

Telling apart black holes and naked singularities with trajectories of massive particles

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ABSTRACT

We performed a numerical study of the dynamics of massive particles orbiting black holes and naked singularities in the Reissner-Nordström geometry. We modeled a stream of particles with a constant angular momentum and with a range of energies. We then solved the geodesic equation of motion and compared the trajectories around black holes and naked singularities by tuning the charge parameter of the metric. The setup that we used can approximate astrophysical scenarios such as tidal disruption events. We discussed differences and similarities in the orbital dynamics and deflection angles. We found that particles reflected by a black hole follow a stream-like family of orbits within a narrow range of deflection angles, whereas in case of naked singularities particles are scattered in all directions on the plane of motion. We explained this behavior as an interplay between the presence of a centrifugal barrier at the location of the unstable circular orbit and an absorbing event horizon in case of a black hole or a reflective zero-gravity sphere in case of a naked singularity. These qualitative differences are expected to impact the observable signatures of tidal disruption events.

Key words: gravitation – black hole physics – methods: numerical – transient: tidal disruption events

1 INTRODUCTION

The cosmic censorship hypothesis (Penrose 1969) proposes that singularities arising from continual gravitational collapse are hidden behind an event horizon, that is, inside black holes (BHs). Thus, the singularity cannot be detected by an external observer. However, direct evidence for the presence of BHs requires the detection of the event horizon, from which no particles or light can escape. Evidence for the presence of event horizons in black hole candidates is strong but inconclusive (e.g., Abramowicz et al. 2002). Studies of gravitational collapse have shown that naked singularities (NkS), hypothetical compact objects in which the singularity is exposed to an external observer, can also form as the final state of continual collapse (see Joshi et al. 2011, 2014). Therefore, studying various aspects of NkS has been a subject of great theoretical interest, particularly in the framework of testing their existence.

The recent remarkable advances in the angular resolution achieved by the Event Horizon Telescope (EHT) allowed for reconstructing event horizon scale images of M87 and Sagittarius A* (Event Horizon Telescope Collaboration et al. 2019a,b, 2022a,b). These images carry vital information regarding spacetime geometry and allow us to directly test theories of gravity and results from simulations. The EHT tests of the spacetime metric (Kocherlakota et al. 2021; Event Horizon Telescope Collaboration et al. 2022c) showed that Reissner-Nordström (RN) NkS predict a significantly smaller shadow size than has been observed but they did not generally rule out the possibility of M87 or Sgr A* spacetime being of a NkS type.

Shaikh & Joshi (2019) studied two classes of NkS and their accretion disks by comparing their images and shadows to those of BHs. They found a NkS with a photon sphere that qualitatively mimicked those of a BH. On the other hand, in the absence of a photon sphere, images of NkSs differ from those of BHs and could be distinguished through future high angular resolution observations (e.g., Vincent et al. 2021; Hudson et al. 2023; Johnson et al. 2024). Additionally, the two classes of NkS considered by Shaikh et al. (2019) were different from each other, allowing to distinguish between them through their images. A number of different tests of existence of NkS have been proposed in the literature (e.g., Chakraborty et al. 2017; Mummery & Ingram 2024).

A star orbiting a supermassive compact object could be ripped apart by tidal forces and produce a luminous transient phenomenon – a “tidal disruption event” (TDE). For a sufficiently heavy central compact object the associated tidal radius may be very small in the mass units of the central object, hence TDEs can be used to probe the strong gravity regime (Stone et al. 2019). During a TDE fragments of the stellar material will be accreted, producing a bright flare lasting at most a few years (Hills 1975; Rees 1988). The dynamics of stellar tidal disruption have been modeled through numerical simulations (e.g., Evans & Kochanek 1989; Haas et al. 2012; Guillochon & Ramirez-Ruiz 2013; Dai et al. 2013; Shiokawa et al. 2015; Bonnerot et al. 2016; Sądowski et al. 2016) as well as with simple analytical models (Rees 1988; Liu et al. 2021) while the observational properties of 56 well-studied TDE candidates have been recently reviewed in Gezari (2021).

The main objective of this work is to investigate the possibility to tell apart central BH from a NkS by observations of the accreting

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TDE debris stream from a tidally disrupted star. To that end, we study the geodesic trajectories of a stream of particles infalling with a constant angular momentum in a space time of a BH and that of a NkS. We use the RN geometry as a model spacetime, allowing us to explore both BH and NkS solutions with a change of a single parameter of the metric.

2 REISSNER-NORDSTRÖM METRIC

Reissner-Nordström (RN) spacetime (Reissner 1916; Nordström 1918) is a static, spherically symmetric solution to Einstein's field equations. It is described by two physical parameters: *mass* M , and *charge* Q . The line element of the RN spacetime, in spherical coordinates (t, r, θ, ϕ) with signature $(-, +, +, +)$, is

$$ds^2 = -c^2 d\tau^2 = -f(r)c^2 dt^2 + \frac{1}{f(r)} dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2), \quad (1)$$

where τ is the proper time and t the time coordinate measured by a stationary clock at infinity. The metric function

$$f(r) = -g_{tt} = 1 - \frac{r_s}{r} + \frac{r_Q^2}{r^2} \quad (2)$$

involves two parameters, the Schwarzschild radius $r_s = 2GM/c^2 = 2r_g$ and the characteristic length scale of charge $r_Q = G^{1/2}Q/(c^2\sqrt{4\pi\epsilon_0})$, with ϵ_0 denoting the dielectric constant. In units where $G = c = 4\pi\epsilon_0 = 1$, which we use throughout the rest of the paper, the metric function can be written as

$$f(r) = -g_{tt} = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}. \quad (3)$$

