET of functions $F:\mathbb{N}\to\mathbb{C}$ WITHOUT G.R.E., thanks to the following.

Theorem 4. *Let* $F:\mathbb{N}\to\mathbb{C}$ DIVERT VALUES IN $\mathbb{Z}^*_{\mathcal{Q}}$, *for some* FIXED $\mathcal{Q}>2$. *Then*

i) $\exists a\in\mathbb{Z}^*_{\mathcal{Q}}\backslash\{1\}: F(a)=F(p_a)\ne F(p_1)=F(1),\ \ \forall p_a\equiv a(\bmod\ \mathcal{Q}), \forall p_1\equiv 1(\bmod\ \mathcal{Q})$ PRIMES

ii) THE TWO SUBSEQUENCES, *in the sequence of primes,* $p_a\equiv a(\bmod\ \mathcal{Q})$ AND $p_1\equiv 1(\bmod\ \mathcal{Q})$ HAVE

$$\lim_{p_a\equiv a(\bmod\ \mathcal{Q})}F'(p_a)\ne\lim_{p_1\equiv 1(\bmod\ \mathcal{Q})}F'(p_1)=0,\ \text{ WHENCE } \lim_p F'(p),\ \text{ OVER ALL PRIMES, DOESN'T EXIST}$$

iii) *in particular,* F' IS NOT INFINITESIMAL, *whence by Theorem 1*

iv) F CAN'T HAVE ANY G.R.E.

Proof. Our *i)* comes from the "DIVERTS VALUES"-DEFINITION. QED
The fact that there are infinitely many primes p_a and p_1 in the above non-trivial arithmetic progressions modulo $\mathcal{Q}$ comes straight from DIRICHLET THEOREM ON PRIMES IN ARITHMETIC PROGRESSIONS, so, using above *i)* and $F'(p)\doteq F(p)-F(1), \forall p\in\mathbb{P}$, we get *ii)* at once. QED
Finally, *iii)* and *iv)* ARE IMMEDIATE. $\square$

3. Application to Correlations

We wish to give our most interesting application, namely for Correlations.

For general definitions about them, see [C1]. There (in second version), we define the $3-$HYPOTHESES CORRELATIONS, see (0), (1), (2) hypotheses there; and, just to give, say, the easiest example of this kind, we build now what we call "COUNTEREXAMPLE 1". This, in fact, is the Counterexample we need to prove that not all the $3-$hp.s Correlations have the R.E.E.F., abbrev.for RAMANUJAN EXACT EXPLICIT FORMULA; this is : point *(iii)* of Theorem 1 [CM] (this Theorem doesn't prove the REEF but gives four equivalent properties for the Ramanujan Expansion of a Correlation, $C_{f,g}(N,a) \stackrel{def}{=} \sum_{n\le N} f(n)g(n+a)$, w.r.t. the *shift* $a \in \mathbb{N}$). Actually, the (R.E.E.F.) is a fixed-length finite Ramanujan Expansion, entailing the Correlation having it to be a Truncated Divisor Sum (see [CM] considerations in sections 2,3 and 6, and compare [C0] Theorem 3, proving that fixed-length RE and TDS are the same,too).

Last but not least, following Counterexample 1 has been my source of inspiration for the "DIVERTING-VALUES" definition. In fact, it diverts values, whence has no G.R.E.; in particular, from this it's easy to prove (we leave it as an exercise for the careful reader) that no (R.E.E.F.), for this Correlation, is possible. (However, it's so easy that this last property will be checked at once.)

We start fixing a prime $p_0 > 2$, in correspondence of which we choose a length $N \in \mathbb{N}$ for our Correlation, with only one requirement : to be large enough, in order to allow the existence of another prime, SAY $n_0 \in \mathbb{P}$, $n_0 > p_0$ and $n_0 \equiv -1 (\bmod\, p_0)$, in the range $n_0 \le N$.

