A CONSTRUCTION OF SIMPLE-MINDED SYSTEMS OVER DOMESTIC BRAUER GRAPH ALGEBRAS I: THE 2-DOMESTIC CASE

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Abstract. Let A be a 2-domestic Brauer graph algebra. We present a construction for a family of objects on A-$\underline{\mathsf{mod}}$ to be a simple-minded system and our construction provides all simple-minded systems on A-$\underline{\mathsf{mod}}$. As a byproduct, we provide a new proof of AR-conjecture for 2-domestic Brauer graph algebras and we prove that a weakly simple-minded system with a finite cardinality is a simple-minded system on A-$\underline{\mathsf{mod}}$. We also prove that an orthogonal system $\mathcal{S}$ which contains at least one object for an Euclidean component extends to a simple-minded system on A-$\underline{\mathsf{mod}}$ and its extension closure $\mathcal{F}(\mathcal{S})$ is functorially finite.

1. Introduction

Koenig and Liu [19] introduced simple-minded systems in the stable module category of any artin algebra, which is a family of objects satisfying orthogonality and generating condition. The authors [16] presented a sufficient and necessary condition for an orthogonal system to be a simple-minded system over a representation-finite self-injective algebra. They also [15] gave an explicit construction of sms's over self-injective Nakayama algebras. For a self-injective algebra, a few of necessary conditions of simple-minded systems were studied, including the finite cardinality of a simple-minded system, invariance of a simple-minded system under stable equivalences (refer to [19]), and Chan, Liu and Zhang [7] proved that any object on a quasi-tube of rank one is not in a simple-minded system over a self-injective algebra. However, there is no full characterization of simple-minded systems for a class of representation-infinite self-injective algebras. In this paper, we shall present a construction for a family of objects on A-$\underline{\mathsf{mod}}$ to be a simple-minded system in the stable module category of a 2-domestic Brauer graph algebra A and this construction provides all simple-minded systems on A-$\underline{\mathsf{mod}}$.

Originated from the modular representation theory of finite groups, Donovan and Freislich [12] defined Brauer graph algebras. Roggenkamp [24] showed that the class of Brauer graph algebras coincides with the class of symmetric special biserial algebras (see also [20]). As a class of tame algebras, self-injective special biserial algebras are known pretty well, including the classification of indecomposable modules (string modules and band modules), irreducible maps between indecomposable modules, and the graph structure of stable Auslander-Reiten quivers (AR-quiver for short), etc. Bocian and Skowronski [6] showed that a domestic Brauer graph algebra is 1-domestic or 2-domestic, and there is no n-domestic Brauer graph algebras for $n \geq 3$. Moreover, a Brauer graph algebra A is 2-domestic if and only if the Brauer graph G of A is a graph with a unique cycle of even length and the multiplicity of each vertices of G is one. Recently, there have been new developments for the study of Brauer graph algebras. Adachi, Aihara and Chan [1] constructed two-term tilting complexes over Brauer graph algebras, Opper and Zvonareva [21] presented a derived equivalence classification of Brauer graph algebras and Antipov and Zvonareva [5] showed that Brauer graph algebras are closed under derived equivalence.

Auslander-Reiten conjecture (AR-conjecture for short) states that stable equivalences preserve the number of isomorphism classes of non-projective simple modules. Pogorzaly [22] proved that, if AR-conjecture is true for a self-injective algebra, then it is true for all algebras. Antipov and A. Zvonareva, [4], Pogorzaly [23] proved that AR-conjecture holds for self-injective special biserial algebras. The study of simple-minded systems is closely related to AR-conjecture. If the cardinality of a simple-minded system over a finite dimensional algebra is a fixed number (the number of non-isomorphic, non-projective simple modules), then AR-conjecture holds for all finite dimensional algebras. In this paper, we give a new proof that AR-conjecture holds for 2-domestic Brauer graph algebras.

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Let R be a self-injective algebra and $\mathcal{M}$ a family of objects on R-$\underline{\mathsf{mod}}$. In general, it is not easy to determine when the extension closure $\mathcal{F}(\mathcal{M})$ of $\mathcal{M}$ is functorially finite on R-$\underline{\mathsf{mod}}$, even when $\mathcal{M}$ consists of only finitely many objects. Let A be a 2-domestic Brauer graph algebra and $\mathcal{S}$ an orthogonal system which contains at least one non-periodic module on A-$\underline{\mathsf{mod}}$. As a byproduct of our main result, we show that $\mathcal{S}$ extends to a simple-minded system and the extension closure $\mathcal{F}(\mathcal{S})$ of $\mathcal{S}$ is functorially finite on A-$\underline{\mathsf{mod}}$. Moreover, we show that a weakly simple-minded system with a finite cardinality is a simple-minded system on A-$\underline{\mathsf{mod}}$.

This paper is organized as follows. In Section 2, we recall simple-minded systems, Brauer graph algebras and some related concepts and conclusions. In Section 3, we study the stable bi-perpendicular category of a stable brick on A-$\underline{\mathsf{mod}}$ over a 2-domestic Brauer graph algebra. In Section 4, we construct orthogonal systems on A-$\underline{\mathsf{mod}}$. In Section 5, we prove our main result Theorem 5.14.

2. Preliminary

Let A be a finite-dimensional k-algebra and Γ_A the Auslander-Reiten quiver of A. An A-module X is said to be **τ-periodic module** if there is a positive integer m such that $X \cong \tau^m(X)$, otherwise, X is called a **non-τ-periodic module**. We say two A-modules X and Y are **orthogonal** on A-mod if $\mathsf{Hom}_A(X,Y) = 0$ and $\mathsf{Hom}_A(Y,X) = 0$. X and Y are called **stable orthogonal** on A-$\underline{\mathsf{mod}}$ if $\underline{\mathsf{Hom}}_A(X,Y) = 0$ and $\underline{\mathsf{Hom}}_A(Y,X) = 0$. For two families $\mathcal{X}$ and $\mathcal{Y}$ of objects are called **mutual stable orthogonal** on A-$\underline{\mathsf{mod}}$, if $\underline{\mathsf{Hom}}_A(X,Y) = 0$ and $\underline{\mathsf{Hom}}_A(Y,X) = 0$ for any $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$.

By a **component** of Γ_A we mean a connected component of Γ_A. A component $\mathcal{C}$ of Γ_A is said to be **acyclic** if $\mathcal{C}$ has no oriented cycles. A subquiver Σ of a component is said to be **right stable** (resp. **left stable**) if τ^{-1} (resp. τ) is defined on all modules over Σ. A subquiver Σ is **closed under successors** (resp. **closed under predecessors**) on Γ_A means that if there is a path whose start point (resp. end point) is on Σ, then the path belongs to Σ.

A connected component Γ is said to be a **stable tube** if it is of the form $\mathbb{Z}A_\infty/(\tau^r)$ for some integer $r \geq 1$. By a **quasi-tube**, we mean a translation quiver Γ such that its full translation subquiver formed by all vertices which are not projective-injective is a tube. We say Γ is a **homogeneous tube** provided that Γ is of the form $\mathbb{Z}A_\infty/(\tau)$. Skowronski proved [26] that every generalized standard family $\mathcal{C}$ of components on Γ_A is almost periodic, that is, all but finitely many τ_A-orbits on $\mathcal{C}$ are periodic. In particular, for a self-injective algebra A, every infinite generalized standard component $\mathcal{C}$ of Γ_A is either acyclic with finitely many τ_A-orbits or a quasi-tube. A component $\mathcal{C}$ is called an **Euclidean component** if $\mathcal{C}$ is of the form $\mathbb{Z}\Delta$, where Δ is an Euclidean quiver in $\{\widetilde{A}_n (n \geq 1), \widetilde{D}_m (m \geq 4), \widetilde{E}_6, \widetilde{E}_7, \widetilde{E}_8\}$.

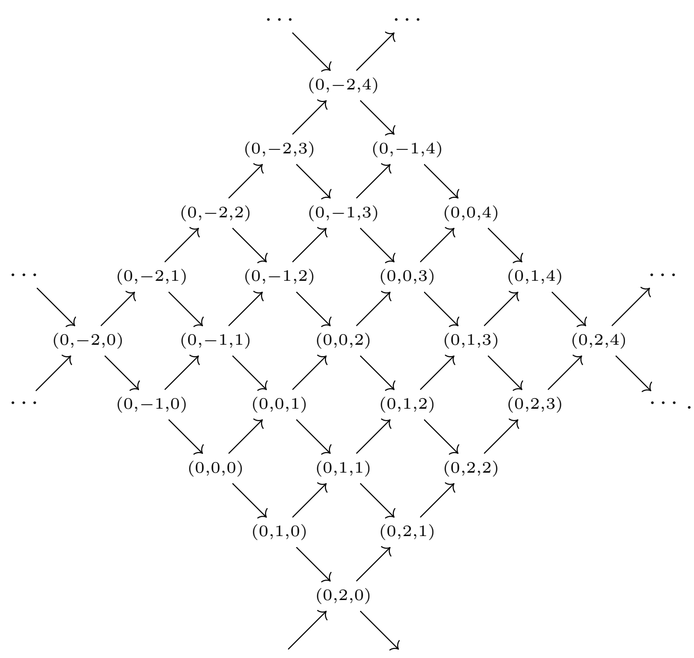
A family of connected components $\{\mathcal{C}_i\}_{i \in J}$ (J an index set) of Γ_A is called **generalized standard**, if given $X \in \mathcal{C}_i$ and $Y \in \mathcal{C}_j$, we have $\mathsf{rad}^\infty(X,Y) = 0$, where $\mathsf{rad}^\infty(-,-)$ means the intersection of $\mathsf{rad}^t(-,-)$ for $t \geq 1$. In particular, a full translation subquiver Σ of Γ_A is called **generalized standard** ([26]), if $\mathsf{rad}^\infty(X,Y) = 0$ for any X and Y on Σ. A connected component $\mathcal{C}$ is called **stable generalized standard**, if $\underline{\mathsf{rad}}^\infty(X,Y) = 0$ for all X and Y on $\mathcal{C}$, where $\underline{\mathsf{rad}}^\infty(X,Y) = \mathsf{rad}^\infty(X,Y)/P(X,Y)$ and $P(X,Y)$ is the subspace of $\mathsf{Hom}_A(X,Y)$ which factors through a projective module. Note that, if a connected component $\mathcal{C}$ is generalized standard, then $\mathcal{C}$ is stable generalized standard, but the converse is not true.

Let $\Delta = (\Delta_0, \Delta_1)$ be a connected and acyclic quiver. Recall that the **infinite translation quiver** $(\mathbb{Z}\Delta, \tau)$ is defined as follows. The set of points of $\mathbb{Z}\Delta$ is $(\mathbb{Z}\Delta)_0 = \mathbb{Z} \times \Delta_0 = \{(n,z) \mid n \in \mathbb{Z}, z \in \Delta_0\}$. For an arrow $\alpha\colon\ x \xrightarrow{a} y$ in Δ_1, there are two arrows

$$(i,\alpha):\ (i,x) \longrightarrow (i,y),\ \ (i,\alpha'):\ (i-1,y) \longrightarrow (i,x)$$

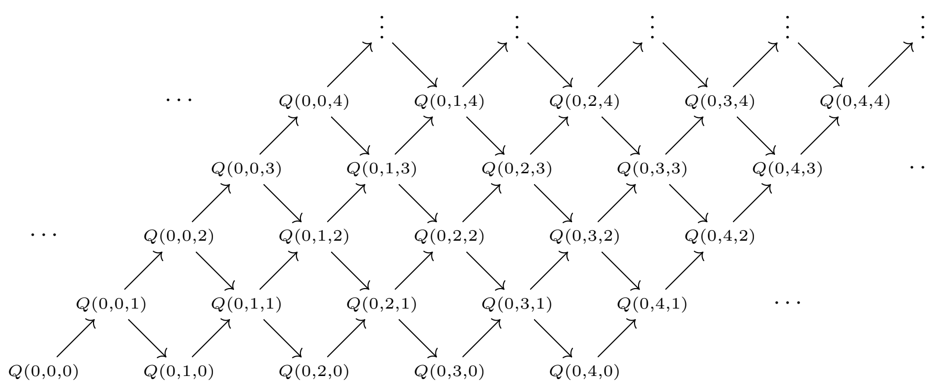
in $(\mathbb{Z}\Delta)_1$ and those are all arrows in $(\mathbb{Z}\Delta)_1$. The translation τ of $\mathbb{Z}\Delta$ is defined by $\tau(i,x) = (i-1,x)$ for $(i,x) \in (\mathbb{Z}\Delta)_0$. Let Γ_A be AR-quiver of a representation-infinite algebra A. In order to distinguish different connected components of Γ_A, we introduce two notations to label vertices on different components of the AR-quiver.

For an acyclic component $\mathcal{C}$ on Γ_A, we use notation (i,j,k) to denote a non-projective vertex on $\mathcal{C}$, where i also distinguishes different acyclic components of AR-quiver, j and k are determined by $\mathbb{Z} \times \Delta_0$. For example, we draw a picture of $\mathbb{Z}A_\infty^\infty$ as follows.



Note that the AR-translation τ on $\mathcal{C}$ satisfies $\tau(i,j,k) = (i, j-1, k-1)$.

For a quasi-tube $\mathcal{Q}$ (its stable part is of the form $\mathbb{Z}A_\infty/(\tau^n)$ for a positive integer n) on Γ_A, we use the notation $Q(i,j,k)$ to denote a non-projective vertex on $\mathcal{Q}$, where i distinguishes different quasi-tubes of Γ_A, j and k are determined by $\mathbb{Z} \times \Delta_0$. For example, we draw a picture of $\mathbb{Z}A_\infty$ as follows.



Note that the AR-translation τ satisfies $\tau Q(i,j,k) = Q(i, j-1, k)$ on the component $\mathcal{Q}$. Please compare with the Euclidean component case. We remind the reader that when we consider a general connected component $\mathcal{D}$ of a AR-quiver, we may still use a pair (i,j) of integers to label a vertex on $\mathcal{D}$.

2.1. **Simple-minded system.** Let $\mathcal{T}$ be a triangulated category with the shift functor [1]. For two families $\mathcal{S}_1, \mathcal{S}_2$ of objects in $\mathcal{T}$, we define

$$\mathcal{S}_1 \star \mathcal{S}_2 := \{X \in \mathcal{T} \mid \text{ There is a triangle } S_1 \longrightarrow X \longrightarrow S_2 \longrightarrow S_1[1], \textit{where } S_1 \in \mathcal{S}_1, S_2 \in \mathcal{S}_2\}.$$

It is routine to check that the operator $\star$ is associative, that is, $(\mathcal{S}_1 \star \mathcal{S}_2) \star \mathcal{S}_3 = \mathcal{S}_1 \star (\mathcal{S}_2 \star \mathcal{S}_3)$ for $\mathcal{S}_1, \mathcal{S}_2$ and $\mathcal{S}_3 \subseteq \mathcal{T}$. One denotes $(\mathcal{S})_0 = \{0\}$ and inductively defines $(\mathcal{S})_n = (\mathcal{S})_{n-1} \star (\mathcal{S} \cup \{0\})$ for $n \in \mathbb{Z}^+$. We say that $\mathcal{S}$ is *extension-closed*, if $\mathcal{S} \star \mathcal{S} \subseteq \mathcal{S}$. We denote the **extension closure** of a family $\mathcal{S}$ of objects in $\mathcal{T}$ as

$$\mathcal{F}(\mathcal{S}) := \bigcup_{n \geq 0} (\mathcal{S})_n,$$

which is the smallest extension-closed full subcategory of $\mathcal{T}$ containing $\mathcal{S}$.

Definition 2.1. *Let $\mathcal{T}$ be an additive k-category. An object M in $\mathcal{T}$ is a* **stable brick** *if $\mathcal{T}(M,M)\cong k$. Moreover, a family $\mathcal{M}$ of stable bricks in $\mathcal{T}$ is an* **orthogonal system** *if $\mathcal{T}(M,N)=0$ for all distinct M,N in $\mathcal{M}$.*

Remark 2.2. *If $\mathcal{S}$ is an orthogonal system, then $\mathcal{F}(\mathcal{M})$ is closed under direct summands. Please refer to* [10, Lemma 2.7] *for more details.*

Definition 2.3. ([19, Definition 2.1],[10, Definition 2.4, 2.5]) *Let $\mathcal{T}$ be a triangulated category. A family of objects $\mathcal{S}$ in $\mathcal{T}$ is a* **simple-minded system** *if the following two conditions are satisfied*:
(1) (Orthogonality) *$\mathcal{S}$ is an orthogonal system in $\mathcal{T}$.*
(2) (Generating condition) *Extension closure $\mathcal{F}(\mathcal{S})$ of $\mathcal{S}$ is equal to $\mathcal{T}$.*

Koenig and Liu [19] introduced a weaker concept than simple-minded system, namely weakly simple-minded system.

Definition 2.4. ([19, Definition 5.3]) *Let $\mathcal{T}$ be a triangulated category. A family of objects $\mathcal{S}$ in $\mathcal{T}$ is a* **weakly simple-minded system** *if the following two conditions are satisfied*:
(1) (Orthogonality) *$\mathcal{S}$ is an orthogonal system in $\mathcal{T}$.*
(2) (Weakly generating condition) *For any non-zero object X in $\mathcal{T}$, there is an object S in $\mathcal{S}$ such that $\mathcal{T}(S,X)\ncong 0$.*

Theorem 2.5. ([7, Theorem 1.3]) *Let A be a self-injective algebra and $\mathcal{C}$ a quasi-tube of rank n. Then the number of elements in a simple-minded system of A lying in $\mathcal{C}$ is strictly less than n. In particular, none of the indecomposable modules in a simple-minded system lie in the homogeneous tubes of the AR-quiver.*

Let $\mathcal{T}$ be an additive category and $\mathcal{X}$ a full subcategory of $\mathcal{T}$. We say that a morphism $f\colon M\to X$ is a **left $\mathcal{X}$-approximation** of M if $X\in\mathcal{X}$, and the abelian group homomorphism

$$\mathcal{T}(f,X')\colon \mathcal{T}(X,X')\to\mathcal{T}(M,X')$$

is surjective for each $X'\in\mathcal{X}$. Dually, a morphism $g\colon Y\to N$ is a **right $\mathcal{X}$-approximation** of N if $Y\in\mathcal{X}$, and the abelian group homomorphism

$$\mathcal{T}(Y',g)\colon \mathcal{T}(Y',Y)\to\mathcal{T}(Y',N)$$

is surjective for each $Y'\in\mathcal{X}$.

Let $\mathcal{S}$ be a full subcategory of $\mathcal{T}$ containing $\mathcal{X}$. $\mathcal{S}$ is said to be **covariantly finite** (resp. **contravariantly finite**) in $\mathcal{X}$ if every $S\in\mathcal{S}$ has a left (resp. right) $\mathsf{add}(\mathcal{X})$-approximation. We say $\mathcal{S}$ is **functorially finite** in $\mathcal{X}$ provided that $\mathcal{S}$ is both covariantly finite and contravariantly finite in $\mathcal{X}$. The following theorem provided us a necessary condition for an extension closure to be functorially finite.

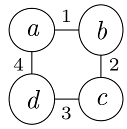
Theorem 2.6. ([10, Theorem 3.3]) *Let $\mathcal{T}$ be a Hom-finite Krull-Schmidt triangulated category. Suppose $\mathcal{X}\subseteq\mathcal{S}$ for a simple-minded system $\mathcal{S}$ in $\mathcal{T}$. Then $({}^{\perp}\mathcal{X},\mathcal{F}(\mathcal{X}))$ and $(\mathcal{F}(\mathcal{X}),\mathcal{X}^{\perp})$ are torsion pairs in $\mathcal{T}$. In particular, $\mathcal{F}(\mathcal{X})$ is a functorially finite subcategory of $\mathcal{T}$.*

2.2. **Brauer graph algebras.** We state the definition of Brauer graph algebras and some related results as follows. Please refer to [25, Section 2] for more details.

Definition 2.7. ([25, Section 2]) *A* **Brauer graph** *$G=(G_0,G_1,m,o)$ is a finite unoriented connected graph, where G_0 is the set of its vertices, G_1 is the set of edges, the map $m\colon G_0\to\mathbb{Z}_{>0}$ is called multiplicity, o is the given orientation, that is, for each vertex $v\in G_0$, each edge incident to vertex v is given a cyclic ordering. We always assume that the cyclic orderings are clockwise. Suppose that the cyclic ordering at vertex v is given by $i_1<i_2<\cdots<i_n$, then we say i_j is the predecessor of i_{j+1}, and i_{j+1} is the successor of i_j.*

Given a Brauer graph $G=(G_0,G_1,m,o)$, we denote *valency* of $v\in G_0$ by $val(v)$, which is defined to be the number of edges in G incident to v. Note that a loop is counted twice. Every Brauer graph determines a finite dimensional basic symmetric algebra, it is called a **Brauer graph algebra**. Let A be the Brauer graph algebra determined by a Brauer graph G. There are a quiver Q_G and an admissible ideal I_G such that $A\cong kQ_G/I_G$. We refer to [25, Section 2] for the construction of $Q_G=(Q_0,Q_1)$ and I_G. We provide a concrete construction in the following example.

Example 2.8. *Take a Brauer graph $G = (G_0, G_1, m, o)$ as follows.*



The multiplicity of each vertex $i \in G_0$ is $m(i) = 1$. Cyclic ordering of the edge incident with vertex a is $1 < 4 < 1$, with vertex b is $2 < 1 < 2$, with vertex c is $3 < 2 < 3$, with vertex d is $4 < 3 < 4$. The valencies of vertices are as follows: $val(a) = val(b) = val(c) = val(d) = 2$.

We now present the algebra $A \cong kQ_G/I_G$ determined by Brauer graph G as follows. We first consider the corresponding quiver $Q_G = (Q_0, Q_1)$. The set of Q_0 is given by the set of edges G_1 of G. We denote the vertex in Q_0 corresponding to the edges j in G_1 also by j for $j = 1, 2, 3, 4$. By the above paragraph, the cyclic orderings in G determine arrows Q_1 of Q in the following way: $\alpha_1 : 1 \to 2$, $\beta_1 : 2 \to 1$, $\alpha_2 : 2 \to 3$, $\beta_2 : 3 \to 2$, $\alpha_3 : 3 \to 4$, $\beta_3 : 4 \to 3$, and $\alpha_4 : 4 \to 1$, $\beta_4 : 1 \to 4$. Hence the quiver Q_G is the following diagram.

$$\begin{tikzcd} 1 \arrow[r, shift left, "\alpha_1"] \arrow[d, shift left, "\beta_4"] & 2 \arrow[l, shift left, "\beta_1"] \arrow[d, shift left, "\alpha_2"] \\ 4 \arrow[u, shift left, "\alpha_4"] \arrow[r, shift left, "\beta_3"] & 3 \arrow[l, shift left, "\alpha_3"] \arrow[u, shift left, "\beta_2"] \end{tikzcd}$$

Then we consider the ideal I_G of kQ_G generated by three types of relations.
Relations of type I: $\beta_1\alpha_1 - \alpha_4\beta_4$, $\beta_2\alpha_2 - \alpha_1\beta_1$, $\beta_3\alpha_3 - \alpha_2\beta_2$, $\beta_4\alpha_4 - \alpha_3\beta_3$.
Relations of type II: $\alpha_1\beta_1\alpha_1$, $\beta_1\alpha_1\beta_1$, $\alpha_2\beta_2\alpha_2$, $\beta_2\alpha_2\beta_2$, $\alpha_3\beta_3\alpha_3$, $\beta_3\alpha_3\beta_3$, $\alpha_4\beta_4\alpha_4$, $\beta_4\alpha_4\beta_4$.
Relations of type III: $\alpha_3\alpha_2\alpha_1$, $\alpha_4\alpha_3\alpha_2$, $\alpha_1\alpha_4\alpha_3$, $\alpha_2\alpha_1\alpha_4$, $\beta_3\beta_4\beta_1$, $\beta_4\beta_1\beta_2$, $\beta_1\beta_2\beta_3$, $\beta_2\beta_3\beta_4$.

Thus $A \cong kQ_G/I_G = \begin{array}{cc} \multicolumn{2}{c}{1} \\ 4 & 2 \\ \multicolumn{2}{c}{1} \end{array} \oplus \begin{array}{cc} \multicolumn{2}{c}{2} \\ 1 & 3 \\ \multicolumn{2}{c}{2} \end{array} \oplus \begin{array}{cc} \multicolumn{2}{c}{3} \\ 2 & 4 \\ \multicolumn{2}{c}{3} \end{array} \oplus \begin{array}{cc} \multicolumn{2}{c}{4} \\ 3 & 1 \\ \multicolumn{2}{c}{4} \end{array}.$

We briefly recall the representation type of an algebra. From the point of view of representation type, according to Drozd's classification [11], an algebra is either tame or wild type, and not both. Let $k[x]$ be the polynomial algebra in one variable. An algebra A is said to be *tame*, if for any dimension d, there exist finitely many $k[x]$-A-bimodules M_i, $1 \leq i \leq n_d$, which are finitely generated and free as left $k[x]$-module, and all but finitely many isomorphism classes of indecomposable modules of dimension d in $ind\,A$ are of the form $k[x]/(x - \lambda) \otimes_{k[x]} M_i$, for some $\lambda \in k$, and some $i \in \{1, 2, \cdots, n_d\}$. Let $\mu_A(d)$ be the least number of $k[x]$-A-bimodules satisfying the above condition for d. Then A is said to be of *polynomial growth* (cf. [28]), if there is a positive integer m such that $\mu_A(d) < d^m$ for all $d > 1$. We say A is of *domestic* type (cf. [8]), if there is a positive integer m such that $\mu_A(d) < m$ for all $d > 1$ and A is said to be *d-domestic* if d is the least such integer m.

Theorem 2.9. ([6]) *Let A be a Brauer graph algebra with Brauer graph G. Then*
(1) *A is 1-domestic if and only if one of the following holds:*
  (*a*) *G is a tree with $m(i) = 2$ for exactly two vertices $i_0, i_1 \in G_0$ and $m(i) = 1$ for all $i \in G_0, i \neq i_0, i_1$.*
  (*b*) *There is exactly one cycle in G and this cycle has odd length, and all multiplicities of edges are one, that is, $m \equiv 1$.*
(2) *A is 2-domestic if and only if there is exactly one cycle in G and this cycle has even length, and $m \equiv 1$.*
(3) *There is no n-domestic Brauer graph algebra for $n \geq 3$.*

Theorem 2.10. ([9, Theorem 4.4, Corollary 4.5]) *Let A be a representation-infinite domestic Brauer graph algebra with Brauer graph G with n edges. If G has a cycle, then it is unique. Let n_1 be the number of* (*additional*) *edges on the inside of the cycle and n_2 the number of* (*additional*) *edges on the outside of the cycle. In the notation above,*
(1) *If A is 1-domestic, then $m = 1$ and $p + q = 2n$. Furthermore,*
  (*a*) *If G is a tree, then $p = q = n$.*
  (*b*) *If G has exactly one cycle in G and this cycle has* (*odd*) *length ℓ, then $p = \ell + 2n_1$ and $q = \ell + 2n_2$.*
(2) *If A is 2-domestic and the unique cycle of Q has* (*even*) *length ℓ, then $m = 2$ and $p = \ell/2 + n_1$ and $q = \ell/2 + n_2$ such that $p + q = n$.*

Note that the Brauer graph algebra in Example 2.7 is 2-domestic. Let A be a 2-domestic Brauer graph algebra. It follows from [13, Chapter 2, Theorem 7.1] that AR-quiver Γ_A of A consists of two stable Euclidean components Γ_0 and Γ_1 of the form $\mathbb{Z}\widetilde{A}_{p,q}$, two quasi-tubes Q_0 and Q_1 of rank q and two quasi-tubes Q_0' and Q_1' of rank p, as well as infinitely many homogeneous tubes. By Theorem 2.10, $p+q=n$. By $\widetilde{A}_{p,q}(p,q\geq 1)$, we mean the quiver

$$\begin{tikzcd}
 & \cdot \arrow[r, "\alpha_2"] & \cdots \arrow[r, "\alpha_{p-1}"] & \cdot \arrow[dr, "\alpha_p"] & \\
\cdot \arrow[ur, "\alpha_1"] \arrow[dr, "\beta_1"'] & & & & \cdot \\
 & \cdot \arrow[r, "\beta_2"'] & \cdots \arrow[r, "\beta_{q-1}"'] & \cdot \arrow[ur, "\beta_q"'] &
\end{tikzcd}$$

2.3. **The property of components over a 2-domestic Brauer graph algebra.** We first study the properties Euclidean components over a 2-domestic Brauer graph algebra. We recall some results about acyclic connected components as follows.

Proposition 2.11. ([29, Corollary 5.7] *Let A be a symmetric k-algebra. Then the following conditions are equivalent.*

(1) *Γ_A admits an acyclic, generalized standard, right stable full translation subquiver which is closed under successors on Γ_A.*
(2) *Γ_A admits an acyclic, generalized standard, left stable full translation subquiver which is closed under predecessors on Γ_A.*
(3) *A is isomorphic to trivial extension algebra $T(B)$, where B is a tilted algebra not of Dynkin type.*

Theorem 2.12. ([6, Theorem 2]) *Let A be a basic indecomposable algebra. Then the following conditions are equivalent.*

(1) *A is isomorphic to a 2-domestic Brauer graph algebra.*
(2) *A is a symmetric algebra of Euclidean type $\widetilde{\mathbb{A}}_m$ and the Cartan matrix of A is singular.*
(3) *A is isomorphic to the trivial extension $T(B)$, where B is a representation-infinite tilted algebra of Euclidean type $\widetilde{\mathbb{A}}_m$.*

Corollary 2.13. *Let A be a 2-domestic Brauer graph algebra and let X and Y be two objects on an Euclidean component. If there is no compositions of finitely many irreducible maps from X to Y, then $\underline{\mathsf{Hom}}_A(X,Y)\cong 0$. In particular, each Euclidean component of AR-quiver Γ_A is stable generalized standard and every indecomposable, non-projective non-periodic module is a stable brick on A-$\underline{\mathsf{mod}}$.*

Proof. Since A is 2-domestic, there are two Euclidean components, denoted by Γ_0 and Γ_1. Note that Γ_0 and Γ_1 have only finitely many τ-orbits and $\Omega({}_s\Gamma_0)={}_s\Gamma_1$. By Proposition 2.11 and Theorem 2.12, there is an acyclic, generalized standard, right stable full translation subquiver which is closed under successors on Γ_A. Without loss of generality, we assume that Γ_0 admits an acyclic, generalized standard, right stable full translation subquiver Σ which is closed under successors Γ_0. Note that $\mathsf{rad}^\infty(M,N)=0$ for any M and N on Σ.

