SPECTRAL TURÁN-TYPE PROBLEMS FOR THE α-SPECTRAL RADIUS OF HYPERGRAPHS WITH DEGREE STABILITY

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Abstract. An r-pattern P is defined as an ordered pair $P = ([l], E)$, where l is a positive integer and E is a set of r-multisets with elements from $[l]$. An r-graph H is said to be P-colorable if there is a homomorphism ϕ: $V(H) \to [l]$ such that the r-multiset $\{\phi(v_1), \ldots, \phi(v_r)\}$ is in E for every edge $\{v_1, \ldots, v_r\} \in E(H)$. Let $Col(P)$ denote the family of all P-colorable r-graphs. This paper establishes spectral extremal results for α-spectral radius of hypergraphs using analytic techniques. We show that for any family $\mathcal{F}$ of r-graphs that is degree-stable with respect to $Col(P)$, spectral Turán-type problems can be effectively reduced to spectral extremal problems within $Col(P)$. As an application, we determine the maximum α-spectral radius ($\alpha \geq 1$) among all n-vertex $F^{(r)}$-free r-graphs, where $F^{(r)}$ represents the r-expansion of the color critical graph F. We also characterize the corresponding extremal hypergraphs. Furthermore, leveraging the spectral method, we derive a corresponding edge Turán extremal result. More precisely, we show that if $\mathcal{F}$ is degree-stable with respect to $Col(P)$, then every $\mathcal{F}$-free edge extremal hypergraph must be a P-colorable hypergraph.

1. Introduction

A *hypergraph* H consists of a set of vertices $V(H)$ and a set of edges $E(H) \subseteq 2^{V(H)}$, where $2^{V(H)}$ is the power set of $V(H)$. The *order* and *size* of H are defined as $\nu(H) := |V(H)|$ and $e(H) := |E(H)|$, respectively. If every edge of H contains exactly r vertices, then H is called an r*-uniform* hypergraph (r-graph, for short). An ordinary graph is precisely a 2-uniform hypergraph. For two r-graphs G and H, we say G is a *subgraph* of H if $V(G) \subseteq V(H)$ and $E(G) \subseteq E(H)$. The subgraph G is *induced* by $V(G)$ if $E(G) = \{e \in E(H) : e \subseteq V(G)\}$. For any vertex $v \in V(H)$, we denote by $H - v$ the subgraph of H induced by $V(H)\backslash\{v\}$.

Let $\mathcal{F}$ be a family of r-graphs. An r-graph G is called $\mathcal{F}$*-free* if no member $F \in \mathcal{F}$ is a subgraph of G. The *Turán number* $ex(n, \mathcal{F})$ is the maximum size of an $\mathcal{F}$-free r-graph on n vertices. The classical Turán's Theorem [40] states that the balanced complete l-partite graph $T_l(n)$ (known as the Turán graph) is the unique extremal graph attaining the maximum size among all graphs of order n excluding the complete graph K_{l+1} as a subgraph. Turán problems constitute a central theme in extremal combinatorics; we refer to [20] for a comprehensive survey. In 2007, Nikiforov [30] proved a spectral analogue of Turán's Theorem, showing that the Turán

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graph is also the unique spectral extremal graph. Since then, the spectral Turán-type problems have attracted great attention; see [21] for recent developments.

In [28, 29], Nikiforov conducted a systematic study of the α-spectral radius of hypergraphs using analytical methods. The α-spectral radius of an r-graph G (see Section 2 for its definition), denoted by $\lambda^{(\alpha)}(G)$, unifies several fundamental concepts: the largest Z-eigenvalue, the largest H-eigenvalue (see [35]), the Lagrangian, and the size of G. Consequently, it provides a unified framework for addressing associated extremal problems. This parameter was first introduced by Keevash, Lenz and Mubayi [14], who established two key criteria for spectral extremal problems. Using one of these criteria, they determined the maximum α-spectral radius over all n-vertex 3-graphs that avoid the Fano plane. Nikiforov [28] further investigated several extremal problems of the following type:

Problem 1.1. *What is the maximum value of $\lambda^{(\alpha)}(G)$ over all r-graphs G satisfying a given property $\mathcal{P}$?*

Erdős, Ko and Rado proved that if G is a t-intersecting n-vertex r-graph, then $e(G) \leq \binom{n-t}{r-t}$, with equality holding if and only if $G = S^r_{t,n}$, where $S^r_{t,n}$ denotes the complete t-star of order n. They further established a stability result: there exists a constant $c = c(r,t) > 0$ such that if $e(G) > c\binom{n-t}{r-t-1}$, then $G \subset S^r_{t,n}$. Building on this stability result, Nikiforov [28] demonstrated that for sufficiently large n, $S^r_{t,n}$ is the unique extremal hypergraph achieving the maximum α-spectral radius among all t-intersecting n-vertex r-graphs. Although Keevash, Lenz and Mubayi [14] had previously obtained this result using another criterion, their approach involved a more complex framework. These foundational contributions have significantly advanced the study of extremal problems for the α-spectral radius. Subsequently, Kang, Nikiforov and Yuan [18] resolved several extremal problems concerning the α-spectral radius of n-vertex k-partite and k-chromatic hypergraphs, generalizing earlier work by Mubayi and Talbot [27]. Ni, Liu and Kang [31] determined the maximum α-spectral radius of all cancellative 3-graphs. Furthermore, Kang, Liu, Lu and Wang [17] investigated the α-spectral radius of Berge-G hypergraphs and determined the 3-graphs with maximum α-spectral radius for $\alpha \geq 1$ among Berge-G hypergraphs when G is a path, a cycle or a star.

Let H be an ordinary graph (i.e., a 2-graph). The *r-expansion* $H^{(r)}$ of H is the r-graph obtained from H by enlarging each edge of H with $r-2$ new vertices disjoint from $V(H)$ such that distinct edges of H are enlarged by distinct vertices. Let $F_{r,l}$ be the r-graph with edges: $\{1, 2, \ldots, r\} =: [r]$ and $E_{ij} \cup \{i, j\}$ over all pairs $\{i, j\} \in \binom{[l]}{2} \setminus \binom{[r]}{2}$, where E_{ij} are pairwise disjoint $(r-2)$-sets disjoint from $[l]$. Generalizing the Turán theorem to hypergraphs, Pikhurko [34] determined the exact Turán number of $K^{(r)}_{l+1}$, while Mubayi and Pikhurko [26] established the Turán number of $F_{r,l+1}$. When sufficiently large n and $\alpha \geq r \geq 2$, Liu et al. [25] proved that the balanced complete l-partite r-graph $T^r_l(n)$ on n vertices is the unique hypergraph maximizing the α-spectral radius among all n-vertex $K^{(r)}_{l+1}$-free (or $F_{r,l+1}$-free) r-graphs. This result generalizes the findings of Pikhurko and Mubayi.

Note that for an r-graph G, the quantity $\lambda^{(r)}(G)/(r-1)!$ is exactly its spectral radius, that is, the maximum modulus of all eigenvalues of the adjacency tensor of G. Due to the development of the spectral theory of nonnegative tensors, the spectral radius of hypergraphs can be understood relatively well, and much research has been conducted on extremal problems for the spectral radius of hypergraphs. Gao, Chang and Hou [11] studied the spectral extremal problem for $K_{r+1}^{(r)}$-free r-graphs among linear hypergraphs and showed that the spectral radius of an n-vertex $K_{r+1}^{(r)}$-free linear r-graph is no more than n/r when n is sufficiently large. She, Fan, Kang and Hou [39] generalized this result by establishing a sharp upper bound on the spectral radius of $F^{(r)}$-free linear r-graphs, where F is a color critical graph with chromatic number $k \geq r+1$. Ellingham, Lu and Wang [6] characterized the extremal hypergraph attaining maximum spectral radius among all outerplanar 3-graphs on n vertices by its 2-shadow graph. There exist numerous results on extremal characterizations of the spectral radius of hypergraphs; for more details, we refer readers to [3, 4, 9, 16, 17, 22, 32, 38, 42–46].

Recently, Zheng, Li and Fan [47] established a spectral stability result for nondegenerate hypergraphs, generalizing the Keevash–Lenz–Mubayi criterion [14, Theorem 1.4]. Their main result can be stated as follows:

Theorem 1.2 ([47]). *Let $r \geq 2$, $\alpha > 1$, $0 < \varepsilon < 1$, and $\mathcal{F}$ be a family of k-graphs with $\pi(\mathcal{F}) > 0$. Let $\mathcal{G}_n$ be the collection of all n-vertex $\mathcal{F}$-free r-graphs with minimum degree at least $(1-\varepsilon)\pi(\mathcal{F})\binom{n}{r-1}$, and let $\lambda^{(\alpha)}(\mathcal{G}_n) = \max\{\lambda^{(\alpha)}(G) : G \in \mathcal{G}_n\}$. Suppose that there exists N such that for all $n \geq N$,*

$$\lambda^{(\alpha)}(\mathcal{G}_n) \geq \lambda^{(\alpha)}(\mathcal{G}_{n-1}) + (r - r/\alpha)(1-\varepsilon')\pi(\mathcal{F})n^{r-r/\alpha-1}, \tag{1.1}$$

where $\varepsilon' = \varepsilon\pi(\mathcal{F})(\alpha-1)/(2r\alpha)$. Then there exists n_0 such that if H is an $\mathcal{F}$-free graph on $n \geq n_0$ vertices, then

$$\lambda^{(\alpha)}(H) \leq \lambda^{(\alpha)}(\mathcal{G}_n).$$

In addition, if the equality holds, then $H \in \mathcal{G}_n$.

By applying this Theorem 1.2, Zheng et al. determined the maximum α-spectral radius for some classes of hypergraphs and characterized the corresponding extremal hypergraphs (see Section 4 in [47]), which demonstrates the power of the spectral stability result. However, the condition (1.1) is generally difficult to verify. Motivated by this challenge, we seek to develop a simpler and more feasible method to address spectral Turán-type problems. The necessary definitions are presented below.

An *r-multiset* is a multiset of cardinality r. An *r-pattern* is an ordered pair $P = ([l], E)$ where l is a positive integer and E is a set of r-multisets with elements from $[l]$. Clearly, the r-pattern is a generalization of the r-graph. For convenience, we use E (or an r-graph G with $E(G) = E$) to represent the pattern P when the context is clear. Given an r-graph H and an r-pattern $P = ([l], E)$, a map ϕ: $V(H) \to [l]$ is a *homomorphism* from H to P if the r-multiset $\{\phi(v_1), \ldots, \phi(v_r)\}$ belongs to E for every edge $\{v_1, \ldots, v_r\} \in E(H)$. We say H is *P-colorable* if there is a homomorphism from H to P. For example, any l-partite r-graph is K_l^r-colorable, where K_l^r is the complete r-graph on l vertices. The notion of pattern was introduced by Pikhurko [33] in the study of hypergraph Turán density and has also been utilized in [13, 24].