With these two primes and $N \in \mathbb{N}$, we are ready to build our CORRELATION, say

$$C_{f_0,g_0}(N,a) \stackrel{def}{=} \sum_{n\le N} f_0(n)g_0(n+a), \qquad \forall a \in \mathbb{N},$$

for arithmetic functions $f_0 \stackrel{def}{=} \mathbf{1}_{\{n_0\}}$ and $g_0(m) \stackrel{def}{=} c_{p_0}(m)$, $\forall m \in \mathbb{N}$; getting

(COUNTEREXAMPLE 1) $$C_{f_0,g_0}(N,a) \stackrel{\bullet}{=} c_{p_0}(a-1), \qquad \forall a \in \mathbb{N},$$

from $n_0 \equiv -1 (\bmod\, p_0)$ and p_0-PERIODICITY OF RAMANUJAN SUM above.

WE MAY WRITE, by definition, $\widehat{g_0}(q) = \mathbf{1}_{\{p_0\}}(q)$ and $g_0(m) = \sum_{q\le p_0} \widehat{g_0}(q)c_q(m)$, SO WE DON'T NEED TO TRUNCATE DIVISORS AT $Q \le N$, compare [C1], since our $Q = p_0 \le N$ (N is "LARGE ENOUGH"), AND OUR CORRELATION IS FAIR, trivially, BECAUSE $a-$DEPENDENCE IS ONLY INSIDE RAMANUJAN SUM above. Thus, hypothesis (0), i.e., BASIC HYPOTHESIS is satisfied; the OTHER TWO HYPOTHESES (1) & (2) in [C1], namely $\widehat{g_0}$ is square-free supported for (1) and hypothesis (2), requiring f_0 supported on primes, are ALSO BOTH FULFILLED, from $p_0, n_o \in \mathbb{P}$. In passing, g_0 IS p_0-PERIODIC, ENTAILING C_{f_0,g_0} IS p_0-PERIODIC.

This is a kind of TOY-MODEL CORRELATION, so its PERIODICITY IS VERY EASY.
From [C1], see section 4 after $(*)$ there, we know that for general, "REAL-WORLD SOUNDING" $3-$hypotheses Correlations, we still have periodicity, linked to the TRUNCATION PARAMETER $Q \in \mathbb{N}$, with PERIOD $\mathtt{T} \in \mathbb{N}$ (EVEN MORE DIFFICULT TO FIND).

We'll come to a general approach to PERIODICITY OF VERY GENERAL CORRELATIONS, namely only having Basic Hypothesis, in future publications regarding Correlations.

Well, C_{f_0,g_0} DIVERTS VALUES IN $\mathbb{Z}^*_{p_0}$. And, being very simple, it does it IN THE SIMPLEST WAY, namely:
for $a = 1 \in \mathbb{Z}^*_{p_0}$, $C_{f_0,g_0}(N,a) = c_{p_0}(a-1) = \varphi(p_0) = p_0 - 1$
and $\forall a \in \mathbb{Z}^*_{p_0} \backslash \{1\}$, $C_{f_0,g_0}(N,a) = c_{p_0}(a-1) = \mu(p_0) = -1$.

Thus Theorem 4 tells us that this Correlation has ERATOSTHENES TRANSFORM

$$C'_{f_0,g_0}(N,d) \stackrel{def}{=} \sum_{a|d} C_{f_0,g_0}(N,a)\mu\left(\frac{d}{a}\right) = \sum_{a|d} c_{p_0}(a-1)\mu\left(\frac{d}{a}\right) \qquad \text{NOT INFINITESIMAL } (as\ d \to \infty)$$

and, in particular, it has no G.R.E. (from Theorem 1). Next considerations exclude the (R.E.E.F.), for it.