When the electric charge Q is set to zero, the metric is reduced to that of the Schwarzschild black hole. Location of the event horizons can be calculated by finding the root of the metric function $f(r_h) = 0$, leading to

$$r_h = M \pm \sqrt{M^2 - Q^2}. \quad (4)$$

For the Schwarzschild case of $Q = 0$, the event horizon is at $r_h = 2M = r_s$. We can identify three cases: $Q < M$, $Q = M$ and $Q > M$. When $Q < M$, the equation $f(r) = 0$ has two real roots, known as the outer horizon and the inner horizon. For $M = Q$, there is only one horizon at $r_h = M$, this case is known as the extreme black hole RN solution. In these two cases, the singularity remains hidden by an event horizon. When $Q > M$ there is no event horizon, and the resulting geometry describes a NkS. A particularly interesting property of RN NkS is that gravitational attraction vanishes on a spherical surface of radius $r_0 = Q^2/M$ (Pugliese et al. 2011). In fact, gravity is repulsive inside this “zero-gravity” sphere, and this allows non-rotating fluid atmospheres to levitate around the RN NkS, and thus to cloak it with a spherical shell (Vieira & Kluźniak 2023). Rotating fluid figures of equilibrium have the topology of a torus (Mishra et al. 2024). This zero-gravity sphere is present in many other models of NkS, although non-vacuum solutions without its presence can be constructed (Joshi et al. 2011). Hence, we expect the presented results to hold qualitatively in a wider range of geometries.

We assume the RN geometry as a simple model of a spacetime and solve the geodesic equation to study the orbital dynamics of massive test particles approaching the central object. The geodesic equation reads

$$\frac{d^2 x^k}{d\tau^2} + \Gamma_{ij}^k \frac{dx^i}{d\tau} \frac{dx^j}{d\tau} = 0, \quad (5)$$

where $u^i = dx^i/d\tau$ is the four-velocity of the particle and Γ_{ij}^k are the

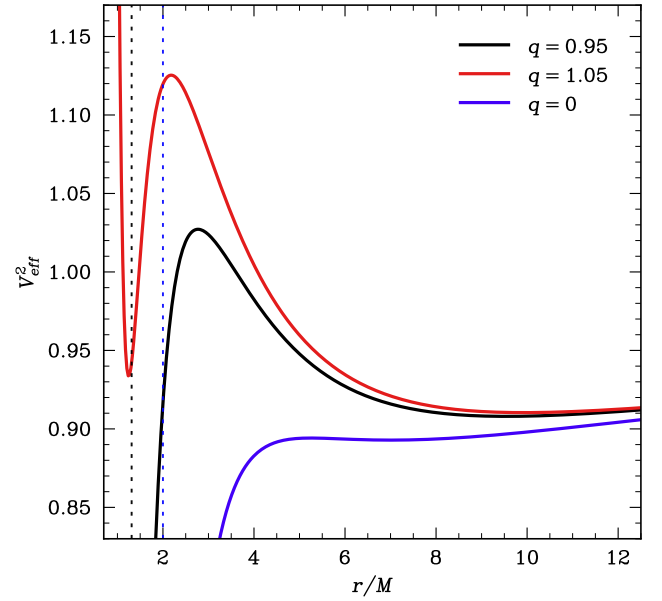


Figure 1. The effective potential for time-like particles in RN metric at charge-to-mass ratio $q \equiv Q/M = 0.95$ (BH; black line), $q = 1.05$ (NkS; red line), and $q = 0$ (Schwarzschild BH; blue line). Dotted vertical lines show the location of BH horizons. The specific angular momentum is fixed to $\ell = 3.5M$.

Christoffel symbols of the second kind, which can be calculated for a general metric tensor g_{ij} using

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl} (\partial_j g_{il} + \partial_i g_{lj} - \partial_l g_{ji}).$$

The behavior around RN BHs is qualitatively similar to the Schwarzschild case, with only quantitative differences. However, as we demonstrate below, trajectories around NkSs are qualitatively different, which should have a significant impact on the course of TDEs.

3 THE EFFECTIVE POTENTIAL

The effective potential determines the radial component of motion and thus its properties dictate the orbital dynamics in a given spacetime. In a spherical spacetime we consider the equatorial plane of $\theta = \pi/2$ without the loss of generality. The planar motion of a test particle is then described with two constants of motion, specific angular momentum $\ell = u_\phi = g_{\phi\phi} d\phi/d\tau$ and specific energy $\epsilon = -u_t = -g_{tt} dt/d\tau$. The radial equation of motion for a massive, neutral (uncharged) particle is

$$\dot{r}^2 + V_{\text{eff}}^2(r) = \epsilon^2, \quad (6)$$

with the effective potential

$$V_{\text{eff}}^2(r) = f(r) \left[1 + \left(\frac{\ell}{r} \right)^2 \right]. \quad (7)$$

Fig. 1 shows the RN effective potential for a BH with $Q/M = 0.95$ (black line) in comparison with a NkS with $Q/M = 1.05$ (red line) and a Schwarzschild BH (blue line). For BHs the vertical dotted lines show the location of the event horizon (2). In the familiar Newtonian case, the centrifugal barrier reflects all test particles with non-zero angular momentum. This is not the case for a BH, whose gravitational