This paper aims to develop efficient methods for solving extremal problems concerning α-spectral radius of hypergraphs. We begin with a systematic review of the necessary terminology, notation, and related results. In Section 3, we prove that spectral extremal hypergraphs exhibit asymptotic regularity within any family of pattern-colorable hypergraphs (see Theorem 3.10). Consequently, spectral extremal k-chromatic r-graphs are asymptotically balanced (see Theorem 3.12), which provides insight into Kang-Nikiforov-Yuan's conjecture [18]. In Section 4, we establish that when a family $\mathcal{F}$ of r-graphs is degree-stable with respect to a family of P-colorable r-graphs for an r-pattern P, every $\mathcal{F}$-free r-graph attaining the maximum α-spectral radius must be P-colorable (see Theorem 4.2). Moreover, for $P = K_l^r$, $\alpha \geq 1$, and sufficiently large n, we deduce that the α-spectral radius of any $\mathcal{F}$-free n-vertex r-graph G is at most that of $T_l^r(n)$. From this we resolve the spectral Turán-type problem for the r-expansion $F^{(r)}$ of a color-critical graph F (see Theorem 4.8). In the final section, we solve edge extremal problems using spectral methods, analyze relations between edge extremal hypergraphs and spectral extremal hypergraphs, and pose an open question.

2. Preliminaries

A property of r-graphs is a family of r-graphs closed under isomorphisms. For a property $\mathcal{P}$, denoted by $\mathcal{P}_n$ the collection of r-graphs in $\mathcal{P}$ of order n. A property is called *monotone* if it is closed under taking subgraphs, and *hereditary* if it is closed under taking induced subgraphs. Given a family $\mathcal{F}$ of r-graphs, the class of all $\mathcal{F}$-free r-graphs constitutes a monotone property, denoted by $Mon(\mathcal{F})$. Similarly, the class of all r-graphs that do not contain any $F \in \mathcal{F}$ as an induced subgraph forms a hereditary property. Evidently, every monotone property is hereditary. Throughout our discussion, we assume that any hereditary property $\mathcal{P}$ of r-graphs satisfies the following conditions:

(a) There exists an r-graph $G \in \mathcal{P}$ that contains at least one edge.
(b) For any $H \in \mathcal{P}$, the disjoint union of H and an isolated vertex belongs to $\mathcal{P}$.

A central topic in extremal hypergraph theory is the following: Given a hereditary property $\mathcal{P}$ of r-graphs, determine

$$ex(\mathcal{P}, n) := \max_{G \in \mathcal{P}_n} e(G).$$

By the Katona-Nemetz-Simonovits averaging argument [15], the ratio $ex(\mathcal{P}, n)/\binom{n}{r}$ is non-increasing in n. Consequently, the limit

$$\pi(\mathcal{P}) := \lim_{n \to \infty} \frac{ex(\mathcal{P}, n)}{\binom{n}{r}} \tag{2.1}$$

exists and is called the *edge density* of $\mathcal{P}$. Thus, determining the asymptotic behavior of $ex(\mathcal{P}, n)$ is equivalent to finding $\pi(\mathcal{P})$. When $\mathcal{P} = Mon(\mathcal{F})$ for a family $\mathcal{F}$ of r-graphs, $ex(\mathcal{P}, n)$ is known as the Turán number of $\mathcal{F}$, and $\pi(Mon(\mathcal{F}))$ corresponds to the well-studied *Turán density* of $\mathcal{F}$.

For an r-graph G on n vertices and $\alpha \geq 1$, the *Lagrangian polynomial* $P_G(\mathbf{x})$ of G is defined as

$$P_G(\mathbf{x}) = r! \sum_{\{i_1,\dots,i_r\}\in E(G)} x_{i_1}\cdots x_{i_r},$$

and the *α-spectral radius* of G is defined as

$$\lambda^{(\alpha)}(G) = \max_{\|\mathbf{x}\|_\alpha=1} P_G(\mathbf{x}),$$

where $\mathbf{x} = (x_1,\dots,x_n) \in \mathbb{R}^n$ and $\|\mathbf{x}\|_\alpha := (|x_1|^\alpha + \cdots + |x_n|^\alpha)^{1/\alpha}$. If $\mathbf{x} \in \mathbb{R}^n$ is a vector such that $\|\mathbf{x}\|_\alpha = 1$ and $\lambda^{(\alpha)}(G) = P_G(\mathbf{x})$, then $\mathbf{x}$ is called an *eigenvector* of G corresponding to $\lambda^{(\alpha)}(G)$. Clearly, there always exists a nonnegative eigenvector corresponding to $\lambda^{(\alpha)}(G)$, called a *principal eigenvector* of G. Moreover, if a principal eigenvector $\mathbf{x}$ is strictly positive (i.e., $x_v > 0$ for all $v \in V(G)$), then we call it a *Perron-Frobenius eigenvector* of G.

The classical Perron-Frobenius theory for $\alpha = r = 2$ guarantees that the principal eigenvector $\mathbf{x}$ of any connected 2-graph H is also its Perron-Frobenius eigenvector. By extending this to r-graphs G, Nikiforov [28] established a comprehensive Perron-Frobenius theory for $\lambda^{(\alpha)}(G)$. Specifically, Nikiforov [28, Theorem 5.10] demonstrated that this holds true if $\alpha \geq r \geq 2$. However, for $1 < \alpha < r$, it is shown by [28, Proposition 4.3] that G may not have Perron-Frobenius eigenvector. Let $\mathbb{PE}_\alpha(G)$ denote the set of all principal eigenvectors for $\lambda^{(\alpha)}(G)$. Define

$$x^{(\alpha)}_{\min}(G) := \inf_{\mathbf{x}\in\mathbb{PE}_\alpha(G)} \mathbf{x}_{\min},$$

where $\mathbf{x}_{\min}$ represents the minimum component of the vector $\mathbf{x}$. We simply write $x_{\min}(G)$ instead of $x^{(\alpha)}_{\min}(G)$ when no confusion occurs.

For $\alpha > 1$, the principal eigenvector $\mathbf{x} = (x_1,\dots,x_n)$ of an r-graph G satisfies the following equations derived from Lagrange's method:

$$\lambda^{(\alpha)}(G)x_i^{\alpha-1} = (r-1)! \sum_{\{i,i_2,\dots,i_r\}\in E(G)} x_{i_2}\cdots x_{i_r}, \text{ for } 1 \leq i \leq n. \tag{2.2}$$

Lemma 2.1 ([28,29])**.** *If $\alpha \geq 1$ and G is an r-graph, then $\lambda^{(\alpha)}(G)$ is an increasing and continuous function in α. Moreover,*

$$\lim_{\alpha\to\infty} \lambda^{(\alpha)}(G) = r!e(G).$$

For a hereditary property $\mathcal{P}$ of r-graphs, define

$$\lambda^{(\alpha)}(\mathcal{P}, n) := \max_{G\in\mathcal{P}_n} \lambda^{(\alpha)}(G).$$

By analogy with the edge density in (2.1), Nikiforov formulated the spectral analogue as follows:

Lemma 2.2 ([28, 29])**.** *Let $\alpha \geq 1$. If $\mathcal{P}$ is a hereditary property of r-graphs, then the limit*

$$\lambda^{(\alpha)}(\mathcal{P}) := \lim_{n\to\infty} \lambda^{(\alpha)}(\mathcal{P}, n)n^{r/\alpha - r}$$

exists. If $\alpha = 1$*, then* $\lambda^{(1)}(\mathcal{P}, n)$ *is increasing, and so*

$$\lambda^{(1)}(\mathcal{P}, n) \leq \lambda^{(1)}(\mathcal{P}).$$

If $\alpha > 1$*, then* $\lambda^{(\alpha)}(\mathcal{P})$ *satisfies*

$$\lambda^{(\alpha)}(\mathcal{P}) \leq \frac{\lambda^{(\alpha)}(\mathcal{P}, n) n^{r/\alpha}}{(n)_r},$$

where $(n)_r = n(n-1)\cdots(n-r+1)$*.*

Remark.

- For any hereditary property $\mathcal{P}$, assumption (a) implies that $\lambda^{(1)}(\mathcal{P}) \geq \lambda^{(1)}(K_r^r) > 0$.
- Consider the family of r-graphs $\mathcal{R} = \{K_r^r\} \cup \{G : E(G) = \emptyset\}$. Although $\mathcal{R}$ is hereditary, for $n > r$, we have $\lambda^{(1)}(\mathcal{R}, r) = \lambda^{(1)}(K_r^r) > \lambda^{(1)}(\mathcal{R}, n) = 0$. Therefore, assumption (b) ensures that $\lambda^{(1)}(\mathcal{P}, n)$ is increasing in n.

For $G \in \mathcal{P}_n$ with $e(G) = ex(\mathcal{P}, n)$, the n-vector $\mathbf{x} = (n^{-1/\alpha}, \ldots, n^{-1/\alpha})$ yields

$$\lambda^{(\alpha)}(G) \geq P_G(\mathbf{x}) = r! e(G)/n^{r/\alpha} = r! ex(\mathcal{P}, n)/n^{r/\alpha}. \tag{2.3}$$

Thus

$$\lambda^{(\alpha)}(\mathcal{P}, n) \geq \lambda^{(\alpha)}(G) \geq r! ex(\mathcal{P}, n)/n^{r/\alpha},$$

which implies

$$\frac{\lambda^{(\alpha)}(\mathcal{P}, n) n^{r/\alpha}}{(n)_r} \geq \frac{ex(\mathcal{P}, n)}{\binom{n}{r}}.$$

Taking $n \to \infty$ and applying Lemma 2.2, we obtain for $\alpha \geq 1$,

$$\lambda^{(\alpha)}(\mathcal{P}) \geq \pi(\mathcal{P}). \tag{2.4}$$

The following theorem establishes that equality in (2.4) necessarily holds for all $\alpha > 1$, demonstrating an asymptotic equivalence between the extremal problem $ex(\mathcal{P}, n)$ and the spectral extremal problem $\lambda^{(\alpha)}(\mathcal{P}, n)$.

Lemma 2.3 ([28, 29])**.** *If* $\mathcal{P}$ *is a hereditary property of* r*-graphs, then for every* $\alpha > 1$*,*

$$\lambda^{(\alpha)}(\mathcal{P}) = \pi(\mathcal{P}).$$

Definition 2.4 (Blow-up hypergraph)**.** Let H be an r-graph on n vertices and $\mathbf{t} = (k_1, \ldots, k_n)$ be a vector of positive integers. The blow-up of H with respect to $\mathbf{t}$, denoted by $H(\mathbf{t})$, is the r-graph obtained by replacing each vertex $i \in V(H)$ with a vertex class V_i of size k_i, and if $\{i_1, \ldots, i_r\} \in E(H)$, then $\{i_{1,j_1}, \ldots, i_{r,j_r}\} \in E(H(\mathbf{t}))$ for every $i_{1,j_1} \in V_{i_1}, \ldots, i_{r,j_r} \in V_{i_r}$.

Definition 2.5 (Multiplicative property)**.** A graph property $\mathcal{M}$ is *multiplicative* if $G \in \mathcal{M}_n$ implies that $G(\mathbf{t}) \in \mathcal{M}$ for every positive vector $\mathbf{t} \in \mathbb{Z}^n$ (equivalently, $\mathcal{M}$ is closed under blow-up).

Definition 2.6 (Flat property)**.** A hereditary property $\mathcal{P}$ is *flat* if $\lambda^{(1)}(\mathcal{P}) = \pi(\mathcal{P}) > 0$ (where $\pi(\mathcal{P}) > 0$ follows from the Remark after Lemma 2.2).

Lemma 2.7 ([28, 29]). *If $\mathcal{P}$ is a hereditary, multiplicative property, then it is flat; that is to say, $\lambda^{(\alpha)}(\mathcal{P}) = \pi(\mathcal{P})$ for every $\alpha \geq 1$.*

The flat property yields significant advantages in extremal hypergraph problems, as shown by the following results.