Let X and Y be two non-projective objects on Γ_0. We assume that there is no compositions of finitely many irreducible maps from X to Y on Γ_0. Thus $\mathsf{Hom}_A(X,Y)\subseteq \mathsf{rad}^\infty(X,Y)$. If both X and Y are in Σ, $\mathsf{Hom}_A(X,Y)\cong 0$, thus $\underline{\mathsf{Hom}}_A(X,Y)\cong 0$. Otherwise, one of X and Y is not in Σ. Since X and Y are not projective and A is symmetric, the τ-orbits of both X and Y contain no projective or injective modules. Since Σ is closed under successors on Γ_0, Σ contains at least one object for each τ-orbits of Γ_0. Thus there is a large number m such that both $\tau^{-m}(X)$ and $\tau^{-m}(Y)$ are contained in Σ. Since there is no compositions of finitely many irreducible maps from X to Y, so are $\tau^{-m}(X)$ and $\tau^{-m}(Y)$. Thus $\underline{\mathsf{Hom}}_A(X,Y)\cong\underline{\mathsf{Hom}}_A(\tau^{-m}(X),\tau^{-m}(Y))\cong 0$. Thus our conclusion holds for Γ_0. Since $\Omega({}_s\Gamma_0)={}_s\Gamma_1$ and Ω is an equivalent functor on A-$\underline{\mathsf{mod}}$, our conclusion also holds for Γ_1. Therefore each Euclidean component is stable generalized standard. Since both Γ_0 and Γ_1 are acyclic, every indecomposable, non-projective non-periodic module is a stable brick on A-$\underline{\mathsf{mod}}$. □

Then we start to study quasi-tubes of a 2-domestic Brauer graph algebra. We recall some conclusions for quasi-tubes as follows.

Proposition 2.14. ([20, Theorem A]) *Let A be a self-injective algebra and Γ a connected component of Γ_A. The following are equivalent.*

(1) *Γ is a quasi-tube.*
(2) *${}_s\Gamma$ is a stable tube.*
(3) *Γ contains an oriented cycle.*

Theorem 2.15. ([27, Section 4, 4.17]) *Let A be a self-injective algebra. Then the following statements are equivalent.*

(1) *A is a symmetric algebra of Euclidean type and has singular Cartan matrix.*
(2) *A is derived equivalent to trivial extension $T(C)$ of a canonical algebra C of Euclidean type.*
(3) *A is stably equivalent to the trivial extension $T(C)$ of a canonical algebra of Euclidean type.*

Theorem 2.16. ([17, Corollary 2]) *Let A be a basic, indecomposable finite dimensional symmetric k-algebra. Then the following statements are equivalent.*

(1) *Γ_A admits a generalized standard family $\mathcal{C} = (\mathcal{C}_\lambda)_{\lambda\in\Lambda}$ of quasi-tubes maximally saturated by simple modules and projective modules.*
(2) *Γ_A admits a generalized standard family $\mathcal{C} = (\mathcal{C}_\lambda)_{\lambda\in\Lambda}$ of stable tubes maximally saturated by simple modules.*
(3) *A is isomorphic to the trivial extension $T(B)$ of a canonical algebra B.*

Remark 2.17. *For a quasi-simple X on a component $\mathcal{C}_\lambda$ of a generalized standard family $\mathcal{C}$, X and Y are orthogonal on A-mod for any quasi-simple which belongs to the family $\mathcal{C}$ of quasi-tube, except $\Omega(X)$, $\Omega^{-1}(X)$ and X.*

Corollary 2.18. *Let A be a 2-domestic Brauer graph algebra and X a quasi-simple on a quasi-tube. Then X and Y are mutual stable orthogonal on A-$\underline{\mathsf{mod}}$ for any quasi-simple Y, except $\Omega(X)$, $\Omega^{-1}(X)$ and X. In particular, each quasi-tube is stable generalized standard.*

Proof. By Theorem 2.12 and Theorem 2.15, A is stably equivalent to the trivial extension $T(C)$ of a canonical algebra C of Euclidean type. Let $F\colon T(C)$-$\underline{\mathsf{mod}} \to A$-$\underline{\mathsf{mod}}$ be a stably equivalence. Thus stable AR-quivers ${}_s\Gamma_A$ and ${}_s\Gamma_{T(C)}$ are isomorphic as stable translation quivers. Note that F preserves quasi-tubes and quasi-simples.

Since AR-quiver Γ_A of A contains two family $\mathcal{T}_1$ and $\mathcal{T}_2$ of quasi-tubes satisfying $\Omega(\mathcal{T}_1) = \mathcal{T}_2$. By Theorem 2.16, one of $\mathcal{T}_1$ and $\mathcal{T}_2$ is the image of a generalized standard family $\mathcal{C} = (\mathcal{C}_\lambda)_{\lambda\in\Lambda}$ of quasi-tubes in $\Gamma_{T(C)}$ under stably equivalence F. Without loss of generality, we assume that $\mathcal{T}_1$ is the image of $\mathcal{C}$ under F, and that X is in $\mathcal{T}_1$. Take a quasi-simple Y in $\mathcal{T}_1$ which is not isomorphic to X. By Remark 2.17, $\underline{\mathsf{Hom}}_A(X,Y) \cong \underline{\mathsf{Hom}}_{T(C)}(F(X),F(Y)) \cong 0$ and $\underline{\mathsf{Hom}}_A(Y,X) \cong \underline{\mathsf{Hom}}_{T(C)}(F(Y),F(X)) \cong 0$. Since $\Omega(\mathcal{T}_1) = \mathcal{T}_2$, $\mathcal{T}_2$ also satisfies the above conclusion.

Take a quasi-simple Y in $\mathcal{T}_2$ which is not isomorphic to $\Omega(X)$. By Serre duality,

$$\underline{\mathsf{Hom}}_A(X,Y) \cong \mathsf{D}\underline{\mathsf{Hom}}_A(Y,\nu\Omega(X)) \cong \mathsf{D}\underline{\mathsf{Hom}}_A(Y,\Omega(X)) \cong 0.$$

Take a quasi-simple Y in $\mathcal{T}_2$ which is not isomorphic to $\Omega^{-1}(X)$. By Serre duality,

$$\underline{\mathsf{Hom}}_A(Y,X) \cong \mathsf{D}\underline{\mathsf{Hom}}_A(X,\nu\Omega(Y)) \cong \mathsf{D}\underline{\mathsf{Hom}}_A(X,\Omega(Y)) \cong 0.$$

Take two objects X and Y on a quasi-tube. If there is no compositions of finitely many irreducible maps from X to Y, by [14, Section 2.2], then $\underline{\mathsf{Hom}}_A(X,Y) \cong 0$. Hence each quasi-tube is stable generalized standard. □

Lemma 2.19. *Let A be a self-injective algebra with a stable generalized standard AR-component Γ and X an indecomposable non-projective module on Γ. If X is a stable brick on A-$\underline{\mathsf{mod}}$, then any sectional path with indecomposable non-projective modules starting at X on Γ is non-zero on A-$\underline{\mathsf{mod}}$.*

Proof. Take a sectional path starting at X on Γ:

$$X = X_1 \xrightarrow{f_1} X_2 \to \cdots \to X_{\ell-1} \xrightarrow{f_\ell} X_\ell.$$

Note that each f_t is an irreducible map on A-mod for $t = 1,2,\cdots,\ell$. We prove that the morphism $\underline{f_t \cdots f_2 f_1}$ is non-zero on A-$\underline{\mathsf{mod}}$ by induction on t. Since X_1 and X_2 are indecomposable non-projective A-module and the morphism f_1 is irreducible, $\underline{f_1}$ is not zero on A-$\underline{\mathsf{mod}}$. Therefore our conclusion holds for $t = 1$.

We assume that our conclusion holds for $t = k$. We prove it is true for $t = k+1$. On the contrary, we assume that the morphism $\underline{f_{k+1}\cdots f_2 f_1}$ is zero on A-$\underline{\mathsf{mod}}$. Take the triangle induced by the almost split sequence ending at X_{k+1} as follows.

$$Z \xrightarrow{(\underline{h_1},\underline{h_2})^t} Y \oplus X_k \xrightarrow{(\underline{g},\underline{f_k})} X_{k+1} \to Z[1].$$

By induction, the morphism $\underline{f_k \cdots f_2 f_1}$ is non-zero on A-$\underline{\mathsf{mod}}$. Take a morphism

$$\alpha = (0, \underline{f_k \cdots f_2 f_1})^t \colon X \to Y \oplus X_k.$$

Then $(\underline{g},\underline{f_k})\alpha = (\underline{f_{k+1} f_k \cdots f_2 f_1}, 0) = (0,0)$ on A-$\underline{\mathsf{mod}}$. Thus there is a morphism $\beta : X \to Z$ such that $(\underline{h_1},\underline{h_2})\beta = \alpha$ on A-$\underline{\mathsf{mod}}$. See the following diagram.

$$\begin{tikzcd}
Z \arrow[r, "(\underline{h_1},\underline{h_2})^t"] & Y \oplus X_k \arrow[r, "(\underline{g},\underline{f_k})"] & X_{k+1} \arrow[r] & Z[1] . \\
X \arrow[u, dashed, "\beta"] \arrow[ur, "\alpha"] & & &
\end{tikzcd}$$

There are two cases to be considered. If Γ has no cycle, then there is no non-zero morphism from X to Z, since Γ is stable generalized standard. Thus $\alpha = (0, \underline{f_k \cdots f_2 f_1}) = 0$ on A-$\underline{\mathsf{mod}}$. It contradicts the induction that the morphism $\underline{f_k \cdots f_2 f_1}$ is non-zero on A-$\underline{\mathsf{mod}}$. If Γ has a cycle, then ${}_s\Gamma$ is a quasi-tube by Proposition 2.14. Thus our conclusion follows from [14, Lemma 2.6]. □

We also state the dual result.

Lemma 2.20. *Let A be a self-injective algebra with a stable generalized standard AR-component Γ and let X be an indecomposable non-projective module on Γ. If X is a stable brick on A-$\underline{\mathsf{mod}}$, then any sectional path with indecomposable non-projective modules ending at X on Γ is non-zero on A-$\underline{\mathsf{mod}}$.*

Lemma 2.21. *Let A be a self-injective algebra and (i,j) an indecomposable non-projective module on a connected component Γ of AR-quiver Γ_A. Then there is a short exact sequence for a projective module P and $k, \ell \in \mathbb{Z}^+$ as follows.*

$$0 \to (i,j) \xrightarrow{(\alpha_\ell, -\beta_k, \varepsilon)^t} (i, j+\ell) \oplus (i+k, j) \oplus P \xrightarrow{(\gamma_k, \delta_\ell, \pi)} (i+k, j+\ell) \to 0, \tag{2.1}$$

where α_ℓ and β_k are the compositions of irreducible maps from $(0,i,j)$ to $(0,i,j+\ell)$ and $(0,i+k,j)$ respectively, γ_k and δ_ℓ are the compositions of irreducible maps from $(i,j+\ell)$ and $(i+k,j)$ to $(i+k,j+\ell)$ respectively.

Proof. Take a set $\{P_1, P_2, \cdots, P_t\}$ consisting of all indecomposable projective modules in the rectangle area with vertices $(0,i,j)$, $(0,i,j+\ell)$, $(0,i+k,j)$ and $(0,i+k,j+\ell)$ on Euclidean component Γ. Then the projective module $P = P_1 \oplus P_2 \oplus \cdots \oplus P_t$ in sequence (2.1). The proof is done by induction on k and ℓ. Please refer to [14, Lemma 2.3.1]. □

The proof of the following result is similar with [7, Lemma 3.4]. We include a proof here.

Lemma 2.22. *Let A be a self-injective algebra with an acyclic stable generalized standard component Γ_0 and let $(0,i,j)$ be an indecomposable non-projective module on $\mathcal{C}$. If $(0,i,j)$ is a stable brick on A-$\underline{\mathsf{mod}}$, then the compositions $\gamma_k\alpha_\ell$ of $\alpha_\ell : (0,i,j) \to (0,i,j+\ell)$ and $\gamma_k : (0,i,j+\ell) \to (0,i+k,j+\ell)$ is non-zero on A-$\underline{\mathsf{mod}}$ for any positive integer ℓ and k, where α_ℓ (resp. γ_k) is the compositions of irreducible maps from $(0,i,j)$ to $(0,i,j+\ell)$ (resp. from $(0,i,j+\ell)$ to $(0,i+k,j+\ell)$).*

Proof Consider the triangle induced by the short exact sequence (2.1) as follows.

$$(0,i,j) \xrightarrow{(\underline{\alpha_\ell}, \underline{-\beta_k})^t} (0,i,j+\ell) \oplus (0,i+k,j) \xrightarrow{(\underline{\gamma_k}, \underline{\delta_\ell})} (0,i+k,j+\ell) \to (0,i,j)[1]. \tag{2.2}$$

Applying $\underline{\mathsf{Hom}}_A((0,i,j), -)$ to the triangle (2.2), we have a long exact sequence as follows.

$$\begin{aligned} \cdots \to \underline{((0,i,j),(0,i,j))} &\xrightarrow{(\underline{\alpha_\ell},\underline{\beta_k})^t_*} \underline{((0,i,j),(0,i,j+\ell)\oplus(0,i+k,j))} \\ &\xrightarrow{(\underline{\gamma_k},\underline{\delta_\ell})_*} \underline{(0,i,j),(0,i+k,j+\ell))} \to \cdots . \end{aligned}$$

Take the map $\alpha := (\underline{\alpha_\ell}, 0)^t : (0,i,j) \to (0,i,j+\ell) \oplus (0,i+k,j)$. Then we have

$$(\underline{\gamma_k}, \underline{\delta_\ell})_*(\alpha) = (\underline{\gamma_k}, \underline{\delta_\ell})(\underline{\alpha_\ell}, 0)^t = \underline{\gamma_k \alpha_\ell}.$$

On the contrary, we assume that $\underline{\gamma_k \alpha_\ell} = 0$. It then follows from the exactness of the above long exact sequence that there is a morphism $\varphi \in \underline{\mathrm{End}}_A((0,i,j))$ such that $(\underline{\alpha_\ell}, -\underline{\beta_k})^t \varphi = \alpha = (\underline{\alpha_\ell}, 0)^t$. Since $\underline{\mathrm{End}}_A((0,i,j)) \cong k$, φ is an isomorphism. This means that $\underline{\beta_k} = 0$ on A-$\underline{\mathsf{mod}}$, which contradicts Lemma 2.19.

$\square$

Remark 2.23. (1) *If $X_n \xrightarrow{f_n} \cdots \xrightarrow{f_3} X_2 \xrightarrow{f_2} X_1 \xrightarrow{f_1} X_0 = X$ be a sectional sequence of irreducible maps between indecomposable modules, then the composition $f_n \cdots f_2 f_1$ is non-zero on A-mod. Please refer to* [3, VII.2, Theorem 2.4].

(2) *There is a similar conclusion with Corollary 2.22 for a quasi-tube of a self-injective algebra. Please refer to* [7, Lemma 3.4].

We visualize the morphism in Lemma 2.22 as follows.

$$\begin{tikzcd}
 & & (0,i,j+\ell) \arrow[dr, dashed, "\gamma_k"] & \\
 & & & (0,i+k,j+\ell) \\
(0,i,j) \arrow[uurr, dashed, "\alpha_\ell"] \arrow[dr, dashed, "\beta_k"'] & & & \\
 & (0,i+k,j) \arrow[uurr, dashed, "\delta_\ell"'] & &
\end{tikzcd}$$

Corollary 2.24. *Let A be a 2-domestic Brauer graph algebra and X an indecomposable non-projective module on a connected component $\mathcal{C}$ of Γ_A. Then*

(1) *If $\mathcal{C}$ is acyclic, then we may assume $X = (0,i,j)$ for integers i and j. Then the compositions $\gamma_k\alpha_\ell$ of $\alpha_\ell : (0,i,j) \to (0,i,j+\ell)$ and $\gamma_k : (0,i,j+\ell) \to (0,i+k,j+\ell)$ is non-zero for any positive integer ℓ and k on A-$\underline{\mathsf{mod}}$, where α_ℓ (resp. γ_k) is the compositions of irreducible maps from $(0,i,j)$ to $(0,i,j+\ell)$ (resp. from $(0,i,j+\ell)$ to $(0,i+k,j+\ell)$).*

(2) *If $\mathcal{C}$ admits a cycle, then we may assume $X = Q(0,i,j)$ for integers i and j. Then the compositions $\gamma_k\alpha_\ell$ of $\alpha_\ell : Q(0,i,j) \to Q(0,i,j+\ell)$ and $\gamma_k : Q(0,i,j+\ell) \to Q(0,i+k,j+\ell)$ is non-zero for any positive integer ℓ and k on A-$\underline{\mathsf{mod}}$, where $0 < \ell < j$, and α_ℓ (resp. γ_k) is the compositions of irreducible maps from $Q(0,i,j)$ to $Q(0,i,j+\ell)$ (resp. from $Q(0,i,j+\ell)$ to $Q(0,i+k,j+\ell)$).*

Proof. The statement (1) follows from Corollary 2.13 and the proof of Lemma 2.22. The statement (2) is a direct consequence of [7, Lemma 3.4]. $\square$

Theorem 2.25. ([31, Theorem 4.17]) *Let A be a domestic Brauer graph algebra and $\mathcal{S}$ an orthogonal system on A-$\underline{\mathsf{mod}}$. Then $\mathcal{S}$ is a simple-minded system if and only if $\mathcal{S}$ satisfies the following conditions.*

(1) *$\mathcal{S}$ contains at least one non-periodic A-module.*

(2) *$\Omega(\mathcal{S})$ (or $\Omega^{-1}(\mathcal{S})$) is in $\mathcal{F}(\mathcal{S})$.*

3. The stable bi-perpendicular category of a stable brick

Starting from this section, we always assume that A is a 2-domestic Brauer graph algebra. Let Γ_0 and Γ_1 be the Euclidean components of the form $\mathbb{Z}\widetilde{A}_{p,q}$ and let Q_0 and Q_1 (resp. Q'_0 and Q'_1) be the quasi-tubes of rank q (resp. p). In this section, we determine the stable bi-perpendicular category of an object on an Euclidean component or on a quasi-tube of rank not one. We shall take three steps to compute the stable bi-perpendicular category.

We list some notations as follows. We use triple (a,b,c) of integers to represent a vertex for an Euclidean component. Note that two Euclidean components Γ_0 and Γ_1 satisfy the condition $\Omega({}_s\Gamma_0) = {}_s\Gamma_1$ and $\Omega({}_s\Gamma_1) = {}_s\Gamma_0$. Without loss of generality, we use the triple $(0,b,c)$ to represent a vertex of ${}_s\Gamma_0$ (resp.

the triple $(1, b, c)$ to represent a vertex of ${}_s\Gamma_1$) and we denote $\Omega((0, b, c))$ by $(1, b, c)$. By the action of AR-translation τ on those triples, $\tau(a, b, c) = (a, b-1, c-1)$. From the shape of Euclidean components, we know that vertices (a, b, c) and $(a, b-p\ell, c+q\ell)$ are equal on A-ind for every integer $\ell \in \mathbb{Z}$. For a family of objects $\mathcal{X}$ on A-$\underline{\mathsf{mod}}$, we define

$$\begin{aligned}
&\mathsf{Rsupp}\mathcal{X}\colon = \{Z \in A\text{-}\underline{\mathsf{mod}} \mid \underline{\mathsf{Hom}}_A(X, Z) \neq 0, \exists X \in \mathcal{X}\},\\
&\mathsf{Lsupp}\mathcal{X}\colon = \{Z \in A\text{-}\underline{\mathsf{mod}} \mid \underline{\mathsf{Hom}}_A(Z, X) \neq 0, \exists X \in \mathcal{X}\},\\
&{}^{\perp}\mathcal{X}^{\perp}\colon = \{Z \in A\text{-}\underline{\mathsf{mod}} \mid \underline{\mathsf{Hom}}_A(X, Z) = 0, \underline{\mathsf{Hom}}_A(Z, X) = 0, \forall X \in \mathcal{X}\}.
\end{aligned}$$

We say $\mathsf{Rsupp}\mathcal{X}$ is the **right support of** $\mathcal{X}$, $\mathsf{Lsupp}\mathcal{X}$ is the **left support of** $\mathcal{X}$ and ${}^{\perp}\mathcal{X}^{\perp}$ is the **stable bi-perpendicular category of** $\mathcal{X}$. Note that ${}^{\perp}\mathcal{X}^{\perp} = A\text{-}\underline{\mathsf{mod}} \backslash (\mathsf{Rsupp}\mathcal{X} \cup \mathsf{Lsupp}\mathcal{X})$. We still use the notations $\mathsf{Rsupp}X$, $\mathsf{Lsupp}X$ and ${}^{\perp}X^{\perp}$ for an object X on A-$\underline{\mathsf{mod}}$.

By Theorem 2.5, any object on a quasi-tube of rank one is not in a simple-minded system. Thus we need not consider the stable bi-perpendicular category of the objects on the quasi-tubes of rank one.

Lemma 3.1. *Let A be a 2-domestic Brauer graph algebra. Then either $\dim_k \underline{\mathsf{Hom}}_A(M, H)$ or $\dim_k \underline{\mathsf{Hom}}_A(H, M)$ is non-zero for any non-periodic, non-projective module M and any module H in a homogeneous tube.*

Proof. On the contrary, we assume that there are a non-periodic, non-projective module M_0 in Γ_0 and a module H_0 in a homogeneous tube such that $\underline{\mathsf{Hom}}_A(M_0, H_0) \cong 0$ and $\underline{\mathsf{Hom}}_A(H_0, M_0) \cong 0$. Since H_0 is in a homogeneous tube, $\tau^i(H_0) \cong H_0$ for every $i \in \mathbb{Z}$,

$$\underline{\mathsf{Hom}}_A(\tau^i(M_0), \tau^i(H_0)) \cong \underline{\mathsf{Hom}}_A(\tau^i(M_0), H_0) \cong 0,\ \underline{\mathsf{Hom}}_A(\tau^i(H_0), \tau^i(M_0)) \cong \underline{\mathsf{Hom}}_A(H_0, \tau^i(M_0)) \cong 0$$

for every $i \in \mathbb{Z}$. It is not hard to see that $\underline{\mathsf{Hom}}_A(M, H_0) \cong 0$ and $\underline{\mathsf{Hom}}_A(H_0, M) \cong 0$ for every object M in Γ_0. By Serre duality, $\underline{\mathsf{Hom}}_A(M', H_0) \cong 0$ and $\underline{\mathsf{Hom}}_A(H_0, M') \cong 0$ for every object M' in Γ_1. Thus H_0 is in the stable bi-perpendicular category of every object on all Euclidean components. By Theorem 2.5, every simple-minded system $\mathcal{S}$ is contained in $\Gamma_0 \cup \Gamma_1 \cup Q_0 \cup Q_1 \cup Q'_0 \cup Q'_1$. By Proposition 2.18, H_0 is in the stable bi-perpendicular category of every object on all quasi-tubes of rank not one. Hence $H_0 \in {}^{\perp}\mathcal{S}^{\perp}$. It is a contradiction. Thus our conclusion is true. $\square$

3.1. **The intersection of the stable bi-perpendicular category of an object (0,a,b) and an Euclidean component.** Take an object $(0, a, b)$ on ${}_s\Gamma_0$. By Lemma 3.1, ${}^{\perp}(0, a, b)^{\perp} \cap {}_s\mathcal{H}$ is an empty set for every homogeneous tube $\mathcal{H}$. Now we compute ${}^{\perp}(0, a, b)^{\perp} \cap {}_s\Gamma_0$, and then consider ${}^{\perp}(0, a, b)^{\perp} \cap {}_s\Gamma_1$ by Serre duality. We first give a definition as follows.

Definition 3.2. *Take two objects $X = (i, m, n)$ and $Y = (i, j, k)$ on Γ_i with $i = 0, 1$, $m \leq j$ and $n \geq k$ and let $\square_{X,Y}\colon = \{(i, e, f) \mid m \leq e \leq j, k \leq f \leq n\}$. $\square_{X,Y}$ is said to be* **the rectangle area of vertices X and Y**.

Note that $\square_{X,Y}$ consists of the vertices which are contained in the rectangle area with vertices $X = (i, m, n)$, (i, m, k), (i, j, n) and $Y = (0, j, k)$.

Step one: Compute ${}^{\perp}(0, a, b)^{\perp} \cap {}_s\Gamma_0$.

By Corollary 2.13 and Corollary 2.24, the right (resp. left) support of vertex $(0, a, b)$ on ${}_s\Gamma_0$ states as follows.

$$\begin{aligned}
\mathsf{Rsupp}(0, a, b) \cap {}_s\Gamma_0 =& \{(0, i, j) \mid i \geq a, j \geqslant b\},\\
\mathsf{Lsupp}(0, a, b) \cap {}_s\Gamma_0 =& \{(0, i, j) \mid i \leq a, j \leq b\}.
\end{aligned} \tag{3.1}$$

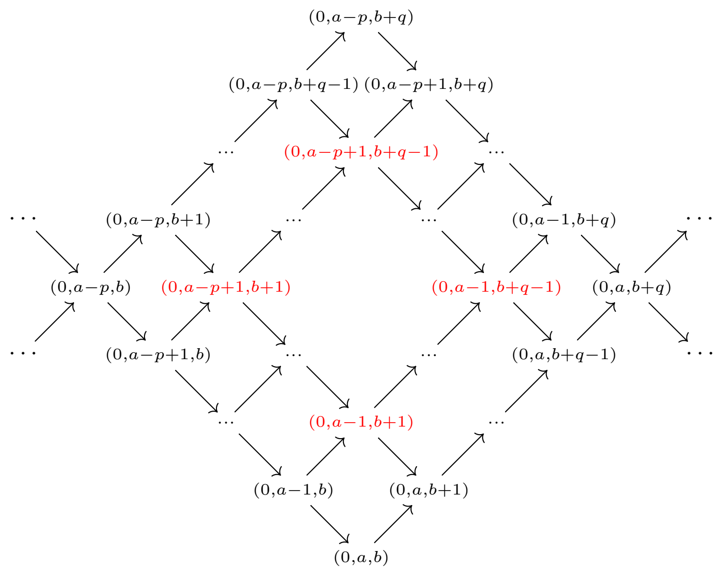
$$\begin{aligned}
\mathsf{Rsupp}(0, a-k, b-\ell) \cap {}_s\Gamma_0 =& \{(0, i, j) \mid i \geq a-k, j \geqslant b-\ell\},\\
\mathsf{Lsupp}(0, a-k, b-\ell) \cap {}_s\Gamma_0 =& \{(0, i, j) \mid i \leq a-k, j \leq b-\ell\}.
\end{aligned} \tag{3.2}$$

Thus the stable bi-perpendicular category of $(0, a, b)$ on ${}_s\Gamma_0$ states as follows.

$$\begin{aligned}
\mathcal{M}^0_{(0,a,b)}\colon =& {}^{\perp}(0, a, b)^{\perp} \cap {}_s\Gamma_0\\
=& {}_s\Gamma_0 \backslash ((\mathsf{Rsupp}(0, a, b) \cup \mathsf{Lsupp}(0, a, b)) \cap {}_s\Gamma_0)\\
=& \{(0, i, j) \mid a-p < i < a, b < j < b+q\}\\
=& \square_{(0,a-p+1,b+q-1),(0,a-1,b+1)}.
\end{aligned} \tag{3.3}$$

$$
\begin{aligned}
\mathcal{M}^0_{(0,a-k,b-\ell)}: \ &= {}^{\perp}(0,a-k,b-\ell)^{\perp} \cap {}_s\Gamma_0 \\
&= {}_s\Gamma_0 \backslash ((\mathsf{Rsupp}(0,a-k,b-\ell) \cup \mathsf{Lsupp}(0,a-k,b-\ell)) \cap {}_s\Gamma_0) \\
&= \{(0,i,j) \mid a-k-p < i < a-k, b-\ell < j < b-\ell+q\} \\
&= \square_{(0,a-k-p+1,b-\ell+q-1),(0,a-k-1,b-\ell+1)}.
\end{aligned} \tag{3.4}
$$

Note that $\mathsf{Rsupp}(0,a,b) = \mathsf{Rsupp}(0,a-p,b+q)$ and $\mathsf{Lsupp}(0,a,b) = \mathsf{Lsupp}(0,a-p,b+q)$. The following diagram provides a visual depiction of ${}^{\perp}(0,a,b)^{\perp} \cap {}_s\Gamma_0$ on ${}_s\Gamma_0$ as follows.



Step two: compute ${}^{\perp}(0,a-k,b-\ell)^{\perp} \cup {}_s\Gamma_1$ for $\ell \in \mathbb{Z}$.