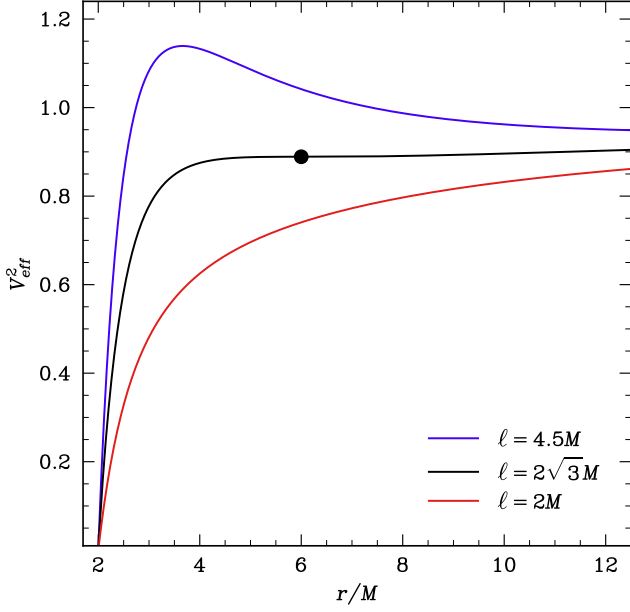


Figure 2. The Schwarzschild effective potential for particles with different specific angular momentum values. Limit value of $\ell = 2\sqrt{3}M$ is shown with the black curve together with the location of ISCO (black filled circle).

attraction overcomes the centrifugal barrier. For any fixed specific angular momentum of the test particle, the BH effective potential is characterized by (at most) a single maximum corresponding to an unstable circular orbit at r_{uo} , and only one minimum at $r_{so} > r_{uo}$, where a stable circular orbit is located.

For $Q = 0.95M$ and $\ell = 3.5M$, as shown in Fig. 1, the location of the peak is at $r_{uo} = 2.77M$, with $V_{\text{eff}}^2(r_{uo}) = 1.027$. Incoming test particles with energy $\epsilon^2 > 1.027$ must fall into the BH, while ones with lower values of energy are reflected back to infinity at the turning point r_{min} defined by $V_{\text{eff}}^2(r_{\text{min}}) = \epsilon^2$. For the NkS case the effective potential goes to plus infinity as $r \rightarrow 0$, so there is always a repulsive core that reflects any particle that penetrates to that core, even one with zero angular momentum. However, for values $1 < Q/M < \sqrt{5}/2$ the effective potential has a maximum, and therefore also two minima (circular orbits), one on each side of the maximum. This is the case illustrated in Fig. 1 for $Q = 1.05M$ and $\ell = 3.5M$, here the peak of the effective potential at $r_{uo} = 2.18M$ with $V_{\text{eff}}^2(2.18M) = 1.125$ is higher than for the BH case, and is located closer to the singularity. In this case, particles with values of energy below the peak of the potential, $\epsilon^2 < 1.125$, have a turning point before they reach the location of the maximum. Particles with energies $\epsilon^2 > 1.125$, sufficient to “clear” the peak, will be reflected by the repulsive core near the zero-gravity sphere r_0 . Thus, the main feature of scattering from NkS in geometries like the RN spacetime is that gravity, being always repulsive sufficiently close to the NkS (specifically, inside the zero-gravity sphere), reflects every sufficiently energetic test particle. This is contrary to the BH case, where for a fixed low angular momentum the most energetic particles are absorbed by the event horizon.

The effective potential for Schwarzschild spacetime is shown in Fig. 2 for three different values of specific angular momentum. The black curve corresponds to $\ell = 2\sqrt{3}M \approx 3.5M$, the Keplerian value of ℓ at the innermost stable circular orbit (ISCO), with the location of the ISCO marked by a black dot at $6M$. For particles with lower specific angular momentum ($\ell < 2\sqrt{3}M$), circular orbits do not exist,