In fact, any Basic Hypothesis Correlation, say, $C_{f,g_Q}(N,a)$, with the

$$\text{(R.E.E.F.)} \qquad C_{f,g_Q}(N,a) = \sum_{q\le Q} \frac{\widehat{g_Q}(q)}{\varphi(q)} \sum_{n\le N} f(n)c_q(n)c_q(a), \qquad \forall a \in \mathbb{N},$$

compare [C1] Theorem 3.1 for the coefficients, has clearly INFINITESIMAL ERATOSTHENES TRANSFORM, since THE (R.E.E.F.) ENTAILS (another exercise for the careful reader, applying (1.7) in [C1]: Kluyver Formula)

$$C'_{f,g_Q}(N,d) = 0, \qquad \forall d > Q.$$

The same result quoted above allows to calculate our Correlation's coefficients; say, Carmichael-Wintner $\ell-$th coeff.'s in general:

$$\frac{\widehat{g_Q}(\ell)}{\varphi(\ell)} \sum_{n\le N} f(n)c_\ell(n)$$

as above; FOR OUR CASE $f = f_0$ & $g_Q = g_0$, IT'S

$$\mathbf{1}_{p_0}(\ell) \cdot \frac{1}{\varphi(\ell)} c_\ell(n_0)$$

giving a (R.E.E.F.)'s RHS(=Right Hand Side), for our Correlation,

$$\sum_{\ell\le p_0} \mathbf{1}_{p_0}(\ell) \cdot \frac{1}{\varphi(\ell)} c_\ell(n_0)c_\ell(a) = \frac{\mu(p_0)}{\varphi(p_0)} c_{p_0}(a).$$

Thus

$$C_{f_0,g_0}(N,a) = c_{p_0}(a-1) = \sum_{\ell\le p_0} \mathbf{1}_{p_0}(\ell) \cdot \frac{1}{\varphi(\ell)} c_\ell(n_0)c_\ell(a) = \frac{\mu(p_0)}{\varphi(p_0)} c_{p_0}(a) \qquad \text{CAN'T HOLD } \forall a \in \mathbb{N}.$$

Namely,

OUR CORRELATION $c_{p_0}(a-1)$ HAS NO (R.E.E.F.), IN PERFECT AGREEMENT WITH THEOREM 4.

In fact, RHS above is

$$-\frac{1}{p_0-1} c_{p_0}(a) = \frac{1}{p_0-1}, \qquad \forall a \not\equiv 0 (\bmod\ p_0),$$

while our Correlation's absolute value is

$$|c_{p_0}(a-1)| \ge 1, \qquad \forall a \in \mathbb{N},$$

since MODULUS p_0 IS SQUARE-FREE (use (1.6) in [C1]: Hölder's Formula).

Just to have fun, say, the "(R.E.E.F.) HOLDS $\forall a \in \mathbb{N}, a \equiv 0(\bmod\ p_0)$". A kind of "RELATIVE REEF".

More in general, OUR THEOREM 4 ENTAILS NOT ONLY THAT DIVERTING VALUES $C_{f,g_Q}(N,a)$ HAVE NO (R.E.E.F.); BUT ALSO THAT THEIR IPPIFICATION $\widetilde{C_{f,g_Q}}(N,a)$ (compare [C1]), WITH SAME ERATOSTHENES TRANSFORM, BUT SUPPORTED ON SQUARE-FREE DIVISORS (see [C1],[C2]), IS STILL DIVERTING-VALUES, SO NO (R.E.E.F.), FOR $\widetilde{C_{f,g_Q}}(N,a)$ TOO.

Again, this is immediate for our Toy-Model Correlation (as the careful reader may check, recalling IPPification definition in [C1,C2]).

4. Final Remarks and coming soon

We start with a Remark about how this paper was born.