By Serre duality,

$$\underline{\mathsf{Hom}}_A(X,Y) \cong \mathsf{D}\underline{\mathsf{Hom}}_A(Y,\nu\Omega(X)) \cong \mathsf{D}\underline{\mathsf{Hom}}_A(Y,\Omega(X)) \cong \mathsf{D}\underline{\mathsf{Hom}}_A(\Omega^{-1}(Y),X).$$

Thus the right support of vertex $(0,a,b)$ on ${}_s\Gamma_1$ is as follows.

$$
\begin{aligned}
\mathsf{Rsupp}(0,a,b) \cap {}_s\Gamma_1 &= \mathsf{Lsupp}\Omega(0,a,b) \cap {}_s\Gamma_1 \\
&= \mathsf{Lsupp}(1,a,b) \cap {}_s\Gamma_1 \\
&= \{(1,i,j) \mid i \leq a, j \leq b\}.
\end{aligned} \tag{3.5}
$$

Similarly, we have the left support of vertex $(0,a,b)$ on ${}_s\Gamma_1$ as follows.

$$
\begin{aligned}
\mathsf{Lsupp}(0,a,b) \cap {}_s\Gamma_1 &= \mathsf{Rsupp}\Omega^{-1}(0,a,b) \cap {}_s\Gamma_1 \\
&= \mathsf{Rsupp}(1,a+1,b+1) \cap {}_s\Gamma_1 \\
&= \{(1,i,j) \mid i \geq a+1, j \geq b+1\}.
\end{aligned} \tag{3.6}
$$

In general,

$$
\begin{aligned}
\mathsf{Rsupp}(0,a-k,b-\ell) \cap {}_s\Gamma_1 &= \mathsf{Lsupp}\Omega(0,a-k,b-\ell) \cap {}_s\Gamma_1 \\
&= \mathsf{Lsupp}(0,a-k,b-\ell) \cap {}_s\Gamma_1 \\
&= \{(1,i,j) \mid i \leq a-k, j \leq b-\ell\}.
\end{aligned} \tag{3.7}
$$

$$
\begin{aligned}
\mathsf{Lsupp}(0,a-k,b-\ell) \cap {}_s\Gamma_1 =& \mathsf{Rsupp}\Omega^{-1}(0,a-k,b-\ell) \cap {}_s\Gamma_1 \\
&= \mathsf{Rsupp}(1,a-k+1,b-\ell+1) \cap {}_s\Gamma_1 \\
&= \{(1,i,j) \mid i \geq a-k+1, j \geq b-\ell+1\}.
\end{aligned} \tag{3.8}
$$

Thus the stable bi-perpendicular category of $(0,a,b)$ on ${}_s\Gamma_1$ states as follows. By equations (3.5) and (3.6),

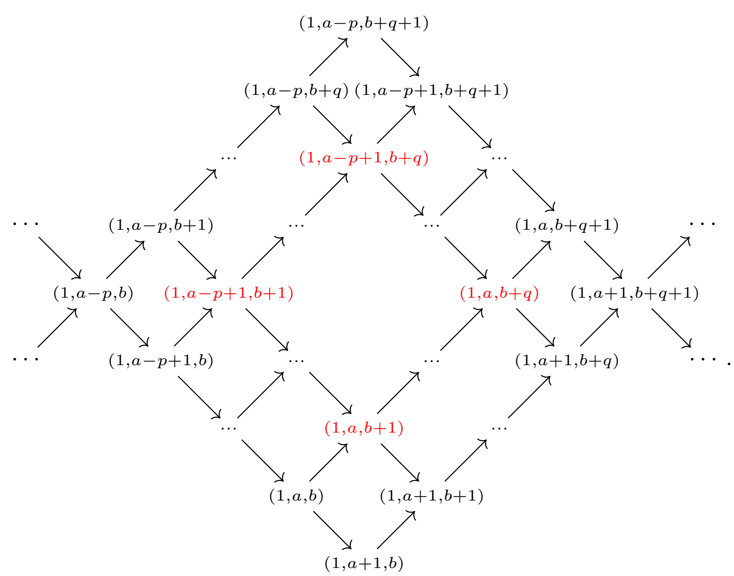
$$
\begin{aligned}
\mathcal{M}^1_{(0,a,b)}: \ &= {}^{\perp}(0,a,b)^{\perp} \cap {}_s\Gamma_1 \\
&= {}_s\Gamma_1 \backslash ((\mathsf{Rsupp}(0,a,b) \cup \mathsf{Lsupp}(0,a,b)) \cap {}_s\Gamma_1) \\
&= \{(1,i,j) \mid a-p+1 \leq i \leq a, b+1 \leq j \leq b+q\} \\
&= \square_{(1,a-p+1,b+q),(1,a,b+1)}.
\end{aligned} \tag{3.9}
$$

Note that $\mathcal{M}^1_{(0,a,b)} = {}^{\perp}(1,a+1,b)^{\perp} \cap {}_s\Gamma_1$.

Thus By equations (3.7) and (3.8),

$$
\begin{aligned}
\mathcal{M}^1_{(0,a-k,b-\ell)}: \ &= {}^{\perp}(0,a-k,b-\ell)^{\perp} \cap {}_s\Gamma_1 \\
&= {}_s\Gamma_1 \backslash ((\mathsf{Rsupp}(0,a-k,b-\ell) \cup \mathsf{Lsupp}(0,a-k,b-\ell)) \cap {}_s\Gamma_1) \\
&= \{(1,i,j) \mid a-k-p+1 \leq i \leq a-k, b-\ell+1 \leq j \leq b-\ell+q\} \\
&= \square_{(1,a-k-p+1,b-\ell+q),(1,a-k,b-\ell+1)}.
\end{aligned} \tag{3.10}
$$

The following diagram shows us the position of $\mathcal{M}^1_{(0,a,b)}$ (the rectangle area with red vertices) on Γ_1.



3.2. **The intersection of the stable bi-perpendicular category of** $(0,a,b)$ **and a quasi-tube.** In this subsection, we determine the intersection of ${}^{\perp}(0,a-k,b-\ell)^{\perp}$ and quasi-tube of rank not one. We need some preparations as follows.

Proposition 3.3. ([18, Corollary 3.5]) *Let A be a self-injective artin algebra, and let $0 \to X \to Y \xrightarrow{f} Z \to 0$ be a short exact sequence on A-*mod*, which is not almost split. Suppose that one of the modules Y and Z is indecomposable and f is irreducible. Then the following conditions are equivalent for a natural number $n > 1$:*

(1) $\alpha(\tau^{-1}(X)) = n$.
(2) *X is simple and* $\mathsf{rad}P/\mathsf{soc}P$ *decomposes into n summands, where P denotes the projective cover of X.*
(3) *X is simple and* $\mathsf{rad}I/\mathsf{soc}I$ *decomposes into n summands, where I denotes the injective envelope of X.*

Moreover, if X satisfies these conditions, then Y is either indecomposable projective or isomorphic to the radical of some indecomposable projective module.

Let A be an algebra and $\mathcal{C}$ a component of AR-quiver of A. A **coray point** of $\mathcal{C}$ is defined to be a point X of $\mathcal{C}$ such that there exists an infinite sectional path in $\mathcal{C}$

$$\cdots \to [r+1]X \to [r]X \to \cdots \to [2]X \to [1]X = X.$$

ending with X and containing all sectional paths ending with X. The corresponding A-module X is called the **coray module.**

Proposition 3.4. ([31, Proposition 4.12]) *Let A be representation-infinite self-injective algebra. Given a triangle*

$$X \to Y \xrightarrow{f} Z \to X[1] \tag{3.11}$$

in A-$\underline{\mathsf{mod}}$, where both Y and Z are indecomposable, non-projective, and $f : Y \to Z$ is an irreducible map on A-mod. Then X (resp. $X[1]$) is a coray point of some stable connected AR-component.

Proof. Note that a point is a ray point if and only it is a coray point in the stable AR-quiver of a self-injective algebra. Let X be a point in the stable connected component ${}_s\Gamma_1$ of stable AR-quiver of A. We only prove that X is a coray point of ${}_s\Gamma_1$, since $[1] : A$-$\underline{\mathsf{mod}} \to A$-$\underline{\mathsf{mod}}$ is an equivalence. We assume that the triangle (3.11) is induced by a short exact sequence as follows.

$$0 \to X \to Y \oplus P \xrightarrow{(f,h)} Z \to 0. \tag{3.12}$$

Where P is a projective A-module. If the sequence (3.12) is an almost split sequence, then Y is isomorphic to $\mathsf{rad}(P)/\mathsf{soc}(P)$ and P is indecomposable. Since Y is indecomposable, $\alpha(\tau^{-1}(X)) = 2$. Thus X is a coray point on ${}_s\Gamma_1$.

Now we assume that the short exact sequence (3.12) is not an almost split sequence.

Case one: P is isomorphic to zero.

We shall show that X is a coray point on ${}_s\Gamma_1$. On the contrary, we assume that X is not a coray point of ${}_s\Gamma_1$. Take $\alpha(\tau^{-1}(X)) = n > 1$. By Proposition 3.3, Y is either indecomposable projective or isomorphic to the radical of some indecomposable projective module. The former case contradicts the condition that Y is non-projective. For the latter case, Y is isomorphic to the radical $\mathsf{rad}P$ of an indecomposable projective P. By the equivanent conditions of Proposition 3.3, X is a simple module and $X \cong \mathsf{soc}P$. Therefore Z is isomorphic to $\mathsf{rad}P/\mathsf{soc}P$ and it is decomposable (with n direct summands). This contradicts the condition that Z is indecomposable. Hence $\alpha(\tau^{-1}(X)) = 1$ and then X is a coray point of ${}_s\Gamma_1$.

Case two: P is not isomorphic to zero.

Since f is an irreducible map, f is a monomorphism or an epimorphism. If f is a monomorphism, then we take a short exact sequence as follows.

$$0 \to Y \xrightarrow{f} Z \to \mathsf{Coker}(f) \to 0. \tag{3.13}$$

It can be proved dually that $\mathsf{Coker}(f)$ is a ray point on ${}_s\Gamma_1$. Considering the triangle induced by short exact sequence (3.13), we have $X[1] \cong \mathsf{Coker}(f)$ on A-$\underline{\mathsf{mod}}$. Thus X is a coray point on ${}_s\Gamma_1$.

If f is an epimorphism, then we take a short exact sequence as follows.

$$0 \to \mathsf{Ker}(f) \to Y \xrightarrow{f} Z \to 0. \tag{3.14}$$

It is similar to the Case one that $\mathsf{Ker}(f)$ is a coray point on ${}_s\Gamma_1$. Considering the triangle induced by short exact sequence (3.14), we have $X \cong \mathsf{Ker}(f)$ on A-$\underline{\mathsf{mod}}$. Thus X is a coray point on ${}_s\Gamma_1$. $\square$

By the shape of Euclidean component, there are precise two irreducible maps ended at vertex $(0, a, b)$ on ${}_s\Gamma_0$.

$$(0, a, b-1) \to (0, a, b), \quad (0, a-1, b) \to (0, a, b).$$

Extending them to triangles as follows.

$$(0, a, b-1) \to (0, a, b) \to Z_0 \to (0, a, b-1)[1],$$
$$(0, a-1, b) \to (0, a, b) \to Z_1 \to (0, a-1, b+1)[1].$$

Dually, there are precise two irreducible maps started at vertex $(0, a, b)$ in ${}_s\Gamma_0$.

$$(0, a, b) \to (0, a, b+1), \quad (0, a, b) \to (0, a+1, b).$$

Extending them to triangles as follows.

$$Z'_0 \to (0, a, b) \to (0, a, b+1) \to Z'_0[1],$$
$$Z'_1 \to (0, a, b) \to (0, a+1, b) \to Z'_1[1].$$

By Proposition 3.4, both Z_0 and Z_1 (resp. Z'_0 and Z'_1) are quasi-simples. Note that one of them is on a quasi-tube of rank p and the other one is on a quasi-tube of rank q. Without loss of generality, we assume Z'_0 (resp. Z'_1) is contained on the quasi-tube Q_0 (resp. Q'_0) of rank q (resp. p). Note that $Z_0 = \Omega(Z'_0)$ and $Z_1 = \Omega(Z'_1)$. We denote Z'_0 (resp. Z'_1) by $Q(0, 0, 0)$ (resp. $Q(0, 0, 0)'$). Thus $Z_0 = \Omega(Z'_0) = Q(1, 0, 0)$ and $Z_1 = \Omega(Z'_1) = Q(1, 0, 0)'$. Take $\tau Q(0, i, 0) = Q(0, i-1, 0)$ and $\tau Q(0, i, 0)' = Q(0, i-1, 0)'$ for any integer i.

Thus every non-projective vertex on a quasi-tube is labeled by $Q(i,j,k)$ or $Q(i,j,k)'$, where $i=0,1$ and $j,k\in\mathbb{Z}$. Note that $Q(i,j,k)=Q(i,j,k+q\ell)$ and $Q(i,j,k)'=Q(i,j,k+p\ell)'$ for $\ell\in\mathbb{Z}$.

Lemma 3.5. ([30, Lemma 3.8]) *Let A be a self-injective algebra and let M, N be two indecomposable, non-projective modules. Let $M_n \xrightarrow{f_n} \cdots \xrightarrow{f_3} M_2 \xrightarrow{f_2} M_1 \xrightarrow{f_1} M_0 = M$ be a sectional path ending at M, and define S_i by extending $M_i \xrightarrow{f_i} M_{i-1}$ to a triangle*

$$M_i \xrightarrow{f_i} M_{i-1} \to S_i \to M_i[1].$$

Then we have $\tau(S_{i-1})=S_i$ for $i=2,\cdots,n$.

Proposition 3.6. ([3, I. Section 5, Corollary 5.7]) *Let A be an artin algebra. Take a short exact sequence on A-mod as follows.*

$$0\to X \xrightarrow{(f_1,f_2)^t} Y_1\oplus Y_2 \xrightarrow{(g_1,g_2)} Z\to 0.$$

Then the following conclusion holds.

(1)

$$\begin{tikzcd} X \arrow[r, "f_1"] \arrow[d, "-f_2"'] & Y_1 \arrow[d, "g_1"] \\ Y_2 \arrow[r, "g_2"'] & Z \end{tikzcd}$$

is both a pushout and a pullback diagram.

(2) $\mathsf{Coker}(f_1)\cong\mathsf{Coker}(g_2)$, $\mathsf{Coker}(f_2)\cong\mathsf{Coker}(g_1)$.

Proposition 3.7. ([2, Proposition 2.2]) *Let A be a symmetric special biserial algebra, M a string module on A-mod and let Φ be the set of quasi-simples which are not band modules. Then*

$$\sum_{X\in\Phi}\mathsf{dim}_{\mathsf{k}}\underline{\mathsf{Hom}}_A(M,X)=\sum_{X\in\Phi}\mathsf{dim}_{\mathsf{k}}\underline{\mathsf{Hom}}_A(X,M)=2.$$

Definition 3.8. *Let A be a self-injective algebra and $X=(j,k)$ a τ-periodic A-module. The* **wing** *of X is the set of isoclasses of indecomposable non-projective modules on a quasi-tube given by*

$$\mathcal{W}_X := \{(m,\ell)\mid m\le j,\ \ j+k\le m+\ell\}.$$

According to the equation of dimension of stable Hom-space provided by Erdmann and Kerner [14, 2.2],

$$\begin{aligned} &\mathsf{Rsupp}(0,a,b)\cap({}_sQ_1\cup{}_sQ_1')=\mathcal{W}_{Q(1,0,0)}\cup\mathcal{W}_{Q(1,0,0)'},\\ &\mathsf{Lsupp}(0,a,b)\cap({}_sQ_1\cup{}_sQ_1')=\{0\}. \end{aligned}\tag{3.15}$$

$$\begin{aligned} &\mathsf{Lsupp}(0,a,b)\cap({}_sQ_0\cup{}_sQ_0')=\mathcal{W}_{Q(0,0,0)}\cup\mathcal{W}_{Q(0,0,0)'},\\ &\mathsf{Rsupp}(0,a,b)\cap({}_sQ_0\cup{}_sQ_0')=\{0\}. \end{aligned}\tag{3.16}$$

Where $\mathcal{W}_{Q(i,j,0)}$ is the wing of quasi-simple $Q(i,j,0)$ for $i=0,1$ and $j\in\mathbb{Z}$. See also [7, Lemma 5.12].

$$\mathsf{Rsupp}(0,a-\ell,b)\cap{}_sQ_1=\mathcal{W}_{Q(1,0,0)},\quad \mathsf{Rsupp}(0,a-\ell,b)\cap{}_sQ_1'=\mathcal{W}_{\tau^\ell Q(1,0,0)'}.\tag{3.17}$$

Thus

$$\mathsf{Rsupp}(0,a-\ell,b)\cap({}_sQ_1\cup{}_sQ_1')=\mathcal{W}_{Q(1,0,0)}\cup\mathcal{W}_{\tau^\ell Q(1,0,0)'}.\tag{3.18}$$

$$\mathsf{Rsupp}(0,a,b-\ell)\cap{}_sQ_1=\mathcal{W}_{\tau^\ell Q(1,0,0)},\quad \mathsf{Rsupp}(0,a,b-\ell)\cap{}_sQ_1'=\mathcal{W}_{Q(1,0,0)'}.\tag{3.19}$$

$$\mathsf{Rsupp}(0,a,b-\ell)\cap({}_sQ_1\cup{}_sQ_1')=\mathcal{W}_{\tau^\ell Q(1,0,0)}\cup\mathcal{W}_{Q(1,0,0)'}.\tag{3.20}$$

Thus

$$\begin{aligned} &\mathsf{Rsupp}(0,a-k,b-\ell)\cap({}_sQ_1\cup{}_sQ_1')=\mathcal{W}_{\tau^\ell Q(1,0,0)}\cup\mathcal{W}_{\tau^k Q(1,0,0)'},\\ &\mathsf{Lsupp}(0,a-k,b-\ell)\cap({}_sQ_1\cup{}_sQ_1')=\{0\}. \end{aligned}\tag{3.21}$$

Similarly, we have description of the left support as follows.

$$\mathsf{Lsupp}(0,a-\ell,b)\cap{}_sQ_0=\mathcal{W}_{Q(0,0,0)},\quad \mathsf{Lsupp}(0,a-\ell,b)\cap{}_sQ_0'=\mathcal{W}_{\tau^\ell Q(0,0,0)'}.\tag{3.22}$$

Thus

$$\mathsf{Lsupp}(0,a-\ell,b)\cap({}_sQ_0\cup{}_sQ_0')=\mathcal{W}_{Q(0,0,0)}\cup\mathcal{W}_{\tau^\ell Q(0,0,0)'}. \tag{3.23}$$

$$\mathsf{Lsupp}(0,a,b-\ell)\cap{}_sQ_0=\mathcal{W}_{\tau^\ell Q(0,0,0)},\quad \mathsf{Lsupp}(0,a,b+\ell)\cap{}_sQ_0'=\mathcal{W}_{Q(0,0,0)'}, \tag{3.24}$$

Thus

$$\mathsf{Lsupp}(0,a,b-\ell)\cap({}_sQ_0\cup{}_sQ_0')=\mathcal{W}_{\tau^\ell Q(0,0,0)}\cup\mathcal{W}_{Q(0,0,0)'}. \tag{3.25}$$

Thus

$$\begin{aligned}&\mathsf{Lsupp}(0,a-k,b-\ell)\cap({}_sQ_0\cup{}_sQ_0')=\mathcal{W}_{\tau^\ell Q(0,0,0)}\cup\mathcal{W}_{\tau^k Q(0,0,0)'},\\ &\mathsf{Rsupp}(0,a-k,b-\ell)\cap({}_sQ_0\cup{}_sQ_0')=\{0\}.\end{aligned} \tag{3.26}$$

Thus by equation (3.15),

$$\begin{aligned}\mathcal{Q}^1_{(0,a,b)}:&={}^{\perp}(0,a,b)^{\perp}\cap({}_sQ_1\cup{}_sQ_1')\\ &=({}_sQ_1\cup{}_sQ_1')\backslash((\mathsf{Rsupp}(0,a,b)\cup\mathsf{Lsupp}(0,a,b))\cap({}_sQ_1\cup{}_sQ_1'))\\ &=\{Q(1,i,j)\mid 1\le i\le q-1,0\le j\le q-2,i+j\le q-1\}\\ &\cup\{Q(1,i,j)'\mid 1\le i\le p-1,0\le j\le p-2,i+j\le p-1\}.\end{aligned} \tag{3.27}$$

Thus by equation (3.16),

$$\begin{aligned}\mathcal{Q}^0_{(0,a,b)}:&={}^{\perp}(0,a,b)^{\perp}\cap({}_sQ_0\cup{}_sQ_0')\\ &=({}_sQ_0\cup{}_sQ_0')\backslash((\mathsf{Rsupp}(0,a,b)\cup\mathsf{Lsupp}(0,a,b))\cap({}_sQ_0\cup{}_sQ_0'))\\ &=\{Q(0,i,j)\mid 1\le i\le q-1,0\le j\le q-2,i+j\le q-1\}\\ &\cup\{Q(0,i,j)'\mid 1\le i\le p-1,0\le j\le p-2,i+j\le p-1\}.\end{aligned} \tag{3.28}$$

Thus by equation (3.21),

$$\begin{aligned}\mathcal{Q}^1_{(0,a-k,b-\ell)}:&={}^{\perp}(0,a-k,b-\ell)^{\perp}\cap({}_sQ_1\cup{}_sQ_1')\\ &=({}_sQ_1\cup{}_sQ_1')\backslash((\mathsf{Rsupp}(0,a-k,b-\ell)\cup\mathsf{Lsupp}(0,a-k,b-\ell))\cap({}_sQ_1\cup{}_sQ_1'))\\ &=\{Q(1,i,j)\mid -\overline{\ell+1}\le i\le -\overline{\ell+q-1},0\le j\le q-2,i+j\le q-1\}\\ &\quad\cup\{Q(1,i,j)'\mid -\overline{k+1}\le i\le -\overline{k+p-1},0\le j\le p-2,i+j\le p-1\}.\end{aligned} \tag{3.29}$$

Thus by equation (3.26),

$$\begin{aligned}\mathcal{Q}^0_{(0,a-k,b-\ell)}:&={}^{\perp}(0,a-k,b-\ell)^{\perp}\cap({}_sQ_0\cup{}_sQ_0')\\ &=({}_sQ_0\cup{}_sQ_0')\backslash((\mathsf{Rsupp}(0,a-k,b-\ell)\cup\mathsf{Lsupp}(0,a-k,b-\ell))\cap({}_sQ_0\cup{}_sQ_0'))\\ &=\{Q(0,i,j)\mid -\overline{\ell+1}\le i\le -\overline{\ell+q-1},0\le j\le q-2,i+j\le q-1\}\\ &\cup\{Q(0,i,j)'\mid -\overline{k+1}\le i\le -\overline{k+p-1},0\le j\le p-2,i+j\le p-1\}.\end{aligned} \tag{3.30}$$

${}^{\perp}(0,a,b)^{\perp}\cap{}_sQ_0$ is depicted as the triangle area with red vertices in the following diagram.

$$\begin{tikzcd}[column sep=tiny]
&&&&& Q(0,0,p) \arrow[dr] &&&&& \\
&&&& Q(0,0,p-1) \arrow[ur] \arrow[dr] && Q(0,1,p-1) \arrow[dr] &&&& \\
&&& \cdots \arrow[ur] \arrow[dr] && {\color{red}Q(0,1,p-2)} \arrow[ur] \arrow[dr] && \cdots \arrow[dr] &&& \\
&& \cdots \arrow[ur] \arrow[dr] && \cdots \arrow[ur] && \cdots \arrow[ur] \arrow[dr] && \cdots \arrow[dr] && \\
& Q(0,0,1) \arrow[ur] \arrow[dr] && Q(0,1,1) \arrow[ur] && \cdots && Q(0,p-2,1) \arrow[ur] \arrow[dr] && Q(0,p-1,1) \arrow[dr] & \\
Q(0,0,0) \arrow[ur] && {\color{red}Q(0,1,0)} \arrow[ur] &&&&&& {\color{red}Q(0,p-1,0)} \arrow[ur] && Q(0,p,0).
\end{tikzcd}$$

Finally,

$$^{\perp}(0,a,b)^{\perp} \cap {}_s\Gamma_A = \mathcal{M}^0_{(0,a,b)} \cup \mathcal{M}^1_{(0,a,b)} \cup \mathcal{Q}^0_{(0,a,b)} \cup \mathcal{Q}^1_{(0,a,b)}, \tag{3.31}$$

$$^{\perp}(0,a-k,b-\ell)^{\perp} \cap {}_s\Gamma_A = \mathcal{M}^0_{(0,a-k,b-\ell)} \cup \mathcal{M}^1_{(0,a-k,b-\ell)} \cup \mathcal{Q}^0_{(0,a-k,b-\ell)} \cup \mathcal{Q}^1_{(0,a-k,b-\ell)}. \tag{3.32}$$

3.3. **The stable bi-perpendicular category of an object on a quasi-tube of rank not 1.** By Theorem 2.5, Take a stable brick $Q(0,c,d)$ on the quasi-tube ${}_sQ_0$. By Theorem 2.5, if $d \geq q-1$, then $Q(0,c,d)$ is not contained in a simple-minded system. Without loss of generality, we assume the non-negative integer $d \leq q-2$. In this subsection, we shall determine the intersection of $^{\perp}Q(0,c,d)^{\perp}$ and quasi-tubes of rank not 1 or Euclidean components.

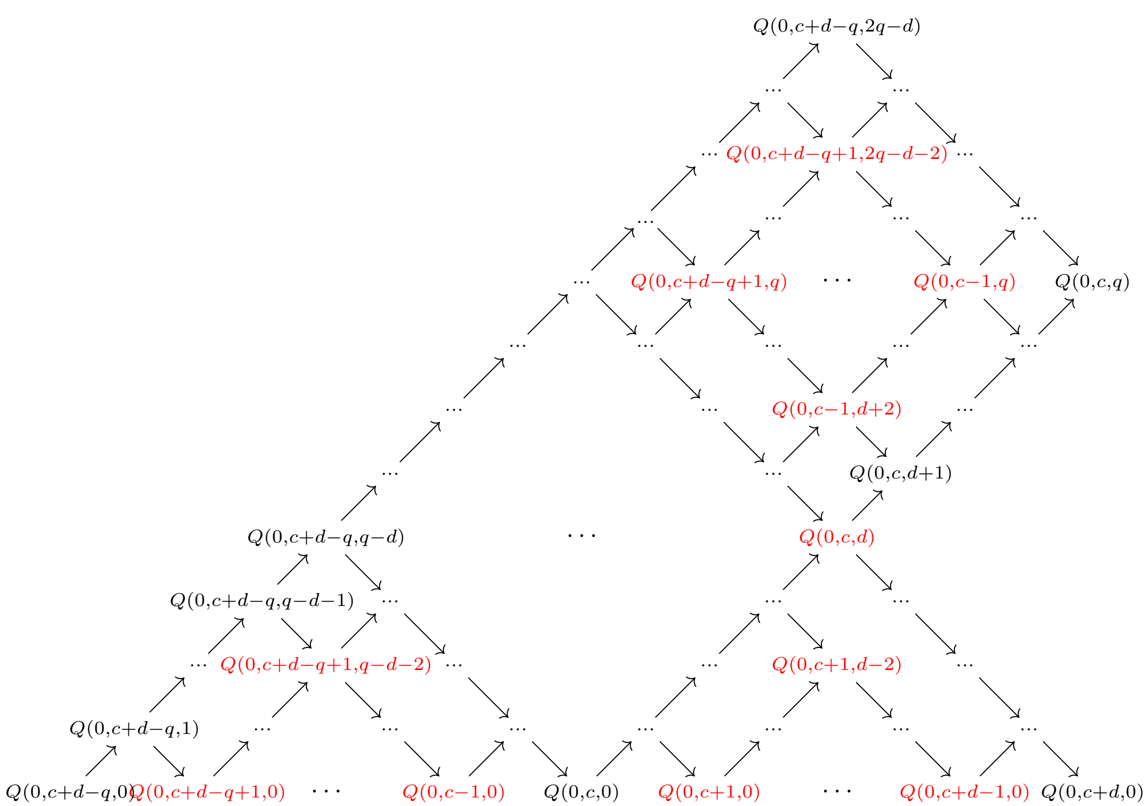
Step one: compute $^{\perp}Q(0,c,d)^{\perp} \cap {}_sQ_0$.

$$\begin{aligned} &\mathsf{Rsupp}Q(0,c,d) \cap {}_sQ_0 = \{Q(0,i,j) \mid c \leq i \leq c+d, c+d \leq i+j\},\\ &\mathsf{Lsupp}Q(0,c,d) \cap {}_sQ_0 = \{Q(0,i,j) \mid i \leq c, c \leq i+j \leq c+d\}. \end{aligned} \tag{3.33}$$

$$\begin{aligned} \mathcal{N}^0_{Q(0,c,d)} :\ &= {}^{\perp}Q(0,c,d)^{\perp} \cap {}_sQ_0\\ &= {}_sQ_0 \backslash ((\mathsf{Rsupp}Q(0,c,d) \cup \mathsf{Lsupp}Q(0,c,d)) \cap {}_sQ_0)\\ &= \{Q(0,i,j) \mid c+1 \leq i \leq c+d-1, i+j \leq c+d-1\}\\ &\quad \cup \{Q(0,i,j) \mid c+d-q+1 \leq i \leq c-1, i+j \leq c-1\}\\ &\quad \cup \{Q(0,i,j) \mid c+d-q+1 \leq i \leq c-1, c+d+1 \leq i+j \leq c+q-1\}. \end{aligned} \tag{3.34}$$

$\mathcal{N}^0_{Q(0,c,d)}$ is a set of vertices which consists of

(1) The rectangle with vertices $Q(1,c+d-q+1,q-d)$, $Q(0,c+d-q+1,2q-d-2)$, $Q(0,c-1,q)$ and $Q(0,c-1,d+2)$.
(2) The triangle with vertices $Q(0,c+d-q+1,0)$, $Q(0,c+d-q+1,q-d-2)$ and $Q(0,c-1,0)$.
(3) The triangle with vertices $Q(0,c+1,0)$, $Q(0,c+1,d-2)$ and $Q(0,c+d-1,0)$.



Note that, by Theorem 2.5, any object in the subset $\{Q(0,i,j) \mid c+d-q+1 \leq i \leq c-1, q \leq j, c+d+1 \leq p+q \leq c-1+q\}$ of (1) is not contained in a simple-minded system on A-$\underline{\mathsf{mod}}$. Please see the location of $\mathcal{N}^0_{Q(0,c,d)}$ in the above diagram.

Step two: compute ${}^{\perp}Q(0,c,d)^{\perp} \cap {}_sQ_1$.

By Serre duality,

$$\underline{\mathsf{Hom}}_A(X,Y) \cong \mathsf{D}\underline{\mathsf{Hom}}_A(Y,\nu\Omega(X)) \cong \mathsf{D}\underline{\mathsf{Hom}}_A(Y,\Omega(X)) \cong \mathsf{D}\underline{\mathsf{Hom}}_A(\Omega^{-1}(Y),X).$$

Thus

$$\begin{aligned} \mathsf{Rsupp}Q(0,c,d) \cap {}_sQ_1 =& \mathsf{Lsupp}\Omega Q(0,c,d) \cap {}_sQ_1 \\ =& \{Q(1,i,j) \mid i \leq c, c \leq i+j \leq c+d\}, \\ \mathsf{Lsupp}Q(0,c,d) \cap {}_sQ_1 =& \mathsf{Rsupp}\Omega^{-1}Q(0,c,d) \cap {}_sQ_1 \\ =& \{Q(1,i,j) \mid c+1 \leq i \leq c+d+1, c+d+1 \leq i+j\}. \end{aligned} \tag{3.35}$$

$$\begin{aligned} \mathcal{N}^1_{Q(0,c,d)} :=& {}^{\perp}Q(0,c,d)^{\perp} \cap {}_sQ_1 \\ =& {}_sQ_1 \backslash ((\mathsf{Rsupp}Q(0,c,d) \cup \mathsf{Lsupp}Q(0,a,b)) \cap {}_sQ_1) \\ =& {}_sQ_1 \backslash ((\mathsf{Lsupp}\Omega(0,c,d) \cup \mathsf{Rsupp}\Omega^{-1}Q(0,c,d)) \cap {}_sQ_1) \\ =& \{Q(1,i,j) \mid c+1 \leq i \leq c+d, i+j \leq c+d\} \\ & \cup \{Q(1,i,j) \mid c+d-q+2 \leq i \leq c-1, i+j \leq c-1\} \\ & \cup \{Q(1,i,j) \mid c+d-q+2 \leq i \leq c-1, c+d+1 \leq i+j \leq c+q-1\}. \end{aligned} \tag{3.36}$$

Step three: compute ${}^{\perp}Q(0,c,d)^{\perp} \cap {}_s\Gamma_0 \cup {}_s\Gamma_1$.