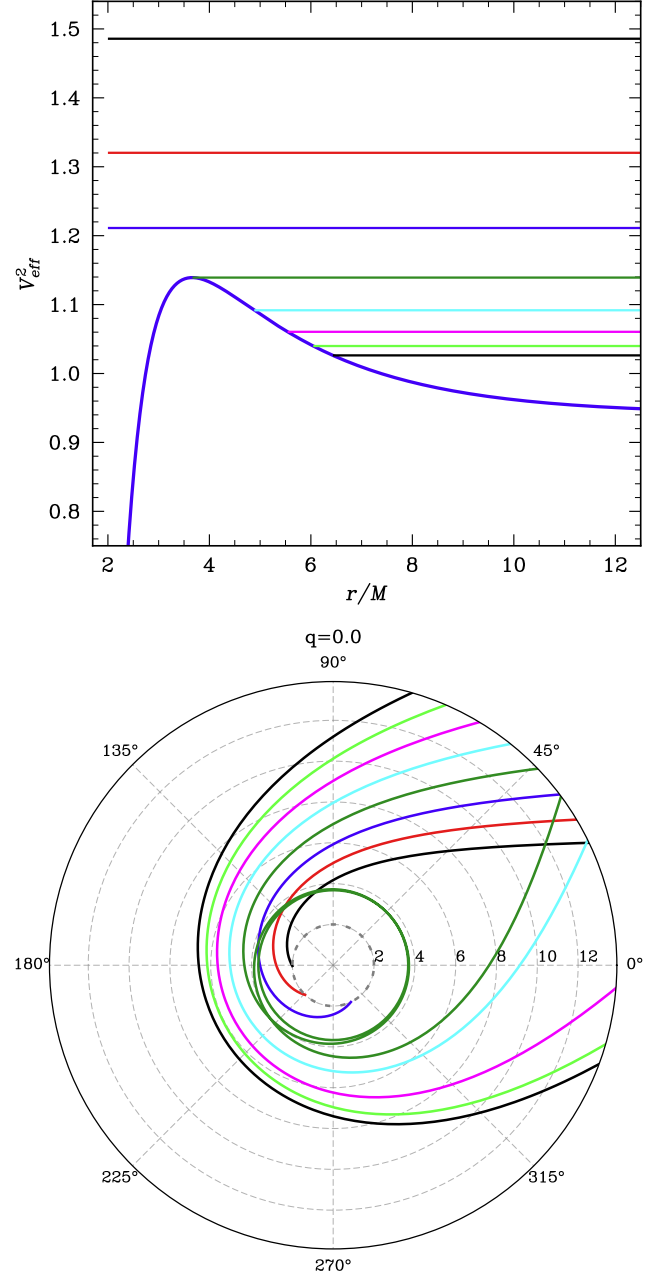


Figure 3. The effective potential (top panel) and the corresponding geodesic orbits around a Schwarzschild BH (bottom panel). Time-like particles have specific angular momentum $\ell = 4.5M$ and their specific energy ϵ^2 is shown with the horizontal lines in the top panel of the figure. They correspond to the trajectories of the same color solid lines of the bottom panel.

and particles fall into the BH. In contrast, for particles with higher specific angular momentum ($\ell > 2\sqrt{3}M$), the stable circular orbits are located above $6M$, and an unstable circular orbit exists at r_{uo} , defining a centrifugal barrier. Particles with energy ϵ^2 lower than the peak of this barrier will be reflected and follow hyperbolic orbits, while those with higher ϵ^2 will be captured by the BH.

Overall, the behavior of the effective potential for Schwarzschild and RN BHs is similar. As the charge-to-mass ratio q increases, the event horizon moves inward from $r_h = 2M$ for $q = 0$ toward $1M$ when $q = 1$. The radius of the ISCO also decreases, from $6M$ to $4M$

and so does the location of the centrifugal barrier (r_{uo}). However, these are only quantitative differences.

4 INITIAL SET-UP

We consider time-like geodesics initiated at a large distance from the compact object, with a Cartesian coordinate $x_0 = 5000 r_g$. The particles have a fixed specific angular momentum value ℓ and an initial velocity v_0 that is inversely proportional to the impact parameter, y_0 . In the limit of small M/r we have

$$y_0 = \frac{\ell}{v_0} = \frac{\ell}{\sqrt{\epsilon^2 - 1}}, \quad (8)$$

from which we can evaluate initial v_0 for each impact parameter y_0 . The velocity components of the particles at distant initial location are defined as:

$$u^r = -v_0 \cos \phi u^t,$$

$$u^\phi = \frac{v_0 \sin \phi}{R_0} u^t,$$

where ϕ is the polar coordinate of the initial location $\phi = \arctan(y_0/x_0)$, and $R_0 = \sqrt{x_0^2 + y_0^2}$ is the initial radial distance. Together with the normalization $u_i u^i = -1$ this fully specifies the initial four-velocity. Hence, by exploring a range of impact parameters y_0 , we can simulate a stream of matter arriving from a distance with a constant specific angular momentum ℓ and varying specific energy ϵ as a model of TDE debris.

5 SCHWARZSCHILD BLACK HOLE

For our analysis of Schwarzschild geometry, we choose the value $\ell = 4.5M$ for which a centrifugal potential barrier exists at $r_{\text{uo}} = 3.66M$ and the value of the effective potential at the local maximum is $V_{\text{eff}}^2(r_{\text{uo}}) = 1.14$. Additionally, a stable circular orbit exists at $r_{\text{so}} \approx 16.6M$ but in our analysis we only consider unbounded trajectories. The chosen value of ℓ ensures the existence of the peak in the effective potential at r_{uo} , with a value above unity, that effectively acts as a centrifugal barrier and reflects infalling particles with $1 < \epsilon^2 < V_{\text{eff}}^2(r_{\text{uo}})$ back to infinity, see the blue line in Fig. 2. The existence and properties of the centrifugal barrier depend on the specific angular momentum of the particle. For low values of specific angular momentum, the effective potential may not have a local maximum, and particles with any energy will be captured by the BH, see the red line in Fig. 2. As the specific angular momentum increases, a local maximum appears, and particles with energies below this maximum will be reflected, while those with higher energies will be captured by the BH.

The geodesic orbits for particles with specific angular momentum $\ell = 4.5M$ arriving with various impact parameters are plotted in Fig. 3. The black, red, blue and dark green trajectories cross the centrifugal barrier and fall into the BH, while the rest of the trajectories are reflected back. Orbits with energy ϵ near the peak of the effective potential (unstable circular orbit at r_{uo}) will rotate around the BH many times until they are either captured or reflected. A similar plot is shown in Fig. 4, but with additional orbits and a different visualization style. Orbits with low impact parameter y_0 (dashed lines) are captured by the BH. As the impact parameter increases, the particles are reflected by the centrifugal barrier and follow hyperbolic orbits. They are deflected roughly in the same direction.