Lemma 2.8 ([28, 29]). *If $\mathcal{P}$ is a flat property of r-graphs, and $G \in \mathcal{P}$, then for every $\alpha \geq 1$,*

$$\lambda^{(\alpha)}(G) \leq \pi(\mathcal{P})^{1/\alpha}(r!e(G))^{1-1/\alpha}.$$

Theorem 2.9 ([28, 29]). *If $\mathcal{P}$ is a flat property of r-graphs, and $G \in \mathcal{P}_n$, then*

$$e(G) \leq \pi(\mathcal{P})n^r/r!,$$

and for every $\alpha \geq 1$,

$$\lambda^{(\alpha)}(G) \leq \pi(\mathcal{P})n^{r-r/\alpha}.$$

Remark. It is worth mentioning that Theorem 2.9 first appeared in [29, Theorem 16]. However, its proof is based on a stronger condition that $\mathcal{P}$ is both hereditary and multiplicative property (see [29, p. 27]). We therefore provide a complete proof in Appendix.

3. Asymptotic regularity of spectral extremal hypergraphs

In this section, we establish a notable property of spectral extremal hypergraphs in pattern-colorable r-graphs. Specifically, given an r-pattern P, if G is a P-colorable r-graph of order n attaining the maximum α-spectral radius for $\alpha > 1$, then G becomes almost regular as $n \to \infty$. To prove this, we first consider a broader family of r-graphs with hereditary property.

3.1. Spectral extremal hypergraphs of hereditary property.

Nikiforov derived the following result through an ingenious and concise method (see Claim A in the proof of [29, Theorem 12]).

Lemma 3.1 ([29]). *Let $\mathcal{P}$ be a hereditary property of r-graphs with $\pi(\mathcal{P}) > 0$. If $\alpha > 1$ and $\lambda_n = \lambda^{(\alpha)}(\mathcal{P}, n)$, then there exist infinitely many n such that*

$$\frac{\lambda_{n-1}(n-1)^{r/\alpha-1}}{(n-2)_{r-1}} - \frac{\lambda_n n^{r/\alpha-1}}{(n-1)_{r-1}} < \frac{1}{n\log n} \cdot \frac{\lambda_n n^{r/\alpha-1}}{(n-1)_{r-1}}.$$

Let $G \in \mathcal{P}_n$ be an r-graph satisfying $\lambda^{(\alpha)}(G) = \lambda^{(\alpha)}(\mathcal{P}, n)$. Applying Lemma 3.1, Nikiforov [29] showed that for $1 < \alpha \leq r$ and n sufficiently large, there exist infinitely many n such that

$$(x_{\min}(G))^\alpha \geq \frac{1}{n}\Big(1 - \frac{1}{(\alpha-1)\log n}\Big).$$

However, a similar argument extends this result to any $\alpha > 1$. We require the following elementary fact before presenting Lemma 3.2.

Fact 1. If $\alpha > 1$ and $r \geq 2$, then the function

$$f(x) = \frac{1 - rx}{(1-x)^{r/\alpha}}$$

is decreasing for $0 \leq x < 1$.

Proof. Computing the derivative of $f(x)$, we obtain

$$\begin{aligned} f'(x) &= \frac{-r(1-x)^{r/\alpha} + (r/\alpha)(1-rx)(1-x)^{r/\alpha-1}}{(1-x)^{2r/\alpha}} \\ &= \frac{r(1-x)^{r/\alpha-1}}{\alpha(1-x)^{2r/\alpha}}\big(-(\alpha-1)-(r-\alpha)x\big). \end{aligned}$$

If $1 < \alpha \leq r$, then $f'(x) < 0$ is immediate. For $\alpha > r$, observe that

$$-(\alpha-1)-(r-\alpha)x < -(\alpha-1)-(r-\alpha) = 1-r < 0$$

since $x < 1$ and $r \geq 2$. Thus, $f'(x) < 0$ holds in all cases. □

For an r-graph H and a vertex $v \in V(H)$, let $E_H(v)$ denote the set of edges in H containing v, and let $d_H(v) = |E_H(v)|$ be the degree of v. The minimum degree of H is denoted by $\delta(H)$. For a vertex subset $U \subseteq V(H)$, we write $x_U = \Pi_{v\in U} x_v$.

Lemma 3.2. *Let $\mathcal{P}$ be a hereditary property of r-graphs with $\pi(\mathcal{P}) > 0$. Suppose that $G_n \in \mathcal{P}_n$ is an r-graph satisfying $\lambda^{(\alpha)}(G_n) = \lambda^{(\alpha)}(\mathcal{P}, n)$ for $\alpha > 1$. Then there exist infinitely many n such that*

$$(x_{\min}(G_n))^\alpha \geq \frac{1}{n}\Big(1 - \frac{\alpha}{(\alpha-1)r\log n}\Big).$$

Proof. Assume, for contradiction, that there exists n_0 such that for all $n > n_0$,

$$(x_{\min}(G_n))^\alpha < \frac{1}{n}\Big(1 - \frac{\alpha}{(\alpha-1)r\log n}\Big).$$

Set $\lambda_n := \lambda^{(\alpha)}(\mathcal{P}, n)$. By Lemma 3.1, we can select $n > n_0$ sufficiently large so that

$$\frac{\lambda_{n-1}(n-1)^{r/\alpha-1}}{(n-2)_{r-1}} - \frac{\lambda_n n^{r/\alpha-1}}{(n-1)_{r-1}} < \frac{1}{n\log n}\cdot\frac{\lambda_n n^{r/\alpha-1}}{(n-1)_{r-1}},$$

which implies

$$\frac{\lambda_{n-1}}{\lambda_n} < \frac{n^{r/\alpha-1}(n-r)}{(n-1)^{r/\alpha}}\Big(1 + \frac{1}{n\log n}\Big). \tag{3.1}$$

By the definition of $x_{\min}(G_n)$, there exists a principal eigenvector $\mathbf{x}$ of G_n such that

$$(\mathbf{x}_{\min})^\alpha < \frac{1}{n}\Big(1 - \frac{\alpha}{(\alpha-1)r\log n}\Big). \tag{3.2}$$

Let $k \in V(G_n)$ be a vertex with $x_k = \mathbf{x}_{\min}$, and let $\mathbf{x}'$ be the $(n-1)$-vector obtained from $\mathbf{x}$ by removing the component x_k. For the subgraph $G_n - k$, we have

$$P_{G_n-k}(\mathbf{x}') = \lambda^{(\alpha)}(G_n) - r!x_k \sum_{e\in E_{G_n}(k)} x_{e\setminus\{k\}} = \lambda_n - r\lambda_n x_k^\alpha.$$

Since $\mathcal{P}$ is hereditary, $G_n - k \in \mathcal{P}_{n-1}$. Therefore,

$$\lambda_n(1 - rx_k^\alpha) = P_{G_n - k}(\mathbf{x}') \le \lambda^{(\alpha)}(G_n - k)(\|\mathbf{x}'\|_\alpha)^r \le \lambda_{n-1}(1 - x_k^\alpha)^{r/\alpha},$$

or equivalently,

$$\frac{\lambda_{n-1}}{\lambda_n} \ge \frac{1 - rx_k^\alpha}{(1 - x_k^\alpha)^{r/\alpha}}. \tag{3.3}$$

Combining (3.1) and (3.3) yields

$$\frac{1 - rx_k^\alpha}{(1 - x_k^\alpha)^{r/\alpha}} \le \frac{\lambda_{n-1}}{\lambda_n} \le \frac{n^{r/\alpha - 1}(n-r)}{(n-1)^{r/\alpha}}\Big(1 + \frac{1}{n\log n}\Big).$$

Applying Fact 1 and incorporating (3.2), we derive

$$\frac{1 - \frac{r}{n}(1 - \frac{\alpha}{(\alpha-1)r\log n})}{\Big(1 - \frac{1}{n}(1 - \frac{\alpha}{(\alpha-1)r\log n})\Big)^{r/\alpha}} \le \frac{1 - rx_k^\alpha}{(1 - x_k^\alpha)^{r/\alpha}} \le \frac{n^{r/\alpha - 1}(n-r)}{(n-1)^{r/\alpha}}\Big(1 + \frac{1}{n\log n}\Big),$$

and then

$$\frac{\Big(n - r + \frac{\alpha}{(\alpha-1)\log n}\Big)n^{r/\alpha - 1}}{\Big(n - 1 + \frac{\alpha}{(\alpha-1)r\log n}\Big)^{r/\alpha}} \le \frac{1 - rx_k^\alpha}{(1 - x_k^\alpha)^{r/\alpha}} \le \frac{n^{r/\alpha - 1}(n-r)}{(n-1)^{r/\alpha}}\Big(1 + \frac{1}{n\log n}\Big).$$

Further simplification leads to

$$1 + \frac{\alpha}{(\alpha-1)(n-r)\log n} \le \Big(1 + \frac{\alpha}{(\alpha-1)r(n-1)\log n}\Big)^{r/\alpha}\Big(1 + \frac{1}{n\log n}\Big).$$

For sufficiently large n, we have

$$\begin{aligned}\Big(1 + \frac{\alpha}{(\alpha-1)r(n-1)\log n}\Big)^{r/\alpha} &= 1 + \frac{1}{(\alpha-1)(n-1)\log n} + O\Big(\frac{1}{(n\log n)^2}\Big) \\ &\le 1 + \frac{1}{(\alpha-1)(n-1)\log n} + \frac{1}{(\alpha-1)(n-1)(n-2)\log n} \\ &= 1 + \frac{1}{(\alpha-1)(n-2)\log n}.\end{aligned}$$

Substituting this bound, we obtain

$$1 + \frac{\alpha}{(\alpha-1)(n-r)\log n} \le \Big(1 + \frac{1}{(\alpha-1)(n-2)\log n}\Big)\Big(1 + \frac{1}{n\log n}\Big).$$

By some cancellations and rearranging, we get

$$\frac{\alpha}{n-r} \le \frac{1}{n-2} + \frac{\alpha-1}{n} + \frac{1}{n(n-2)\log n}.$$

Noting that $\frac{\alpha}{n-r} \ge \frac{\alpha}{n-2}$, this inequality reduces to

$$2(\alpha - 1) \le \frac{1}{\log n},$$

which is impossible for sufficiently large n. □

Furthermore, we show that if the minimum component of a principal eigenvector of the spectral extremal hypergraph approaches $n^{-1/\alpha}$, then its minimum degree approximates its average degree, implying that the spectral extremal hypergraph is asymptotically regular.