Now we determine ${}^{\perp}Q(0,c,d)^{\perp} \cap {}_s\Gamma_0$ for $Q(0,c,d)$. Take a set of quasi-simples

$$\Sigma_{Q(0,c,d)} := \{Q(0,c,0), Q(0,c+1,0), \cdots, Q(0,c+d,0)\}.$$

Take an object $(0,i,j)$ on ${}_s\Gamma_0$. By [14, 2.2],

$$\dim_k \underline{\mathsf{Hom}}_A((0,i,j),Q(0,c,d)) = \Sigma_{\ell=0}^{d} \dim_k \underline{\mathsf{Hom}}_A((0,i,j),Q(0,c+\ell,0)). \tag{3.37}$$

Thus

$$\mathsf{Lsupp}Q(0,c,d) \cap {}_s\Gamma_0 = (\bigcup_{\ell=0}^{d} \mathsf{Lsupp}Q(0,c+\ell,0)) \cap {}_s\Gamma_0. \tag{3.38}$$

By equation (3.26), $Q(0,c+\ell,0) \in \mathsf{Rsupp}(0,a-t,b-c-\ell)$ for every $\ell,t \in \mathbb{Z}$. Thus

$$(0,a-t,b-c-\ell) \in \mathsf{Lsupp}Q(0,c+\ell,0)$$

for every $\ell,t \in \mathbb{Z}$. For an integer h, if $\overline{h} \equiv h(mod\ p) \notin \{\overline{b-c-d}, \overline{b-c-d+1}, \cdots, \overline{b-c}\}$, then $\underline{\mathsf{Hom}}_A((0,t,h),Q(0,c+\ell,0)) = 0$ for $\ell \in \{1,2,\cdots,d\}$ and $t \in \mathbb{Z}$ by equations (3.37) and (3.26). Therefore,

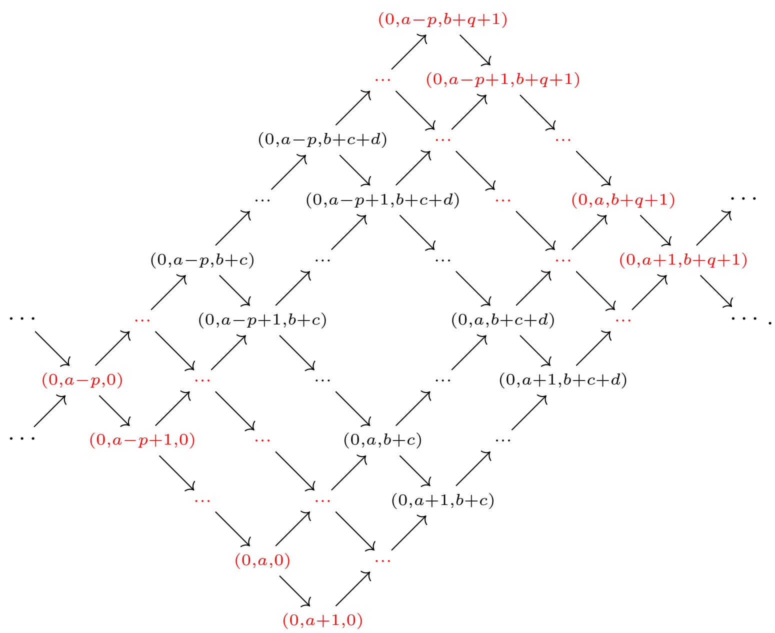
$$\begin{aligned} \mathsf{Rsupp}Q(0,c,d) \cap {}_s\Gamma_0 &= \{(0,i,b+c+\ell) \mid i \in \mathbb{Z}, \ell = 0,1,2,\cdots,d\}, \\ \mathsf{Lsupp}Q(0,c,d) \cap {}_s\Gamma_0 &= \mathsf{Rsupp}\Omega Q(0,c,d) \cap {}_s\Gamma_0 \\ &= \mathsf{Rsupp}Q(1,c,d) \cap {}_s\Gamma_0 = \{0\}. \end{aligned} \tag{3.39}$$

$$\begin{aligned} \mathsf{Lsupp}Q(0,c,d) \cap {}_s\Gamma_1 &= \{(1,i,b+c-1+\ell) \mid i \in \mathbb{Z}, \ell = 0,1,2,\cdots,d\}, \\ \mathsf{Rsupp}Q(0,c,d) \cap {}_s\Gamma_1 &= \mathsf{Lsupp}\Omega Q(0,c,d) \cap {}_s\Gamma_1 \\ &= \mathsf{Lsupp}Q(1,c,d) \cap {}_s\Gamma_1 = \{0\}. \end{aligned} \tag{3.40}$$

Thus

$$\begin{aligned} \mathsf{Rsupp}Q(0,c,d) \cap ({}_s\Gamma_0 \cup {}_s\Gamma_1) &= \{(0,i,b+c+\ell) \mid i \in \mathbb{Z}, \ell = 0,1,2,\cdots,d\}, \\ \mathsf{Lsupp}Q(0,c,d) \cap ({}_s\Gamma_0 \cup {}_s\Gamma_1) &= \{(1,i,b+c-1+\ell) \mid i \in \mathbb{Z}, \ell = 0,1,2,\cdots,d\}. \end{aligned} \tag{3.41}$$

$$
\begin{aligned}
\mathcal{P}_{Q(0,c,d)} :\; &= {}^{\perp}Q(0,c,d)^{\perp} \cap ({}_s\Gamma_0 \cup {}_s\Gamma_1) \\
&= {}_s\Gamma_0 \cup {}_s\Gamma_1 \backslash ((\mathsf{Lsupp}Q(0,c,d) \cup \mathsf{Rsupp}Q(0,c,d)) \cap ({}_s\Gamma_0 \cup {}_s\Gamma_1)) \\
&= \{(0,i,\overline{b+c+\ell}) \mid i \in \mathbb{Z}, \ell = d+1, d+2, \ldots, p-1\} \\
&\cup \{(1,i,\overline{b+c+\ell}) \mid i \in \mathbb{Z}, \ell = d, d+1, \ldots, p\}.
\end{aligned} \tag{3.42}
$$



Thus ${}^{\perp}Q(0,c,d)^{\perp} \cap {}_s\Gamma_A = \mathcal{N}^0_{Q(0,c,d)} \cup \mathcal{N}^1_{Q(0,c,d)} \cup \mathcal{P}_{Q(0,c,d)}$.

4. The construction of orthogonal systems

The following conclusion provides us a necessary condition for an orthogonal system to be a simple-minded system on A-$\underline{\mathsf{mod}}$.

Lemma 4.1. *Let A be a 2-domestic Brauer graph algebra and $\mathcal{S}$ a simple-minded system on A-$\underline{\mathsf{mod}}$. Then $\mathcal{S}$ contains at least one non-periodic module.*

Proof. By Theorem 2.5, any object in a homogeneous tube is not in $\mathcal{S}$. We show that $\mathcal{S}$ contains at least one non-periodic module. On the contrary, if the set $\mathcal{S} \cap ({}_s\Gamma_0 \cup {}_s\Gamma_1)$ is empty, then $\mathcal{S}$ is contained in ${}_sQ_0 \cup {}_sQ_1 \cup {}_sQ'_0 \cup {}_sQ'_1$. By Corollary 2.18, any object on a homogeneous tube is contained in the stable bi-perpendicular category of ${}_sQ_0 \cup {}_sQ_1 \cup {}_sQ'_0 \cup {}_sQ'_1$. Thus every object on a homogeneous tube is in ${}^{\perp}\mathcal{S}^{\perp}$. It is a contradiction. Hence our conclusion holds. $\square$

According to Lemma 4.1, we shall mainly consider the orthogonal systems which contain at least one object for an Euclidean component.

4.1. **The construction of orthogonal systems on Euclidean components.** Without loss of generality, we always assume $p \le q$ for any Euclidean component in this section. Now we shall present all orthogonal systems on ${}_s\Gamma_0 \cup {}_s\Gamma_1$ containing the vertex $(0,a,b)$ as follows.

Take $(0,a_1,b_1) := (0,a,b)$. By equation (3.3), one vertex $(0,a_2,b_2) \in \mathcal{M}^0_{(0,a_1,b_1)}$ if and only if $a_1 - p < a_2 < a_1, b_1 < b_2 < b_1 + q$. Thus the set $\{(0,a_1,b_1),(0,a_2,b_2),\cdots,(0,a_k,b_k)\}$ is an orthogonal system on A-$\underline{\mathsf{mod}}$ if and only if $a_1 - p < a_i < a_1, b_1 < b_i < b_1 + q$, $i = 2,3,\cdots,k$, and if $a_i < a_j$, then $b_j < b_i$ for any $i,j = 1,2,\cdots,k$. Thus

$$
\begin{aligned}
\mathcal{O}^0_{(0,a_1,b_1)} := \{(0,a_i,b_i) \mid\; & a_1 - p < a_i < a_1, b_1 < b_i < b_1 + q \text{ for } i = 2,\cdots,k, \text{and}, \\
& \text{if } a_i < a_j, \text{then } b_j < b_i \text{ for any } i,j = 1,2,\cdots,k\}.
\end{aligned} \tag{4.1}
$$

is an orthogonal system on A-$\underline{\mathsf{mod}}$ containing $(0,a,b)$.

Similarly, the set $\{(1,c_1,d_1),(1,c_2,d_2),\cdots,(1,c_\ell,d_\ell)\}$ is an orthogonal system on A-$\underline{\mathsf{mod}}$ if and only if $c_1-p<c_i<c_1$ for $i=2,3,\cdots,\ell$, $d_1<d_i<d_1+q$ for any $i=1,2,\cdots,\ell$, and if $c_i<c_j$, then $d_j<d_i$ for any $i,j=1,2,\cdots,\ell$. Thus

$$\begin{aligned}\mathcal{O}^1_{(1,c_1,d_1)}:\ =\{(1,c_i,d_i)\mid\ & c_1-p<c_i<c_1, d_1<d_i<d_1+q \text{ for } i=2,\cdots,\ell,\\ & \text{and if } c_i<c_j,\ \text{then } d_j<d_i \text{ for any } i,j=1,2,\cdots,\ell\}.\end{aligned} \tag{4.2}$$

is also an orthogonal system on A-$\underline{\mathsf{mod}}$.

By equation (3.9), one vertex $(1,c,d)\in\mathcal{M}^1_{(0,a,b)}$ if and only if $(1,c,d)\in{}^{\perp}(1,a+1,b)^{\perp}\cap{}_s\Gamma_1$ if and only if $a_1-p+1\le c\le a_1, b_1+1\le d\le b_1+q$. Therefore the set $\mathcal{O}^0_{(0,a_1,b_1)}\cup\{(1,c_1,d_1)\}$ is an orthogonal system on A-$\underline{\mathsf{mod}}$ if and only if $a_i-p+1\le c_1\le a_i, b_i+1\le d_1\le b_i+q$ for any $i\in\{1,2,\cdots,k\}$. Therefore the set $\mathcal{O}^0_{(1,a_1,b_1)}\cup\mathcal{O}^1_{(0,c_1,d_1)}$ is an orthogonal system on A-$\underline{\mathsf{mod}}$ if and only if $a_j-p+1\le c_i\le a_j, b_j+1\le d_i\le b_j+q$. Thus we have the following proposition.

Proposition 4.2. *Let A be a 2-domestic Brauer graph algebra and let Γ_0 and Γ_1 be two Euclidean components. Take a subset $\mathcal{M}_0=\{(0,a_i,b_i)\mid 1\le i\le k\}$ on ${}_s\Gamma_0$ and a subset $\mathcal{M}_1=\{(1,c_i,d_i)\mid 1\le i\le \ell\}$ on ${}_s\Gamma_1$, respectively. Then*

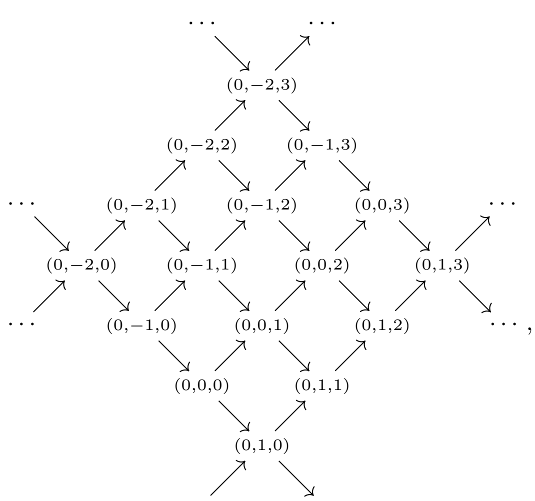
(1) $\mathcal{M}_0$ is an orthogonal system if and only if $a_1-p<a_i<a_1, b_1<b_i<b_1+q$ for $i=2,3,\cdots,k$, and if $a_i<a_j$, then $b_j<b_i$ for any $i,j=1,2,\cdots,k$.

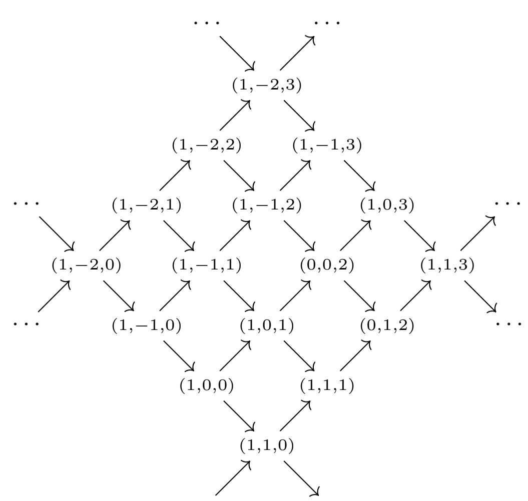
(2) $\mathcal{M}_1$ is an orthogonal system if and only if $c_1-p<c_i<c_1$, $d_1<d_i<d_1+q$ for $i=2,3,\cdots,\ell$, and if $c_i<c_j$, then $d_j<d_i$ for any $i,j=1,2,\cdots,\ell$.

(3) We assume that $\mathcal{M}_0$ and $\mathcal{M}_1$ are two orthogonal systems. $\mathcal{M}_0\cup\mathcal{M}_1$ is an orthogonal system if and only if $a_j-p+1\le c_i\le a_j, b_j+1\le d_i\le b_j+q$. for any $i,j=1,2,\cdots,k$.

Note that the cardinality of a maximal orthogonal system on an Euclidean component $\mathbb{Z}\widetilde{A}_{p,q}$ may be $2,3,\cdots,p$. We present the following example for an Euclidean component $\mathbb{Z}\widetilde{A}_{3,3}$ over a 2-domestic Brauer graph algebra.

Example 4.3. *Take the following two Euclidean components ${}_s\Gamma_0$ and ${}_s\Gamma_1$ of the form $\mathbb{Z}\widetilde{A}_{3,3}$ over a 2-domestic Brauer graph algebra. Note that $(0,-2,3)$ and $(0,1,0)$ are identical.*





By (1) of Proposition 4.2, the sets $\mathcal{U}_0=\{(0,1,0),(0,-1,1)\}$ and $\mathcal{U}_1=\{(0,1,0),(0,-1,2),(0,0,1)\}$ are all maximal orthogonal systems containing $(0,1,0)$ on ${}_s\Gamma_0$. Now we list all maximal orthogonal systems containing $(0,1,0)$ on ${}_s\Gamma_0\cup{}_s\Gamma_1$.

Case one: *All maximal orthogonal systems containing $\mathcal{U}_1$ on ${}_s\Gamma_0\cup{}_s\Gamma_1$.*

By Serre duality,

$$\mathcal{M}_1:\ ={}^{\perp}\mathcal{U}_0^{\perp}\cap{}_s\Gamma_1=\{(1,-1,2),(1,-1,3),(1,0,1),(1,1,1)\}.$$

By (2) of Proposition 4.2, there are four maximal orthogonal systems on $\mathcal{M}$ as follows.

$$\mathcal{V}_1=\{(1,-1,3),(1,0,1)\},\mathcal{V}_2=\{(1,-1,3),(1,1,1)\},\mathcal{V}_3=\{(1,-1,2),(1,0,1)\},\mathcal{V}_4=\{(1,-1,2),(1,1,1)\}.$$

Thus any maximal orthogonal system containing $\mathcal{U}_0$ on ${}_s\Gamma_0 \cup {}_s\Gamma_1$ is of the form $\mathcal{U}_0 \cup \mathcal{V}_i$ for $i = 1, 2, 3, 4$. Note that the cardinality of $\mathcal{U}_0 \cup \mathcal{V}_i$ is 4 for each $i = 1, 2, 3, 4$.

Case two: *All maximal orthogonal systems containing $\mathcal{U}_1$ on ${}_s\Gamma_0 \cup {}_s\Gamma_1$.*

By Serre duality,

$$\mathcal{M}_1' :={}^{\perp}\mathcal{U}_1^{\perp} \cap {}_s\Gamma_1 = \{(1, -1, 3), (1, 0, 2), (1, 1, 1)\}.$$

By (2) of Proposition 4.2, $\mathcal{M}_1'$ is an orthogonal system. Hence $\mathcal{U}_1 \cup \mathcal{M}_1'$ is the only maximal orthogonal system containing $\mathcal{U}_1$ on ${}_s\Gamma_0 \cup {}_s\Gamma_1$. Note that the cardinality of $\mathcal{U}_1 \cup \mathcal{M}_1'$ is 6.

4.2. **The construction of orthogonal systems on quasi-tubes.** In this subsection, we shall construct orthogonal systems on stable generalized standard quasi-tubes.

Definition 4.4. *Let A be a self-injective algebra, and let Q be a quasi-tube and (a, b) a vertex on the quasi-tube Q. Take the set*

$$\triangle_{(a,b)} := \{(i, j) \mid a \leq i \leq a + b, a \leq i + j \leq a + b\}.$$

$\triangle_{(a,b)}$ is called **triangle area of vertices** *(a, b) and we call the number b the* **height** *of $\triangle_{(a,b)}$.*

Note that $\triangle_{(a,b)}$ is the set of vertices contained in the triangle with vertices $(a, 0)$, (a, b) and $(a + b, 0)$ and that, if $b \leq -1$, then we always assume that $\triangle_{(a,b)} = \varnothing$. We shall show that the intersection between stable bi-perpendicular category of an orthogonal system on an Euclidean component and a quasi-tube is a union of triangle areas of some vertices.

Take a vertex $(0, a, b) \in {}_sQ_0$. By equation (3.30), ${}^{\perp}(0, a, b)^{\perp} \cap {}_sQ_0 = \triangle_{Q(0,1,q-2)}$. First, we shall construct all orthogonal systems on triangle area $\triangle_{Q(0,1,q-2)}$ by induction on the height j of triangle area $\triangle_{Q(0,i,j)}$. Without loss of generality, We shall always fix the orthogonal system appeared in this subsection with the second coordinate in an ascending order. For example, the orthogonal system $\{Q(0, 1, 0), Q(0, 2, 0), \cdots, Q(0, q - 1, 0)\}$ on $\triangle_{Q(0,1,q-2)}$ satisfies the rule.

Example 4.5. *We list all orthogonal systems on triangle area $\triangle_{Q(0,1,q-2)}$ for $q - 2 = 0, 1, 2, 3$ as follows.*

(1) *The case $q - 2 = 0$. There is only one vertex $Q(0, 1, 0)$ on triangle area $\triangle_{Q(0,1,0)}$. Thus $\{Q(0, 1, 0)\}$ is the only orthogonal system on triangle area $\triangle_{Q(0,1,0)}$.*
(2) *The case $q - 2 = 1$. There are four orthogonal systems on triangle area $\triangle_{Q(0,1,1)}$ as follows.*
$$\{Q(0, 1, 0)\}, \{Q(0, 2, 0)\}, \{Q(0, 1, 0), Q(0, 2, 0)\}, \{Q(0, 1, 1)\}.$$
(3) *The case $q - 2 = 2$. We list all orthogonal systems on triangle area $\triangle_{Q(0,1,2)}$ as follows.*
   (*a*) *Orthogonal systems with one object: Any subset of $\triangle_{Q(0,1,1)}$ containing only one object.*
   (*b*) *Orthogonal systems with two objects:*
$$\{Q(0, 1, 0), Q(0, 2, 0)\}, \{Q(0, 1, 0), Q(0, 2, 1)\}, \{Q(0, 1, 0), Q(0, 3, 0)\},$$
$$\{Q(0, 1, 1), Q(0, 3, 0)\}, \{Q(0, 1, 2), Q(0, 2, 0)\}, \{Q(0, 2, 0), Q(0, 3, 0)\}.$$
   (*c*) *Orthogonal systems with three objects:*
$$\{Q(0, 1, 0), Q(0, 2, 0), Q(0, 3, 0)\}.$$
(4) *The case $q - 2 = 3$. We only list all maximal orthogonal systems on triangle area $\triangle_{Q(0,1,q-2)}$, since there are a lot of orthogonal systems in this case. Note that every orthogonal system is a subset of a maximal orthogonal system.*
   (*a*) *Maximal orthogonal systems with two objects:*
$$\{Q(0, 1, 1), Q(0, 3, 1)\}, \{Q(0, 1, 3), Q(0, 2, 1)\}.$$
   (*b*) *Maximal orthogonal systems with three objects:*
$$\{Q(0, 1, 0), Q(0, 2, 0), Q(0, 3, 1)\}, \{Q(0, 1, 0), Q(0, 2, 1), Q(0, 4, 0)\}, \{Q(0, 1, 0), Q(0, 2, 2), Q(0, 3, 0)\},$$
$$\{Q(0, 1, 1), Q(0, 3, 0), Q(0, 4, 0)\}, \{Q(0, 1, 2), Q(0, 2, 0), Q(0, 4, 0)\}, \{Q(0, 1, 3), Q(0, 2, 0), Q(0, 3, 0)\}.$$
   (*c*) *Maximal orthogonal systems with four objects:*
$$\{Q(0, 1, 0), Q(0, 2, 0), Q(0, 3, 0), Q(0, 4, 0)\}.$$

Now we construct orthogonal systems on the triangle area $\triangle_{Q(0,1,q-2)}$ by induction as follows. Take a vertex $Q(0, 1, i_1)$ on $\triangle_{Q(0,1,q-2)}$, where $0 \leq i_1 \leq q-2$. By equation (3.34), we consider two cases as follows.

(1) If $i_1 = 0$, then ${}^{\perp}Q(0, 1, i_1)^{\perp} \cap \triangle_{Q(0,1,q-2)} = \triangle_{Q(0,2,q-3)}$.

(2) If $i_1 \geq 1$, then ${}^{\perp}Q(0,1,i_1)^{\perp} \cap \triangle_{Q(0,1,q-2)} = \triangle_{Q(0,2,i_1-2)} \cup \triangle_{Q(0,i_1+2,q-i_1-3)}$.
Note that, the triangle area $\triangle_{Q(0,2,i_1-2)} = \varnothing$ when $i_1 = 1$. Note that both $i_1 - 2$ and $q - i_1 - 3$ are less than $p-2$ strictly, and that the triangle areas $\triangle_{Q(0,2,i_1-2)}$ and $\triangle_{Q(0,i_1+2,q-i_1-3)}$ are disjoint and they are mutual stable orthogonal on A-$\underline{\mathsf{mod}}$.

We add one more object on the set $\{Q(0,1,i_1)\}$ from ${}^{\perp}Q(0,1,i_1)^{\perp} \cap \triangle_{Q(0,1,q-2)}$ as follows.

(a) For (1), we take a vertex $(0,2,i_2)$ on triangle area $\triangle_{Q(0,2,q-3)}$. By equation (3.34),
$$ {}^{\perp}\{Q(0,1,0),Q(0,2,i_2)\}^{\perp} \cap \triangle_{Q(0,1,q-2)} = \triangle_{Q(0,3,i_2-2)} \cup \triangle_{Q(0,i_2+3,q-i_2-4)}. $$
(b) For $i_1 \geq 2$ in (2), we take a vertex $(0,2,i_2)$ on triangle area $\triangle_{Q(0,2,i_1-2)}$. By equation (3.34),
$$ {}^{\perp}\{Q(0,1,i_1),Q(0,2,i_2)\}^{\perp} \cap \triangle_{Q(0,1,q-2)} = \triangle_{Q(0,3,i_2-2)} \cup \triangle_{Q(0,i_2+3,i_1-i_2-3)} \cup \triangle_{Q(0,i_1+2,q-i_1-3)}. $$
Note that $i_2 < i_1$ in this case.

(c) For $i_1 = 1$ in (2), we take a vertex $(3,i_3)$ on triangle area $\triangle_{Q(0,3,q-4)}$.
(i) If $i_3 = 0$, then ${}^{\perp}\{Q(0,1,i_1),Q(0,3,0)\}^{\perp} \cap \triangle_{Q(0,1,q-2)} = \triangle_{Q(0,4,q-5)}$.
(ii) If $i_3 \geq 1$, then
$$ {}^{\perp}\{Q(0,1,i_1),Q(0,3,i_3)\}^{\perp} \cap \triangle_{Q(0,1,q-2)} = \triangle_{Q(0,4,i_3-2)} \cup \triangle_{Q(0,i_3+4,q-i_3-5)}. $$

Note that the heights of triangle areas for above cases are less than the height of triangle areas on ${}^{\perp}Q(0,1,i_1)^{\perp} \cap \triangle_{Q(0,1,q-2)}$, respectively. They are disjoint union of triangle areas and those triangle areas are mutual stable orthogonal on A-$\underline{\mathsf{mod}}$.

Take a vertex $Q(0,\ell,i_\ell)$ on triangle area $\triangle_{Q(0,1,q-2)}$, where $0 \leq i_\ell \leq q-2$. Let $\mathcal{S}_\ell$ be an orthogonal system containing the vertex $Q(0,\ell,i_\ell)$ such that $i \geq \ell$ for each $Q(0,i,j)$ in $\mathcal{S}_\ell$. By equation (3.34), $\mathcal{S}_\ell$ is contained in the disjoint of two triangle areas of vertices $Q(0,\ell+1,i_\ell-2)$ and $Q(0,\ell+i_\ell,q-\ell-i_\ell-2)$. If the stable bi-perpendicular category ${}^{\perp}\mathcal{S}_\ell^{\perp} \cap \triangle_{Q(0,1,q-2)}$ is non-zero, then by induction we choose a non-zero element on ${}^{\perp}\mathcal{S}_\ell^{\perp} \cap \triangle_{Q(0,1,q-2)}$, which is contained in a union triangle areas of some vertices. Note that the height of those vertices is lower than $i_\ell - 2$ or $q-\ell-i_\ell-2$. Continuing this operation, with finitely many steps, we get a maximal orthogonal system with the second coordinate is larger than or equals to ℓ. In this way, we can construct all maximal orthogonal systems, and in particular, we get all orthogonal systems on the triangle area $\triangle_{Q(0,1,q-2)}$.

Now we determine orthogonal systems on the triangle areas $\triangle_{Q(0,1,q-2)} \cup \triangle_{Q(1,1,q-2)}$. By equation (3.27), ${}^{\perp}(0,a,b)^{\perp} \cap {}_sQ_1 = \triangle_{Q(1,1,q-2)}$. Thus
$$ {}^{\perp}(0,a,b)^{\perp} \cap ({}_sQ_0 \cup {}_sQ_1) = \triangle_{Q(0,1,q-2)} \cup \triangle_{Q(1,1,q-2)}. $$
Note that $\triangle_{Q(1,1,q-2)} = \Omega(\triangle_{Q(0,1,q-2)})$. Using a similar method with the above, we may construct all orthogonal systems on ${}^{\perp}(0,a,b)^{\perp} \cap {}_sQ_1$.

By equation (3.36),
$$ {}^{\perp}Q(0,i,j)^{\perp} \cap {}_sQ_1 = \Omega(Q(0,i,j))^{\perp} \cap {}^{\perp}\Omega^{-1}(Q(0,i,j)) \cap {}_sQ_1 = Q(1,i,j)^{\perp} \cap {}^{\perp}Q(1,i+1,j) \cap {}_sQ_1. $$
Take a vertex $Q(0,1,i_1)$ on triangle areas $\triangle_{Q(0,1,q-2)} \cup \triangle_{Q(1,1,q-2)}$. Then
$$ \begin{aligned} {}^{\perp}Q(0,1,i_1)^{\perp} \cap (\triangle_{Q(0,1,q-2)} \cup \triangle_{Q(1,1,q-2)}) =& (\triangle_{Q(0,2,i_1-2)} \cup \triangle_{Q(0,2+i_1,q-i_1-3)}) \\ & \cup (\triangle_{Q(1,2,i_1-1)} \cup \triangle_{Q(1,3+i_1,q-i_1-4)}) \\ & \cup \{Q(1,1,j) \mid i_1 < j < q-1\}. \end{aligned} \tag{4.3} $$
We shall construct an orthogonal system $\mathcal{S}$ containing the vertex $Q(0,1,i_1)$. According to equation (4.3), we consider two cases.

Case one: If $\mathcal{S}$ contains an object of the form $Q(1,1,j)$ for $i_1 < j < q-1$. Without loss of generality, we assume that $Q(1,1,i_h)(i_1 < i_h < q-1)$ is in $\mathcal{S}$. Thus
$$ \begin{aligned} & {}^{\perp}\{Q(0,1,i_1),Q(1,1,i_h)\}^{\perp} \cap (\triangle_{Q(0,1,q-2)} \cup \triangle_{Q(1,1,q-2)}) \\ & = (\triangle_{Q(0,2,i_1-2)} \cup \triangle_{Q(0,2+i_1,i_h-i_1-2)} \cup \triangle_{Q(0,2+i_h,q-i_h-3)}) \\ & \cup (\triangle_{Q(1,2,i_1-1)} \cup \triangle_{Q(1,3+i_1,i_h-i_1-3)} \cup \triangle_{Q(1,2+i_h,q-i_h-3)}). \end{aligned} \tag{4.4} $$
Case two: If $\mathcal{S}$ contains no object of the form $Q(1,1,j)$ for $i_1 < j < q-1$. By equation (4.3), $\mathcal{S}$ is contained in the following triangle areas of some vertices:
$$ (\triangle_{Q(0,2,i_1-2)} \cup \triangle_{Q(0,2+i_1,q-i_1-3)}) \cup (\triangle_{Q(1,2,i_1-1)} \cup \triangle_{Q(1,3+i_1,q-i_1-4)}). $$

In conclusion, it is not hard to know that the stable bi-perpendicular category of an orthogonal systems on the triangle areas $\triangle_{Q(0,1,q-2)} \cup \triangle_{Q(1,1,q-2)}$ is contained in triangle areas of some vertices. By induction on the height of vertices, we may construct all maximal orthogonal systems on triangle areas $\triangle_{Q(0,1,q-2)} \cup \triangle_{Q(1,1,q-2)}$, thus all orthogonal systems on $\triangle_{Q(0,1,q-2)} \cup \triangle_{Q(1,1,q-2)}$. We shall compute the cardinality of a maximal orthogonal system on triangle areas $\triangle_{Q(0,1,q-2)} \cup \triangle_{Q(1,1,q-2)}$.

