On concatenations of two k-generalized Lucas numbers

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Abstract

For an integer $k \geq 2$, the sequence of k-generalized Lucas numbers is defined by the recurrence relation $L_n^{(k)} = L_{n-1}^{(k)} + \cdots + L_{n-k}^{(k)}$ for all $n \geq 2$, with initial conditions $L_0^{(k)} = 2$, $L_1^{(k)} = 1$ for all $k \geq 2$, and $L_{2-k}^{(k)} = \cdots = L_{-1}^{(k)} = 0$ for $k \geq 3$. In this paper, we determine all k-generalized Lucas numbers that are concatenations of two terms of the same sequence and completely solve this problem for $k \geq 3$. Our approach combines nonzero lower bounds for linear forms in logarithms, reduction techniques based on the Baker–Davenport method and the LLL-algorithm, together with continued fraction analysis and computational verification using SageMath.

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1 Introduction

1.1 Background

Let $k \geq 2$ be an integer. The sequence of k-generalized Lucas numbers, also known as the k-Lucas number sequence, is defined by the recurrence relation,

$$L_n^{(k)} = L_{n-1}^{(k)} + \cdots + L_{n-k}^{(k)}, \quad \text{for all } n \geq 2,$$

with initial conditions

$$L_0^{(k)} = 2, \quad L_1^{(k)} = 1 \quad \text{for all } k \geq 2, \text{ and } L_{-1}^{(k)} = L_{-2}^{(k)} = \cdots = L_{2-k}^{(k)} = 0 \quad \text{for } k \geq 3.$$

For $k = 2$, this is the well-known classical sequence of Lucas numbers, and in this case, we omit the superscript $^{(k)}$ in the notation. In table 1.1, for some small values of k, the first few values $L_n^{(k)}$ of are given.

k	Name	First non-zero terms
2	Lucas	2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, $\cdots$
3	3-Lucas	2, 1, 3, 6, 10, 19, 35, 64, 118, 217, 399, 734, $\cdots$
4	4-Lucas	2, 1, 3, 6, 12, 22, 43, 83, 160, 308, 594, 1145, $\cdots$
5	5-Lucas	2, 1, 3, 6, 12, 24, 46, 91, 179, 352, 692, 1360, $\cdots$
6	6-Lucas	2, 1, 3, 6, 12, 24, 48, 94, 187, 371, 736, 1460, $\cdots$
7	7-Lucas	2, 1, 3, 6, 12, 24, 48, 96, 190, 379, 755, 1504, $\cdots$
8	8-Lucas	2, 1, 3, 6, 12, 24, 48, 96, 192, 382, 763, 1523, $\cdots$
9	9-Lucas	2, 1, 3, 6, 12, 24, 48, 96, 192, 384, 766, 1531, $\cdots$

Table 1.1: First non-zero k-Lucas numbers.

In [4], the authors determined all Fibonacci numbers that are concatenations of two Fibonacci numbers. In particular, they proved that $F_{10} = 55$ is the greatest Fibonacci number with this property. They also proved that for any binary recurrent sequence of integers t_n, there exists only finitely many terms of that sequence which can be written as concatenations of two or more terms of t_n under the same mild hypothesis. In [2], the authors determined all Fibonacci numbers that are concatenations of a

Fibonacci number and a Lucas number. Similarly, in [1], they determined all k-Fibonacci numbers that are concatenations of two terms of the same sequence. Particularly, they proved that the Diophantine equation

$$F_n^{(k)} = F_m^{(k)} \cdot 10^d + F_\ell^{(k)}, k \geq 3,$$

has solutions only in the cases $F_7^{(3)} = 24, F_8^{(3)} = 44$ and $F_{10}^{(8)} = 16128$. In this paper, we consider a similar but modified problem in which we take into account k-generalized Lucas numbers that are concatenations of two terms of the same sequence. We completely solve this problem for all $k \geq 3$. As a mathematical expression of this problem, we solve the Diophantine equation

$$L_n^{(k)} = L_m^{(k)} \cdot 10^d + L_p^{(k)}, \quad k \geq 3. \tag{1.1}$$

in non-negative integers $m, n \geq 0$, and p, where d is the number of digits of $L_p^{(k)}$. In [17], it was found out that for $k = 2$, $L_6^{(2)} = 11 = \overline{L_2^{(2)} L_2^{(2)}}$, and $L_9^{(2)} = 47 = \overline{L_4^{(2)} L_5^{(2)}}$, are the only solutions to the Diophantine equation (1.1). We therefore state Theorem 1.1 for cases $k \geq 3$.