Lemma 3.3. *Let $\mathcal{P}$ be a hereditary property of r-graphs with $\pi(\mathcal{P}) > 0$, and $G_n \in \mathcal{P}_n$ satisfy $\lambda^{(\alpha)}(G_n) = \lambda^{(\alpha)}(\mathcal{P}, n)$ for $\alpha > 1$. Let $0 < \varepsilon < 1$, $0 \le \varepsilon' < \varepsilon\pi(\mathcal{P})/(r-1)$, and $\mathbf{x} \in \mathbb{R}^n$ be a principal eigenvector of G_n. If n is sufficiently large and*

$$(\mathbf{x}_{\min})^\alpha \ge \frac{1}{n}(1-\varepsilon'),$$

then

$$\delta(G_n) \ge (1-\varepsilon)\pi(\mathcal{P})\binom{n}{r-1}.$$

Proof. Set $V = V(G_n)$, $\lambda = \lambda^{(\alpha)}(G_n)$, $\delta = \delta(G_n)$, $\mu_n = \mathbf{x}_{\min}$, and let $k \in V$ be a vertex attaining minimum degree δ. Considering the eigenequation for $\lambda^{(\alpha)}(G_n)$ at vertex k:

$$\lambda\mu_n^{\alpha-1} \le \lambda x_k^{\alpha-1} = (r-1)! \sum_{e \in E_{G_n}(k)} x_{e\setminus\{k\}}.$$

By Hölder's inequality, we have

$$\left(\frac{\lambda\mu_n^{\alpha-1}}{(r-1)!}\right)^\alpha \le \delta^{\alpha-1} \sum_{e \in E_{G_n}(k)} x^\alpha_{e\setminus\{k\}}. \tag{3.4}$$

First, observe that

$$\begin{aligned}\sum_{e \in E_{G_n}(k)} x^\alpha_{e\setminus\{k\}} &= \sum_{S \in \binom{V}{r-1}} x_S^\alpha - \sum_{T \in \binom{V}{r-1} \text{ and } T\cup\{k\} \notin E_{G_n}(k)} x_T^\alpha \\ &\le \sum_{S \in \binom{V}{r-1}} x_S^\alpha - \sum_{T \in \binom{V}{r-1} \text{ and } T\cup\{k\} \notin E_{G_n}(k)} \mu_n^{\alpha(r-1)} \\ &= \sum_{S \in \binom{V}{r-1}} x_S^\alpha - \left(\binom{n}{r-1} - \delta\right)\mu_n^{\alpha(r-1)}.\end{aligned} \tag{3.5}$$

By Maclaurin's inequality, we have

$$\sum_{S \in \binom{V}{r-1}} x_S^\alpha \le \binom{n}{r-1}\left(n^{-1}\sum_{i \in V} x_i^\alpha\right)^{r-1} = \frac{\binom{n}{r-1}}{n^{r-1}}. \tag{3.6}$$

Suppose for contradiction that $\delta(G_n) < (1-\varepsilon)\pi(\mathcal{P})\binom{n}{r-1}$. Combining (3.5) and (3.6), we obtain

$$\begin{aligned}\sum_{e \in E_{G_n}(k)} x^\alpha_{e\setminus\{k\}} &\le \frac{\binom{n}{r-1}}{n^{r-1}} - \big(1-(1-\varepsilon)\pi(\mathcal{P})\big)\binom{n}{r-1}\mu_n^{\alpha(r-1)} \\ &\le \frac{\binom{n}{r-1}}{n^{r-1}} - \big(1-(1-\varepsilon)\pi(\mathcal{P})\big)\binom{n}{r-1}\frac{(1-\varepsilon')^{r-1}}{n^{r-1}} \\ &= \frac{\binom{n}{r-1}}{n^{r-1}}\Big(1-\big(1-(1-\varepsilon)\pi(\mathcal{P})\big)\big(1-\varepsilon'\big)^{r-1}\Big).\end{aligned} \tag{3.7}$$

By Bernoulli's inequality and the definition of ε', we have

$$(1-\varepsilon')^{r-1} \geq 1-(r-1)\varepsilon' \geq 1-\varepsilon\pi(\mathcal{P}).$$

Therefore,

$$\begin{aligned}\sum_{e\in E_{G_n}(k)} x^{\alpha}_{e\setminus\{k\}} &\leq \frac{\binom{n}{r-1}}{n^{r-1}}\Big(1-\big(1-(1-\varepsilon)\pi(\mathcal{P})\big)\big(1-\varepsilon\pi(\mathcal{P})\big)\Big)\\ &= \frac{\binom{n}{r-1}}{n^{r-1}}\Big(\pi(\mathcal{P})-\varepsilon(1-\varepsilon)\pi(\mathcal{P})^2\Big)\\ &\leq \frac{\pi(\mathcal{P})\binom{n}{r-1}}{n^{r-1}}.\end{aligned} \tag{3.8}$$

On the other hand, by Lemmas 2.2 and 2.3, for sufficiently large n,

$$\lambda \geq (1-r^2/n)\lambda^{(\alpha)}(\mathcal{P})n^{r(\alpha-1)/\alpha} = (1-r^2/n)\pi(\mathcal{P})n^{r(\alpha-1)/\alpha}.$$

Combining this with (3.4) and (3.8), we obtain

$$\frac{(1-r^2/n)^{\alpha}\pi(\mathcal{P})^{\alpha}n^{r(\alpha-1)}}{((r-1)!)^{\alpha}}\mu_n^{\alpha(\alpha-1)} \leq \left(\frac{\lambda\mu_n^{\alpha-1}}{(r-1)!}\right)^{\alpha} \leq \frac{\pi(\mathcal{P})\delta^{\alpha-1}\binom{n}{r-1}}{n^{r-1}}.$$

Since $\mu_n^{\alpha} \geq (1-\varepsilon')/n$ and $\delta < (1-\varepsilon)\pi(\mathcal{P})\binom{n}{r-1}$, we arrive at

$$\frac{(1-\varepsilon')^{\alpha-1}(1-r^2/n)^{\alpha}\pi(\mathcal{P})^{\alpha}n^{(r-1)(\alpha-1)}}{((r-1)!)^{\alpha}} \leq \frac{(1-\varepsilon)^{\alpha-1}\pi(\mathcal{P})^{\alpha}((n)_{r-1})^{\alpha}}{((r-1)!)^{\alpha}n^{r-1}}.$$

After some eliminations and scalings, we have

$$(1-\varepsilon')^{\alpha-1}(1-r^2/n)^{\alpha} \leq (1-\varepsilon)^{\alpha-1},$$

that is

$$1-\frac{r^2}{n} \leq \Big(\frac{1-\varepsilon}{1-\varepsilon'}\Big)^{\frac{\alpha-1}{\alpha}}. \tag{3.9}$$

Thus,

$$0 < 1-\Big(\frac{1-\varepsilon}{1-\varepsilon'}\Big)^{\frac{\alpha-1}{\alpha}} \leq \frac{r^2}{n},$$

which is a contradiction. This completes the proof of Lemma 3.3. □

3.2. **Properties of spectral extremal hypergraphs in pattern-colorable hypergraphs.** Building upon properties of spectral extremal hypergraphs for the hereditary property, we establish lower bounds on the minimum component of the principal eigenvector and the minimum degree of spectral extremal hypergraphs in pattern-colorable hypergraphs. The following concepts and results are fundamental.

Definition 3.4 (Clonal family [4])**.** For an r-graph G and vertices $u,v \in V(G)$, let $G_{u\to v}$ denote the r-graph obtained by first deleting all edges incident to u, and then adding, for every edge $e\in E(G)$ such that $v\in e$ and $u\notin e$, the edge $e+u-v$. A family of r-graphs $\mathcal{F}$ is called *clonal* if $G_{u\to v}\in\mathcal{F}$ for all $G\in\mathcal{F}$ and $u,v\in V(G)$.

Definition 3.5 (Principal ratio)**.** For $\alpha > 1$, and a principal eigenvector $\mathbf{x}$ of an r-graph G, the *principal ratio* of $(G, \mathbf{x})$ is

$$\gamma(G, \mathbf{x}) := \frac{\mathbf{x}_{\max}}{\mathbf{x}_{\min}},$$

where $\mathbf{x}_{\max}$ is the maximum component of the vector $\mathbf{x}$. If $\mathbf{x}$ has a zero component, then we set $\gamma(G, \mathbf{x}) = \infty$.

In [4], Cooper, Desai, and Sahay studied the principal eigenvector of extremal graphs for the clonal family of r-graphs, and showed that the principal ratio is close to 1.

Lemma 3.6 ([4])**.** *Let $\alpha > 1$ and $\mathcal{F}$ be a clonal family of r-graphs with $\pi(\mathcal{F}) > 0$. Suppose that $G \in \mathcal{F}_n$ such that $\lambda^{(\alpha)}(G) = \lambda^{(\alpha)}(\mathcal{F}, n)$. If $\mathbf{x}$ is a Perron-Frobenius eigenvector of G, then*

$$\gamma(G, \mathbf{x}) = 1 + O(n^{-1}).$$

Remark. The definitions of parameters $\pi(\mathcal{F})$, $\mathcal{F}_n$ and $\lambda^{(\alpha)}(\mathcal{F}, n)$ for the clonal (or general) family of r-graphs are the same as those for the hereditary property of r-graphs.

Lemma 3.7 ([18])**.** *Let G be an r-graph of order n with at least one edge, and let u and v be vertices of G such that transposing u and v is an automorphism. If $\alpha > 1$, and $\mathbf{x}$ is an eigenvector corresponding to $\lambda^{(\alpha)}(G)$, then $x_u = x_v$.*

Lemma 3.8 ([18])**.** *Let $\alpha \geq 1$, and let G be an r-graph such that every nonnegative eigenvector corresponding to $\lambda^{(\alpha)}(G)$ is positive. If H is a subgraph of G, then $\lambda^{(\alpha)}(H) < \lambda^{(\alpha)}(G)$, unless $H = G$.*

Let P be an r-pattern. Denote by $Col(P)$ the family of all P-colorable r-graphs. It is not hard to show that $Col(P)$ is hereditary and multiplicative (see Fact A in Appendix). Lemma 2.7 implies that $Col(P)$ is flat, and hence $\pi(Col(P)) > 0$. Some spectral extremal problems for hypergraph classes can be reduced to the corresponding problems in $Col(P)$ for suitable P (cf. Theorem 3.12). To characterize the properties of the spectral extremal hypergraphs in $Col(P)$, we require the following lemma.

Lemma 3.9. *Let $\alpha > 1$, $r \geq 2$ and $1 \leq i \leq r$. If m is sufficiently large, then*

$$\binom{m+1}{i}\Big(\frac{m}{m+1}\Big)^{i/\alpha} - \binom{m}{i} \geq \binom{m-1}{i-1}\Big(1 - \frac{1}{\alpha} - \frac{1}{\alpha(m-r+1)}\Big).$$

Proof. For enough large m, we have

$$\begin{aligned}
\Big(\frac{m}{m+1}\Big)^{i/\alpha} &= \Big(1 - \frac{1}{m+1}\Big)^{i/\alpha} \\
&= 1 - \frac{i}{\alpha(m+1)} + \frac{\binom{i/\alpha}{2}}{(m+1)^2} + o\big(m^{-2}\big) \\
&= 1 - \frac{i}{\alpha m} + \frac{i}{\alpha m(m+1)} + \frac{\binom{i/\alpha}{2}}{(m+1)^2} + o\big(m^{-2}\big).
\end{aligned}$$

Since $\frac{i}{\alpha}+\binom{i/\alpha}{2}>0$, it follows that

$$\frac{i}{\alpha m(m+1)}+\frac{\binom{i/\alpha}{2}}{(m+1)^2}+o\big(m^{-2}\big)>0.$$

Therefore,

$$\Big(\frac{m}{m+1}\Big)^{i/\alpha}>1-\frac{i}{\alpha m}. \tag{3.10}$$

On the other hand, we have

$$\begin{aligned}\binom{m+1}{i}\Big(1-\frac{i}{\alpha m}\Big)-\binom{m}{i}&=\binom{m}{i-1}-\frac{m+1}{\alpha(m-i+1)}\binom{m-1}{i-1}\\&=\frac{m}{m-i+1}\binom{m-1}{i-1}-\frac{m+1}{\alpha(m-i+1)}\binom{m-1}{i-1}\\&=\binom{m-1}{i-1}\Big(1-\frac{1}{\alpha}+\frac{\alpha(i-1)-i}{\alpha(m-i+1)}\Big)\\&\geq\binom{m-1}{i-1}\Big(1-\frac{1}{\alpha}-\frac{1}{\alpha(m-i+1)}\Big)\\&\geq\binom{m-1}{i-1}\Big(1-\frac{1}{\alpha}-\frac{1}{\alpha(m-r+1)}\Big).\end{aligned} \tag{3.11}$$

The conclusion follows immediately from (3.10) and (3.11). □

Theorem 3.10. *Let $P=([l],E)$ be an r-pattern, and $G_n\in Col(P)_n$ satisfy $\lambda^{(\alpha)}(G_n)=\lambda^{(\alpha)}(Col(P),n)$. If $\alpha>1$, then there exist $n_1\in\mathbb{N}$ and a constant $M>0$ such that for any $n\geq n_1$, the following hold:*

(1) $\lambda^{(\alpha)}(G_{n+1})\geq\big(1+r\big(1-\frac{1}{\alpha}-\frac{l}{\alpha(n-lr+l)}\big)(x_{\min}(G_n))^\alpha\big)\lambda^{(\alpha)}(G_n)$.