Theorem 4.6. *Let A be a symmetric algebra and let Q_0 and Q_1 be generalized standard quasi-tubes of rank q satisfying $\Omega({}_sQ_0) = {}_sQ_1$.*

(1) *Let $\mathcal{S}_1$ be a maximal orthogonal system on triangle areas $\triangle_{Q(0,x,y-1)} \cup \triangle_{Q(1,x,y)}$ over ${}_sQ_0 \cup {}_sQ_1$ for $y \leq q-2$. Then the cardinality of $\mathcal{S}_1$ is $y+1$ for $y = 0, 1, \cdots, q-2$.*
(2) *Let $\mathcal{S}_2$ be a maximal orthogonal system on the triangle areas $\triangle_{Q(0,v,w)} \cup \triangle_{Q(1,v+1,w-1)}$ over ${}_sQ_0 \cup {}_sQ_1$ for $w \leq q-2$. Then the cardinality of $\mathcal{S}_2$ is $w+1$ for $w = 0, 1, \cdots, q-2$.*
(3) *Let $\mathcal{S}$ be a maximal orthogonal system on the triangle areas $\triangle_{Q(0,j,i)} \cup \triangle_{Q(1,j,i)}$ over ${}_sQ_0 \cup {}_sQ_1$ for $i \leq q-2$. Then the cardinality of $\mathcal{S}$ is $i+1$ for $i = 0, 1, \cdots, q-2$.*

Proof. We prove (1), (2) and (3) by induction on the heights y, w and i, respectively. For (1) and (2), we only prove (1), since the proof of (2) is dual. For (3), we proof only the case for $j = 1$, since $\tau(Q(0,j,i)) = Q(0,j-1,i)$ and τ is a stable self-equivalence on A-$\underline{\mathsf{mod}}$. Therefore we assume that $j = 1$.

We first induct on the height y for conclusion (1). If $y = 0$, then $\triangle_{Q(0,x,-1)} \cup \triangle_{Q(1,x,0)} = \{Q(1,x,0)\}$ is a maximal orthogonal system on triangle areas $\triangle_{Q(0,x,-1)} \cup \triangle_{Q(1,x,0)}$. Thus conclusion (1) is true for $y = 0$. If $y = 1$, then

$$\triangle_{Q(0,x,0)} \cup \triangle_{Q(1,x,1)} = \{Q(0,x,0), Q(1,x,0), Q(1,x,1), Q(1,x+1,0)\}.$$

It is easy to know that there are two maximal orthogonal systems on triangle areas $\triangle_{Q(0,x,0)} \cup \triangle_{Q(1,x,1)}$ as follows.

$$\{Q(0,x,0), Q(1,x,1)\}, \{Q(1,x,0), Q(1,x+1,0)\}.$$

Thus conclusion (1) is true for $y = 1$.

Then we induct on the height i for conclusion (3). If $i = 0$, then

$$\triangle_{Q(0,1,0)} \cup \triangle_{Q(1,1,0)} = \{Q(0,1,0), Q(1,1,0)\}$$

and $Q(1,1,0) = \Omega(Q(0,1,0))$. It is easy to know that $\{Q(0,1,0)\}$ and $\{Q(1,1,0)\}$ are all maximal orthogonal systems on triangle areas $\triangle_{Q(0,1,0)} \cup \triangle_{Q(1,1,0)}$. Therefore conclusion (3) is true for $i = 0$.

If $i = 1$, then

$$\triangle_{Q(0,1,1)} \cup \triangle_{Q(1,1,1)} = \{Q(0,1,0), Q(0,1,1), Q(0,2,0), Q(1,1,0), Q(1,1,1), Q(1,2,0)\}.$$

There are two maximal orthogonal systems $\{Q(0,1,0), Q(0,2,0)\}$ and $\{Q(0,1,1)\}$ on triangle area $\triangle_{Q(0,1,1)}$. According to equation (3.36), the set $\{Q(0,1,0), Q(0,2,0)\}$ is still maximal on triangle areas $\triangle_{Q(0,1,1)} \cup \triangle_{Q(1,1,1)}$. Thus the set $\{Q(1,1,0), Q(1,2,0)\}$ is maximal on triangle areas $\triangle_{Q(0,1,1)} \cup \triangle_{Q(1,1,1)}$. By equation (3.36),

$${}^{\perp}\{Q(0,1,1)\}^{\perp} \cap (\triangle_{Q(0,1,1)} \cup \triangle_{Q(1,1,1)}) = \{Q(1,2,0)\}.$$

Thus the set $\{Q(0,1,1), Q(1,2,0)\}$ is a maximal orthogonal system on triangle areas $\triangle_{Q(0,1,1)} \cup \triangle_{Q(1,1,1)}$. Using similar method, we know that the sets $\{Q(0,1,0), Q(1,1,1)\}$ and $\{Q(0,2,0), Q(1,1,0)\}$ are also maximal orthogonal systems on $\triangle_{Q(0,1,1)} \cup \triangle_{Q(1,1,1)}$. Hence there are five maximal orthogonal systems on triangle areas $\triangle_{Q(0,1,1)} \cup \triangle_{Q(1,1,1)}$, which have the same cardinality 2. Therefore conclusion (3) is true for $i = 1$.

Now we assume that conclusions (1) and (2) hold for $y = k$ and $w = k$ respectively, and that conclusion (3) is true for $i = k$. We prove that (1), (2) and (3) are true for $y = k+1, w = k+1$, and $i = k+1$, respectively. Let $\mathcal{S}$ be a maximal orthogonal system on triangle areas $\triangle_{Q(0,1,k+1)} \cup \triangle_{Q(1,1,k+1)}$. Take

$$\mathcal{L}_1 \colon = \{Q(0,1,j) \mid 0 \leq j \leq k+1\} \cup \{Q(1,1,j) \mid 0 \leq j \leq k+1\}.$$

Claim one: $\mathcal{L}_1 \cap \mathcal{S} \neq \varnothing$.

Otherwise, $\mathcal{S} \subseteq \triangle_{Q(0,2,k)} \cup \triangle_{Q(1,2,k)}$ and it is a maximal orthogonal system on triangle areas $\triangle_{Q(0,2,k)} \cup \triangle_{Q(1,2,k)}$. Take

$$\mathcal{L}_2 \colon = \{Q(0,2,j) \mid 0 \leq j \leq k\} \cup \{Q(1,2,j) \mid 0 \leq j \leq k\}.$$

By induction, the cardinality of $\mathcal{S}$ is $k+1$. We claim that $\mathcal{L}_2 \cap \mathcal{S} \neq \varnothing$. Otherwise, $\mathcal{S} \subseteq \triangle_{Q(0,3,k-1)} \cup \triangle_{Q(1,3,k-1)}$. By induction, the cardinality of $\mathcal{S}$ is k. It is a contradiction. Thus there is an object in $\mathcal{S}$

which is contained in $\mathcal{L}_2$, denoted by S. Then we assume that $S = (s, 2, t)$ for $s = 0, 1$ and $0 \le t \le k$. By equations (3.34) and (3.36), we always have $Q(1, 1, 0) \in {}^{\perp}\mathcal{S}^{\perp}$. Therefore $\mathcal{S}$ is not maximal on triangle areas $\triangle_{Q(0,1,k+1)} \cup \triangle_{Q(1,1,k+1)}$. It is a contradiction. Therefore $\mathcal{L}_1 \cap \mathcal{S} \neq \varnothing$. Thus Claim one is proved.

Let $\mathcal{S}_1$ be a maximal orthogonal system on triangle areas $\triangle_{Q(0,x,k)} \cup \triangle_{Q(1,x,k+1)}$.

Claim two: $\mathcal{S}_1 \cap \{Q(1, r, x + k + 1 - r) \mid x \le r \le x + k + 1\} \neq \varnothing$.

Otherwise, $\mathcal{S}_1 \cap \{Q(1, r, x+k+1-r) \mid x \le r \le x+k+1\} = \varnothing$, then $\mathcal{S}_1$ is a maximal orthogonal system on triangle areas $\triangle_{Q(0,x,k)} \cup \triangle_{Q(1,x,k)}$. Take

$$\mathcal{H}_1 = \{Q(0, r, x + k - r) \mid x \le r \le x + k\} \cup \{Q(1, r, x + k - r) \mid x \le r \le x + k\}.$$

Dual with Claim one, $\mathcal{S}_1 \cap \mathcal{H}_1 \neq \varnothing$. Take $M \in \mathcal{S}_1 \cap \mathcal{H}_1 \neq \varnothing$. If $M = Q(1, x, k)$, by equations (3.34) and (3.36), then $Q(0, x + k + 1, 0) \in {}^{\perp}\mathcal{S}_1^{\perp}$. It is a contradiction. If $M = \mathcal{H}_1 \backslash \{Q(1, x, k)\}$, then $Q(0, x, k + 1) \in {}^{\perp}\mathcal{S}_1^{\perp}$. It contradicts the condition that $\mathcal{S}_1$ is maximal on triangle areas $\triangle_{Q(0,x,k)} \cup \triangle_{Q(1,x,k+1)}$. Therefore $\mathcal{S}_1 \cap \{Q(1, r, x + k + 1 - r) \mid x \le r \le x + k + 1\} \neq \varnothing$. Thus Claim two is proved.

Take $M_1 = Q(1, s, x + k + 1 - s) \in \mathcal{S}_1 \cap \{Q(1, r, x + k + 1 - r) \mid x \le r \le x + k + 1\}$. Then

$$\begin{aligned} &{}^{\perp}\mathcal{S}_1^{\perp} \cap (\triangle_{Q(0,x,k)} \cup \triangle_{Q(1,x,k+1)} \\ &= (\triangle_{Q(0,x,s-x-2)} \cup \triangle_{Q(0,s,x+k-s)}) \cup (\triangle_{Q(1,x,s-x-1)} \cup \triangle_{Q(1,s+1,x+k-s-1)}) \\ &= (\triangle_{Q(0,x,s-x-2)} \cup \triangle_{Q(1,x,s-x-1)}) \cup (\triangle_{Q(0,s,x+k-s)} \cup \triangle_{Q(1,s+1,x+k-s-1)}). \end{aligned} \tag{4.5}$$

Note that $s - x - 1 \le k$ and $x + y - s - 2 \le k - 1$. By induction of (1) and (2), the cardinality of a maximal orthogonal system on triangle areas $\triangle_{Q(0,x,s-x-2)} \cup \triangle_{Q(1,x,s-x-1)}$ is $(s-x-1)+1 = s-x$ and the cardinality of a maximal orthogonal system on triangle areas $\triangle_{Q(0,s,x+k-s)} \cup \triangle_{Q(1,s+1,x+k-s-1)}$ is $(x + k - s) + 1 = x + k - s + 1$. Since triangle areas $\triangle_{Q(0,x,s-x-2)} \cup \triangle_{Q(1,x,s-x-1)}$ and $\triangle_{Q(0,s,x+k-s)} \cup \triangle_{Q(1,s+1,x+k-s-1)}$ are mutual orthogonal, the cardinality of $\mathcal{M}$ is $1 + (s - x) + (x + k - s + 1) = k + 2$. Thus conclusion (1) is proved.

Now we prove (3). By Claim one, $\mathcal{L}_1 \cap \mathcal{S} \neq \varnothing$. We consider two cases as follows.

Case one: $\mathcal{S} \cap \{Q(0, 1, j) \mid 0 \le j \le k+1\} \neq \varnothing$. We assume that $S_1 = Q(0, 1, j_1)$ in $\mathcal{S}$. By equation (3.34),

$$\begin{aligned} &{}^{\perp}S_1^{\perp} \cap (\triangle_{Q(0,1,k+1)} \cup \triangle_{Q(1,1,k+1)} \\ &= (\triangle_{Q(0,2,j_1-2)} \cup \triangle_{Q(0,j_1+2,k-j_1)}) \cup (\triangle_{Q(1,2,j_1-1)} \cup \triangle_{Q(0,j_1+3,k-j_1-1)}) \\ &= (\triangle_{Q(0,2,j_1-2)} \cup \triangle_{Q(1,2,j_1-1)}) \cup (\triangle_{Q(0,j_1+2,k-j_1)} \cup \triangle_{Q(0,j_1+3,k-j_1-1)}). \end{aligned} \tag{4.6}$$

Note that $j_1 - 1 \le k$ and $k - j_1 \le k$. By induction of (1) and (2), the cardinality of a maximal orthogonal system on triangle areas $\triangle_{Q(0,2,j_1-2)} \cup \triangle_{Q(1,2,j_1-1)}$ is $(j_1 - 1) + 1 = j_1$ and the cardinality of a maximal orthogonal system on triangle areas $\triangle_{Q(0,j_1+2,k-j_1)} \cup \triangle_{Q(0,j_1+3,k-j_1-1)}$ is $(k - j_1) + 1$. Since triangle areas $\triangle_{Q(0,2,j_1-2)} \cup \triangle_{Q(1,2,j_1-1)}$ and $\triangle_{Q(0,j_1+2,k-j_1)} \cup \triangle_{Q(0,j_1+3,k-j_1-1)}$ are mutual orthogonal, the cardinality of $\mathcal{S}$ is $1 + j_1 + (k - j_1 + 1) = k + 2$. Thus our conclusion (3) holds in this case.

Case two: $\mathcal{S} \cap \{Q(1, 1, j) \mid 0 \le j \le k+1\} \neq \varnothing$. We assume that $S_1 = Q(0, 1, k_1)$ in $\mathcal{S}$. By equation (3.34),

$$\begin{aligned} &{}^{\perp}S_1^{\perp} \cap (\triangle_{Q(0,1,k+1)} \cup \triangle_{Q(1,1,k+1)} \\ &= (\triangle_{Q(1,2,k_1-2)} \cup \triangle_{Q(1,k_1+2,k-k_1)}) \cup (\triangle_{Q(0,1,k_1-1)} \cup \triangle_{Q(0,k_1+2,k-k_1)}) \\ &= (\triangle_{Q(1,2,k_1-2)} \cup \triangle_{Q(0,1,k_1-1)}) \cup (\triangle_{Q(1,k_1+2,k-k_1)} \cup \triangle_{Q(0,k_1+2,k-k_1)}). \end{aligned} \tag{4.7}$$

Note that $k_1 - 1 \le k$ and $k - k_1 \le k$. By induction of (1) and (2), the cardinality of a maximal orthogonal system on triangle areas $\triangle_{Q(1,2,k_1-2)} \cup \triangle_{Q(0,1,k_1-1)}$ is $(k_1 - 1) + 1 = k_1$ and the cardinality of a maximal orthogonal system on triangle areas $\triangle_{Q(1,k_1+2,k-k_1)} \cup \triangle_{Q(0,k_1+2,k-k_1)}$ is $(k - k_1) + 1$. Since triangle areas $\triangle_{Q(1,2,k_1-2)} \cup \triangle_{Q(0,1,k_1-1)}$ and $\triangle_{Q(1,k_1+2,k-k_1)} \cup \triangle_{Q(0,k_1+2,k-k_1)}$ are mutual orthogonal, the cardinality of $\mathcal{S}$ is $1 + k_1 + (k - k_1 + 1) = k + 2$. Thus our conclusion (3) holds in this case.

Hence conclusion (3) is proved by summarizing the above proof. $\square$

4.3. **A maximal orthogonal systems on A-$\underline{\mathsf{mod}}$.** Let $\mathcal{M}$ be an orthogonal system on the stable Euclidean component ${}_s\Gamma_0$ over A-$\underline{\mathsf{mod}}$. Let $\mathcal{P}_i$ be the set of quasi-simples on Q_i and $\mathcal{Q}_i$ the set of quasi-simples on Q_i' for $i = 0, 1$. Without loss of generality, by equation (4.1), we assume that $\mathcal{M}$ equals to

$$\begin{aligned} \mathcal{O}^0_{(0,a_1,b_1)} :&= \{(0, a_i, b_i) \mid a_1 - p < a_i < a_1, b_1 < b_i < b_1 + q \text{ for } i = 2, \cdots, k, \\ &\qquad \text{and, if } a_i < a_j, \text{then } b_j < b_i \text{ for any } i, j = 1, 2, \cdots, k\}. \end{aligned} \tag{4.8}$$

and we assume that $(0, a_1, b_1) = (0, a, b)$.

Proposition 4.7. *Let $\mathcal{M}$, ${}_sQ_0$ and ${}_sQ_1$ (resp. ${}_sQ'_0$ and ${}_sQ'_1$) be as above. Then the cardinality of a maximal orthogonal system on ${}^{\perp}\mathcal{M}^{\perp} \cap {}_sQ_0 \cup {}_sQ_1$ (resp. ${}^{\perp}\mathcal{M}^{\perp} \cap {}_sQ'_0 \cup {}_sQ'_1$) is $q-k$ (resp. $p-k$).*

Proof. We only prove that the cardinality of a maximal orthogonal system on ${}^{\perp}\mathcal{M}^{\perp} \cap {}_sQ_0 \cup {}_sQ_1$ is $q-k$, since the other case is similar to deal with. By Proposition 3.4 and the construction of an orthogonal system on an Euclidean component, if $i \neq j$, then $(\mathsf{Lsupp}(0, a_i, b_i) \cap \mathcal{P}_0) \cap (\mathsf{Lsupp}(0, a_j, b_j) \cap \mathcal{P}_0) = \varnothing$. Thus $\mid \mathsf{Lsupp}(\mathcal{M}) \cap \mathcal{P}_0 \mid = k$. Without loss of generality, we may assume that

$$\mathsf{Lsupp}(\mathcal{M}) \cap \mathcal{P}_0 = \{Q(0, b_i, 0) \mid i = 1, 2, \cdots, k\}$$

with the second coordinate in an ascending way. Note that $\mathsf{Rsupp}(\mathcal{M}) \cap \mathcal{P}_1 = \Omega(\mathsf{Lsupp}(\mathcal{M}) \cap \mathcal{P}_0)$ and that $(\mathsf{Rsupp}(\mathcal{M}) \cap \mathcal{P}_1) \cap (\mathsf{Lsupp}(\mathcal{M}) \cap \mathcal{P}_0) = \varnothing$.

Take

$$\mathcal{R} = \{Q(j, b_i + 1, b_{i+1} - b_i - 2) \mid i = 1, \cdots, k-1, j = 0, 1\} \cup \{Q(j, \overline{b_k + 1}, b_1 + p - b_k - 2) \mid j = 0, 1\},$$

where $\overline{b_k + 1}$ is the congruence class of modulo q. By equation (3.30), ${}^{\perp}\mathcal{M}^{\perp} \cap ({}_sQ_0 \cup {}_sQ_1)$ is a disjoint union of $\triangle_v$ for $v \in \mathcal{R}$.

By Subsection 3.2, we may construct a maximal orthogonal system for each triangle area as follows.

$\triangle_{Q(0,b_i+1,b_{i+1}-b_i-2)} \cup \triangle_{Q(1,b_i+1,b_{i+1}-b_i-2)}$ for $i = 1, \cdots, k$, and $\triangle_{Q(0,\overline{b_k+1},b_1+q-b_k-2)} \cup \triangle_{Q(1,\overline{b_k+1},b_1+q-b_k-2)}$.

Note that if $b_{i+1} = b_i + 1$, then $b_{i+1} - b_i - 2 = -1$. In this case, we assume that $\triangle_{Q(0,b_i+1,b_{i+1}-b_i-2)} = \varnothing$. Put those maximal orthogonal systems together, denoted by $\mathcal{S}_1$. Therefore $\mathcal{S}_1$ is a maximal orthogonal system on ${}^{\perp}\mathcal{M}^{\perp} \cap ({}_sQ_0 \cup {}_sQ'_0)$.

Now we determine the cardinality of $\mathcal{S}_1$. Since $(0, a, b)$ is in $\mathcal{M}$, $\mathcal{S}_1$ is contained in triangle areas $\triangle_{Q(0,1,q-2)} \cup \triangle_{Q(1,1,q-2)}$. Since $\mathsf{Lsupp}(\mathcal{S}_1) \cap \mathcal{P}_0$ is an orthogonal system on $\triangle_{Q(0,1,q-2)} \cup \triangle_{Q(1,1,q-2)}$. By the construction of $\mathcal{S}_1$, $\mathsf{Lsupp}(\mathcal{M}) \cap \mathcal{P}_0$ are mutual orthogonal with $\mathcal{S}_1$ on A-$\underline{\mathsf{mod}}$. Thus $(\mathsf{Lsupp}(\mathcal{M}) \cap (\mathcal{P}_0 \backslash \{Q(0,0,0)\})) \cup \mathcal{S}_1$ is a maximal orthogonal system on triangle areas $\triangle_{Q(0,1,q-2)} \cup \triangle_{Q(1,1,q-2)}$. By Theorem 4.6, $\mid \mathcal{S}_1 \mid = q - 1 - (k-1) = q - k$. By the similar method, we may also construct a maximal orthogonal system $\mathcal{S}_2$ on ${}^{\perp}\mathcal{M}^{\perp} \cap ({}_sQ'_0 \cup {}_sQ'_1)$, and the cardinality of $\mathcal{S}_2$ is $p - m$. □

Let $\mathcal{M}$, $\mathcal{S}_1$ and $\mathcal{S}_2$ be as in the proof of Proposition 4.7. By equations (3.10) and (3.42), ${}^{\perp}(\mathcal{M} \cup \mathcal{S}_1 \cup \mathcal{S}_2)^{\perp} \cap {}_s\Gamma_1$ is a disjoint union of some rectangle areas of some vertices. By Subsection 3.1, we may construct a maximal orthogonal system for each rectangle area. Put those maximal orthogonal systems together, denoted by $\mathcal{M}'$. $\mathcal{M}'$ is a maximal orthogonal system on ${}^{\perp}(\mathcal{M} \cup \mathcal{S}_1 \cup \mathcal{S}_2)^{\perp} \cap {}_s\Gamma_1$. Take $\mathcal{S} = \mathcal{M} \cup \mathcal{M}' \cup \mathcal{S}_1 \cup \mathcal{S}_2$. Hence $\mathcal{S}$ is a maximal orthogonal system on A-$\underline{\mathsf{mod}}$. Since $\mathcal{S}_1 \cup \mathcal{S}_2$ is a maximal orthogonal system on ${}^{\perp}\mathcal{M}^{\perp} \cap ({}_sQ_0 \cup {}_sQ_1 \cup {}_sQ'_0 \cup {}_sQ'_1)$, $\mathcal{M}$ is maximal on ${}^{\perp}(\mathcal{S}_1 \cup \mathcal{S}_2)^{\perp} \cap {}_s\Gamma_0$ and $\mathcal{M}'$ is maximal on ${}^{\perp}(\mathcal{S}_1 \cup \mathcal{S}_2)^{\perp} \cap {}_s\Gamma_1$. $\mathcal{S}_1 \cup \mathcal{S}_2$ is a maximal orthogonal system on ${}^{\perp}\mathcal{M}'^{\perp} \cap ({}_sQ_0 \cup {}_sQ_1 \cup {}_sQ'_0 \cup {}_sQ'_1)$. The cardinality of a maximal orthogonal system on ${}^{\perp}\mathcal{M}^{\perp} \cap ({}_sQ_0 \cup {}_sQ_1 \cup {}_sQ'_0 \cup {}_sQ'_1)$ equals to the cardinality of a maximal orthogonal system on ${}^{\perp}\mathcal{M}'^{\perp} \cap ({}_sQ_0 \cup {}_sQ_1 \cup {}_sQ'_0 \cup {}_sQ'_1)$.

Proposition 4.8. *Let $\mathcal{M}$ and $\mathcal{M}'$ be as above. Then the cardinality of $\mathcal{M}$ equals to the cardinality of $\mathcal{M}'$.*

Proof. Otherwise, $\mid \mathcal{M} \mid \neq \mid \mathcal{M}' \mid$. Without loss of generality, we assume that $\mid \mathcal{M} \mid > \mid \mathcal{M}' \mid$. Then $\mid \mathsf{Lsupp}(\mathcal{M}) \cap \mathcal{P}_0 \mid > \mid \mathsf{Lsupp}(\mathcal{M}') \cap \mathcal{P}_0 \mid$ and $\mid \mathsf{Rsupp}(\mathcal{M}) \cap \mathcal{P}_1 \mid > \mid \mathsf{Rsupp}(\mathcal{M}') \cap \mathcal{P}_1 \mid$. Similar with the above construction, we know that ${}^{\perp}\mathcal{M}'^{\perp} \cap ({}_sQ_0 \cup {}_sQ_1)$ is a disjoint union of $\triangle_{v'}$ for v' in some set $\mathcal{R}'$(compared with $\mathcal{R}$ in Proposition 4.7) and $\mid \mathcal{R} \mid > \mid \mathcal{R}' \mid$. Thus it is not hard to know that the cardinality of ${}^{\perp}\mathcal{M}'^{\perp} \cap ({}_sQ_0 \cup {}_sQ_1 \cup {}_sQ'_0 \cup {}_sQ'_1)$ is strictly less than the cardinality of ${}^{\perp}\mathcal{M}^{\perp} \cap ({}_sQ_0 \cup {}_sQ_1 \cup {}_sQ'_0 \cup {}_sQ'_1)$. It is a contradiction. □

Corollary 4.9. *Let A be a 2-domestic Brauer graph algebra and $\mathcal{S}$ a maximal orthogonal system on A-$\underline{\mathsf{mod}}$ containing at least one object on an Euclidean component. Then the cardinality of $\mathcal{S}$ is the number of non-projective simple A-modules.*

Proof. By Proposition 4.8, $\mid \mathcal{S} \cap {}_s\Gamma_0 \mid = \mid \mathcal{S} \cap {}_s\Gamma_1 \mid = k$. By Proposition 4.7, $\mid \mathcal{S} \cap ({}_sQ_0 \cup {}_sQ_1) \mid = q - k$ and $\mid \mathcal{S} \cap ({}_sQ'_0 \cup {}_sQ'_1) \mid = p - k$. Thus the cardinalty of $\mathcal{S}$ is $2k + (q-k) + (p-k) = q + p = n$ by Theorem 2.10, the number of non-projective simple A-modules. □

Thus every maximal orthogonal system containing at least one object for an Euclidean component on A-$\underline{\mathsf{mod}}$ has the same cardinality, the number of non-projective simple A-modules.

5. A sufficient and necessary condition for an orthogonal system to be a simple-minded system

Let A be a 2-domestic Brauer graph algebra and $\mathcal{S}$ a maximal orthogonal system on A-$\underline{\mathsf{mod}}$ containing at least one object for an Euclidean component. Our main task in this section is to prove that $\mathcal{S}$ is a simple-minded system on A-$\underline{\mathsf{mod}}$, see Theorem 5.14. We assume that $\mathcal{M} = \{M_1, \cdots, M_\ell\}$ is the subset of $\mathcal{S}$ which is contained in the Euclidean component ${}_s\Gamma_0$ and we may assume that $\mathcal{M}' = \{M'_1, \cdots, M'_\ell\}$ is the subset of $\mathcal{S}$ contained in ${}_s\Gamma_1$ by Proposition 4.8. Without loss of generality, we assume that the object $M_1 = (0, a_1, b_1) = (0, a, b)$ (resp. $M'_1 = (1, a'_1, b'_1)$) and $M_i = (0, a_i, b_i)$ (resp. $M'_i = (1, a'_i, b'_i)$) for $\imath = 2, \cdots, \ell$. By equations (3.3) and (3.4), ${}^\perp\mathcal{M}^\perp \cap {}_s\Gamma_0$ consists of a disjoint of ℓ rectangle areas. By equation (3.30), ${}^\perp\mathcal{M}^\perp \cap ({}_sQ'_0 \cup {}_sQ'_1)$ is a disjoint of several triangle areas and $\mathcal{S} \cap ({}_sQ'_0 \cup {}_sQ'_1) \subseteq {}^\perp\mathcal{M}^\perp \cap ({}_sQ'_0 \cup {}_sQ'_1)$.

We first consider a special case that every object in $\mathcal{S} \cap ({}_sQ'_0 \cup {}_sQ'_1)$ and $\mathcal{S} \cap ({}_sQ_0 \cup {}_sQ_1)$ is a quasi-simple. Since ${}^\perp\mathcal{M}^\perp \cap ({}_sQ'_0 \cup {}_sQ'_1)$ is a disjoint of several triangle areas and they are mutual stable orthogonal on A-$\underline{\mathsf{mod}}$, without loss of generality, we need only figure out how can we construct a maximal orthogonal system which contains only quasi-simples on one of the triangle areas, such as ${}^\perp({}^\perp\{M_1, M_2\}^\perp \cap \square_{M_1,M_2})^\perp \cap ({}_sQ'_0 \cup {}_sQ'_1)$. Note that ${}^\perp({}^\perp\{M_1, M_2\}^\perp \cap \square_{M_1,M_2})^\perp \cap ({}_sQ'_0 \cup {}_sQ'_1) = \triangle_{Q(0,1,a_2-a_1-2)'} \cup \triangle_{Q(1,1,a_2-a_1-2)'}$ and all quasi-simples on triangle areas $\triangle_{Q(0,1,a_2-a_1-2)'} \cup \triangle_{Q(1,1,a_2-a_1-2)'}$ state as follows.

$$\mathcal{Q}\colon = \{Q(0, i, 0)' \mid 1 \leq i \leq a_2 - a_1 - 1\} \cup \{Q(1, i, 0)' \mid 1 \leq i \leq a_2 - a_1 - 1\}.$$

Let $\mathcal{W} \subseteq \mathcal{Q}$ be a maximal orthogonal system on triangle areas $\triangle_{Q(0,1,a_2-a_1-2)'} \cup \triangle_{Q(1,1,a_2-a_1-2)'}$. By Theorem 4.6, the cardinality of $\mathcal{W}$ is $a_2 - a_1 - 1$. The following proposition characterizes the maximal orthogonal system $\mathcal{W}$.