1.2 Main Result

Theorem 1.1. *Let $k \geq 3$, the Diophantine equation* (1.1) *has only the following solutions.*

$$L_4^{(k)} = 12 = \overline{L_1^{(k)} L_0^{(k)}}, \text{ for } k \geq 4, \text{ and } L_5^{(4)} = 22 = \overline{L_0^{(4)} L_0^{(4)}}.$$

Effective methods for tackling Diophantine equations similar to (1.1) include approaches like nonzero lower bounds for linear forms in logarithms of algebraic numbers, as developed by Matveev [20] and the reduction algorithm proposed by Dujella and Pethő [12], which is a variation of Baker and Davenport. For our analysis, we employed these techniques in conjunction with specific properties of k-Lucas numbers and the use of continued fractions. All computations were conducted using SageMath, with precision up to 1000 digits. The next section presents details of these methods used.

2 Methods

2.1 Some properties of k-generalized Lucas numbers

It is well established for k-Lucas numbers that

$$L_n^{(k)} = 3 \cdot 2^{n-2}, \qquad \text{for all} \qquad 2 \leq n \leq k. \tag{2.1}$$

These numbers generate a linearly recurrent sequence with the characteristic polynomial

$$\Psi_k(x) = x^k - x^{k-1} - \cdots - x - 1,$$

which is irreducible over $\mathbb{Q}[x]$. The polynomial $\Psi_k(x)$ has a unique real root $\alpha(k) > 1$, with all other roots lying outside the unit circle, as noted in [21]. The root $\alpha(k) := \alpha$ satisfies the bound

$$2(1 - 2^{-k}) < \alpha < 2 \quad \text{for all} \quad k \geq 2, \tag{2.2}$$

as established in [27]. Similar to the classical case where $k = 2$, it was proved in [6] that

$$\alpha^{n-1} \leq L_n^{(k)} \leq 2\alpha^n, \quad \text{for all } n \geq 1, \quad k \geq 2. \tag{2.3}$$

For any $k \geq 2$, define the function

$$f_k(x) := \frac{x-1}{2 + (k+1)(x-2)}. \tag{2.4}$$

From the bound in (2.2), it follows that

$$\frac{1}{2} = f_k(2) < f_k(\alpha) < f_k(2(1 - 2^k)) \leq \frac{3}{4}, \tag{2.5}$$

for all $k \geq 3$ since $f_k(x)$ is reducing for all $x > 0$. It is straightforward to check that this inequality is also valid for $k = 2$. Moreover, one can verify that for all $2 \leq i \leq k$, where α_i represents the remaining roots of $\Psi_k(x)$, the condition $|f_k(\alpha_i)| < 1$ holds, see [6]. The following lemma plays a crucial role in applying Baker's theory.

Lemma 2.1 (Lemma 2, [18])**.** *For all $k \geq 2$, the number $f_k(\alpha)$ is not an algebraic integer.*

Additionally, it was proven in [6] that

$$L_n^{(k)} = \sum_{i=1}^{k} (2\alpha_i - 1) f_k(\alpha_i)\alpha_i^{n-1}, \quad \text{and} \quad \left| L_n^{(k)} - f_k(\alpha)(2\alpha - 1)\alpha^{n-1} \right| < \frac{3}{2}, \tag{2.6}$$

for all $k \geq 2$ and $n \geq 2 - k$. This leads to the expression

$$L_n^{(k)} = f_k(\alpha)(2\alpha - 1)\alpha^{n-1} + e_k(n), \quad \text{where} \quad |e_k(n)| < 1.5. \tag{2.7}$$

The left-hand expression in (2.6) is commonly referred to as the Binet-like formula for $L_n^{(k)}$. Furthermore, the inequality on the right in (2.7) indicates that the contribution of roots inside the unit circle to $L_n^{(k)}$ is negligible. A better estimate than (2.6) appears in Section 3.3 of [5], but with a more restricted range of n in terms of k. It states that

$$\left| f_k(\alpha)(2\alpha - 1)\alpha^{n-1} - 3 \cdot 2^{n-2} \right| < 3 \cdot 2^{n-2} \cdot \frac{36}{2^{k/2}}, \qquad \text{provided} \qquad n < 2^{k/2}. \tag{2.8}$$

Lastly, the following result from [3] will be used in our proof.