(2) $(x_{\min}(G_n))^\alpha\geq\frac{1}{n}\big(1-\frac{\pi(Col(P))M}{rn}\big)$.

(3) $\delta(G_n)\geq\pi(Col(P))\left(1-\frac{M}{n}\right)\binom{n}{r-1}$.

Proof. (1) For any given $n\in\mathbb{N}$, select $G_n\in Col(P)_n$ satisfying $\lambda^{(\alpha)}(G_n)=\lambda^{(\alpha)}(Col(P),n)$. Clearly, there exists an edge-maximal r-graph $H_n\in Col(P)_n$ such that $G_n\subseteq H_n$. Observe that $\lambda^{(\alpha)}(G_n)\leq\lambda^{(\alpha)}(H_n)$, and thus $\lambda^{(\alpha)}(G_n)=\lambda^{(\alpha)}(H_n)$. Let ϕ be a homomorphism from H_n to P. For any $i\in[l]$, and vertices u, $v\in\phi^{-1}(i)$, if an edge $e\in E(H_n)$ contains u but not v, then $e-u+v\in E(H_n)$ by edge-maximality. Consequently, transposing u and v induces an automorphism of H_n.

Clearly, there exists some $j\in[l]$ such that $m:=|\phi^{-1}(j)|\geq n/l$. For sufficiently large n, m is correspondingly large. Fix a vertex $w\in\phi^{-1}(j)$ and construct an r-graph $H_n\circ k$ of order $n+1$ obtained from H_n by adding a vertex k, with a homomorphism ϕ' extending ϕ such that $\phi'(k)=\phi(w)$. The edge set is given by:

$$E(H_n\circ k)=E(H_n)\cup\{e-v+k:e\in E(H_n),v\in e\cap\phi^{-1}(j)\}.$$

This construction ensures that $H_n\circ k$ is P-colorable and edge-maximal, and transposing k and w yields an automorphism of $H_n\circ k$.

Let $\mathbf{x} \in \mathbb{R}^n$ be a principal eigenvector of H_n and $\mu_n := \mathbf{x}_{\min}$. Define a vector $\mathbf{y} \in \mathbb{R}^{n+1}$ as follows:

$$y_v = \begin{cases} \left(\frac{mx_w^\alpha}{m+1}\right)^{1/\alpha}, & \text{if } v \in \phi^{-1}(j) \cup \{k\}; \\ x_v, & \text{else.} \end{cases}$$

By Lemma 3.7, $x_i = x_w$ for all $i \in \phi^{-1}(j)$, which implies $\sum_{v \in V(H_n \circ k)} y_v^\alpha = 1$. Set $\lambda_n := \lambda^{(\alpha)}(Col(P), n)$; then $\lambda_{n+1} \geq \lambda^{(\alpha)}(H_n \circ k)$.

Claim 1. *There exists $n_0 \in \mathbb{N}$ such that $\lambda_{n+1} \geq \big(1 + r(1 - \frac{1}{\alpha} - \frac{l}{\alpha(n-lr+l)})\mu_n^\alpha\big)\lambda_n$ for all $n > n_0$.*

Proof. Partition the edges incident to vertex w into sets $E_{H_n}^{(i)}(w)$ defined by

$$E_{H_n}^{(i)}(w) = \{e \in E_{H_n}(w) : |e \cap \phi^{-1}(j)| = i\}$$

(which may be empty). For $1 \leq i \leq r$, define $M_i = \{e \backslash \phi^{-1}(j) : e \in E_{H_n}^{(i)}(w)\}$. It follows that $E_{H_n}(w) = \cup_{i=1}^r E_{H_n}^{(i)}(w)$, and for each i $(1 \leq i \leq r)$,

$$\sum_{e \in E_{H_n}^{(i)}(w)} x_{e\backslash\{w\}} = \binom{m-1}{i-1} x_w^{i-1} \sum_{F \in M_i} x_F.$$

By Lemmas 3.7 and 3.9, there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$,

$$\begin{aligned}
\lambda_{n+1} - \lambda_n &\geq P_{H_n \circ k}(\mathbf{y}) - P_{H_n}(\mathbf{x}) \\
&= r! \sum_{i=1}^r \left(\binom{m+1}{i} \Big(\frac{mx_w^\alpha}{m+1}\Big)^{\frac{i}{\alpha}} - \binom{m}{i} x_w^i \right) \sum_{F \in M_i} x_F \\
&\geq r! \sum_{i=1}^r \binom{m-1}{i-1} \Big(1 - \frac{1}{\alpha} - \frac{1}{\alpha(m-r+1)}\Big) x_w^i \sum_{F \in M_i} x_F \\
&= r! \sum_{i=1}^r x_w \Big(1 - \frac{1}{\alpha} - \frac{1}{\alpha(m-r+1)}\Big) \binom{m-1}{i-1} x_w^{i-1} \sum_{F \in M_i} x_F \\
&= r! x_w \Big(1 - \frac{1}{\alpha} - \frac{1}{\alpha(m-r+1)}\Big) \sum_{i=1}^r \sum_{e \in E_{H_n}^{(i)}(w)} x_{e\backslash\{w\}} \\
&= r! x_w \Big(1 - \frac{1}{\alpha} - \frac{1}{\alpha(m-r+1)}\Big) \sum_{e \in E_{H_n}(w)} x_{e\backslash\{w\}} \\
&= r! x_w \Big(1 - \frac{1}{\alpha} - \frac{1}{\alpha(m-r+1)}\Big) \cdot \frac{\lambda_n x_w^{\alpha-1}}{(r-1)!} \\
&= r \Big(1 - \frac{1}{\alpha} - \frac{1}{\alpha(m-r+1)}\Big) \lambda_n x_w^\alpha \\
&\geq r \Big(1 - \frac{1}{\alpha} - \frac{l}{\alpha(n-lr+l)}\Big) \lambda_n \mu_n^\alpha.
\end{aligned}$$

This yields that $\lambda_{n+1} \geq \big(1 + r(1 - \frac{1}{\alpha} - \frac{l}{\alpha(n-lr+l)})\mu_n^\alpha\big)\lambda_n$ as desired. $\square$

Claim 2. *There exists n_1 such that for all $n > n_1$, every principal eigenvector of H_n is strictly positive.*

Proof. Recall that $\lambda^{(\alpha)}(H_n) = \lambda^{(\alpha)}(Col(P), n)$ and $\pi(Col(P)) > 0$. By Lemma 3.2, there exists sufficiently large $n_1 > n_0$ such that

$$\mu_{n_1}^{\alpha} \geq (x_{\min}(H_{n_1}))^{\alpha} \geq \frac{1}{n_1}\Big(1 - \frac{\alpha}{(\alpha-1)r\log n_1}\Big) > 0.$$

By Claim 1, we have

$$\lambda_{n_1+1} - \lambda_{n_1} \geq r\Big(1 - \frac{1}{\alpha} - \frac{l}{\alpha(n_1 - lr + l)}\Big)\lambda_{n_1}\mu_{n_1}^{\alpha} > 0.$$

We now show that every principal eigenvector of H_{n_1+1} is strictly positive. Suppose, for contradiction, that there exist a principal eigenvector $\mathbf{z}$ of H_{n_1+1} and a vertex $v \in V(H_{n_1+1})$ such that $z_v = 0$. Let $\mathbf{z}'$ be the vector obtained from $\mathbf{z}$ by omitting z_v. Then $\|\mathbf{z}'\|_\alpha = 1$, and

$$\lambda_{n_1} \geq P_{H_{n_1+1}-v}(\mathbf{z}') = P_{H_{n_1+1}}(\mathbf{z}) = \lambda_{n_1+1},$$

which contradicts $\lambda_{n_1+1} > \lambda_{n_1}$. Proceeding by induction, assume that the claim holds for some $n \geq n_1 + 1$, so that every principal eigenvector of H_n must be strictly positive (hence $\mu_n^{\alpha} > 0$). Claim 1 implies $\lambda_{n+1} - \lambda_n > 0$. The same contradiction argument as above shows that every principal eigenvector of H_{n+1} is strictly positive. By induction, the claim holds for all $n > n_1$. $\square$

Claim 2 and Lemma 3.8 imply $G_n = H_n$ for $n > n_1$. By Claim 1, we have

$$\begin{aligned}
\lambda_{n+1} &\geq \Big(1 + r\Big(1 - \frac{1}{\alpha} - \frac{l}{\alpha(n - lr + l)}\Big)\mu_n^{\alpha}\Big)\lambda_n \\
&\geq \Big(1 + r\Big(1 - \frac{1}{\alpha} - \frac{l}{\alpha(n - lr + l)}\Big)(x_{\min}(H_n))^{\alpha}\Big)\lambda_n \\
&= \Big(1 + r\Big(1 - \frac{1}{\alpha} - \frac{l}{\alpha(n - lr + l)}\Big)(x_{\min}(G_n))^{\alpha}\Big)\lambda_n.
\end{aligned}$$

(2) Cooper, Desai and Sahay [4, Theorem 4.6] established that any hereditary and multiplicative family $\mathcal{F}$ is clonal. Consequently, $Col(P)$ is clonal. For $n > n_1$, Lemma 3.6 and Claim 2 give

$$\gamma(G_n, \mathbf{x}) = \frac{\mathbf{x}_{\max}}{\mathbf{x}_{\min}} = 1 + O(n^{-1}).$$

Since $\mathbf{x}_{\max} \geq n^{-1/\alpha}$, we have

$$(\mathbf{x}_{\min})^{\alpha} = \frac{(\mathbf{x}_{\max})^{\alpha}}{1 + O(n^{-1})} = (\mathbf{x}_{\max})^{\alpha}(1 - O(n^{-1})) \geq \frac{1}{n}(1 - O(n^{-1})).$$

This shows that there exists $M = \alpha r M_0/(\alpha-1)(r-1)$ with $M_0 > r^2$ such that $(\mathbf{x}_{\min})^{\alpha} \geq \frac{1}{n}(1 - \frac{\pi(Col(P))M}{rn})$, and so $(\mathbf{x}_{\min}(G_n))^{\alpha} \geq \frac{1}{n}(1 - \frac{\pi(Col(P))M}{rn})$.