Proposition 5.1. *Let $\mathcal{Q}$ and $\mathcal{W}$ be as the above. Then $\mathcal{W}$ is of the form $\{Q(1, i, 0)' \mid 1 \leq i \leq \ell\} \cup \{Q(0, i, 0)' \mid \ell + 1 \leq i \leq a_2 - a_1 - 1\}$ for an integer $0 \leq \ell \leq a_2 - a_1 - 1$.*

Proof. there are two cases to be considered as follows.
Case one: $Q(0, 1, 0)'$ is in $\mathcal{W}$.
Then $Q(1, 1, 0)'$ and $Q(1, 2, 0)'$ are not in $\mathcal{W}$ by equation (3.34). Consider a subset of $\mathcal{Q}$ as follows.

$$\mathcal{Q}_1\colon = \{Q(0, i, 0)' \mid 2 \leq i \leq a_2 - a_1 - 1\} \cup \{Q(1, i, 0)' \mid 3 \leq i \leq a_2 - a_1 - 1\}.$$

By equation (3.34), $\mathcal{W} \subseteq \mathcal{Q}_1$. We claim that $Q(0, 2, 0)'$ is in $\mathcal{W}$. Otherwise,

$$\mathcal{W} = \{Q(0, 1, 0)'\} \cup (\mathcal{W} \cap (\triangle_{Q(0,3,a_2-a_1-4)'} \cup \triangle_{Q(1,3,a_2-a_1-4)'})).$$

By Theorem 4.6, the cardinality of $\mathcal{W} \cap (\triangle_{Q(0,3,a_2-a_1-4)'} \cup \triangle_{Q(1,3,a_2-a_1-4)'})$ is $a_2 - a_1 - 4 + 1 = a_2 - a_1 - 3$. Then the cardinality of $\mathcal{W}$ is $a_2 - a_1 - 3 + 1 = a_2 - a_1 - 2 < a_2 - a_1 - 1$. It is a contradiction. Thus $Q(0, 2, 0)'$ is in $\mathcal{W}$. Proceeding this process, $\mathcal{W} = \{Q(0, i, 0)' \mid 1 \leq i \leq a_2 - a_1 - 1\}$.
Case two: $Q(0, 1, 0)'$ is not in $\mathcal{W}$.

Since $Q(0, 1, 0)'$ is not in $\mathcal{W}$, by Theorem 4.6, $Q(1, 1, 0)'$ is in $\mathcal{W}$. By equation (3.34),

$$\mathcal{W} = \{Q(1, 1, 0)'\} \cup (\mathcal{W} \cap (\triangle_{Q(0,2,a_2-a_1-3)'} \cup \triangle_{Q(1,2,a_2-a_1-3)'})).$$

Then $Q(1, 2, 0)' \in \mathcal{W}$ or $Q(0, 2, 0)' \in \mathcal{W}$. If $Q(0, 2, 0)' \in \mathcal{W}$, by Case one,

$$\mathcal{W} = \{Q(1, 1, 0)'\} \cup \{Q(0, k, 0)' \mid 2 \leq k \leq a_2 - a_1 - 1\}.$$

If $Q(1, 2, 0)' \in \mathcal{W}$, then

$$\mathcal{W} = \{Q(1, 1, 0)', Q(1, 2, 0)'\} \cup (\mathcal{W} \cap (\triangle_{Q(0,3,a_2-a_1-4)'} \cup \triangle_{Q(1,3,a_2-a_1-4)'})).$$

Proceeding this process, there exists an positive integer r such that

$$\mathcal{W} = \{Q(1, k, 0)' \mid 1 \leq k \leq r\} \cup \{Q(0, k, 0)' \mid r + 1 \leq k \leq a_2 - a_1 - 1\}.$$

By combining Case one and Case two, our conclusion holds. □

We state the general form of Proposition 5.1 as follows and we will not give a proof here.

Corollary 5.2. *Let $\mathcal{W}$ be a maximal orthogonal system on A-$\underline{\mathsf{mod}}$ which consists of quasi-simples on ${}^\perp({}^\perp\{M_k, M_{k+1}\}^\perp \cap \square_{M_k,M_{k+1}})^\perp \cap ({}_sQ'_0 \cup {}_sQ'_1)$ for $1 \leq k \leq \ell$. Then $\mathcal{W}$ is of the form $\{Q(1, m, 0)' \mid a_k - a_1 + 1 \leq m \leq r\} \cup \{Q(0, m, 0)' \mid r + 1 \leq m \leq a_{k+1} - a_1 - 1\}$ for $0 \leq r \leq a_{k+1} - a_1 - 1$.*

Note that we assume that $M_{\ell+1}$ is isomorphic to M_1 and $a_{\ell+1} = a_1 + p$ on Corollary 5.2. By Proposition 5.1 and Corollary 5.2, we may assume that $\mathcal{S} \cap ({}_sQ_0 \cup {}_sQ_1)$ is of the form

$$\begin{aligned}
&\{Q(1,1,0)', Q(1,2,0)', \cdots, Q(1,t_1-1,0)'\} \cup \{Q(0,t_1,0)', \cdots, Q(0,a_2-a_1-1,0)'\} \cup \\
&\{Q(1,a_2-a_1+1,0)', \cdots, Q(1,t_2-1,0)'\} \cup \{Q(0,t_2,0)', \cdots, Q(0,a_3-a_1-1,0)'\} \\
&\cup, \cdots, \cup \\
&\{Q(1,a_r-a_1+1,0)', \cdots, Q(1,t_r-1,0)'\} \cup \{Q(0,t_r,0)', \cdots, Q(0,a_{r+1}-a_1-1,0)'\} \\
&\cup, \cdots, \cup \\
&\{Q(1,a_\ell-a_1+1,0)', \cdots, Q(1,t_\ell-1,0)'\} \cup \{Q(0,t_\ell,0)', \cdots, Q(0,p-1,0)'\},
\end{aligned} \tag{5.1}$$

where $a_r - a_1 + 2 \leq t_r \leq a_{r+1} - a_1 - 1$ for $1 \leq r \leq \ell - 1$ and $a_\ell - a_1 + 2 \leq t_\ell \leq p - 1$.

Proposition 5.3. *Let $\mathcal{S}$, $\mathcal{M}$, $\mathcal{M}'$ and $\{t_k \mid 1 \leq k \leq \ell\}$ be as the stated above. Then the following subsets of objects are contained in $\mathcal{F}(\mathcal{S})$.*

(1) $\{(0, a_1 + j, b_1) \mid j = 0, 1, \cdots, t_1 - 1\} \cup \{(0, a_2 - j, b_2) \mid j = 0, 1, \cdots, a_2 - a_1 - t_1\}$.
(2) $\{(0, a_r + j, b_r) \mid j = 0, 1, \cdots, t_r - a_r + a_1 - 1\} \cup \{(0, a_{r+1} - j, b_{r+1}) \mid j = 0, 1, \cdots, a_{r+1} - a_1 - t_r\}$ *for* $r = 2, 3, \cdots, \ell - 1$.
(3) $\{(0, a_\ell + j, b_\ell) \mid j = 0, 1, \cdots, t_\ell - a_\ell + a_1 - 1\} \cup \{(0, a_1 + p - j, b_1 - q) \mid j = 0, 1, \cdots, p + 1 - t_\ell\}$.

Proof. Without loss of generality, we only prove (1), since other cases are similar to handle.

Consider the following triangles:

$$(0, a_1, b_1) \to (0, a_1 + 1, b_1) \to Q(1, 1, 0)' \to (1, a_1 + 1, b_1 + 1),$$

$$(0, a_1 + k, b_1) \to (0, a_1 + k + 1, b_1) \to Q(1, k + 1, 0)' \to (1, a_1 + k + 1, b_1 + 1) \tag{5.2}$$

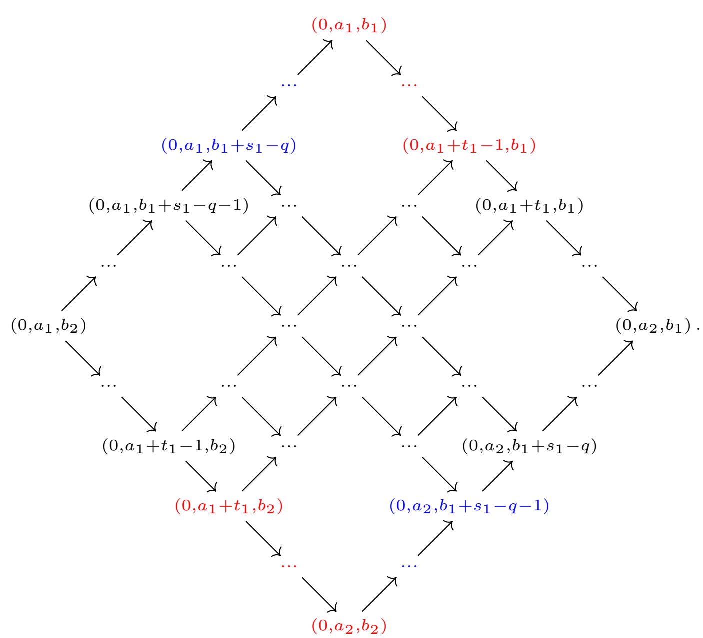
for $1 \leq k \leq t_1 - 3$. Since $(0, a_1, b_1)$ and $Q(1, 1, 0)'$ are in $\mathcal{S}$, $(0, a_1 + 1, b_1) \in \mathcal{F}(\mathcal{S})$. By triangle (5.2), it follows from induction that $(0, a_1 + j, b_1) \in \mathcal{F}(\mathcal{S})$ for $j = 0, 1, \cdots, t_1 - 1$.

Dually, consider triangles as follows.

$$Q(0, a_2 - a_1 - 1, 0)' \to (0, a_2 - 1, b_2) \to (0, a_2, b_2) \to Q(1, a_2 - a_1, 0)',$$

$$Q(0, a_2 - a_1 - k - 1, 0)' \to (0, a_2 - k - 1, b_2) \to (0, a_2 - k, b_2) \to Q(1, a_2 - a_1 - k, 0)' \tag{5.3}$$

for $1 \leq k \leq a_2 - a_1 - t_1 - 1$. Since $(0, a_2, b_2)$ and $Q(0, a_2 - a_1 - 1, 0)'$ are in $\mathcal{S}$, $(0, a_2 - 1, b_2) \in \mathcal{F}(\mathcal{S})$. By the triangle (5.3), it follows from induction that $\{Q(0, t_1, 0)', \cdots, Q(0, a_2 - a_1 - 1, 0)'\} \subseteq \mathcal{F}(\mathcal{S})$. $\square$



The above diagram describes the location of objects in (1) of Proposition 5.3 and (1) of Proposition 5.4. Specifically, the red one is the objects in (1) of Proposition 5.3 and the blue one is the objects in (1) of Proposition 5.4. Note that both objects $(0, a_1, b_1)$ and $(0, a_2, b_2)$ are in $\mathcal{S}$.

Dually, we may assume that $\mathcal{S} \cap ({}_sQ_0 \cup {}_sQ_1)$ is of the form:

$$\begin{aligned}
&\{Q(0,q-1,0), Q(0,q-2,0), \cdots, Q(0,s_1,0)\} \cup \{Q(1,s_1-1,0), \cdots, Q(1,q-(b_1-b_2-1),0)\} \cup \\
&\{Q(0,q-(b_1-b_2+1),0), \cdots, Q(0,s_2,0)\} \cup \{Q(1,s_2-1,0), \cdots, Q(1,q-(b_1-b_3-1),0)\} \\
&\cup, \cdots, \cup \\
&\{Q(0,q-(b_1-b_r+1),0), \cdots, Q(0,s_r,0)\} \cup \{Q(1,s_r-1,0), \cdots, Q(1,q-(b_1-b_{r+1}-1),0)\} \\
&\cup, \cdots, \cup \\
&\{Q(0,q-(b_1-b_\ell+1),0), \cdots, Q(0,s_\ell,0)\} \cup \{Q(1,s_\ell-1,0), \cdots, Q(1,1,0)\}.
\end{aligned} \tag{5.4}$$

where $q-(b_1-b_{r+1}+1) \le s_r \le q-(b_1-b_r+1)$ for $r = 1, \cdots, \ell-1$ and $1 \le s_\ell \le q-(b_1-b_\ell+1)$. Thus

Proposition 5.4. *Let $\mathcal{S}$, $\mathcal{M}$ and $\{s_k \mid 1 \le k \le \ell\}$ be as the stated above. Then the following subsets of objects are contained in $\mathcal{F}(\mathcal{S})$.*

(1) $\{(0,a_1,b_1-j) \mid j = 0,1,\cdots,q-s_1\} \cup \{(0,a_2,b_2+j) \mid j = 0,1,\cdots,b_1-b_2+s_1-q-1\}$,
(2) $\{(0,a_i,b_i-j) \mid j = 0,1,\cdots,q+b_i-b_1-s_i\} \cup \{(0,a_{i+1},b_{i+1}+j) \mid j = 0,1,\cdots,b_1-b_{i+1}+s_i-q-1\}$ *for* $i = 2,3,\cdots,\ell-1$.
(3) $\{(0,a_\ell,b_\ell-j) \mid j = 0,1,\cdots,q+b_\ell-b_1-s_\ell\} \cup \{(0,a_1+p,b_1-q+j) \mid j = 0,1,\cdots,s_i-1\}$.

We hope to generalize Propositions 5.3 and 5.4 for a maximal orthogonal system on $\mathcal{S} \cap ({}_sQ_0 \cup {}_sQ_1)$ or $\mathcal{S} \cap ({}_sQ'_0 \cup {}_sQ'_1)$. We study properties of a maximal orthogonal system on triangle areas $\triangle_{Q(0,j,i)'} \cup \triangle_{Q(1,j,i)'}$ for $i \le p-2$. Let $\mathcal{W}$ be a maximal orthogonal system on triangle areas $\triangle_{Q(0,j,i)'} \cup \triangle_{Q(1,j,i)'}$. Without loss of generality, we assume $\mathcal{W} = \mathcal{U} \cup \mathcal{V}$, where $\mathcal{U} = \{U_1, U_2, \cdots, U_k\} \subseteq \triangle_{Q(0,j,i)'}$, $\mathcal{V} = \{V_1, V_2, \cdots, V_\ell\} \subseteq \triangle_{Q(1,j,i)'}$ and the second coordinates of objects of $\mathcal{U}$ and $\mathcal{V}$ are in an ascending way, respectively. Note that $k + \ell = i + 1$ by Theorem 4.6. Then we have the following observations.

Proposition 5.5. *Let $\mathcal{W}$, $\mathcal{U}$ and $\mathcal{V}$ be subsets on triangle areas $\triangle_{Q(0,j,i)'} \cup \triangle_{Q(1,j,i)'}$ as stated above. Then*

(1) *V_1 in $\mathcal{V}$ is of the form $Q(1,j,m)'$ or U_1 in $\mathcal{U}$ is of the form $Q(0,j,k)'$ for integers m and k.*
(2) *If V_1 is of the form $Q(1,j,m)'$ with the height $m > 0$, then U_1 in $\mathcal{U}$ is of the form $Q(0,j,k)'$ with $k < m$.*
(3) *If U_1 is of the form $Q(0,j,n)'$ with the height $n > 0$, then there is an object in $\mathcal{V}$ of the form $Q(0,j+i_0,n-i_0)'$.*
(4) *If there is an object $V_r = Q(1,j_1,i_1)'$ in $\mathcal{V}$ with the height $i_1 > 0$, then there is an object U_s in $\mathcal{U}$ of the form $Q(0,j_1,i'_1)'$ with $i'_1 < i_1$.*
(5) *If there is an object $U_e = Q(0,j_2,i_2)'$ in $\mathcal{U}$ with the height $i_2 > 0$, then there is an object V_f in $\mathcal{V}$ of the form $Q(1,j_2+t,i_2-t)'$.*

Proof. We prove only (1) and (2), since the proof of (3) is dual to (2) and (4) (resp. (5)) is similar to (2)(resp. (3)).

(1) If there is no object in the set $\{Q(0,j,t)' \mid 0 \le t \le i\}$ contained in $\mathcal{U}$ and no object in $\{Q(1,j,t)' \mid 0 \le t \le i\}$ contained in $\mathcal{V}$, then the cardinality of a maximal orthogonal system on triangle areas $\triangle_{Q(0,j,i)'} \cup \triangle_{Q(1,j,i)'}$ equals to the cardinality of a maximal orthogonal system on triangle areas $\triangle_{Q(0,j+1,i-1)'} \cup \triangle_{Q(1,j+1,i-1)'}$. It contradicts Theorem 4.6.

(2) By Theorem 4.6, the cardinality of a maximal orthogonal system on triangle areas $\triangle_{Q(1,j,m)'} \cup \triangle_{Q(1,j,m)'}$ is $m+1$. By equations (3.34) and (3.36), ${}^{\perp}Q(1,j,m)'^{\perp} \cap (\triangle_{Q(0,j,m)'} \cup \triangle_{Q(1,j,m)'}) = \{Q(0,j,t)' \mid 0 \le t \le m-1\} \cup (\triangle_{Q(0,j+1,m-2)'} \cup \triangle_{Q(1,j+1,m-2)'})$. By Theorem 4.6, the cardinality of a maximal orthogonal system on triangle areas $\triangle_{Q(0,j+1,m-2)'} \cup \triangle_{Q(1,j+1,m-2)'}$ is $m-1$. Since $m > 0$, the set $\{Q(0,j,t)' \mid 0 \le t \le m-1\}$ is not empty. Let $\mathcal{U}_1$ be a maximal orthogonal system on triangle areas $\triangle_{Q(0,j,m)'} \cup \triangle_{Q(1,j,m)'}$ containing $Q(1,j,m)'$. If $\mathcal{U}_1$ contains no object in $\{Q(0,j,t)' \mid 0 \le t \le m-1\}$, then the cardinality $\mathcal{U}_1$ is $1+(m-1) = m$. It contradicts Theorem 4.6. Therefore there is an object $Q(0,j,k)'$ in $\{Q(0,j,t)' \mid 0 \le t \le m-1\}$ contained in $\mathcal{V}$ and $k < m$. Thus $V_1 = Q(0,j,k)'$. □

By equation (3.30), we know that ${}^{\perp}\mathcal{M}^{\perp} \cap ({}_sQ_0 \cup {}_sQ_1)$ is a disjoint union of triangle areas of some vertices. Without loss of generality, we only consider a maximal orthogonal system on triangle areas $\triangle_{Q(0,1,r_1)'} \cup \triangle_{Q(1,1,r_1)'}$, where $r_1 = a_2 - a_1 - 2$. By Theorem 4.6, the cardinality of a maximal orthogonal system on triangle areas $\triangle_{Q(0,1,r_1)'} \cup \triangle_{Q(1,1,r_1)'}$ is $r_1 + 1$. Take $\mathcal{U}_1 := \{U_1, \cdots, U_s\}$ and $\mathcal{V}_1 := \{V_1, \cdots, V_t\}$ such that $\mathcal{U}_1$ is a maximal subset of $\mathcal{U}$ contained in $\triangle_{Q(0,1,r_1)'}$, where s is a positive integer and $t = r_1+1-s$. Note that $\mathcal{V}_1$ is a maximal subset of $\mathcal{V}$ contained in $\triangle_{Q(1,1,r_1)'}$.

Proposition 5.6. *Let $\mathcal{U}_1$, $\mathcal{V}_1$ and related notations be as stated above. Then there are a subset $\{Q(1,k_m,\ell_m)' \mid 1\le m\le v_1\}$ of $\mathcal{V}_1$ and a subset $\{Q(0,k_n,\ell_n)' \mid v_1+1\le n\le v_2\}$ of $\mathcal{U}_1$ for non-negative integers v_1 and v_2 such that $\mathcal{U}_1\cap\mathcal{V}_1$ is contained in a disjoint union of some triangle areas as follows.*

$$\bigcup_{m=1}^{v_1}(\triangle_{Q(0,k_m,\ell_m)'}\cup\triangle_{Q(1,k_m,\ell_m)'})\cup\bigcup_{n=v_1+1}^{v_2}(\triangle_{Q(0,k_n,\ell_n)'}\cup\triangle_{Q(1,k_n,\ell_n)'}).$$

Proof. **Case one:** There is a maximal non-empty subset $\{U_1,\cdots,U_{s_1}\}$ of $\mathcal{U}_1$ such that each object $U_k=Q(0,k,0)'$ is a quasi-simple for $1\le k\le s_1$.

By equations (3.34) and (3.36),

$$\begin{aligned}&{}^{\perp}\{U_1,\cdots,U_{s_1}\}^{\perp}\cap(\triangle_{Q(0,1,r_1)'}\cup\triangle_{Q(1,1,r_1)'})\\&=\triangle_{Q(0,s_1,r_1-s_1+1)'}\cup\triangle_{Q(1,s_1+1,r_1-s_1)'}\cup\{Q(1,1,s_1+k)' \mid -1\le k\le r_1-s_1\}.\end{aligned}\tag{5.5}$$

(1) If the object V_1 is in the set $\{Q(1,1,s_1+k)' \mid -1\le k\le r_1-s_1\}$. We assume $V_1=Q(1,1,\ell)'$, where $s_1-1\le\ell\le r_1$. By equations (3.28) and (3.30),

$$\begin{aligned}&{}^{\perp}\{U_1,\cdots,U_{s_1},V_1\}^{\perp}\cap(\triangle_{Q(0,1,r_1)'}\cup\triangle_{Q(1,1,r_1)'})\\&=(\triangle_{Q(0,s_1,\ell-s_1)'}\cup\triangle_{Q(0,\ell+2,r_1-\ell-1)'})\cup(\triangle_{Q(1,s_1+1,\ell-s_1-1)'}\cup\triangle_{Q(1,\ell+2,r_1-\ell-1)'})\\&=(\triangle_{Q(0,s_1,\ell-s_1)'}\cup\triangle_{Q(1,s_1+1,\ell-s_1-1)'})\cup(\triangle_{Q(0,\ell+2,r_1-\ell-1)'}\cup\triangle_{Q(1,\ell+2,r_1-\ell-1)'}).\end{aligned}\tag{5.6}$$

Thus $\mathcal{U}_1\cup\mathcal{V}_1\subseteq(\triangle_{Q(0,1,\ell)'}\cup\triangle_{Q(1,1,\ell)'})\cup(\triangle_{Q(0,\ell+2,r_1-\ell-1)'}\cup\triangle_{Q(1,\ell+2,r_1-\ell-1)'})$ and $Q(1,1,\ell)'\in\mathcal{U}_1$. Furthermore, it is easy to see that $V_2\in\{Q(0,\ell+2,k)' \mid 0\le k\le r_1-\ell-1\}$ or $U_{s_1+1}\in\{Q(1,\ell+2,k)' \mid 0\le k\le r_1-\ell-1\}$ on $\triangle_{Q(0,\ell+2,r_1-\ell-1)'}\cup\triangle_{Q(1,\ell+2,r_1-\ell-1)'}$. We may repeat Case one on triangle areas $\triangle_{Q(0,\ell+2,r_1-\ell-1)'}\cup\triangle_{Q(1,\ell+2,r_1-\ell-1)'}$.

(2) If the object V_1 is not in the set $\{Q(1,1,s_1+k)' \mid -1\le k\le r_1-s_1\}$, then $U_{s_i+1}=Q(0,s_1,\ell)'$ for $1\le\ell\le r_1-s_1+1$. Thus

$$\begin{aligned}&{}^{\perp}(\mathcal{U}_1\cup\{U_{s_1+1}\})^{\perp}\cap(\triangle_{Q(0,1,r_1)'}\cup\triangle_{Q(1,1,r_1)'})\\&\subseteq\bigcup_{k=1}^{s_i-1}(\triangle_{Q(0,k,0)'}\cup\triangle_{Q(1,k,0)'})\cup(\triangle_{Q(0,s_1,\ell)'}\cup\triangle_{Q(1,s_1,\ell)'})\\&\quad\cup(\triangle_{Q(0,s_1+\ell-1,r_1-s_1-\ell)'}\cup\triangle_{Q(1,s_1+\ell+2,r_1-s_1-\ell-1)'}).\end{aligned}\tag{5.7}$$

Hence

$$\begin{aligned}\mathcal{U}_1\cup\mathcal{V}_1\subseteq&\bigcup_{k=1}^{s_i-1}(\triangle_{Q(0,k,0)'}\cup\triangle_{Q(1,k,0)'})\cup(\triangle_{Q(0,s_1,\ell)'}\cup\triangle_{Q(1,s_1,\ell)'})\\&\cup(\triangle_{Q(0,s_1+\ell-1,r_1-s_1-\ell)'}\cup\triangle_{Q(1,s_1+\ell+2,r_1-s_1-\ell-1)'}),\end{aligned}\tag{5.8}$$

where $Q(0,k,0)'\in\mathcal{U}_1$ for $1\le k\le s_1-1$ and $U_{s_1+1}=Q(0,s_1,\ell)'\in\mathcal{U}_1$. Furthermore, it is easy to see that $U_{s_1+2}\in\{Q(0,s_1+\ell+1,k)' \mid 0\le k\le r_1-\ell-1\}$. Then we may repeat (2) of Case one on triangle areas $\triangle_{Q(0,s_1+\ell-1,r_1-s_1-\ell)'}\cup\triangle_{Q(1,s_1+\ell+2,r_1-s_1-\ell-1)'}$.

Case two: There is a maximal non-empty subset $\{V_1,\cdots,V_{t_1}\}$ of $\mathcal{V}_1$ such that each $V_k=Q(1,k,0)'$ is a quasi-simple for $k=1,\cdots,t_1$.

By equations (3.34) and (3.36),

$$\begin{aligned}&{}^{\perp}\{V_1,\cdots,V_{t_1}\}^{\perp}\cap(\triangle_{Q(0,1,r_1)'}\cup\triangle_{Q(1,1,r_1)'})\\&=\triangle_{Q(0,t_1,r_1-t_1+1)'}\cup\triangle_{Q(1,t_1,r_1-t_1+1)'}\cup\{Q(0,1,t_1+k)' \mid -2\le k\le r_1-t_1\}.\end{aligned}\tag{5.9}$$

(1) If the object U_1 is in the set $\{Q(0,1,t_1+k)' \mid -2\le k\le r_1-t_1\}$. We assume $U_1=Q(0,1,\ell)'$, where $t_1-2\le\ell\le r_1$. By equations (3.28) and (3.30),

$$\begin{aligned}&{}^{\perp}\{V_1,\cdots,V_{t_1},U_1\}^{\perp}\cap(\triangle_{Q(0,1,r_1)'}\cup\triangle_{Q(1,1,r_1)'})\\&=(\triangle_{Q(0,t_1,\ell-t_1)'}\cup\triangle_{Q(0,\ell+2,r_1-\ell-1)'})\cup(\triangle_{Q(1,t_1,\ell-t_1+1)'}\cup\triangle_{Q(1,\ell+3,r_1-\ell-2)'})\\&=(\triangle_{Q(0,t_1,\ell-t_1)'}\cup\triangle_{Q(1,t_1,\ell-t_1+1)'})\cup(\triangle_{Q(0,\ell+2,r_1-\ell-1)'}\cup\triangle_{Q(1,\ell+3,r_1-\ell-2)'}).\end{aligned}\tag{5.10}$$

Therefore $\mathcal{U}_1\cup\mathcal{V}_1\subseteq(\triangle_{Q(0,1,\ell)'}\cup\triangle_{Q(1,1,\ell)'})\cup(\triangle_{Q(0,\ell+2,r_1-\ell-1)'}\cup\triangle_{Q(1,\ell+3,r_1-\ell-2)'})$, $V_k=Q(1,k,0)'\in\mathcal{V}_1$ for $1\le k\le t_1-1$ and $U_1=Q(0,1,\ell)'\in\mathcal{U}_1$. Furthermore, there is an object $U_{s'}$ in $\mathcal{U}_1$ contained in the set $\{Q(0,\ell+2,k)' \mid 0 \le k \le r_1-\ell-1\}$. Thus we may repeat (1) of Case one on $\triangle_{Q(0,\ell+2,r_1-\ell-1)'} \cup \triangle_{Q(1,\ell+3,r_1-\ell-2)'}$.

(2) If the object U_1 is not in the set $\{Q(0,1,t_1+k)' \mid -2 \le k \le r_1-t_1\}$, then $U_{t_i+1} = Q(0,t_1,\ell)'$ for $1 \le \ell \le r_1-t_1+1$ or $V_1 = Q(1,t_1,\ell')'$ for $1 \le \ell' \le r_1-t_1+1$. Thus

$$\begin{aligned} &{}^{\perp}\mathcal{V}_1^{\perp} \cap (\triangle_{Q(0,1,r_1)'} \cup \triangle_{Q(1,1,r_1)'}) \\ &\subseteq \bigcup_{k=1}^{t_i-1} (\triangle_{Q(0,k,0)'} \cup \triangle_{Q(1,k,0)'}) \cup (\triangle_{Q(0,t_1,r_1-\ell-1)'} \cup \triangle_{Q(1,t_1,r_1-\ell-1)'}). \end{aligned} \tag{5.11}$$

Therefore

$$\mathcal{U}_1 \cup \mathcal{V}_1 \subseteq \bigcup_{k=1}^{t_i-1} (\triangle_{Q(0,k,0)'} \cup \triangle_{Q(1,k,0)'}) \cup (\triangle_{Q(0,t_1,r_1-\ell-1)'} \cup \triangle_{Q(1,t_1,r_1-\ell-1)'}),$$

$V_k = Q(1,k,0)' \in \mathcal{V}_1$ for $1 \le k \le t_1-1$. Furthermore $U_1 \in \{Q(0,\ell+2,k)' \mid 0 \le k \le r_1-\ell-1\}$ or $V_1 \in \{Q(1,\ell+2,k)' \mid 0 \le k \le r_1-\ell-1\}$. Thus we may repeat Case one or Case two on $\triangle_{Q(0,t_1,r_1-\ell-1)'} \cup \triangle_{Q(1,t_1,r_1-\ell-1)'}$.

Summarizing the above two cases, it follows from induction that the set $\mathcal{U}_1 \cap \mathcal{V}_1$ may be divided into a disjoint union of some subsets as follows.

$$\bigcup_{m=1}^{v_1} (\triangle_{Q(0,k_m,\ell_m)'} \cup \triangle_{Q(1,k_m,\ell_m)'}) \cup \bigcup_{m=v_1+1}^{v_2} (\triangle_{Q(0,k_m,\ell_m)'} \cup \triangle_{Q(1,k_m,\ell_m)'}),$$

where $Q(x,k_m,\ell_m)' \notin \triangle_{Q(0,k_n,\ell_n)'} \cup \triangle_{Q(1,k_n,\ell_n)'}$ for any $x=0,1$ and $m \ne n$.

For (1) of Case one and (2) of Case two, we have $Q(1,k_m,\ell_m)' \in \mathcal{V}_1$ for $1 \le m \le v_1$ and $Q(0,k_m,\ell_m)' \in \mathcal{U}_1$ for $v_1+1 \le m \le v_2$. For (2) of Case one and (1) if Case two, we have $Q(0,k_m,\ell_m)' \in \mathcal{V}_1$ for $1 \le m \le v_2$. Thus the proof is finished. □

Now we consider the triangle areas $\triangle_{Q(0,k_m,\ell_m)'} \cup \triangle_{Q(1,k_m,\ell_m)'}$ with $Q(0,k_m,\ell_m)' \in \mathcal{U}_1$ and the triangle areas $\triangle_{Q(0,k_n,\ell_n)'} \cup \triangle_{Q(1,k_n,\ell_n)'}$ with $Q(1,k_n,\ell_n)' \in \mathcal{V}_1$. Without loss of generality, we simplify the above notations and prove the following Theorem.