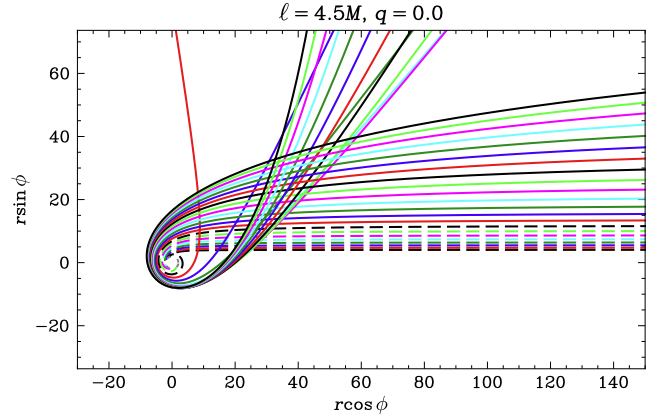


Figure 4. Trajectories for Schwarzschild BH with the same initial set up as in Fig. 3 but with a finer distribution of impact parameter. Orbits with the dashed lines are captured at the event horizon.

6 COMPARISON BETWEEN RN BH AND NKS

In contrast to BHs, which are characterized by an attractive gravitational force that traps everything, including light, within their event horizon, RN NkSs exhibit repulsive effective gravitational force inside an imaginary spherical surface of zero gravity, see Section 2. This fundamental difference in the spacetime structure leads to distinct orbital dynamics around BHs and NkSs.

We consider two values of $q = 0.95, 1.05$ as for the analysis of the effective potential presented in Section 3, see Fig. 5. We set $\ell = 3.5$ and the resulting trajectories shown in Fig. 6. Before reaching the vicinity of the central singularity, the orbits exhibit similar behavior. In the BH case, orbits with a smaller impact parameter (larger ϵ^2 assuming constant ℓ , Eq. 8) are captured by the BH. As the impact parameter increases under fixed specific angular momentum, the trajectories approach the $\epsilon = 1$ orbit, which is a consequence of Eq. 8. Unlike the BH case, all orbits around the NkS are reflected back to infinity. For high impact parameters, these orbits also converge to the orbit with $\epsilon = 1$, following Eq. 8. This is also visible in the Fig. 7 where we compare these two different configurations using the same initial setup but showing additional orbits and using a finer distribution of impact parameter y_0 (or specific energy ϵ^2) values.

Overall, we observe a significant difference in the deflection angles between the two cases. Notably, particles reflected by NkS are scattered in all directions. This is also evident in Fig. 8, where we show the NkS orbits with ϵ^2 values above the peak of the centrifugal barrier at r_{uo} . These are the orbits with low impact parameter that could be scattered at all angles. Orbits with $\epsilon^2 \approx V_{\text{eff}}^2(r_{\text{uo}})$ will rotate around the singularity many times (e.g., the light green orbit in Fig. 8). Although the centrifugal barrier also exists in the BH case, orbits above the peak are always captured, which is the reason for this distinct feature of NkS.

7 DIFFERENT SPECIFIC ANGULAR MOMENTUM VALUES

In this subsection, we repeat our previous investigations for additional values of the specific angular momentum ℓ . In Fig. 9 we indicate the effective potential curves for the two compared spacetimes of a BH and of a NkS, with four distinct values of the specific angular momentum ℓ shown in different colors. For a low ℓ maximum of V_{eff}^2 corresponds to a value below 1. For a BH case this implies that all

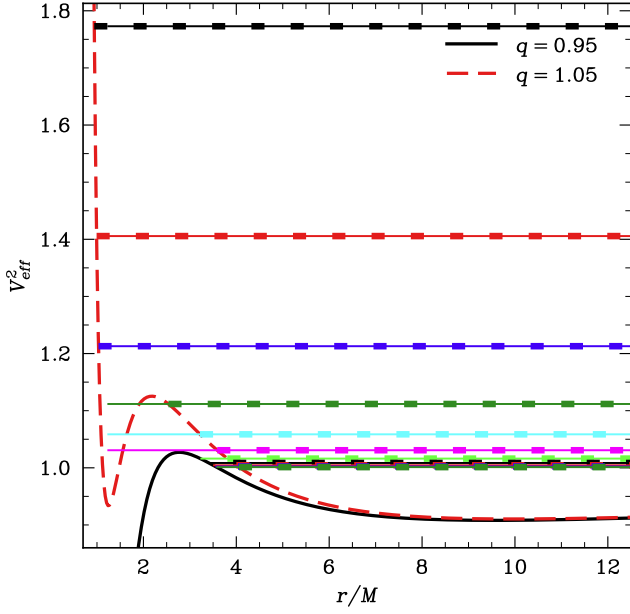


Figure 5. The effective potential for particles with $\ell = 3.5M$. The black continuous curve corresponds to the BH case ($q = 0.95$) and the red dashed curve to the NkS ($q = 1.05$). The horizontal lines correspond to the values of ϵ^2 for the trajectories plotted in Fig. 6 with the same color.

particles are absorbed, irrespectively of their ϵ . TDE debris in such a range of parameters would entirely disappear inside the BH, but it would be reflected and scattered by a NkS.