Remark 6. Regarding the Correlations, in our paper with Ram Murty [CM] about the finite Ramanujan expansions, we assumed that the coefficients in their *shift-Ramanujan expansion* had an extremely fast decay, gaining, as we assumed, powers of the index q (NOT log −POWERS, like here). Then, playing with this assumption, I was wondering the effect, over an arbitrary arithmetic function, of having a much slower decay, namely the ($\eta-$DECAY) above. Subsequently, I realized Theorem 1, namely that it descends, from this decay for the Ramanujan Coefficients $G(q)$ of our F, that $F'(d)$ is infinitesimal as $d \to \infty$, compare Th.m 1. In particular, $F'(p) \stackrel{\bullet}{=} F(p)-F(1)$ is infinitesimal as $p \to \infty$, along primes p; but, playing around with formulæ, for (3−Hypotheses-)Correlations in [C1], I saw a high probability to get oscillating such $F'(p)$ along primes $p \to \infty$; last but not least, again from [C1], I knew that a Basic Hypothesis Correlation is $\mathcal{Q}-$periodic, see [C1] : looking at residue classes in $\mathbb{Z}^*_{\mathcal{Q}}$ and applying Theorem 4 arguments, I got the non-existence, in diverting-values case, of $F'(p)$ limit. In fact, $F'(p)$ was **diverting values** of limits for it, in the subsequences for different residue classes (1 and $a \neq 1$, if $F(a) \neq F(1)$) : Thanks very much, to our Peter Gustav Lejeune Dirichlet ! (Otherwise, no subsequences!). $\diamond$

By the way, as it's clear from Theorem 4 above, **in case the Correlation diverts values** in $\mathbb{Z}^*_{\mathcal{Q}}$, we may forget about the Possibility, not only to have it in First Class (see [CM]), but also, of course, to get a Good Ramanujan Expansion, for it : sorry Ram ! However, see next Remarks 10 & 11.

We get back, by the way, to the (η − DECAY), since we get an interesting consequence of Theorem 2.

Remark 7. From Th.m 2, we have decay for the $q-$th Wintner Coefficient, now $\mathrm{Win}_q F = G(q)$, that is, from (η − DECAY), $\mathrm{Win}_q F \ll_\eta (\log q)^{-1-\eta}/q$, of course; however, since Th.m 1 gives: $F'(d) \ll_\eta (\log d)^{-\eta}$, the series defining $\mathrm{Win}_q F$ has the upper bound, for all $q \geq 2$, (compare above Theorems' Proofs' calculations)

$$|\mathrm{Win}_q F| \stackrel{\bullet}{=} \frac{1}{q}\left|\sum_{m=1}^{\infty}\frac{F'(qm)}{m}\right| \leq \frac{1}{q}\sum_{m=1}^{\infty}\frac{|F'(qm)|}{m} \ll_\eta \frac{1}{q}\sum_{m\leq q}\frac{1}{m(\log q)^\eta} + \frac{1}{q}\sum_{m>q}\frac{1}{m(\log m)^\eta} \ll_\eta (\log q)^{1-\eta}/q,$$

easily getting a log −SQUARED GAIN FOR THE DECAY, motivated by the sign-change (in case $F : \mathbb{N} \to \mathbb{R}$) or, more generally, phase-change of our Eratosthenes Transform, within these series. Say, our Th.m 2 hypotheses "FORCE THE CANCELLATION", coming from $F'(d)$, in these series defining Wintner coefficients. $\diamond$

We continue with two remarks about the (G.R.E.) definition.

Remark 8. From the bound $|c_q(a)| \leq (a, q)$, COMPARE Th.m 3 PROOF, we get, applying (η − DECAY),

$$\forall a \in \mathbb{N}, \quad \sum_{q=1}^{\infty}|G(q)c_q(a)| \leq \sum_{q=1}^{\infty}(a,q)\,|G(q)| \ll_\eta 1 + d(a),$$

COMPARE Th.m 3 BOUNDS, whence we get THE ABSOLUTE CONVERGENCE of our (G.R.E.). Then, IN PARTICULAR, ALL SUMMATION METHODS FOR THE (RAMANUJAN EXPANSION) GIVE <u>BOTH</u> CONVERGENCE AND TO THE SAME SUM. $\diamond$