Theorem 5.7. *Let $\mathcal{S}$ be an orthogonal system on A-$\underline{\mathsf{mod}}$ and the subset $\mathcal{X}$ of $\mathcal{S}$ an orthogonal system on triangle areas $\triangle_{Q(0,m,\ell)'} \cup \triangle_{Q(1,m,\ell)'}$ with $Q(0,m,\ell)' \in \mathcal{X}$ for $\ell \le q-2$. If there is an object $(0,c,d)$ contained in $\mathcal{F}(\mathcal{S})$ such that there is a triangle as follows.*

$$Q(0,m+\ell,0)' \to (0,c-1,d) \to (0,c,d) \to Q(1,m+\ell+1,0)',$$

then the subset $\{(0,c-k,d) \mid 0 \le k \le \ell\} \subseteq \mathcal{F}(\mathcal{S})$.

Note that the triangle in Theorem 5.7 is from Proposition 3.4. Dually,

Theorem 5.8. *Let $\mathcal{S}$ be an orthogonal system on A-$\underline{\mathsf{mod}}$ and the subset $\mathcal{X}$ of $\mathcal{S}$ an orthogonal system on triangle areas $\triangle_{Q(0,r,s)'} \cup \triangle_{Q(1,r,s)'}$ with $Q(1,r,s)' \in \mathcal{X}$ for $s \le q-2$. If there is an object $(0,e,f)$ contained in $\mathcal{F}(\mathcal{S})$ such that there is a triangle as follows.*

$$(0,e,f) \to (0,e+1,f) \to Q(1,r,0)' \to (1,e+1,f+1),$$

then the subset $\{(0,e+t,f) \mid 0 \le t \le r\} \subseteq \mathcal{F}(\mathcal{S})$.

Before proving the above Theorem 5.7, we state the following key proposition.

Proposition 5.9. *Let $\mathcal{M}$ be an orthogonal system on A-$\underline{\mathsf{mod}}$ and $\mathcal{X} \subseteq \mathcal{M}$ be a maximal orthogonal system on triangle areas $\triangle_{Q(0,x,y-1)'} \cup \triangle_{Q(1,x,y)'}$. We assume that there is a triangle*

$$(0,b-1,c) \to (0,b,c) \to Q(1,x,0)' \to (1,b,c+1).$$

If $(0,b-1,c) \in \mathcal{F}(\mathcal{M})$, then $(0,b+k,c) \in \mathcal{F}(\mathcal{M})$ for integers $k=0,1,\cdots,y$.

Dually, we assume that $\mathcal{Z} \subseteq \mathcal{M}$ is a maximal orthogonal system on triangle areas $\triangle_{Q(0,v,w)'} \cup \triangle_{Q(1,v+1,w-1)'}$ and that there is a triangle

$$Q(0,v+w,0)' \to (0,e,f) \to (0,e+1,f) \to Q(1,v+w+1,0)'.$$

If $(0,e+1,f) \in \mathcal{F}(\mathcal{M})$, then $(0,e-k,f) \in \mathcal{F}(\mathcal{M})$ for integers $k=0,1,\cdots,w$.

Proof. We prove it by induction on the height y and w. We first consider the case $y = 1$ and $w = 1$. Thus $\triangle_{Q(0,x,0)'} = \{Q(0,x,0)'\}$ and $\triangle_{Q(1,x,1)'} = \{Q(1,x,0)', Q(1,x,1)', Q(1,x+1,0)'\}$. By equations (3.28) and (3.30), there are two maximal orthogonal systems on triangle areas $\triangle_{Q(0,x,0)'} \cup \triangle_{Q(1,x,1)'}$ as follows.

$$\mathcal{X}_1 = \{Q(1,x,0)', Q(1,x+1,0)'\}, \quad \mathcal{X}_2 = \{Q(1,x,1)', Q(0,x,0)'\}.$$

Case one: $\mathcal{X} = \mathcal{X}_1 = \{Q(1,x,0)', Q(1,x+1,0)'\}$.

Consider the triangles as follows.

$$(0,b-1,c) \to (0,b,c) \to Q(1,x,0)' \to (1,b,c+1),$$
$$(0,b,c) \to (0,b+1,c) \to Q(1,x+1,0)' \to (1,b+1,c+1).$$

Since $(0,b-1,c) \in \mathcal{F}(\mathcal{M})$ and $Q(1,x,0)' \in \mathcal{X}_1 \subseteq \mathcal{M}$, $(0,a,b) \in \mathcal{F}(\mathcal{M})$. Since $(0,a,b) \in \mathcal{F}(\mathcal{M})$ and $Q(1,x+1,0)' \in \mathcal{X}_1 \subseteq \mathcal{M}$, $(0,b+1,c) \in \mathcal{F}(\mathcal{M})$.

Case two: $\mathcal{X} = \mathcal{X}_2 = \{Q(1,x,1)', Q(0,x,0)'\}$.

Consider the triangle as follows.

$$(0,b-1,c) \to (0,b+1,c) \to Q(1,x,1)' \to (1,b,c+1),$$
$$Q(0,x,0)' \to (0,b,c) \to (0,b+1,c) \to Q(1,x+1,0)'.$$

Since $(0,b-1,c) \in \mathcal{F}(\mathcal{M})$ and $Q(1,x,1)' \in \mathcal{X}_2 \subseteq \mathcal{F}(\mathcal{M})$, $(0,b+k,c) \in \mathcal{F}(\mathcal{M})$ for $k = 0,1$. Thus our conclusion holds for $y = 1$.

Dually, we show the conclusion is true for $w = 1$. Thus $\triangle_{Q(0,v,1)'} = \{Q(0,v,0)', Q(0,v,1)', Q(0,v+1,0)'\}$ and $\triangle_{Q(1,v+1,0)'} = \{Q(1,v+1,0)'\}$. By equations (3.28) and (3.30), there are two maximal orthogonal systems on triangle areas $\triangle_{Q(0,v,1)'} \cup \triangle_{Q(1,v+1,0)'}$ as follows.

$$\mathcal{Z}_1 = \{Q(0,v,0)', Q(0,v+1,0)'\}, \quad \mathcal{Z}_2 = \{Q(0,v,1)', Q(1,v+1,0)'\}.$$

Case one: $\mathcal{Z} = \mathcal{Z}_1 = \{Q(0,v,1)', Q(0,v+1,0)'\}$.

Consider the triangles as follows.

$$Q(0,v+1,0)' \to (0,e,f) \to (0,e+1,f) \to Q(1,v+2,0)'.$$
$$Q(0,v,0)' \to (0,e-1,f) \to (0,e,f) \to Q(1,v+1,0)'.$$

Since $(0,e+1,f) \in \mathcal{F}(\mathcal{M})$ and $Q(0,v+1,0)' \in \mathcal{Z}_1 \subseteq \mathcal{M}$, $(0,e,f) \in \mathcal{F}(\mathcal{M})$. Since $Q(0,v,0)' \in \mathcal{Z}_1$ and $(0,e,f) \in \mathcal{F}(\mathcal{M})$, $(0,e-1,f) \in \mathcal{F}(\mathcal{M})$.

Case two: $\mathcal{Z} = \mathcal{Z}_2 = \{Q(0,v,1)', Q(1,v+1,0)'\}$.

Consider the triangles as follows.

$$Q(0,v,1)' \to (0,e-1,f) \to (0,e+1,f) \to Q(1,v+1,1)',$$
$$(0,e-1,f) \to (0,e,f) \to Q(1,v+1,0)' \to (1,e,f+1).$$

Since $Q(0,v,1)' \in \mathcal{Z}_2 \subseteq \mathcal{M}$ and $(0,e+1,f)$ is in $\mathcal{F}(\mathcal{M})$, $(0,e-k,f) \in \mathcal{F}(\mathcal{M})$ for $k = 0,1$. Thus our conclusion holds for $w = 1$.

We assume our conclusion is true for $y = k$ and $w = k$, and we only prove it is true for $y = k+1$, since the other case is dual. By Theorem 4.6, there is an object $Q(1,x_1,y_1)'$ on the set $\{Q(1,x+s,k+1-s)' \mid 0 \le s \le k+1\}$ belonging to $\mathcal{X}$, where $x_1 + y_1 = k+1+x$ and $y_1 \le k+1$. By equations (3.34) and (3.36),

$$\begin{aligned}(5.12)\qquad &(\triangle_{Q(0,x,k)'} \cup \triangle_{Q(1,x,k+1)'}) \cap \mathcal{X} \\ &= ((\triangle_{Q(0,x_1,y_1-1)'} \cup \triangle_{Q(1,x_1+1,y_1-2)'}) \cap \mathcal{X}) \cup ((\triangle_{Q(0,x,x_1-x-1)'} \cup \triangle_{Q(1,x,x_1-x)'}) \cap \mathcal{X})\end{aligned}$$

Since $x_1 - x < k$ and $(0,b-1,c) \in \mathcal{F}(\mathcal{M})$, $(0,b+k,c) \in \mathcal{F}(\mathcal{M})$ for $k = 0,1,\cdots,x_1-x$ by induction.

Consider the triangle as follows.

$$(0,b+x_1-x,0) \to (0,b+k+1,0) \to Q(1,x_1,y_1)' \to (1,b-x_1-x+1,1).$$

Since $(0,b-x_1-x,0) \in \mathcal{F}(\mathcal{M})$ and $Q(1,x_1,y_1)' \in \mathcal{X} \subseteq \mathcal{M}$, $(0,b+k+1,0) \in \mathcal{F}(\mathcal{M})$. Since $y_1 - 1 \le y$ and $(0,b+x_1+y_1,0) \in \mathcal{F}(\mathcal{M})$, $(0,b+k+1-s,0) \in \mathcal{F}(\mathcal{M})$ for $s = 0,\cdots,y_1$ by induction. Combining $(0,b+k,c) \in \mathcal{F}(\mathcal{M})$ for $k = 0,1,\cdots,x_1-x$ and $(0,b+k+1-s,0) \in \mathcal{F}(\mathcal{M})$ for $s = 0,\cdots,y_1$, we have $(0,b+r,c) \in \mathcal{F}(\mathcal{M})$ for $r = 0,\cdots,k+1$. Note that $b+k+1-y_1 = b+x_1-x$. $\square$

Proof of Theorem 5.7:

Proof. Note that there is a non-zero morphism from $Q(0,m,0)'$ to $(0,c-\ell,d)$ by Lemma 3.5.

Consider a triangle as follows.

$$Q(0,m,\ell)' \to (0,c-\ell,0) \to (0,c,d) \to Q(1,m+1,\ell)'. \tag{5.13}$$

Since $(0,c,d) \in \mathcal{F}(\mathcal{S})$ and $Q(0,m,\ell)' \in \mathcal{X} \subseteq \mathcal{S}$, $(0,c-\ell,0) \in \mathcal{F}(\mathcal{S})$. By (3) of Proposition 5.5, there is an object $Q(1,m_1,\ell_1)'$ in $\mathcal{X}$ such that $m_1+\ell_1 = m+\ell$. Thus by equations (3.34) and (3.36),

$$(\triangle_{Q(0,m,\ell)'} \cup \triangle_{Q(1,m,\ell)'}) \cap \mathcal{X} = \{Q(0,m,\ell)'\} \cup (\triangle_{Q(0,m+1,\ell-2)'} \cup \triangle_{Q(1,m,\ell-1)'}) \cap \mathcal{X}. \tag{5.14}$$

Considering an orthogonal system on triangle areas $\triangle_{Q(0,m+1,\ell-2)'} \cup \triangle_{Q(1,m,\ell-1)'}$, we have $\{(0,c-k,d) \mid 0 \le k \le \ell\} \subseteq \mathcal{F}(\mathcal{S})$ by Proposition 5.9. □

Theorem 5.10. *Let A be a 2-domestic Brauer graph algebra and $\mathcal{S}$ a maximal orthogonal system which contains at least one object for an Euclidean component. We assume $\mathcal{M} = \{M_1, \cdots, M_\ell\}$ is the subset of $\mathcal{S}$ which is contained in the stable Euclidean component ${}_s\Gamma_0$, where $M_k = (0,a_k,b_k)$ for $1 \le k \le \ell$. Then*

(*a*) *There is a set $\{t_i \mid 1 \le i \le \ell\}$ of integers such that the following subsets are contained in $\mathcal{F}(\mathcal{S})$.*
  (1) $\{(0,a_1+j,b_1) \mid j = 0,1,\cdots,t_1-1\} \cup \{(0,a_2-j,b_2) \mid j=0,1,\cdots,a_2-a_1-t_1\}$.
  (2) $\{(0,a_r+j,b_r) \mid j=0,1,\cdots,t_r-a_r+a_1-1\} \cup \{(0,a_{r+1}-j,b_{r+1}) \mid j=0,1,\cdots,a_{r+1}-a_1-t_r\}$ *for* $r = 2,3,\cdots,\ell-1$.
  (3) $\{(0,a_\ell+j,b_\ell) \mid j=0,1,\cdots,t_\ell-a_\ell+a_1-1\} \cup \{(0,a_1+p-j,b_1-q) \mid j=0,1,\cdots,p-t_\ell\}$.

(*b*) *Dually, there is a set $\{s_i \mid 1 \le i \le \ell\}$ of integers such that the following subsets are contained in $\mathcal{F}(\mathcal{S})$.*
  (1) $\{(0,a_1,b_1-j) \mid j=0,1,\cdots,q-s_1\} \cup \{(0,a_2,b_2+j) \mid j=0,1,\cdots,b_1-b_2+s_1-q-1\}$,
  (2) $\{(0,a_i,b_i-j) \mid j=0,1,\cdots,q+b_i-b_1-s_i\} \cup \{(0,a_{i+1},b_{i+1}+j) \mid j=0,1,\cdots,b_1-b_{i+1}+s_i-q-1\}$ *for* $i=2,3,\cdots,\ell-1$.
  (3) $\{(0,a_\ell,b_\ell-j) \mid j=0,1,\cdots,q+b_\ell-b_1-s_\ell\} \cup \{(0,a_1+p,b_1-q+j) \mid j=0,1,\cdots,s_\ell-1\}$.

(*c*) $\mathcal{S} \cap {}_s\Gamma_1 = \{M'_k = (1,a_1+t_k,b_1-s_k-q) \mid 1 \le k \le \ell\}$.

Proof. We only prove (1) of (a), (1) of (b) and $(1,a_1+t_1,b_1-s_1-q) \in \mathcal{S}$ in (c), since other cases are similar.

(1) of (a) and (1) of (b):

By Proposition 5.6, there are a subset $\{Q(1,k_m,\ell_m)' \mid 1 \le m \le v_1\}$ of ${}_sQ'_1 \cap \mathcal{S}$ and a subset $\{Q(0,k_n,\ell_n)' \mid v_1+1 \le n \le v_2\}$ of ${}_sQ'_0 \cap \mathcal{S}$ for non-negative integers v_1 and v_2 such that $\mathcal{S} \cap (\triangle_{Q(0,1,a_2-a_1-2)'} \cup \triangle_{Q(1,1,a_2-a_1-2)'})$ is contained in a disjoint union of some triangle areas as follows.

$$\bigcup_{m=1}^{v_1} (\triangle_{Q(0,k_m,\ell_m)'} \cup \triangle_{Q(1,k_m,\ell_m)'}) \cup \bigcup_{n=v_1+1}^{v_2} (\triangle_{Q(0,k_n,\ell_n)'} \cup \triangle_{Q(1,k_n,\ell_n)'}).$$

Consider the triangle areas $\triangle_{Q(0,k_{v_1},\ell_{v_1})'} \cup \triangle_{Q(1,k_{v_1},\ell_{v_1})'}$ and $\triangle_{Q(0,k_{v_1+1},\ell_{v_1+1})'} \cup \triangle_{Q(1,k_{v_1+1},\ell_{v_1+1})'}$, where $Q(1,k_{v_1},\ell_{v_1})' \in \mathcal{S}$ and $Q(0,k_{v_1+1},\ell_{v_1+1})' \in \mathcal{S}$. By the orthogonality of objects on quasi-tubes, $k_{v_1}+\ell_{v_1}+1 = k_{v_1+1}$. Take $t_1 = k_{v_1+1}$. It is not hard to see that $\mathcal{W}_{Q(0,t_1-1,0)'} \cap \mathcal{S} = \varnothing$ and $\mathcal{W}_{Q(1,t_1,0)'} \cap \mathcal{S} = \varnothing$. Note that $Q(0,t_1-1,0)' = \Omega(Q(1,t_1,0)')$ and $\mathcal{W}_{Q(0,t_1-1,0)'}$ is the wing of object $Q(0,t_1-1,0)'$.

By Lemma 3.5, there are two irreducible maps:

$$Q(1,t_1,0)' \to (1,t_1-a_1,b_1),\ (1,t_1-a_1,b_1) \to Q(0,t_1-1,0)'.$$

Since $\mathcal{W}_{Q(0,t_1-1,0)'} \cap \mathcal{S} = \varnothing$ and $\mathcal{W}_{Q(1,t_1,0)'} \cap \mathcal{S} = \varnothing$, by equation (3.42),

$$\{(1,t_1-a_1,k) \mid k \in \mathbb{Z}\} \subseteq {}^{\perp}(\mathcal{S} \cap ({}_sQ'_0 \cup {}_sQ'_1))^{\perp}. \tag{5.15}$$

Since the objects $(0,a_1,b_1)$ and $(0,a_2,b_2)$ are in $\mathcal{F}(\mathcal{S})$, by Theorem 5.7 and Theorem 5.8,

$$\{(0,a_1+j,b_1) \mid j=0,1,\cdots,t_1-1\} \cup \{(0,a_2-j,b_2) \mid j=0,1,\cdots,a_2-a_1-t_1\} \subseteq \mathcal{F}(\mathcal{S}).$$

Similarly, by Proposition 5.6, there are a subset $\{Q(1,x_m,y_m) \mid 1 \le m \le u_1\}$ of ${}_sQ_1 \cap \mathcal{S}$ and a subset $\{Q(0,x_n,y_n) \mid u_1+1 \le n \le u_2\}$ of ${}_sQ_0 \cap \mathcal{S}$ for non-negative integers u_1 and u_2 such that $\mathcal{S} \cap (\triangle_{Q(0,b_2-b_1+p,b_1-b_2-1)} \cup \triangle_{Q(1,b_2-b_1+p,b_1-b_2-1)})$ is contained in a disjoint union of some triangle areas as follows.

$$\bigcup_{m=1}^{u_1} (\triangle_{Q(0,x_m,y_m)} \cup \triangle_{Q(1,x_m,y_m)}) \cup \bigcup_{n=u_1+1}^{u_2} (\triangle_{Q(0,x_n,y_n)} \cup \triangle_{Q(1,x_n,y_n)}).$$

Consider the triangle areas $\triangle_{Q(0,x_{u_1},y_{u_1})} \cup \triangle_{Q(1,x_{u_1},y_{u_1})}$ and $\triangle_{Q(0,x_{u_1+1},y_{u_1+1})} \cup \triangle_{Q(1,x_{u_1+1},y_{u_1+1})}$, where $Q(1,x_{u_1},y_{u_1}) \in \mathcal{S}$ and $Q(0,x_{u_1+1},y_{u_1+1}) \in \mathcal{S}$. By the orthogonality of objects on quasi-tubes, $x_{u_1}+y_{u_1}+1 =$

x_{u_1+1}. Take $s_1 = x_{u_1+1}$. It is not hard to see that $\mathcal{W}_{Q(0,s_1-1,0)} \cap \mathcal{S} = \varnothing$ and $\mathcal{W}_{Q(1,s_1,0)} \cap \mathcal{S} = \varnothing$ and $Q(0, s_1-1, 0) = \Omega(Q(1, s_1, 0))$.

By Lemma 3.5, there are two irreducible maps:

$$Q(1, s_1, 0) \to (1, a_1, b_1 + s_1 - q),\ (1, a_1, b_1 + s_1 - q) \to Q(0, s_1 - 1, 0).$$

By equation (3.42),

$$\{(1, k, b_1 + s_1 - q) \mid k \in \mathbb{Z}\} \subseteq {}^{\perp}(\mathcal{S} \cap ({}_sQ_0 \cup {}_sQ_1))^{\perp}. \tag{5.16}$$

Since $(0, a_1, b_1)$ and $(0, a_2, b_2)$ are in $\mathcal{F}(\mathcal{S})$, by Theorem 5.7 and Theorem 5.8,

$$\{(0, a_1, b_1 - j) \mid j = 0, 1, \cdots, q - s_1\} \cup \{(0, a_2, b_2 + j) \mid j = 0, 1, \cdots, b_1 - b_2 + s_1 - q - 1\} \subseteq \mathcal{F}(\mathcal{S}).$$

(c) Since $\{(0, a_1 + j, b_1) \mid j = 0, 1, \cdots, t_1 - 1\} \cup \{(0, a_2 - j, b_2) \mid j = 0, 1, \cdots, a_2 - a_1 - t_1\} \subseteq \mathcal{F}(\mathcal{S})$,

$$\begin{aligned}\{(1, a_1 + j, b_1) \mid j = 0, 1, \cdots, t_1 - 1\} \cup \{(1, a_2 - j, b_2) \mid j = 0, 1, \cdots, a_2 - a_1 - t_1\}\\ \nsubseteq {}^{\perp}(\mathcal{S} \cap ({}_s\Gamma_0 \cup {}_sQ_0 \cup {}_sQ_1))^{\perp}.\end{aligned} \tag{5.17}$$

Thus $(1, a_1+j, b_1)$ and $(1, a_2-k, b_2)$ are not in $\mathcal{S}$ for any $1 \le j \le t_1-1$ and $1 \le k \le a_2-a_1-t_1$, respectively. Since $\{(0, a_1, b_1 - j) \mid j = 0, 1, \cdots, q - s_1\} \cup \{(0, a_2, b_2 + j) \mid j = 0, 1, \cdots, b_1 - b_2 + s_1 - q - 1\} \subseteq \mathcal{F}(\mathcal{S})$,

$$\begin{aligned}\{(1, a_1, b_1 - j) \mid j = 0, 1, \cdots, q - s_1\} \cup \{(1, a_2, b_2 + j) \mid j = 0, 1, \cdots, b_1 - b_2 + s_1 - q - 1\}\\ \nsubseteq {}^{\perp}(\mathcal{S} \cap ({}_s\Gamma_0 \cup {}_sQ_0 \cup {}_sQ_1))^{\perp}.\end{aligned} \tag{5.18}$$

Thus the objects $(1, a_1, b_1 - j)$ and $(1, a_2, b_2 + k)$ are not in $\mathcal{S}$ for any $1 \le j \le q - s_1$ and $1 \le k \le b_1 - b_2 + s_1 - q - 1$, respectively.

By equations (5.15) and (5.16),

$$\{(1, a_1 + t_1, k) \mid b_2 + 1 \le k \le b_1\} = {}^{\perp}(\mathcal{S} \cap ({}_s\Gamma_0 \cup {}_sQ_0' \cup {}_sQ_1'))^{\perp} \cap \square_{(1,a_1+1,b_1),(1,a_2,b_2+1)}, \tag{5.19}$$

$$\{(1, k, b_1 + s_1 - q) \mid a_1 + 1 \le k \le a_2\} = {}^{\perp}(\mathcal{S} \cap ({}_s\Gamma_0 \cup {}_sQ_0 \cup {}_sQ_1))^{\perp} \cap \square_{(1,a_1+1,b_1),(1,a_2,b_2+1)}. \tag{5.20}$$

By equations (5.19) and (5.20),

$$(1, a_1 + t_1, b_1 + s_1 - q) = {}^{\perp}(\mathcal{S} \cap ({}_s\Gamma_0 \cup {}_sQ_0 \cup {}_sQ_1 \cup {}_sQ_0' \cup {}_sQ_1'))^{\perp} \cap \square_{(1,a_1+1,b_1),(1,a_2,b_2+1)}. \tag{5.21}$$

We claim that $(1, a_1 + t_1, b_1 + s_1 - q) \in \mathcal{S}$. Otherwise, we assume that $(1, a_1 + t_1, b_1 + s_1 - q)$ is not in $\mathcal{S}$. Therefore, by equation (3.42), $\mathcal{S} \cap {}_s\Gamma_1 \subseteq \square_{(1,a_2+1,b_2),(1,a_1-p,b_1+1-q)}$. Thus $(1, a_1 + t_1, b_1 + s_1 - q) \in {}^{\perp}(\mathcal{S} \cap {}_s\Gamma_1)^{\perp}$. By equation (3.10), $(1, a_1 + t_1, b_1 + s_1 - q) \in {}^{\perp}\mathcal{M}^{\perp}$. By equations (5.15) and (5.16),

$$(1, a_1 + t_1, b_1 + s_1 - q) \in {}^{\perp}(\mathcal{S} \cap ({}_sQ_0' \cup {}_sQ_1'))^{\perp} \cup {}^{\perp}(\mathcal{S} \cap ({}_sQ_0 \cup {}_sQ_1))^{\perp}.$$

Since $\mathcal{S} = \mathcal{M} \cup (\mathcal{S} \cap {}_s\Gamma_1) \cup (\mathcal{S} \cap ({}_sQ_0' \cup {}_sQ_1')) \cup (\mathcal{S} \cap ({}_sQ_0 \cup {}_sQ_1))$, $(1, a_1 + t_1, b_1 + s_1 - q) \in {}^{\perp}\mathcal{S}^{\perp}$. It is a contradiction. Hence $(1, a_1 + t_1, b_1 + s_1 - q) \in \mathcal{S}$. $\square$

$$\begin{tikzcd}[column sep=tiny, row sep=small]
 & & & & & (0,a_1,b_1) \arrow[dr] & & & & & \\
 & & & & \cdots \arrow[ur] & & \cdots \arrow[dr] & & & & \\
 & & & (0,a_1,b_1') \arrow[ur] \arrow[dr] & & & & (0,a_1'-1,b_1) \arrow[dr] & & & \\
 & & (0,a_1,b_1'-1) \arrow[ur] & & \cdots \arrow[dr] & & \cdots \arrow[ur] & & (0,a_1',b_1) \arrow[dr] & & \\
 & \cdots \arrow[ur] & & & & (0,a_1'-1,b_1') \arrow[ur] \arrow[dr] & & & & \cdots \arrow[dr] & \\
(0,a_1,b_2) \arrow[ur] \arrow[dr] & & & & (0,a_1'-1,b_1'-1) \arrow[ur] \arrow[dr] & & (0,a_1',b_1') & & & & (0,a_2,b_1). \\
 & \cdots \arrow[dr] & & & & (0,a_1',b_1'-1) \arrow[ur] \arrow[dr] & & & & \cdots \arrow[ur] & \\
 & & (0,a_1'-1,b_2) \arrow[dr] & & \cdots \arrow[ur] & & \cdots \arrow[dr] & & (0,a_2,b_1') \arrow[ur] & & \\
 & & & (0,a_1',b_2) \arrow[ur] \arrow[dr] & & & & (0,a_2,b_1'-1) \arrow[ur] & & & \\
 & & & & \cdots \arrow[dr] & & \cdots \arrow[ur] & & & & \\
 & & & & & (0,a_2,b_2) \arrow[ur] & & & & &
\end{tikzcd}$$

where $(0, a'_1, b'_1) = (0, a_1 - t_1, b_1 + s_1 - q) = \Omega^{-1}((1, a_1 - t_1, b_1 + s_1 - q))$. The above diagram shows us the position of some objects on Theorem 5.10.

Proposition 5.11. *Under the above notations on Theorem 5.10, we have the following results.*
(1) *Rectangle areas* $\square_{M_k,(0,a'_k-1,b'_k)} \cup \square_{(0,a'_k,b'_k-1),M_{k+1}} \subseteq \mathcal{F}(\mathcal{S})$ *for each* $1 \leq k \leq \ell$.
(2) *Rectangle areas* $\square_{M'_k,(1,a_{k+1},b_{k+1}+1)} \cup \square_{(1,a_{k+1}+1,b_{k+1}),M'_{k+1}} \subseteq \mathcal{F}(\mathcal{S})$ *for each* $1 \leq k \leq \ell$.
(3) $\Omega(M_k) = (1, a_k, b_k)$ *and* $\Omega^{-1}(M_k) = (1, a_k + 1, b_k + 1)$ *are in* $\mathcal{F}(\mathcal{S})$ *for each* $1 \leq k \leq \ell$.
(4) $\Omega(M'_k) = (1, a'_k, b'_k)$ *and* $\Omega^{-1}(M'_k) = (1, a'_k + 1, b'_k + 1)$ *are in* $\mathcal{F}(\mathcal{S})$ *for each* $1 \leq k \leq \ell$.

Proof. For (1) and (2), we only prove that $k = 1$ for (1), Since other cases are similar to prove. For (3) and (4), we only prove $k = 1$ for (3), that is, $\Omega(M_1) = (1, a_1, b_1)$ and $\Omega^{-1}(M_1) = (1, a_1 + 1, b_1 + 1)$ are in $\mathcal{F}(\mathcal{S})$.
(1) Consider the following triangles:

$$(0, a_1, b_1 - j) \to (0, a_1, b_1) \oplus (0, a_1 + i, b_1 - j) \to (0, a_1 + i, b_1) \to (1, a_1 + 1, b_1 + 1 - j), \tag{5.22}$$

where $j = 1, \cdots, q - s_1$ and $i = 1, \cdots, t_1 - 1$.

$$(0, a_2 - i, b_2) \to (0, a_2, b_2) \oplus (0, a_1 - i, b_2 + j) \to (0, a_2, b_2 + j) \to (1, a_2 - i + 1, b_2 + 1), \tag{5.23}$$

where $j = 1, \cdots, b_1 - b_2 + s_1 - q - 1$ and $i = 1, \cdots, a_2 - a_1 - t_1$.

By (1) of (b) and (1) of (a) in Theorem 5.10, $(0, a_1, b_1 - j), (0, a_1, b_1)$ and $(0, a_1 + i, b_1)$ are contained in $\mathcal{F}(\mathcal{S})$ in triangle (5.22). Since $\mathcal{F}(\mathcal{S})$ is closed under direct summands, $(0, a_1 - i, b_2 + j) \in \mathcal{F}(\mathcal{S})$ for $j = 1, \cdots, q - s_1$ and $i = 1, \cdots, t_1 - 1$. Thus $\square_{M_1,Y_1} \subseteq \mathcal{F}(\mathcal{S})$, where $Y_1 = (0, a_1 + t_1 - 1, b_1 + s_1 - q)$.