Fig. 10 presents trajectories around the NkS for particles with $\ell = 2.7M, 3M, 3.3M$. The main difference between each panel of Fig. 10 and the results obtained with $\ell = 3.5M$ (Fig. 7) is the angle of deflection and the radius of the closest approach. As shown in Fig. 9, the peak of the potential barrier $V_{\text{eff}}^2(r_{\text{uo}})$ in NkS with $q = 1.05$ is below 1 for sufficiently small ℓ . Thus, for the two lowest values of ℓ considered in Fig. 10 (two bottom panels), all particles arriving from infinity are reflected by the repulsive core around the NkS rather than by the centrifugal barrier. In the top panel, for $\ell = 3.3M$, some particles are reflected by the centrifugal barrier and some by the repulsive NkS core related to the zero-gravity sphere.

8 DISCUSSION AND CONCLUSIONS

In this study, we investigated the orbital dynamics of particles around black holes and naked singularities in the Reissner-Nordström spacetime. We analyzed the effective potential for massive particles with different specific angular momenta and we examined the orbits around a Schwarzschild BH ($q = 0$), charged BH ($q = 0.95$), and NkS ($q = 1.05$). We discussed the specific angular momentum values for which the centrifugal barrier exists. The initial setup of our simulations consisted of timelike particles with a fixed specific angular momentum value ℓ , placed initially at a large distance from the central BH or NkS. Under fixed ℓ their initial velocity (and hence the energy ϵ) decreases as the impact parameter increases. For BHs, particles with low impact parameter are captured by the event horizon, while those with higher impact parameter converge to orbits with $\epsilon = 1$.

In the accepted picture of TDEs, post-disruption stellar debris acquire a spread of specific energy and angular momentum. Roughly half of the stellar debris is unbound and escape the system and

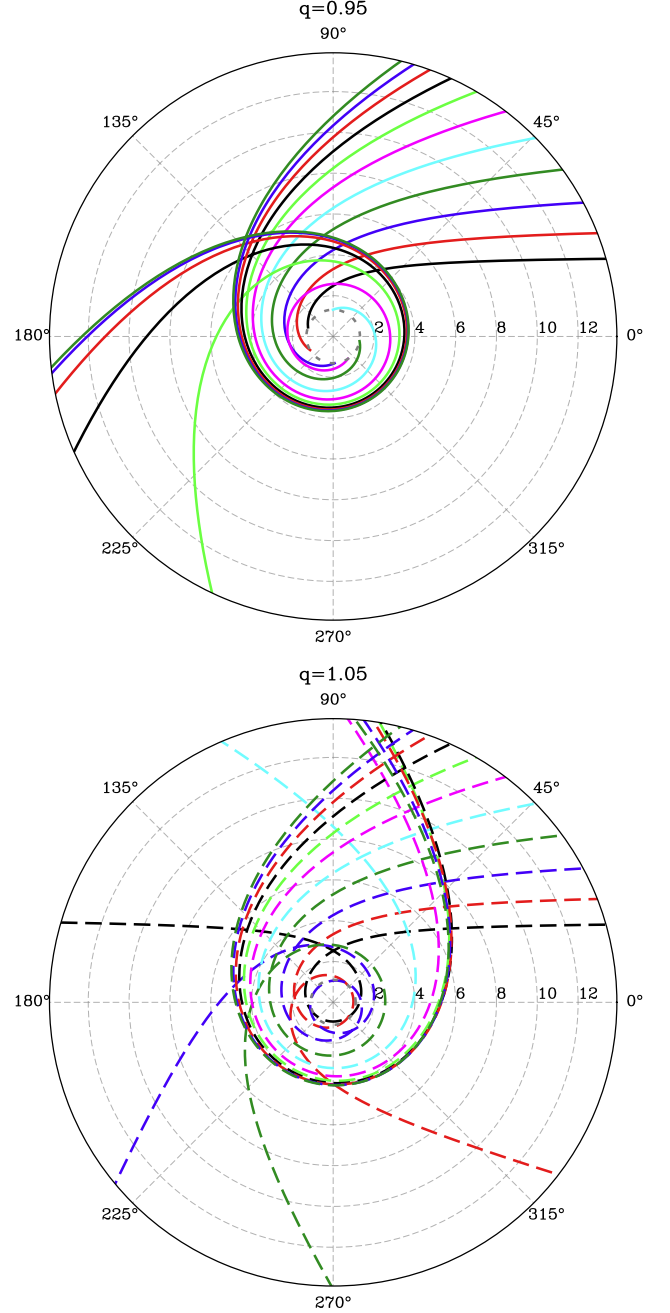


Figure 6. Trajectories around a charged BH ($q = 0.95$, top panel) and a NkS with $q = 1.05$ (bottom panel). The colors correspond to different values of ϵ^2 shown with horizontal lines in Fig. 5. Specific angular momentum is fixed to $\ell = 3.5M$.

the other half remain bound to the central object. In this work, we considered only unbound orbits moving toward the central singularity. For a direct and easier comparison of the trajectories with the effective potential that determines the radial motion, we set the specific angular momentum at the same value for all particles.