Remark 9. In the (G.R.E.) definition we have included (LUCHT EXPANSION), because it's not immediately descending from (RAMANUJAN EXPANSION), as APPLYING KLUYVER FORMULA (quoted (1.7) in [C1])

$$F(a) = \sum_{q=1}^{\infty}G(q)c_q(a) = \sum_{d|a} d \sum_{\substack{q=1 \\ q\equiv 0 (\mathrm{mod}\ d)}}^{\infty} G(q)\mu\left(\frac{q}{d}\right) = \sum_{d|a} d \sum_{K=1}^{\infty}\mu(K)G(dK),$$

in general DOESN'T IMPLY (LUCHT EXPANSION), because IN CASE G DEPENDS ON a, WE CAN'T APPLY MÖBIUS INVERSION. $\diamond$

We come now to other two remarks, pertaining to Correlations.

Remark 10. In the case of COUNTEREXAMPLE 1, we recognize at once the DIVERTING-VALUES PROPERTY, thanks to the very simple structure of this Correlation; actually, reducing to a, say, SHIFTED RAMANUJAN SUM, see §3; for more complicated Correlations (say, "real-life ones"), the check of this property is NOT EASY & MAY REQUIRE SOME NEW HYPOTHESES. ◇

Remark 11. For those related to the HARDY-LITTLEWOOD CORRELATION (for $a = 2k-$twin primes, say):

$$C_{\Lambda,\Lambda}(N,a) \stackrel{def}{=} \sum_{n\le N} \Lambda(n)\Lambda(n+a), \quad \forall a \in \mathbb{N}$$

where Λ is the von-Mangoldt function (see [C1]) AND its "TRUNCATION" [C1] :

$$C_{\Lambda,\Lambda_N}(N,a) \stackrel{def}{=} \sum_{n\le N} \Lambda(n)\Lambda_N(n+a), \quad \forall a \in \mathbb{N},$$

that is EASIER TO STUDY, compare Theorem 3.1 [C1], the check to verify the MONOCHROMATICITY is not easy at all. However, this problem is very interesting, thanks to Theorem 4, as in case of a DIVERTING-VALUES CORRELATION, we know that (see §3 considerations, above) not only, say, $C_{f,g_Q}(N,a)$ MAY NOT HAVE THE (R.E.E.F.), but also $\widetilde{C_{f,g_Q}}(N,a)$ HAS NO (R.E.E.F.). ◇

Our last remark regards the existence of a (RAMANUJAN EXPANSION) (without ensuring any decay feature), for all the Arithmetic Functions.

Remark 12. Once we have FIXED ANY arbitrary $F : \mathbb{N} \to \mathbb{C}$, we know, from a result of Hildebrand dated 1984, that THERE EXISTS A (RAMANUJAN EXPANSION) for F; actually, it is a FINITE Ramanujan Expansion (see [C2]), BUT NOT A FIXED-LENGTH ONE (see [C0]); for a complete exposition, with full details, of this very interesting result, see Appendix in [C3]. ◇

From these Remarks and the results in this paper, it is evident that IT'S WORTHY TO STUDY THE DIVERTING-VALUES CORRELATIONS <u>AND</u> THE $\mathcal{Q}-$MONOCHROMATIC ONES, that MAY HAVE A PARTICULAR STRUCTURE, USEFUL TO PROVE ASYMPTOTIC FORMULÆ.

This we will do in future papers.

Acknowledgment. I profit, here, to thank once again Ram Murty, starting from his survey [M] introducing me to the Ramanujan Expansions; and for the patience, in working with me [CM] on Finite Ramanujan Expansions applied to Correlations, namely shifted convolution sums. And this [CM], also, was an introduction for me, to the world of Ramanujan Expansions & Correlations !

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`Giovanni Coppola - Università degli Studi di Salerno (affiliation)`
`Home address : Via Partenio 12 - 83100, Avellino (AV) - ITALY`
`e-mail : giocop70@gmail.com`
`e-page : www.giovannicoppola.name`
`e-site : www.researchgate.net`