Lemma 2.2 ([3])**.** *For any positive integer $n > 1$ with $n \neq 6$, the number $2^n - 1$ has a prime divisor that does not divide $2^k - 1$ for any $k < n$.*

2.2 Linear forms in logarithms

Definition 2.1. *Let θ be an algebraic number, and consider its minimal polynomial over $\mathbb{Z}$, given by*

$$u_0 x^d + u_1 x^{d-1} + \cdots + u_t = u_0 \prod_{i=1}^{d} (x - \theta^{(i)}),$$

where the coefficients u_i are relatively prime integers with $u_0 > 0$, and the numbers $\theta^{(i)}$ represent the conjugates of θ. The logarithmic height of θ is defined as

$$h(\theta) = \frac{1}{d} \left(\log u_0 + \sum_{i=1}^{d} \log \left(\max\{|\theta^{(i)}|, 1\} \right) \right).$$

In particular, for a rational number r/s, where r and $s > 0$ are coprime integers, the logarithmic height is given by $h(r/s) = \log \max\{|r|, s\}$. The following properties are useful and shall be applied without reference:

(*a*) $h(\theta_1 \pm \theta_2) \leq h(\theta_1) + h(\theta_2) + \log 2$.
(*b*) $h(\theta_1 \theta_2^{\pm 1}) \leq h(\theta_1) + h(\theta_2)$.
(*c*) $h(\theta^s) = |s| h(\theta)$, for any integer $s \in \mathbb{Z}$.

It was computed in Section 3, equation (12) in [7] that $h(f_k(\alpha)) < 3 \log k$ for all $k \geq 2$. Theorem 2.1 is a very important theorem proved in [20] that is fundamental in obtaining bounds on variables in the Diophantine equation (1.1). First, the general setting in which it holds is introduced. Let $\mathbb{K}$ be a number field of degree D, let $\alpha_1, \ldots, \alpha_t$ be non-zero elements of $\mathbb{K}$ and $b_1, \ldots, b_n$ be rational integers. Set

$$B = \max\{|b_1|, \ldots, |b_t|\},$$

and

$$\Gamma = \eta_1^{b_1} \cdots \eta_t^{b_t} - 1.$$

Let h denote the absolute logarithmic height and let $A_1, \ldots, A_t$ be real numbers with

$$A_j \geq h'(\eta_j) := \max\{D h(\eta_j), |\log \eta_j|, 0.16\}, \quad 1 \leq j \leq t.$$

We call h' the modified height (with respect to the field $\mathbb{K}$). With this notation, the main result in [20] implies the following estimate.

Theorem 2.1 (Theorem 9.4 in [8])**.** *Assume that Λ is non-zero and $\mathbb{K}$ is real, we have*

$$\log |\Gamma| > -1.4 \cdot 30^{t+3} t^{4.5} D^2 (1 + \log D)(1 + \log B) A_1 \cdots A_t.$$

2.3 Reduction methods

2.3.1 Lattice-based tools

The bounds obtained from applying Theorem 2.1 are often too large for practical computations. A reduction method based on the LLL-algorithm will be applied to refine the bounds obtained from applying Theorem 2.1. First, definitions and explanations are provided to facilitate an understanding of the LLL-algorithm.

Definition 2.2 (Lattice)**.** *Let k be a positive integer. A subset L of the real vector space $\mathbb{R}^k$ is called a* lattice *if there exists a basis $\{b_1, \ldots, b_k\}$ of $\mathbb{R}^k$ such that*

$$L = \left\{ \sum_{i=1}^{k} r_i b_i \;\middle|\; r_i \in \mathbb{Z} \right\}.$$

The vectors $b_1, \ldots, b_k$ are said to form a basis for L, and k is termed the rank of L. The *determinant* $\det(L)$ is defined as

$$\det(L) = |\det(b_1, \ldots, b_k)|,$$

where the b_i are represented as column vectors. This value is independent of the choice of basis, (see [13], Section 1.2). For linearly independent vectors $b_1, \ldots, b_k$ in $\mathbb{R}^k$, we employ the Gram-Schmidt orthogonalization process to define vectors b_i^* and coefficients $\mu_{i,j}$ via

$$b_i^* = b_i - \sum_{j=1}^{i-1} \mu_{i,j} b_j^*, \quad \mu_{i,j} = \frac{\langle b_i, b_j^* \rangle}{\langle b_j^*, b_j^* \rangle},$$

where $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbb{R}^k$.