The results are qualitatively the same for Schwarzschild and charged BHs: as the charge-to-mass ratio increases, the radius of the ISCO, the location of the peak of the centrifugal barrier and the event horizon are shifted inwards. Similarly, the trajectories of particles are also qualitatively the same with only difference in radius of

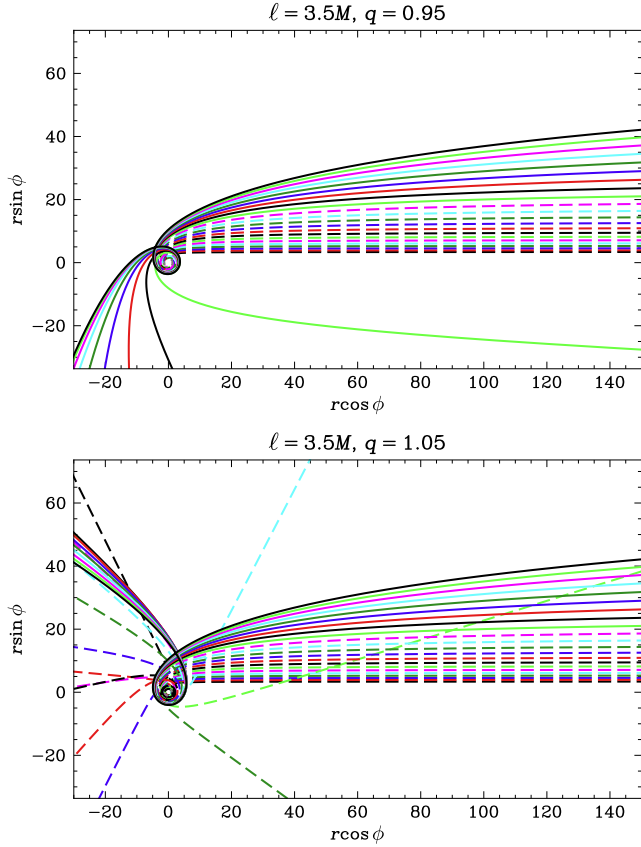


Figure 7. Trajectories around a charged BH ($q = 0.95$, top panel) and a NkS with $q = 1.05$ (bottom panel). Colors correspond to different values of ϵ^2 . Dashed lines correspond to the orbits with impact parameter $y_0 \lesssim 20M$ (or $\epsilon^2 > 1.027$), that are captured by a BH, but reflected and scattered by a NkS. Specific angular momentum is fixed to $\ell = 3.5M$.

the closed approach (since the radius of the potential peak decreases) and in the value of the deflection angle. For values of the specific angular momentum lower than the value obtained at ISCO (for Schwarzschild: $\ell = 2\sqrt{3}M$) all particles falling from infinity ($\epsilon > 1$) will be captured by the BH. A centrifugal barrier appears for larger specific angular momenta and reflects particles with $\epsilon^2 < V_{\text{eff}}^2(r_{\text{uo}})$, where r_{uo} is the radius of the unstable circular orbit.

The behavior of particles around NkSs is qualitatively different from that in the BH case. For NkS all particles are reflected, either by the centrifugal barrier, or in the vicinity of the zero-gravity sphere appearing in the close vicinity of the central singularity. We compared the behavior of particles in RN spacetimes with a similar charge parameter but distinct topologies – around charged ($q = 0.95$) BHs and NkSs with $q = 1.05$. Our results show that the stream of particles is reflected by the BH roughly in the same direction while in NkS it is scattered in all directions. Self intersection of the fallback stream is often considered necessary to form a circular accretion disk in a classic TDE framework (Rees 1988). If the returning stream in NkS case is scattered broadly, it may lead to inefficient circularization and a different geometry of the accretion disk (e.g., Liu et al. 2021). However, predicting the aftermath of interaction of the relativistic streams is a complex problem that will likely require numerical general relativistic magnetohydrodynamic simulations of a TDE around a NkS.