(3) Define $\varepsilon(n) = M/n$ and $\varepsilon'(n) = \pi(Col(P))M/rn$. Then $0 \leq \varepsilon'(n) < \varepsilon(n)\pi(Col(P))/(r-1)$. Suppose for contradiction that for some $n > n_1$, $\delta(G_n) < \pi(Col(P))(1 - \varepsilon(n))\binom{n}{r-1}$. Note that

$\pi(Col(P)) > 0$ and $(\mathbf{x}_{\min})^\alpha \geq \frac{1}{n}(1-\varepsilon'(n))$. We replicate the methodology of Lemma 3.3 to derive the following analogy of inequality (3.9):

$$1-\frac{r^2}{n} \leq \Big(\frac{1-\varepsilon(n)}{1-\varepsilon'(n)}\Big)^{\frac{\alpha-1}{\alpha}} = \Big(1-\frac{\varepsilon(n)-\varepsilon'(n)}{1-\varepsilon'(n)}\Big)^{\frac{\alpha-1}{\alpha}} \leq \Big(1-(\varepsilon(n)-\varepsilon'(n))\Big)^{\frac{\alpha-1}{\alpha}}.$$

Substituting $\varepsilon(n)-\varepsilon'(n) = \frac{(r-\pi(Col(P)))M}{rn}$ and applying Bernouli's inequality,

$$\begin{aligned}\big(1-(\varepsilon(n)-\varepsilon'(n))\big)^{\frac{\alpha-1}{\alpha}} &\leq 1-\frac{(\alpha-1)(r-\pi(Col(P)))M}{\alpha rn}\\ &\leq 1-\frac{(\alpha-1)(r-1)M}{\alpha rn}\\ &= 1-\frac{M_0}{n}.\end{aligned}$$

Thus $1-\frac{r^2}{n} \leq 1-\frac{M_0}{n}$, implying $M_0 \leq r^2$. This contradicts $M_0 > r^2$, completing the proof. $\square$

3.3. **Characterizing spectral extremal hypergraphs in k-chromatic r-graphs.** Let G be an r-graph on n vertices. Assume that $V(G)$ has a partition $V(G) = V_1 \cup \ldots \cup V_l$, and this partition is called *balanced* if $V_1, \ldots, V_l$ have almost equal sizes, namely $\lfloor n/l \rfloor \leq |V_i| \leq \lceil n/l \rceil$ for all $i \in [l]$. Then G is called *l-partite* (resp. *l-chromatic*) if for any edge $e \in E(G)$, $|e \cap V_i| \leq 1$ (resp. $|e \cap V_i| \leq r-1$) for all $i \in [l]$; furthermore, an l-partite (resp. l-chromatic) r-graph G is called *complete* if it is edge-maximal with respect to the given partition constraints (i.e. $\forall e \in \binom{[n]}{r}$, $e \in E(G)$ whenever $|e \cap V_i| \leq 1$ (resp. $|e \cap V_i| \leq r-1$) for $i \in [l]$). Note that if an l-partite r-graph has at least one edge, then $l \geq r$. We denote by $T_l^r(n)$ (resp. $Q_l^r(n)$) the balanced complete l-partite (resp. l-chromatic) r-graph on n vertices. Clearly, $T_l^2(n)$ and $Q_l^2(n)$ coincide, and both are isomorphic to Turán graph $T_l(n)$.

In [18], Kang, Nikiforov and Yuan proposed the following conjecture:

Conjecture 3.11. Let $k \geq 2$, and let G be a k-chromatic r-graph of order $n > (r-1)k$. For every $\alpha \geq 1$,

$$\lambda^{(\alpha)}(G) < \lambda^{(\alpha)}(Q_k^r(n)),$$

unless $G = Q_k^r(n)$.

This conjecture has been verified for $r = 3$ in [18]. We state that for $\alpha > 1$ and $k \geq 2$, the spectral extremal k-chromatic r-graphs asymptotically approach $Q_k^r(n)$ as $n \to \infty$. Hereafter, the notation $x = y \pm m$ denotes $y - m \leq x \leq y + m$.

Theorem 3.12. *Let $\alpha > 1$, k, $r \geq 2$, and let $\mathcal{F}$ be the set of k-chromatic r-graphs. Let G_n be a k-chromatic r-graph on n vertices with $\lambda^{(\alpha)}(G_n) = \lambda^{(\alpha)}(\mathcal{F}, n)$, and let $V_1, \ldots, V_k$ be the partition sets of $V(G_n)$. Then there exist $n_0 \in \mathbb{N}$ and $m > 0$ such that for all $n \geq n_0$, $|V_i| = \frac{n}{k} \pm m$ for each $i \in [k]$. In particular, $\lim\limits_{n\to\infty} \frac{|V_i|}{n} = \frac{1}{k}$.*

Proof. Let $P = ([k], E)$ be an r-pattern with $E = [k]^r \setminus \cup_{i=1}^k \{\{i, \dots, i\}\}$, where $[k]^r$ is the set of all r-multisets of elements from $[k]$. Then $Col(P) = \mathcal{F}$. By a straight calculation, we get

$$\pi(Col(P)) = \lim_{n\to\infty} \frac{e(Q_k^r(n))}{\binom{n}{r}} = 1 - \frac{1}{k^{r-1}}.$$

By part (3) of Theorem 3.10, there exist $M > 0$ and sufficiently large n_0 such that for all $n \geq n_0$,

$$\delta(G_n) \geq \Big(1 - \frac{1}{k^{r-1}}\Big)\Big(1 - \frac{M}{n}\Big)\binom{n}{r-1}.$$

Assume that $|V_i| = n_i$ for $i \in [k]$ and $n_1 \geq \cdots \geq n_k \geq 0$. From the proof of Theorem 3.10, we know that G_n must be a complete k-chromatic r-graph, and so

$$\delta(G_n) = \binom{n-1}{r-1} - \binom{n_1 - 1}{r-1} \geq \Big(1 - \frac{1}{k^{r-1}}\Big)\Big(1 - \frac{M}{n}\Big)\binom{n}{r-1}.$$

This implies that

$$\begin{aligned}
\binom{n_1-1}{r-1} &\leq \binom{n-1}{r-1} - \Big(1 - \frac{1}{k^{r-1}}\Big)\Big(1 - \frac{M}{n}\Big)\binom{n}{r-1} \\
&= \frac{1}{(r-1)!}(n^{r-1} - O(n^{r-2})) - \frac{1 - 1/k^{r-1}}{(r-1)!}(n^{r-1} - O(n^{r-2})) \\
&\leq \frac{n^{r-1}}{k^{r-1}(r-1)!}(1 + O(n^{-1})),
\end{aligned}$$

and consequently

$$n_1^{r-1} \leq \frac{n^{r-1}}{k^{r-1}}(1 + O(n^{-1})).$$

Note that $n_1 \geq n/k$, we have

$$\frac{n}{k} \leq n_1 \leq \frac{n}{k} + O(1).$$

On the other hand, we have

$$n = n_1 + \cdots + n_k \leq (k-1)n_1 + n_k \leq \frac{(k-1)n}{k} + O(1) + n_k,$$

and hence,

$$\frac{n}{k} - O(1) \leq n_k \leq \cdots \leq n_1 \leq \frac{n}{k} + O(1).$$

This completes the proof of Theorem 3.12. □

4. Spectral Turán-type problems with degree stability

Building on Theorem 3.10, certain spectral Turán-type extremal problems admit resolution through degree stability. In subsequent discussions, when addressing spectral Turán-type or classical Turán extremal problems for a family $\mathcal{F}$ of r-graphs, we consistently assume that all members of $\mathcal{F}$ contain no isolated vertices.

Definition 4.1 (Degree stability [23])**.** Let $\mathcal{F}$ be a family of r-graphs with $\pi(Mon(\mathcal{F})) > 0$, where $r \geq 2$, and let $\mathfrak{H}$ be a family of $\mathcal{F}$-free r-graphs. We say $\mathcal{F}$ is *degree-stable* with respect to $\mathfrak{H}$ if there exist constants n_0 and $\varepsilon > 0$ such that every $\mathcal{F}$-free r-graph $\mathcal{H}$ on $n \geq n_0$ vertices with $\delta(\mathcal{H}) \geq (\pi(Mon(\mathcal{F}))/(r-1)! - \varepsilon)n^{r-1}$ is a member of $\mathfrak{H}$.

The classical Andrásfai-Erdős-Sós Theorem [1] states that for any integer $l \geq 2$, the complete graph K_{l+1} is degree-stable with respect to $Col(K_l)$. Turán problems are closely related to the phenomenon of stability, and usually can be solved by stability results of the corresponding graphs or hypergraphs. The stability properties of hypergraphs have attracted much attention; for details, see [3, 10, 19, 23, 26, 37]. Now the following result can be viewed as another demonstration of the power of stability phenomenon in spectral Turán-type extremal problems.

Theorem 4.2. *Let $\alpha > 1$ and $r \geq 2$. Let P be an r-pattern, and let $\mathcal{F}$ be a family of r-graphs that is degree-stable with respect to $Col(P)$. Then for all sufficiently large n and any n-vertex $\mathcal{F}$-free r-graph G, we have*

$$\lambda^{(\alpha)}(G) \leq \lambda^{(\alpha)}(Col(P), n),$$

where equality holds only if G is P-colorable.

Proof. Let $H_n \in Mon(\mathcal{F})_n$ with $\lambda^{(\alpha)}(H_n) = \lambda^{(\alpha)}(Mon(\mathcal{F}), n)$, and $G_n \in Col(P)_n$ with $\lambda^{(\alpha)}(G_n) = \lambda^{(\alpha)}(Col(P), n)$. We claim that for all sufficiently large n, $H_n \in Col(P)_n$, and hence $\lambda^{(\alpha)}(H_n) = \lambda^{(\alpha)}(G_n)$.

Since $\mathcal{F}$ is degree-stable with respect to $Col(P)$, we have $\pi(Mon(\mathcal{F})) \geq \pi(Col(P)) > 0$. Therefore, there exist n_0 and $\varepsilon \in (0, 1)$ such that every $\mathcal{F}$-free r-graph $\mathcal{H}$ on $n \geq n_0$ vertices with $\delta(\mathcal{H}) \geq (\pi(Mon(\mathcal{F}))/(r-1)! - \varepsilon)n^{r-1}$ is contained in $Col(P)$. Choose a constant $\varepsilon' \in (0, \varepsilon\pi(Mon(\mathcal{F}))/2(r-1))$. By Lemma 3.2, there exists a sufficiently large $n_1 > n_0$ such that $(x_{\min}(H_{n_1}))^\alpha \geq (1-\varepsilon')/n_1$. Lemma 3.3 then yields

$$\delta(H_{n_1}) \geq (1-\varepsilon/2)\pi(Mon(\mathcal{F}))\binom{n_1}{r-1} \geq (\pi(Mon(\mathcal{F}))/(r-1)! - \varepsilon)n_1^{r-1}.$$

This implies that $H_{n_1} \in Col(P)_{n_1}$.