Definition 2.3. *A basis $b_1, \ldots, b_n$ for the lattice L is called* reduced *if*

$$|\mu_{i,j}| \leq \frac{1}{2} \text{ for } 1 \leq j < i \leq n,$$

$$\|b_i^* + \mu_{i,i-1} b_{i-1}^*\|^2 \geq \frac{3}{4} \|b_{i-1}^*\|^2 \text{ for } 1 < i \leq n,$$

where $\| \cdot \|$ is the Euclidean norm. The constant $\frac{3}{4}$ can be replaced by any fixed $y \in (1/4, 1)$, (see [19], section 1).

For a k-dimensional lattice $L \subset \mathbb{R}^k$ with reduced basis $b_1, \ldots, b_k$, let B be the matrix with columns $b_1, \ldots, b_k$. Define

$$\ell(L, y) = \begin{cases} \min_{x \in L} \|x - y\| & \text{if } y \notin L, \\ \min_{0 \neq x \in L} \|x\| & \text{if } y \in L, \end{cases}$$

where $\| \cdot \|$ is the Euclidean norm. The LLL algorithm efficiently computes a lower bound $\ell(L, y) \geq c_1$ in polynomial time, (see [24]., Section V.4)

Lemma 2.3 ([24], Section V.4)**.** *Let $y \in \mathbb{R}^k$ and $z = B^{-1} y$ with $z = (z_1, \ldots, z_k)^T$.*

(i) If $y \notin L$, let i_0 be the largest index with $z_{i_0} \neq 0$ and set $\lambda := \{z_{i_0}\}$, where $\{\cdot\}$ denotes the fractional part.

(ii) If $y \in L$, set $\lambda := 1$.

Then,

$$c_1 := \max_{1 \leq j \leq k} \left\{ \frac{\|b_1\|}{\|b_j^*\|} \right\}, \quad \delta := \lambda \frac{\|b_1\|}{c_1}.$$

In applications, one encounters real numbers $\eta_0, \eta_1, \ldots, \eta_k$, linearly independent over $\mathbb{Q}$, and constants $c_3, c_4 > 0$ such that

$$|\eta_0 + x_1 \eta_1 + \cdots + x_k \eta_k| \leq c_3 \exp(-c_4 H),$$

where $|x_i| \leq X_i$ for given bounds X_i $(1 \leq i \leq k)$. Let $X_0 := \max_{1\leq i\leq k}\{X_i\}$. Following [14], the inequality is approximated using a lattice generated by the columns of

$$\mathcal{A} = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \\ \lfloor C\eta_1 \rfloor & \lfloor C\eta_2 \rfloor & \dots & \lfloor C\eta_{k-1} \rfloor & \lfloor C\eta_k \rfloor \end{pmatrix},$$

where C is a large constant (typically $\approx X_0^k$). For an LLL-reduced basis $b_1, \dots, b_k$ of L and $y = (0, \dots, -\lfloor C\eta_0 \rfloor)$, Lemma 2.3 yields a lower bound $\ell(L, y) \geq c_1$.

Lemma 2.4 (Lemma VI.1 in [24])**.** *Let $S := \sum_{i=1}^{k-1} X_i^2$ and $T := \frac{1+\sum_{i=1}^{k} X_i}{2}$. If $\delta^2 \geq T^2 + S$, then the above inequality implies either $x_1 = \dots = x_{k-1} = 0$ and $x_k = -\lfloor C\eta_0 \rfloor / \lfloor C\eta_k \rfloor$, or*

$$H \leq \frac{1}{c_4}\left(\log(Cc_3) - \log\left(\sqrt{\delta^2 - S} - T\right)\right).$$

2.4 Continued fractions

The following lemma also helps in refining upper bounds on certain variables.

Lemma 2.5 ([7], Lemma 4)**.** *Let M be a positive integer, and let p/q be a convergent of the continued fraction of the irrational number γ such that $q > 6M$. Suppose A, B, μ are real numbers satisfying $A > 0$ and $B > 1$. If*

$$\epsilon := ||\mu q|| - M||\gamma q|| > 0,$$

then the inequality

$$0 < |u\gamma - v + \mu| < AB^{-w},$$

has no solutions in positive integers u, v, and w satisfying

$$u \leq M \quad \textit{and} \quad w \geq \frac{\log(Aq/\epsilon)}{\log B}.$$

Lemma 2.5 cannot be applied where $\mu = 0$ or when μ is a linear combination of 1 and γ with integer coefficients, as in such cases ϵ becomes negative for sufficiently large M. When this happens, a classical result due to Legendre (Lemma 2.5,[22]) is applied instead.