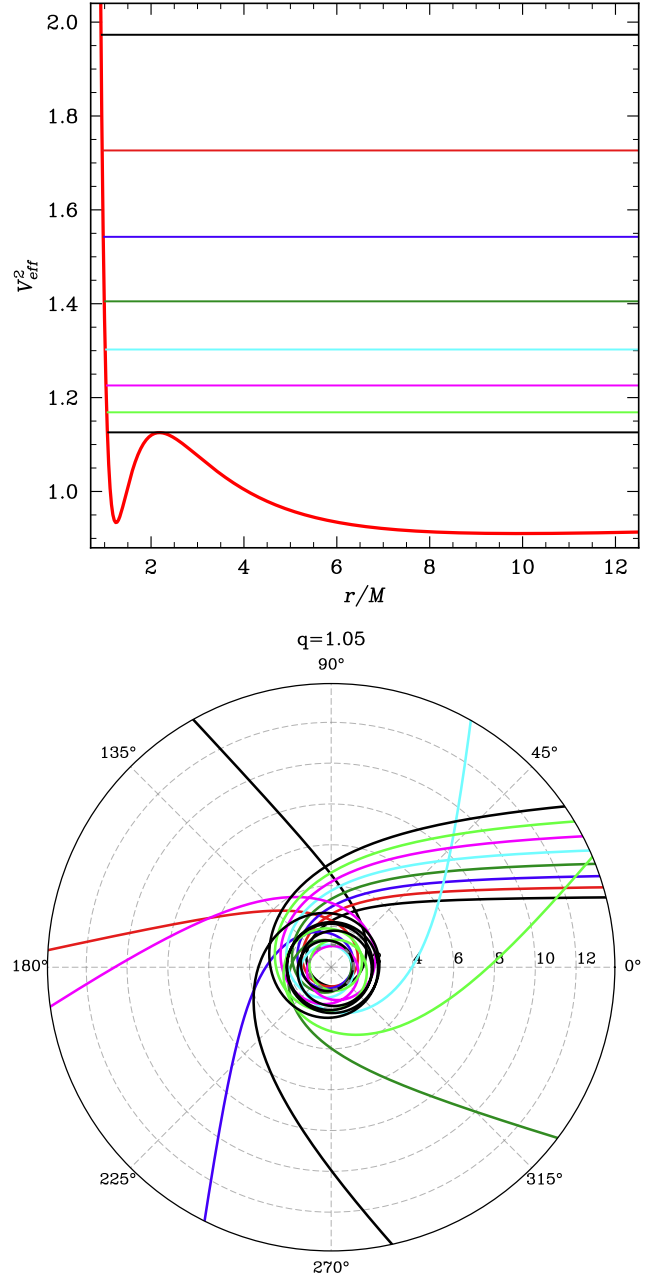


Figure 8. Trajectories around a NkS ($q = 1.05$) for orbits initialized at low values of y_0 , corresponding to large values of ϵ . The top panel presents the effective potential with horizontal lines indicating the value ϵ^2 of orbits plotted in the bottom panel. The specific angular momentum is fixed to $\ell = 3.5M$.

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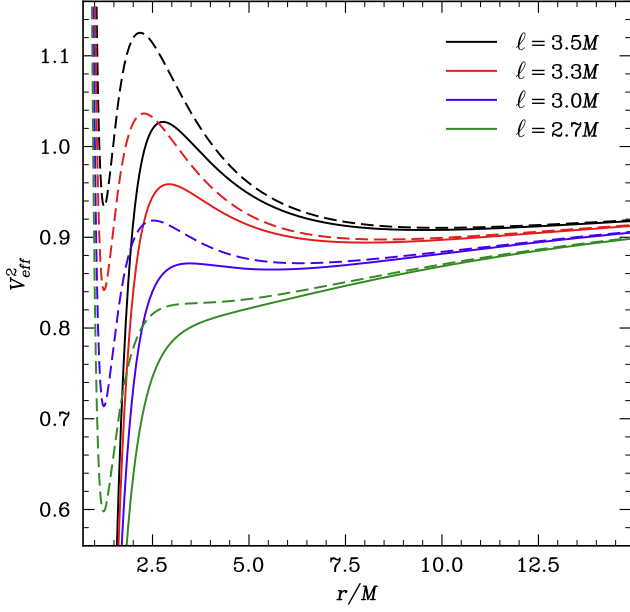


Figure 9. The effective potential for particles with different values of specific angular momentum (ℓ , represented by the various colors). The solid lines correspond to charged BH ($q = 0.95$) while, dashed lines show the NkS case with $q = 1.05$.

DATA AVAILABILITY

The data used in the work presented in this article are available upon request to the corresponding author.

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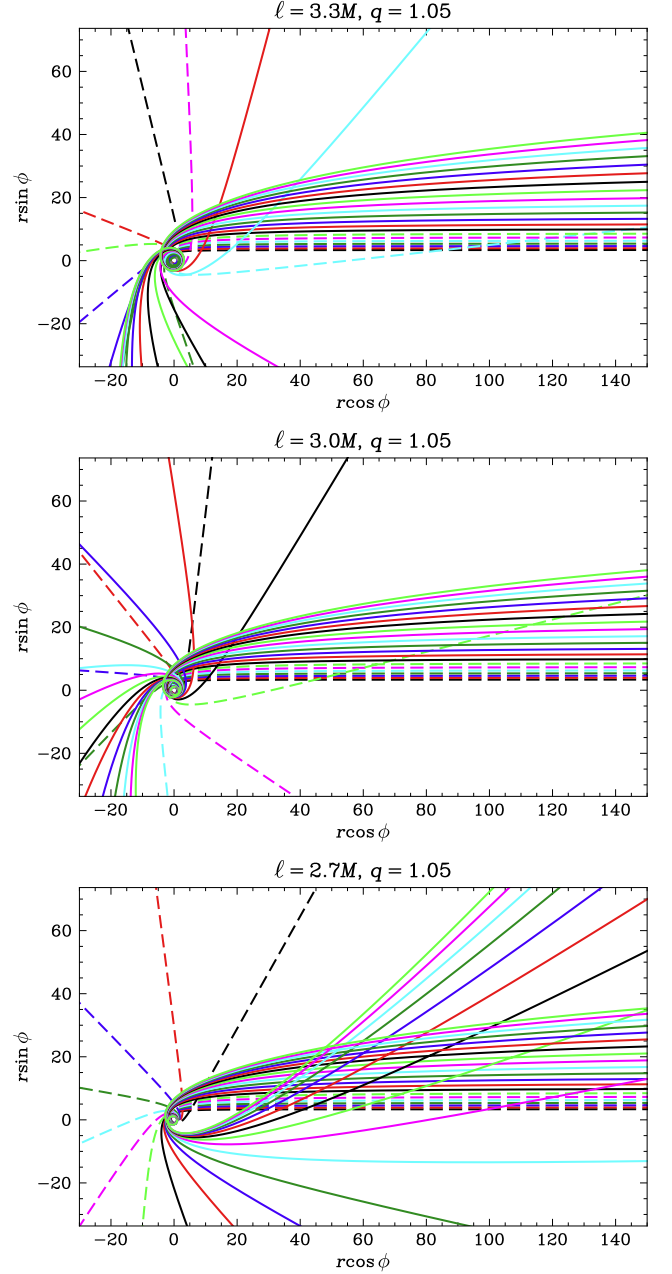


Figure 10. Geodesic orbits around NkS for particles of the same angular momentum. From top to bottom, we compare three values of specific angular momentum $\ell/M = (3.3, 3.0, 2.7)$. Dashed lines show orbits with low values in impact parameter.

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