Now, assume that for some $n \geq n_1$, $H_n \in Col(P)_n$, and hence $\lambda^{(\alpha)}(H_n) = \lambda^{(\alpha)}(G_n)$. By part (2) of Theorem 3.10 (since n_1 was chose sufficiently large), we have $(x_{\min}(H_n))^\alpha \geq (1-\varepsilon'/2)/n$. Furthermore, part (1) of Theorem 3.10 gives

$$\begin{aligned} \lambda^{(\alpha)}(H_{n+1}) &\geq \lambda^{(\alpha)}(G_{n+1}) \\ &\geq \left(1 + r\left(1 - \frac{1}{\alpha} - \frac{l}{\alpha(n-lr+l)}\right)(x_{\min}(H_n))^\alpha\right)\lambda^{(\alpha)}(H_n) \\ &\geq \left(1 + \frac{r}{n}\left(1 - \frac{1}{\alpha} - \frac{l}{\alpha(n-lr+l)}\right)(1-\varepsilon'/2)\right)\lambda^{(\alpha)}(H_n). \end{aligned} \tag{4.1}$$

Next we show that

$$(x_{\min}(H_{n+1}))^\alpha \geq \frac{1}{n+1}(1-\varepsilon').$$

Suppose, for contradiction, that

$$(x_{\min}(H_{n+1}))^{\alpha} < \frac{1}{n+1}(1-\varepsilon').$$

Then, there exists a principal eigenvector $\mathbf{x}$ of H_{n+1} such that

$$(\mathbf{x}_{\min})^{\alpha} < \frac{1}{n+1}(1-\varepsilon'). \tag{4.2}$$

Applying (3.3) and (4.2), Fact 1, and Bernoulli's inequality, we obtain

$$\begin{aligned}
\frac{\lambda^{(\alpha)}(H_n)}{\lambda^{(\alpha)}(H_{n+1})} &\geq \frac{1-r(\mathbf{x}_{\min})^{\alpha}}{(1-(\mathbf{x}_{\min})^{\alpha})^{r/\alpha}} \\
&\geq \Big(1-\frac{r(1-\varepsilon')}{n+1}\Big)\Big(1-\frac{1-\varepsilon'}{n+1}\Big)^{-r/\alpha} \\
&\geq \Big(1-\frac{r(1-\varepsilon')}{n+1}\Big)\Big(1+\frac{r(1-\varepsilon')}{\alpha(n+1)}\Big) \\
&= 1-\frac{r(1-1/\alpha)(1-\varepsilon')}{n+1}-\frac{r^2(1-\varepsilon')^2}{\alpha(n+1)^2} \\
&= 1-\frac{r(1-1/\alpha)(1-\varepsilon')}{n+1}-o(n^{-1}).
\end{aligned} \tag{4.3}$$

From (4.1),

$$\begin{aligned}
\frac{\lambda^{(\alpha)}(H_n)}{\lambda^{(\alpha)}(H_{n+1})} &\leq \frac{1}{1+r\big(1-1/\alpha-l/\alpha(n-lr+l)\big)\big(1-\varepsilon'/2\big)n^{-1}} \\
&= 1-r\big(1-1/\alpha-l/\alpha(n-lr+l)\big)\big(1-\varepsilon'/2\big)n^{-1}+o(n^{-1}) \\
&= 1-r(1-1/\alpha)(1-\varepsilon'/2)n^{-1}+o(n^{-1}) \\
&\leq 1-r(1-1/\alpha)(1-\varepsilon'/2)(n+1)^{-1}+o(n^{-1}).
\end{aligned} \tag{4.4}$$

Combining (4.3) and (4.4) gives

$$\frac{r(1-1/\alpha)(1-\varepsilon'/2)}{n+1} \leq \frac{r(1-1/\alpha)(1-\varepsilon')}{n+1}+o(n^{-1}),$$

which implies

$$\frac{r(1-1/\alpha)\varepsilon'}{2(n+1)} < o(n^{-1}).$$

Consequently,

$$\frac{r(1-1/\alpha)\varepsilon'}{2} < o(1).$$

This is a contradiction. Hence,

$$(x_{\min}(H_{n+1}))^{\alpha} \geq \frac{1}{n+1}(1-\varepsilon').$$

By Lemma 3.3 again, we deduce that

$$\delta(H_{n+1}) \geq (1-\varepsilon/2)\pi(Mon(\mathcal{F}))\binom{n+1}{r-1} \geq (\pi(Mon(\mathcal{F}))/(r-1)!-\varepsilon)(n+1)^{r-1}.$$

Thus $H_{n+1} \in Col(P)_{n+1}$. By induction on n, we prove that $H_n \in Col(P)_n$ for all $n \geq n_1$, and so $\lambda^{(\alpha)}(H_n) = \lambda^{(\alpha)}(G_n)$. This shows that $\lambda^{(\alpha)}(G) \leq \lambda^{(\alpha)}(H_n) = \lambda^{(\alpha)}(Col(P), n)$ for all $n \geq n_1$, completing the proof. □

By a similar method, Theorem 4.2 can be generalized to the case for multiple r-patterns as follows.

Theorem 4.3. *Let $\alpha > 1$ and $r \geq 2$. Let P_i be r-patterns for $i \in [t]$, and $\mathcal{F}$ be a family of r-graphs that is degree-stable with respect to $\cup_{i=1}^t Col(P_i)$. Then for all sufficiently large n and any n-vertex $\mathcal{F}$-free r-graph G, we have $\lambda^{(\alpha)}(G) \leq \max_{i\in[t]} \lambda^{(\alpha)}(Col(P_i), n)$. Moreover, if equality holds then G is P_i-colorable for some $i \in [t]$.*

The following corollary is an immediate consequence of Lemma 3.6, Theorems 3.10 and 4.2.

Corollary 4.4. *Let $\alpha > 1$ and $r \geq 2$. Let P be an r-pattern, and $\mathcal{F}$ be a family of r-graphs that is degree-stable with respect to $Col(P)$. Suppose that G is an $\mathcal{F}$-free r-graph on n vertices satisfying $\lambda^{(\alpha)}(G) = \lambda^{(\alpha)}(Mon(\mathcal{F}), n)$, and $\mathbf{x}$ is a principle eigenvector of G. Then for sufficiently large n,*

$$\gamma(G, \mathbf{x}) = 1 + O(n^{-1}).$$

Corollary 4.5. *Let $\alpha \geq 1$ and $r \geq 2$. Let P be an r-pattern, and $\mathcal{F}$ be a family of r-graphs that is degree-stable with respect to $Col(P)$. Then for all sufficiently large n and any $\mathcal{F}$-free n-vertex r-graph G, we have $\lambda^{(\alpha)}(G) \leq \pi(Col(P))n^{r-r/\alpha}$. Moreover, for $\alpha > 1$,*

$$\lambda^{(\alpha)}(Mon(\mathcal{F}), n) = \pi(Col(P))n^{r-r/\alpha} - |O(1)|n^{r-r/\alpha-1}.$$

Proof. Theorems 2.9 and 4.2 imply that for $\alpha > 1$,

$$\lambda^{(\alpha)}(G) \leq \pi(Col(P))n^{r-r/\alpha}. \tag{4.5}$$

Applying Lemma 2.1 and continuity arguments to (4.5), we have

$$\lambda^{(1)}(G) = \lim_{\alpha\to 1^+} \lambda^{(\alpha)}(G) \leq \lim_{\alpha\to 1^+} \pi(Col(P))n^{r-r/\alpha} = \pi(Col(P)).$$

Thus the first statement holds true.

Furthermore, combining Lemmas 2.2 and 2.3 yields for $\alpha > 1$,

$$\begin{aligned}\lambda^{(\alpha)}(Mon(\mathcal{F}), n) &\geq \frac{\lambda^{(\alpha)}(Col(P))(n)_r}{n^{r/\alpha}} = \frac{\pi(Col(P))(n)_r}{n^{r/\alpha}}\\ &\geq \pi(Col(P))n^{r-r/\alpha} - |O(1)|n^{r-r/\alpha-1}.\end{aligned}$$

Thus the second statement follows from the above result and (4.5). □

Lemma 4.6 ([18])**.** *Let $l \geq r \geq 2$, and let G be an l-partite r-graph of order n. For all $\alpha > 1$,*

$$\lambda^{(\alpha)}(G) < \lambda^{(\alpha)}(T_l^r(n)),$$

unless $G = T_l^r(n)$.

Next, we characterize spectral extremal hypergraphs among all $\mathcal{F}$-free r-graphs, where $\mathcal{F}$ is degree-stable with respect to $Col(K_l^r)$. This result generalizes Theorem 3.3 in [47].

Corollary 4.7. *Let $\alpha \geq 1$ and $l \geq r \geq 2$. Let $\mathcal{F}$ be a family of r-graphs that is degree-stable with respect to $Col(K_l^r)$. Then, for all sufficiently large n and any n-vertex $\mathcal{F}$-free r-graph G, we have $\lambda^{(\alpha)}(G) \leq \lambda^{(\alpha)}(T_l^r(n))$. The equality holds if and only if $G = T_l^r(n)$ for $\alpha > 1$, and if $G \supseteq K_l^r$ for $\alpha = 1$.*

Proof. Note that $Col(K_l^r)$ consists of all l-partite r-graphs and satisfies $\pi(Col(K_l^r)) = (l)_r/l^r$. For $\alpha > 1$, the conclusion follows directly from Theorem 4.2 and Lemma 4.6. Consider now the case $\alpha = 1$. By Corollary 4.5, we have $\lambda^{(1)}(G) \leq (l)_r/l^r$. If G contains a copy of K_l^r, then

$$\lambda^{(1)}(G) \geq \lambda^{(1)}(K_l^r) \geq P_{K_l^r}((1/l, \dots, 1/l)) = (l)_r/l^r,$$

which implies $\lambda^{(1)}(G) = (l)_r/l^r$. Therefore, $\lambda^{(1)}(G) \leq \lambda^{(1)}(T_l^r(n)) = (l)_r/l^r$, completing the proof. □

Let H be an ordinary graph with $\chi(H) = l$. If there exists an edge e of H such that $\chi(H - e) = l - 1$, then H is called *l-color critical*. As established in Table 1 of [12], the r-expansion of any $(l+1)$-color critical graph is degree-stable with respect to $Col(K_l^r)$. Consequently, Corollary 4.7 yields the following result.

Theorem 4.8. *Let $\alpha \geq 1$, $l \geq r \geq 2$, and F be an $(l+1)$-color critical graph. Then there exists n_0, such that for any $F^{(r)}$-free r-graph G on $n > n_0$ vertices, $\lambda^{(\alpha)}(G) \leq \lambda^{(\alpha)}(T_l^r(n))$. The equality holds if and only if $G = T_l^r(n)$ for $\alpha > 1$, and if $G \supseteq K_l^r$ for $\alpha = 1$.*

5. Application of spectral method to classical Turán problems

As noted by Nikiforov [28], the α-spectral radius provides a unified framework incorporating the Lagrangian, the number of edges, the spectral radius, and related concepts. This enables a cohesive approach to addressing these problems. We demonstrate this principle by resolving classical edge-Turán extremal problems for specified families of r-graphs using the α-spectral radius, thereby providing further validation of this idea.

Theorem 5.1. *Let P be an r-pattern, where $r \geq 2$. Let $\mathcal{F}$ be a family of r-graphs that is degree-stable with respect to $Col(P)$. Then for all sufficiently large n and any n-vertex $\mathcal{F}$-free r-graph G, we have $e(G) \leq ex(Col(P), n)$. Moreover, if equality holds then G is P-colorable.*

Proof. By Theorem 4.2, for any $\alpha > 1$ and n sufficiently large, we have

$$\lambda^{(\alpha)}(G) \leq \lambda^{(\alpha)}(Col(P), n) = \max_{H \in Col(P)_n} \lambda^{(\alpha)}(H),$$

and thus,

$$\begin{aligned} r!e(G) &= \lim_{\alpha\to\infty} \lambda^{(\alpha)}(G) \le \overline{\lim_{\alpha\to\infty}} \max_{H\in Col(P)_n} \lambda^{(\alpha)}(H) \\ &= \max_{H\in Col(P)_n} \overline{\lim_{\alpha\to\infty}} \lambda^{(\alpha)}(H) \\ &= \max_{H\in Col(P)_n} r!e(H). \end{aligned}$$

This implies

$$e(G) \le ex(Col(P), n),$$

completing the proof. □

The following generalization follows similarly, and its proof is omitted.