By (1) of (b) and (1) of (a) in Theorem 5.10, $(0, a_2 - i, b_2), (0, a_2, b_2)$ and $(0, a_2, b_2 + j)$ are contained in $\mathcal{F}(\mathcal{S})$ in triangle (5.23). Since $\mathcal{F}(\mathcal{S})$ is closed under direct summands, $(0, a_1 - i, b_2 + j) \in \mathcal{F}(\mathcal{S})$ for $j = 1, \cdots, b_1 - b_2 + s_1 - q - 1$ and $i = 1, \cdots, a_2 - a_1 - t_1$. Thus $\square_{X_1,M_2} \subseteq \mathcal{F}(\mathcal{S})$, where $X_1 = (0, a_1 + t_1, b_1 - s_1 - q - 1)$. By (c) of Theorem 5.10,

$$\mathcal{S} \cap \square_{\Omega(M_1),\Omega(M_2)} = \mathcal{S} \cap \square_{(1,a_1,b_1),(1,a_2,b_2)} = M'_1 = (1, a_1 + t_1, b_1 + s_1 - q) = (1, a'_1, b'_1).$$

Therefore $Y_1 = (0, a_1 + t_1 - 1, b_1 + s_1 - q) = (0, a'_1 - 1, b'_1)$. $X_1 = (0, a_1 + t_1, b_1 - s_1 - q - 1) = (0, a'_1, b'_1 - 1)$. Thus rectangle areas $\square_{M_1,(0,a'_1-1,b'_1)} \cup \square_{(0,a'_1,b'_1-1),M_2} \subseteq \mathcal{F}(\mathcal{S})$.
(3) Consider the following triangle:

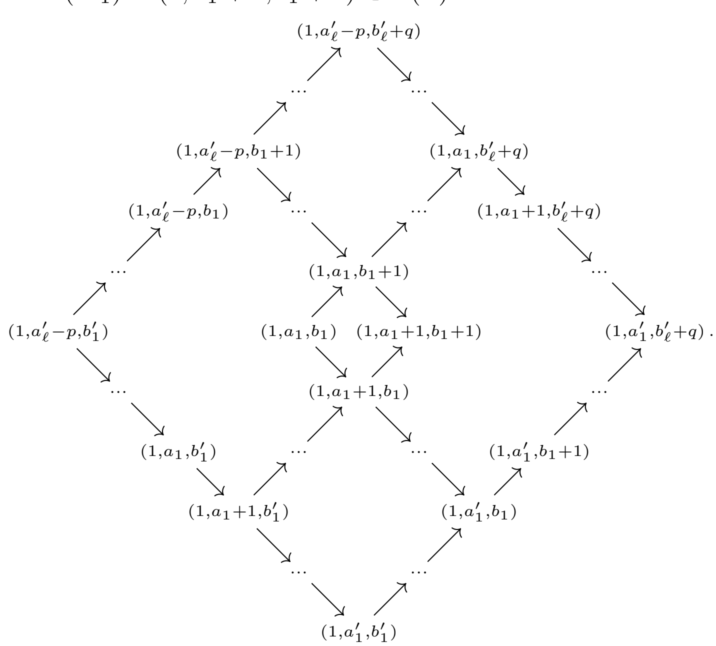
$$(0, a_1, b_1) \to (1, a_1, b_1) \to (1, a_1, b_1 + 1) \oplus (1, a_1 + 1, b_1) \to (1, a_1 + 1, b_1 + 1), \tag{5.24}$$

By conclusion (2), $(1, a_1, b_1 + 1) \oplus (1, a_1 + 1, b_1) \in \mathcal{F}(\mathcal{S})$. Thus $\Omega(M_1) = (1, a_1, b_1) \in \mathcal{F}(\mathcal{S})$.

Dually, consider the triangle as follows.

$$(1, a_1, b_1 + 1) \oplus (1, a_1 + 1, b_1) \to (1, a_1 + 1, b_1 + 1) \to (0, a_1, b_1) \to (0, a_1, b_1 + 1) \oplus (0, a_1 + 1, b_1). \tag{5.25}$$

It is easy to know that $\Omega^{-1}(M_1) = (1, a_1 + 1, b_1 + 1) \in \mathcal{F}(\mathcal{S})$. $\square$



The above diagram show us the position of some rectangle areas on (2) of Proposition 5.11.

Proposition 5.12. *Let A be a 2-domestic Brauer graph algebra and $\mathcal{S}$ a maximal orthogonal system which contains at least one object on an Euclidean component. We assume $\mathcal{M} = \{M_1, \cdots, M_\ell\}$ (resp, $\mathcal{M}' = \{M'_1, \cdots, M'_\ell\}$) is the subset of $\mathcal{S}$ which is contained in the stable Euclidean component ${}_s\Gamma_0$ (resp. ${}_s\Gamma_1$), where $M_k = (0, a_k, b_k)$ (resp. $M'_k = (1, a'_k, b'_k)$) for $1 \le k \le \ell$. Then*

(a) *The following subsets on ${}_s\Gamma_1$ are contained $\mathcal{F}(\mathcal{S})$.*
  1. $\{(1, a_1+1+j, b_1+1) \mid j = 0, 1, \cdots, a'_1 - a_1 - 1\} \cup \{(1, a'_\ell - p + j, b_1) \mid j = 0, 1, \cdots, a_1 + p - a'_\ell\} \cup \{(1, a_1, b'_1 + j) \mid j = 0, 1, \cdots, b_1 - b'_1\} \cup \{(1, a_1+1, b_1+1+j) \mid j = 0, 1, \cdots, b'_\ell + q - b_1 - 1\}$.
  2. $\{(1, a_k+1+j, b_k+1) \mid j = 0, 1, \cdots, a'_k - a_k - 1\} \cup \{(1, a'_{k-1} + j, b_k) \mid j = 0, 1, \cdots, a_k - a'_{k-1}\} \cup \{(1, a_k, b_k - j) \mid j = 0, 1, \cdots, b_k - b'_k\} \cup \{(1, a_k+1, b_k+1+j) \mid j = 0, 1, \cdots, b'_{k-1} - b_k - 1\}$ *for* $2 \le k \le \ell$.
  3. *Rectangle areas* $\square_{M'_k,(1,a_{k+1}+1,b_{k+1})} \cup \square_{(1,a_{k+1},b_{k+1}+1),M'_{k+1}}$ *for any integer* $1 \le k \le \ell$.

(b) *Dually, the following subsets on ${}_s\Gamma_0$ are contained in $\mathcal{F}(\mathcal{S})$.*
  1. $\{(0, a'_1+j, b'_1) \mid j = 0, 1, \cdots, a_2 - a'_1\} \cup \{(0, a_1+j, b'_1 - 1) \mid j = 0, 1, \cdots, a'_1 - 1 - a_1\} \cup \{(1, a'_1, b'_1 + j) \mid j = 0, 1, \cdots, b_1 - b'_1\} \cup \{(1, a'_1 - 1, b_2 + j) \mid j = 0, 1, \cdots, b'_1 - 1 - b_2\}$.
  2. $\{(0, a'_k+j, b'_k) \mid j = 0, 1, \cdots, a_{k+1} - a'_k\} \cup \{(0, a_k+j, b'_k - 1) \mid j = 0, 1, \cdots, a'_k - 1 - a_k\} \cup \{(1, a'_k, b'_k + j) \mid j = 0, 1, \cdots, b_k - b'_k\} \cup \{(1, a'_k - 1, b_{k+1} + j) \mid j = 0, 1, \cdots, b'_k - 1 - b_{k+1}\}$ *for* $2 \le k \le \ell$. *Note that* $a_{\ell+1} = a_1 - p$ *and* $b_{\ell+1} = b_1 + q$ *(resp.* $a'_{\ell+1} = a'_1 - p$ *and* $b'_{\ell+1} = b'_1 + q$*).*
  3. *Rectangle areas* $\square_{M_k,(0,a'_k,b'_k-1)} \cup \square_{(0,a'_k-1,b'_k),M_{k+1}}$ *for any integer* $1 \le k \le \ell$.

Proof. We only prove (1) of (a), since (2) of (a) (resp. (b)) are similar to prove, (3) of (a) (resp. (b)) is a direct consequence of (1) and (2) of (a) (resp. (b)). We divide (1) of (a) into four claims as follows.

Claim one: $\{(1, a_1+1+j, b_1+1) \mid j = 0, 1, \cdots, a'_1 - a_1 - 1\} \subseteq \mathcal{F}(\mathcal{S})$.

Consider the triangle as follows.

$$(1, A_j, B) \oplus (1, A_j+1, B-1) \to (1, A_j+1, B) \to (0, A_j, B-1) \to (0, A_j, B) \oplus (0, A_j+1, B-1). \tag{5.26}$$

where $a_1 + j = A_j$ and $b_1 + 1 = B$ for $0 \le j \le a'_1 - a_1 - 1$.

By (1) of (a) in Theorem 5.10,

$$(0, A_j, B-1) = (0, a_1+j, b_1) \in \mathcal{F}(\mathcal{S})$$

for $0 \le j \le t_1 - 1$. Note that $t_1 = a'_1 - a_1$. By (2) of Proposition 5.11,

$$(1, A_j+1, B-1) = (1, a_1+j+1, b_1) \in \square_{(1,a_1+1,b_1),M'_1} \subseteq \mathcal{F}(\mathcal{S})$$

for $0 \le j \le a'_1 - a_1 - 1$. By (3) of Proposition 5.11, $\Omega^{-1}(M_1) = (1, a_1+1, b_1+1) \in \mathcal{F}(\mathcal{S})$. By the triangle (5.26) and by induction,

$$(1, A_j+1, B) = (1, a_1+j+1, b_1+1) \in \mathcal{F}(\mathcal{S})$$

for $j = 0, 1, \cdots, a'_1 - a_1 - 1$.

Claim two: $\{(1, a'_\ell - p + j, b_1) \mid j = 0, 1, \cdots, a_1 + p - a'_\ell\} \subseteq \mathcal{F}(\mathcal{S})$.

Note that $\{(1, a'_\ell - p + j, b_1) \mid j = 0, 1, \cdots, a_1 + p - a'_\ell\} = \{(1, a_1 - j, b_1) \mid j = 0, 1, \cdots, a_1 - a'_\ell + p\}$. Consider the triangle as follows.

$$(0, a_1 - j, b_1) \to (1, a_1 - j, b_1) \to (1, a_1 - j, b_1+1) \oplus (1, a_1 - j + 1, b_1) \to (1, a_1 - j + 1, b_1 + 1) \tag{5.27}$$

for $0 \le j \le a_1 - a'_\ell + p$.

By (3) of (a) in Theorem 5.10, $(0, a_1 - j, b_1)$ is in $\mathcal{F}(\mathcal{S})$ for $0 \le j \le a_1 - a'_\ell + p$. By (2) of Proposition 5.11,

$$(1, a_1 - j, b_1+1) \in \square_{M'_\ell,(1,a_1,b_1+1)} \subseteq \mathcal{F}(\mathcal{S})$$

for $0 \le j \le a_1 - a'_\ell + p$. By (3) of Proposition 5.11, $\Omega(M_1) = (1, a_1, b_1) \in \mathcal{F}(\mathcal{S})$. Thus, by induction and triangle (5.27), $(1, a_1 - j, b_1) \in \mathcal{F}(\mathcal{S})$ for $j = 0, 1, \cdots, a_1 - a'_\ell + p$.

Claim three: $\{(1, a_1, b'_1 + j) \mid j = 0, 1, \cdots, b_1 - b'_1\} \subseteq \mathcal{F}(\mathcal{S})$.

Note that $\{(1, a_1, b'_1 + j) \mid j = 0, 1, \cdots, b_1 - b'_1\} = \{(1, a_1, b_1 - j) \mid j = 0, 1, \cdots, b_1 - b'_1\}$. Consider the triangle as follows.

$$(0, a_1, b_1 - j) \to (1, a_1, b_1 - j) \to (1, a_1, b_1 - j - 1) \oplus (1, a_1+1, b_1 - j) \to (1, a_1+1, b_1 - j + 1). \tag{5.28}$$

for $0 \le j \le b_1 - b'_1$.

By (1) of (b) in Theorem 5.10, $(0, a_1, b_1 - j)$ is in $\mathcal{F}(\mathcal{S})$ for $0 \le j \le b_1 - b'_1$. By (2) of Proposition 5.11,

$$(1, a_1+1, b_1 - j) \in \square_{(1,a_1+1,b_1),M'_1} \subseteq \mathcal{F}(\mathcal{S})$$

for $0 \leq j \leq b_1 - b_1'$. By (3) of Proposition 5.11, $\Omega^(M_1) = (1, a_1, b_1) \in \mathcal{F}(\mathcal{S})$. Thus, by induction and triangle (5.28), $(1, a_1, b_1 - j) \in \mathcal{F}(\mathcal{S})$ for $j = 0, 1, \cdots, b_1 - b_1'$.

Claim four: $\{(1, a_1 + 1, b_1 + 1 + j) \mid j = 0, 1, \cdots, b_\ell' + q - b_1 - 1\} \subseteq \mathcal{F}(\mathcal{S})$.

Consider the triangle as follows.

$$(1, A, B_j) \oplus (1, A-1, B_j+1) \to (1, A, B_j+1) \to (0, A-1, B_j) \to (0, A, B_j) \oplus (0, A-1, B_j+1). \tag{5.29}$$

for $0 \leq j \leq b_\ell' + q - b_1 - 1$, where $A = a_1 + 1, B_j = b_1 + j$.

By (2) of Proposition 5.11,

$$(1, A-1, B_j+1) = (1, a_1, b_1 + j + 1) \in \square_{M_\ell', (1, a_1, b_1+1)} \subseteq \mathcal{F}(\mathcal{S})$$

for $0 \leq j \leq b_\ell' + q - b_1 - 1$. By Proposition 5.11 (1),

$$(0, A-1, B_j) = (0, a_1, b_1 + j) \in \square_{(0, a_\ell' - p, b_\ell' + q - 1), M_1} \subseteq \mathcal{F}(\mathcal{S})$$

for $0 \leq j \leq b_\ell' + q - b_1 - 1$.

By (3) of Proposition 5.11, $\Omega^{-1}(M_1) = (1, a_1+1, b_1+1) \in \mathcal{F}(\mathcal{S})$. By the triangle (5.29) and by induction,

$$(1, A, B_j + 1) = (1, a_1 + 1, b_1 + j + 1) \in \mathcal{F}(\mathcal{S})$$

for $j = 0, 1, \cdots, b_\ell' + q - b_1 - 1$. $\square$

Corollary 5.13. *Under the above notations on Proposition 5.12, we have the following results.*

(1) *Rectangle areas* $\square_{(0, a_k' - 1, b_{k+1}), (0, a_{k+1}, b_{k+1}' - 1)} \cup \square_{(0, a_{k+1}, b_k'), (0, a_{k+1}', b_{k+1})} \subseteq \mathcal{F}(\mathcal{S})$ *for any integer* $1 \leq k \leq \ell$.
(2) *Rectangle areas* $\square_{(1, a_k, b_k'), (1, a_k', b_{k+1})} \cup \square_{(1, a_k', b_k+1), (1, a_{k+1}+1, b_k')} \subseteq \mathcal{F}(\mathcal{S})$ *for any integer* $1 \leq k \leq \ell$.
(3) *Rectangle areas* $\square_{(0, a_k' - 1, b_k'), (0, a_{k+1}', b_{k+1}' - 1)} \cup \square_{(1, a_k, b_k+1), (1, a_{k+1}+1, b_{k+1})} \subseteq \mathcal{F}(\mathcal{S})$ *for any integer* $1 \leq k \leq \ell$.

Proof. We only prove conclusion (1) for $k = \ell$, since conclusion (2) is similar to prove and conclusion (3) is a direct consequence of (1) and (2).

Claim one: Rectangle area $\square_{(0, a_\ell' - 1, b_1), (0, a_1, b_1' - 1)} \subseteq \mathcal{F}(\mathcal{S})$.

Consider the triangle as follows.

$$(1, a_1, b_1) \to (0, A_m, B_n) \to (0, A_m, b_1) \oplus (0, a_1, B_n) \to (0, a_1, b_1). \tag{5.30}$$

where $A_m = a_\ell' - p - 1 + m$ for $0 \leq m \leq a_1 + p - a_\ell' + 1$ and $B_n = b_1' - 1 + n$ for $0 \leq n \leq b_1 - b_1' + 1$.

By (3) of Proposition 5.11, $\Omega(M_1) = (1, a_1, b_1) \in \mathcal{F}(\mathcal{S})$. By (b) of Proposition 5.12,

$$(0, A_m, b_1) = (0, a_\ell' - p - 1 + m, b_1) \in \square_{(0, a_\ell' - p, b_\ell' + q - 1), M_1} \subseteq \mathcal{F}(\mathcal{S})$$

for $0 \leq m \leq a_1 + p - a_\ell' + 1$. By (b) of Proposition 5.12,

$$(0, a_1, B_n) = (0, a_1, b_1' - 1 + n) \in \square_{M_1, (0, a_1', b_1' - 1)} \subseteq \mathcal{F}(\mathcal{S})$$

for $0 \leq n \leq b_1 - b_1' + 1$. Therefore, by triangle (5.30),

$$(0, A_m, B_n) = (0, a_\ell' - p - 1 + m, b_1' - 1 + n) \in \mathcal{F}(\mathcal{S})$$

for $0 \leq m \leq a_1 + p - a_\ell' + 1$ and $0 \leq n \leq b_1 - b_1' + 1$. Thus $\square_{(0, a_\ell' - 1, b_1), (0, a_1, b_1' - 1)} \subseteq \mathcal{F}(\mathcal{S})$.

Claim two: Rectangle area $\square_{(0, a_1, b_\ell'), (0, a_1', b_1)} \subseteq \mathcal{F}(\mathcal{S})$.

Consider the following triangle.

$$(0, a_1, b_1) \to (0, a_1 + m, b_1) \oplus (0, a_1, b_1 + n) \to (0, a_1 + m, b_1 + n) \to (1, a_1 + 1, b_1 + 1). \tag{5.31}$$

where $0 \leq m \leq a_1' - a_1$ and $0 \leq n \leq b_\ell' + p - b_1$.

By (3) of Proposition 5.11,

$$\Omega^{-1}(M_1) = (1, a_1 + 1, b_1 + 1) \in \mathcal{F}(\mathcal{S}).$$

By (b) of Proposition 5.12,

$$(0, a_1 + m, b_1) \in \square_{M_1, (0, a_1', b_1' - 1)} \subseteq \mathcal{F}(\mathcal{S})$$

$0 \leq m \leq a_1' - a_1$. By (b) of Proposition 5.12,

$$(0, a_1, b_1 + n) \in \square_{(0, a_\ell' - 1, b_\ell'), M_1} \subseteq \mathcal{F}(\mathcal{S})$$

for $0 \leq n \leq b_\ell' + p - b_1$. Thus $\square_{(0, a_1, b_\ell'), (0, a_1', b_1)} \subseteq \mathcal{F}(\mathcal{S})$. $\square$

The following diagrams indicate the location of some rectangle areas in Corollary 5.13.

$$\begin{tikzcd}[column sep=tiny, row sep=small]
& & & & & (0,a_1,b_1) \arrow[dr] & & & & & \\
& & & & \cdots \arrow[ur] & & \cdots \arrow[dr] & & & & \\
& & & (0,a_1,b_1') \arrow[ur] \arrow[dr] & & & & (0,a_1'-1,b_1) \arrow[dr] & & & \\
& & (0,a_1,b_1'-1) \arrow[ur] \arrow[dr, dashed] & & \cdots \arrow[dr] & & \cdots \arrow[ur] & & (0,a_1',b_1) \arrow[dr] & & \\
& \cdots \arrow[ur] & & \cdots \arrow[dr, dashed] & & (0,a_1'-1,b_1') \arrow[ur] \arrow[dr] & & \cdots \arrow[ur, dashed] & & \cdots \arrow[dr] & \\
(0,a_1,b_2) \arrow[ur] \arrow[dr] & & & & (0,a_1'-1,b_1'-1) \arrow[ur] \arrow[dr] & & (0,a_1',b_1') \arrow[ur, dashed] \arrow[dr, dashed] & & & & (0,a_2,b_1)\,. \\
& \cdots \arrow[dr] & & \cdots \arrow[ur, dashed] & & (0,a_1',b_1'-1) \arrow[ur] \arrow[dr] & & \cdots \arrow[dr, dashed] & & \cdots \arrow[ur] & \\
& & (0,a_1'-1,b_2) \arrow[ur, dashed] \arrow[dr] & & \cdots \arrow[ur] & & \cdots \arrow[dr] & & (0,a_2,b_1') \arrow[ur] & & \\
& & & (0,a_1',b_2) \arrow[ur] \arrow[dr] & & & & (0,a_2,b_1'-1) \arrow[ur] & & & \\
& & & & \cdots \arrow[dr] & & \cdots \arrow[ur] & & & & \\
& & & & & (0,a_2,b_2) \arrow[ur] & & & & &
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=tiny, row sep=small]
& & & & & (1,a_\ell'-p,b_\ell'+q) \arrow[dr] & & & & & \\
& & & & \cdots \arrow[ur] & & \cdots \arrow[dr] & & & & \\
& & & (0,a_\ell'-p,b_1+1) \arrow[ur] \arrow[dr] & & & & (1,a_1,b_\ell'+q) \arrow[dr] & & & \\
& & (0,a_\ell'-p,b_1) \arrow[ur] \arrow[dr, dashed] & & \cdots \arrow[dr] & & \cdots \arrow[ur] & & (1,a_1+1,b_\ell'+q) \arrow[dr] & & \\
& \cdots \arrow[ur] & & \cdots \arrow[dr, dashed] & & (0,a_1,b_1+1) \arrow[ur] \arrow[dr] & & \cdots \arrow[ur, dashed] & & \cdots \arrow[dr] & \\
(1,a_\ell'-p,b_1') \arrow[ur] \arrow[dr] & & & & (1,a_1,b_1) \arrow[ur] \arrow[dr] & & (1,a_1+1,b_1+1) \arrow[ur, dashed] \arrow[dr, dashed] & & & & (1,a_1',b_\ell'+q)\,. \\
& \cdots \arrow[dr] & & \cdots \arrow[ur, dashed] & & (0,a_1+1,b_1) \arrow[ur] \arrow[dr] & & \cdots \arrow[dr, dashed] & & \cdots \arrow[ur] & \\
& & (1,a_1,b_1') \arrow[ur, dashed] \arrow[dr] & & \cdots \arrow[ur] & & \cdots \arrow[dr] & & (0,a_1',b_1+1) \arrow[ur] & & \\
& & & (1,a_1+1,b_1') \arrow[ur] \arrow[dr] & & & & (1,a_1',b_1) \arrow[ur] & & & \\
& & & & \cdots \arrow[dr] & & \cdots \arrow[ur] & & & & \\
& & & & & (1,a_1',b_1') \arrow[ur] & & & & &
\end{tikzcd}$$

Theorem 5.14. *Let A be a 2-domestic Brauer graph algebra and $\mathcal{S}$ a maximal orthogonal system which contains at least one object on an Euclidean component. Then $\Omega^{-1}(\mathcal{S}) \subseteq \mathcal{F}(\mathcal{S})$, in particular, $\mathcal{S}$ is a simple-minded system on A-$\underline{\mathrm{mod}}$.*

Proof. Take $\mathcal{M} = \mathcal{S} \cap_s \Gamma_0$ and $\mathcal{M}' = \mathcal{S} \cap_s \Gamma_1$. We assume that $\mathcal{M} = \{M_1, \cdots, M_\ell\}$ and $\mathcal{M}' = \{M_1', \cdots, M_\ell'\}$, where $M_i = (0, a_i, b_i)$ and $M_i' = (1, a_i', b_i')$. Without loss of generality, we assume that $(0, a_1, b_1) = (0, a, b)$. By (3) and (4) of Proposition 5.11, $\Omega(M_k) = (1, a_k, b_k)$ and $\Omega^{-1}(M_k) = (1, a_k + 1, b_k + 1)$ are in $\mathcal{F}(\mathcal{S})$ for any $1 \leq k \leq \ell$. and $\Omega(M_k') = (1, a_k', b_k')$ and $\Omega^{-1}(M_k') = (1, a_k' + 1, b_k' + 1)$ are in $\mathcal{F}(\mathcal{S})$ for any $1 \leq k \leq \ell$.

Now we prove that, for every τ-periodic module $T \in \mathcal{S}$, $\Omega^{-1}(T) \in \mathcal{F}(\mathcal{S})$. By Theorem 5.10, for τ-periodic module $T \in \mathcal{S}$, there are triangles as follows.

$$T \to W_1 \to W_2 \to \Omega^{-1}(T), \tag{5.32}$$

or

$$W_1 \to W_2 \to T \to \Omega^{-1}(W_1) \tag{5.33}$$

satisfying that both W_1 and W_2 are in $\mathcal{F}(\mathcal{S})$.

Rotating triangle (5.32) to the right once, we have

$$W_1 \to W_2 \to \Omega^{-1}(T) \to \Omega^{-1}(W_1), \tag{5.34}$$

Acting the functor Ω^{-1} on the triangle (5.33), we have

$$\Omega^{-1}(W_1) \to \Omega^{-1}(W_2) \to \Omega^{-1}(T) \to \Omega^{-2}(W_1). \tag{5.35}$$

We only consider the case that W_1 and W_2 are in (1) of (a) over Theorem 5.10, since other cases are similar to handle.

Case one: W_1 and W_2 are in the set $\{(0, a_1 + j, b_1) \mid j = 0, 1, \cdots, t_1 - 1\}$.

In triangle (5.34), by (a) of Proposition 5.12,

$$\Omega^{-1}(W_1) \in \Box_{(1,a_1,b_1+1),M_1'} \subseteq \Box_{(1,a_1,b_1+1),(1,a_2+1,b_2)} \subseteq \mathcal{F}(\mathcal{S}).$$

Since $W_2 \in \mathcal{F}(\mathcal{S})$, $\Omega^{-1}(T) \in \mathcal{F}(\mathcal{S})$.

In triangle (5.35), by (a) of Proposition 5.12, $\Omega^{-1}(W_2) \in \mathcal{F}(\mathcal{S})$. By Corollary 5.13,

$$\Omega^{-2}(W_1) \in \{(0, a_1 + j + 1, b_1 + 1) \mid j = 0, 1, \cdots, t_1 - 1\} \subseteq \Box_{(0,a_\ell'-1,b_\ell'),(0,a_1'-1,b_1')} \subseteq \mathcal{F}(\mathcal{S}).$$

Thus $\Omega^{-1}(T) \in \mathcal{F}(\mathcal{S})$.

Case two: W_1 and W_2 are in the set $\{(0, a_2 - j, b_2) \mid j = 0, 1, \cdots, a_2 - a_1 - t_1\}$.

For every object $W \in \{(0, a_2 - j, b_2) \mid j = 0, 1, \cdots, a_2 - a_1 - t_1\}$,

$$\Omega^{-1}(W) \in \Box_{(1,a_1,b_1+1),(1,a_2+1,b_2)} \subseteq \mathcal{F}(\mathcal{S}).$$

Thus $\Omega^{-1}(W_1)$ and $\Omega^{-1}(W_2)$ are in $\mathcal{F}(\mathcal{S})$.

In triangle (5.34), $\Omega^{-1}(W_1) \in \mathcal{F}(\mathcal{S})$. Since $W_2 \in \mathcal{F}(\mathcal{S})$, $\Omega^{-1}(T) \in \mathcal{F}(\mathcal{S})$.

In triangle (5.35), $\Omega^{-1}(W_2) \in \mathcal{F}(\mathcal{S})$. By Corollary 5.13,

$$\Omega^{2}(W_1) \in \{(0, a_2 - j + 1, b_2 + 1) \mid 0 \leq j \leq a_2 - a_1 - t_1\} \subseteq \Box_{(0,a_1'-1,b_1'),(0,a_2',b_2'-1)} \subseteq \mathcal{F}(\mathcal{S}).$$

Thus $\Omega^{-1}(T) \in \mathcal{F}(\mathcal{S})$.

Therefore, for every $M \in \mathcal{S}$, we have $\Omega^{-1}(M) \in \mathcal{F}(\mathcal{S})$. Thus $\mathcal{S}$ is a simple-mined system on A-$\underline{\mathsf{mod}}$ by Theorem 2.25. $\square$

Corollary 5.15. *Let A be a 2-domestic Brauer graph algebra and $\mathcal{S}$ an orthogonal system on an Euclidean component. Then $\mathcal{S}$ extends to a simple-minded system on A-$\underline{\mathsf{mod}}$. In particular, $\mathcal{F}(\mathcal{S})$ is functorially finite on A-$\underline{\mathsf{mod}}$.*

Proof. It follows from Theorem 5.14 and Theorem 2.6. $\square$

Corollary 5.16. *Let A be a 2-domestic Brauer graph algebra and $\mathcal{S}$ a simple-minded system on A-$\underline{\mathsf{mod}}$. Then the cardinality of $\mathcal{S}$ is the number of non-projective simple A-modules.*

Proof. It is a direct consequence of Theorem 5.14 and Corollary 4.9. $\square$

Corollary 5.17. *Let A be a 2-domestic Brauer graph algebra and $\mathcal{S}$ a weakly simple-minded system with a finite cardinality. Then $\mathcal{S}$ is a simple-minded system on A-$\underline{\mathsf{mod}}$.*

Proof. By Corollary 2.18, any quasi-simple in a homogeneous tube is mutual stable orthogonal with each quasi-simple on ${}_sQ_0 \cup {}_sQ_1 \cup {}_sQ_0' \cup {}_sQ_1'$. Thus if $\mathcal{S}$ does not contain any object on Euclidean components, then there are quasi-simples of infinitely many homogeneous tubes contained in $\mathcal{S}$. Thus the cadrinality of $\mathcal{S}$ is infinite. Hence every weakly simple-minded system with a finite cardinality contains at least one object for an Euclidean component. By Theorem 2.6, $\mathcal{S}$ is a simple-minded system on A-$\underline{\mathsf{mod}}$.

$\square$

We [30] provided a class of weakly simple-minded systems which are not simple-minded systems over a 2-domestic Brauer graph algebra. Note that those weakly simple-minded systems contains objects in homogeneous tubes and they have a infinite cardinality. We conjecture that a weakly simple-minded system $\mathcal{S}$ with a finite cardinality is a simple-midned system over a self-injective algerbra.

Conjecture 5.18. *Let A be a self-injective algebra and $\mathcal{S}$ a weakly simple-minded system with a finite cardinality. Then $\mathcal{S}$ is a simple-minded system on A-$\underline{\mathsf{mod}}$.*

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