Theorem 5.2. *Let P_i be r-patterns for $i \in [t]$, where $r \ge 2$. Let $\mathcal{F}$ be a family of r-graphs that is degree-stable with respect to $\cup_{i=1}^{t} Col(P_i)$. Then for all sufficiently large n and any n-vertex $\mathcal{F}$-free r-graph G, we have $e(G) \le \max_{i\in[t]} ex(Col(P_i), n)$. Moreover, if equality holds then G is P_i-colorable for some $i \in [t]$.*

By letting $\alpha \to \infty$ in Corollary 4.7, we obtain the following result.

Corollary 5.3. *Let $l \ge r \ge 2$, and $\mathcal{F}$ be a family of r-graphs that is degree-stable with respect to $Col(K_l^r)$. Then, for all sufficiently large n and any n-vertex $\mathcal{F}$-free r-graph G, we have $e(G) \le e(T_l^r(n))$. The equality holds if and only if $G = T_l^r(n)$.*

The Turán number of the r-expansion of any color critical graph can be derived from Theorem 4.8 by taking $\alpha \to \infty$.

Corollary 5.4. *Let $l \ge r \ge 2$, and F be an $(l+1)$-color critical graph. Then, for all sufficiently large n and any n-vertex $\mathcal{F}$-free r-graph G, we have $e(G) \le e(T_l^r(n))$. The equality holds if and only if $G = T_l^r(n)$.*

The following result, corresponding to Corollary 4.5, follows directly from Theorems 2.9 and 5.1.

Corollary 5.5. *Let P be an r-pattern, $r \ge 2$. Let $\mathcal{F}$ be a family of r-graphs that is degree-stable with respect to $Col(P)$. Then for all sufficiently large n and any n-vertex $\mathcal{F}$-free r-graph G, we have $e(G) \le \pi(Col(P))n^r/r!$.*

Let $\mathcal{P}$ be a hereditary property of r-graphs. Denote by $EX(\mathcal{P}, n)$ (resp. $SPEX_\alpha(\mathcal{P}, n)$) the set of n-vertex r-graphs in $\mathcal{P}_n$ achieving the maximum size (resp. maximum α-spectral radius) . A natural problem is to investigate the relation between $SPEX_\alpha(\mathcal{P}, n)$ and $EX(\mathcal{P}, n)$. Due to the high symmetry of the P-colorable r-graphs, we propose the following question.

Problem 5.6. *Let $\alpha > 1$ and $r \ge 2$. For all r-patterns P, is it true that for sufficiently large n,*

$$SPEX_\alpha(Col(P), n) \subseteq EX(Col(P), n)?$$

This appears challenging in general, but we can address certain cases.

Theorem 5.7. *Let $\alpha \geq 1$ and $r \geq 2$. Let $\mathcal{P}$ be a flat property of r-graphs. If $ex(\mathcal{P}, n) = \pi(\mathcal{P})n^r/r!$, then*

$$SPEX_\alpha(\mathcal{P}, n) = EX(\mathcal{P}, n).$$

Proof. For any $G \in EX(\mathcal{P}, n)$, by inequality (2.3), we have

$$\lambda^{(\alpha)}(G) \geq r!e(G)/n^{r/\alpha} = \pi(\mathcal{P})n^{r-r/\alpha}.$$

On the other hand, Theorem 2.9 yields $\lambda^{(\alpha)}(G) \leq \pi(\mathcal{P})n^{r-r/\alpha}$. Consequently, $\lambda^{(\alpha)}(G) = \lambda^{(\alpha)}(\mathcal{P}, n) = \pi(\mathcal{P})n^{r-r/\alpha}$. Therefore, $EX(\mathcal{P}, n) \subseteq SPEX_\alpha(\mathcal{P}, n)$.

For any $H \in SPEX_\alpha(\mathcal{P}, n)$, we have $\lambda^{(\alpha)}(H) = \lambda^{(\alpha)}(\mathcal{P}, n) = \pi(\mathcal{P})n^{r-r/\alpha}$. Applying Lemma 2.8, we obtain

$$\pi(\mathcal{P})^{1/\alpha}(r!e(H))^{1-1/\alpha} \geq \pi(\mathcal{P})n^{r-r/\alpha},$$

which implies $e(H) \geq \pi(\mathcal{P})n^r/r!$. This shows that $SPEX_\alpha(\mathcal{P}, n) \subseteq EX(\mathcal{P}, n)$. □

Remark. The result above generalizes Theorem 6.1 of [25]. This extension follows from the fundamental observation that an r-graph H is $\mathcal{M}$-hom-free (as defined in [25]) if and only if $H(\mathbf{t})$ is $\mathcal{M}$-free for every $\mathbf{t} \in \mathbb{Z}^{\nu(H)}$, which implies that $Mon(\mathcal{M})$ is multiplicative. Further details are omitted here.

Corollary 5.8. *Let $\alpha \geq 1$, and let P be an r-pattern. If $ex(Col(P), n) = \pi(Col(P))n^r/r!$, then*

$$SPEX_\alpha(Col(P), n) = EX(Col(P), n).$$

Applying Theorem 4.2 yields the following result directly.

Proposition 5.9. *Let $\alpha > 1$, $r \geq 2$, and let P be an r-pattern. Let $\mathcal{F}$ be a family of r-graphs that is degree-stable with respect to $Col(P)$. If the answer to Problem 5.6 is affirmative, then*

$$SPEX_\alpha(Mon(\mathcal{F}), n) \subseteq EX(Mon(\mathcal{F}), n)$$

for sufficiently large n.

Data availability

No data was used for the research described in the article.

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Appendix

Proof of Lemma 2.2. For every integer $n > r$, define $\lambda_n^{(\alpha)} := \lambda^{(\alpha)}(\mathcal{P}, n)$. Let $G \in \mathcal{P}_n$ be an r-graph satisfying $\lambda^{(\alpha)}(G) = \lambda^{(\alpha)}(\mathcal{P}, n)$, and let $\mathbf{x} = (x_1, \ldots, x_n)$ be a principal eigenvector for $\lambda^{(\alpha)}(G)$ with $\|\mathbf{x}\|_\alpha = 1$.

For $\alpha = 1$, by assumption (b), we have $\lambda_n^{(1)} \geq \lambda_{n-1}^{(1)}$. Moreover, Maclaurin's inequality implies

$$\lambda_n^{(1)} = P_G(\mathbf{x}) \leq r! \sum_{1 \leq i_1 < \cdots < i_r \leq n} x_{i_1} \cdots x_{i_r} \leq (x_1 + \cdots + x_n)^r = 1.$$

Thus, the sequence $\left\{\lambda_n^{(1)}\right\}_{n=1}^{\infty}$ converges to a limit λ, and we conclude

$$\lambda = \lim_{\alpha \to \infty} \lambda_n^{(1)} n^{r-r} = \lambda^{(1)}(\mathcal{P}).$$

For $\alpha > 1$, noting that $(\mathbf{x}_{\min})^\alpha \le 1/n$, by (3.3) and Fact 1, we have

$$\frac{\lambda_{n-1}^{(\alpha)}}{\lambda_n^{(\alpha)}} \ge \frac{1 - r(\mathbf{x}_{\min})^\alpha}{(1-(\mathbf{x}_{\min})^\alpha)^{r/\alpha}} \ge \frac{1-r/n}{(1-1/n)^{r/\alpha}}.$$

This implies that

$$\frac{\lambda_{n-1}^{(\alpha)}(n-1)^{r/\alpha}}{(n-1)_r} \ge \frac{\lambda_n^{(\alpha)} n^{r/\alpha}}{(n)_r}.$$

Therefore, the sequence $\left\{\frac{\lambda_n^{(\alpha)} n^{r/\alpha}}{(n)_r}\right\}_{n=1}^{\infty}$ is decreasing and hence convergent. This completes the proof. □

Proof of Theorem 2.9. Let $\mathbf{x} = (x_1, \ldots, x_n)$ be a principal eigenvector for $\lambda^{(\alpha)}(G)$ with $\|\mathbf{x}\|_\alpha = 1$. Then

$$\frac{\lambda^{(\alpha)}(G)}{n^{r-r/\alpha}} \le \frac{P_G(\mathbf{x})}{n^{r-r/\alpha}} = P_G((x_1/n^{1-1/\alpha}, \ldots, x_n/n^{1-1/\alpha})).$$

By Power-Mean inequality, for any $\alpha \ge 1$,

$$\frac{x_1 + \cdots + x_n}{n^{1-1/\alpha}} \le (x_1^\alpha + \cdots + x_n^\alpha)^{1/\alpha} = 1.$$

From Lemma 2.2 it follows that

$$\frac{\lambda^{(\alpha)}(G)}{n^{r-r/\alpha}} \le \lambda^{(1)}(G) \le \lambda^{(1)}(\mathcal{P}, n) \le \lambda^{(1)}(\mathcal{P}) = \pi(\mathcal{P}).$$

Since $\lambda^{(\alpha)}(G) \ge r! e(G)/n^{r/\alpha}$, we conclude

$$e(G) \le \pi(\mathcal{P}) n^r / r!,$$

completing the proof. □

Fact A. Let $P = ([l], E)$ be an r-pattern. Then $Col(P)$ is hereditary and multiplicative.

Proof. We first show that $Col(P)$ is hereditary. Let $H \in Col(P)$ and let H' be an induced subgraph of H. There exists a homomorphism $\phi : V(H) \to [l]$ such that for every edge $\{v_1, \ldots, v_r\} \in E(H)$, the multiset $\{\phi(v_1), \ldots, \phi(v_r)\} \in E$. Restrict ϕ to $V(H')$. For any edge $\{v_1, \ldots, v_r\} \in E(H')$, since H' is induced, this edge is also in $E(H)$, so $\{\phi(v_1), \ldots, \phi(v_r)\} \in E$. Thus, $\phi|_{V(H')}$ is a homomorphism from H' to P, so $H' \in Col(P)$.

Next, we prove that $Col(P)$ is multiplicative. Let $H \in Col(P)_n$ and let $\mathbf{t} \in \mathbb{Z}^n_{>0}$. Consider the blow-up $H(\mathbf{t})$. There exists a homomorphism $\phi : V(H) \to [l]$. Define $\psi : V(H(\mathbf{t})) \to [l]$ by mapping every vertex in the blow-up of $v \in V(H)$ to $\phi(v)$. For any edge $\{w_1, \ldots, w_r\} \in E(H(\mathbf{t}))$, each w_i lies in the blow-up of some $v_i \in V(H)$, and $\{v_1, \ldots, v_r\}$ is an edge in H (by definition of blow-up). Then $\{\psi(w_1), \ldots, \psi(w_r)\} = \{\phi(v_1), \ldots, \phi(v_r)\} \in E$. Thus, ψ is a homomorphism from $H(\mathbf{t})$ to P, so $H(\mathbf{t}) \in Col(P)$.

Hence, $Col(P)$ is hereditary and multiplicative. □

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