Lemma 2.6 ([22])**.** *Let τ, be an irrational number, M be a positive integer and $\frac{p_k}{q_k}(k = 0, 1, 2, \cdots)$ be all the convergents of the continued fraction $[a_0, a_1, \cdots]$ of τ. let N be such that $q_N > M$. Then putting $a_M := max\{a_i := 0, 1, \cdots, N\}$, the inequality*

$$|m\tau - n| > \frac{1}{(a_M + 2)m},$$

holds for all pairs $(n, m)$of integers with $0 < m < M$.

All computations in this work were carried out using SageMath 10.6.

3 Proof of Theorem 1.1

3.1 Preliminaries

Assume that (1.1) holds, let us examine the relations between the variables in (1.1). The number of digits of $L_p^{(k)}$ can be written as

$$d = \lfloor \log_{10} L_p^{(k)} \rfloor + 1, \tag{3.1}$$

where $\lfloor \Phi \rfloor$ is the greatest integer less than or equal to Φ, the floor function. Therefore,

$$\begin{aligned} d &= \lfloor \log_{10} L_p^{(k)} \rfloor + 1 \leq 1 + \log_{10} L_p^{(k)} \leq 1 + \log_{10}(2\alpha^p) \\ &= 1 + \log_{10} 2 + p \log_{10} \alpha < 2p \log_{10} 2 < p + 2, \end{aligned}$$

and

$$\begin{aligned} d &= \lfloor \log_{10} L_p^{(k)} \rfloor + 1 > \log_{10} L_p^{(k)} \geq \log_{10} \alpha^{p-1} \\ &= (p-1) \log_{10} \alpha \geq \frac{p-1}{5}. \end{aligned}$$

We can conclude that

$$\frac{p-1}{5} < d < p+2. \tag{3.2}$$

One can also see from (3.1) that

$$L_p^{(k)} < 10^d < 10 L_p^{(k)}. \tag{3.3}$$

From the relations (3.2), (2.3) and (1.1), we rewrite

$$\begin{aligned} \alpha^{n-1} &\leq L_n^{(k)} = L_m^{(k)} \cdot 10^d + L_p^{(k)} \leq L_m^{(k)} \cdot 10 \cdot L_p^{(k)} + L_p^{(k)} \\ &\leq 11 L_m^{(k)} L_p^{(k)} \leq 44 \alpha^{m+p} < \alpha^{m+p+8}, \end{aligned}$$

and

$$2\alpha^n \geq L_n^{(k)} = L_m^{(k)} \cdot 10^d + L_p^{(k)} > L_m^{(k)} \cdot L_p^{(k)} + L_p^{(k)} > L_m^{(k)} L_p^{(k)} \geq \alpha^{m+p-2},$$

hence, we get the relation

$$m + p - 2 < n < m + p + 8. \tag{3.4}$$

Lastly here, since we are interested in a concatenation of two k-Lucas numbers, we disregard all cases when $n \leq 3$, Moreover, when $n = 4$, we see from Table1.1 that $L_4^{(3)} = 10$ and $L_4^{(k)} = 12$ whenever $k \geq 4$. Clearly $L_4^{(3)} = 10$ is not a concatenation of two k-Lucas numbers, so it does not solve (1.1). On the other hand, $L_4^{(k)} = 12$ solves (1.1), so we state it in the main result. From now on, we work with $n \geq 5$.

3.2 The Case $n \leq k$

Here, we assume that $5 \leq n \leq k$. This means that $L_n^{(k)} = 3 \cdot 2^{n-2}$. Thus, Equation (1.1) becomes

$$3 \cdot 2^{n-2} = 3 \cdot 2^{m-2} \cdot 10^d + 3 \cdot 2^{p-2}. \tag{3.5}$$

Now, if $p \leq m$, then (3.5) becomes

$$2^{n-p} = 2^{m-p} \cdot 10^d + 1,$$

which implies that $0 \equiv 1 \pmod 2$, a contradiction. Next, if $p > m$, then (3.5) becomes

$$2^{n-m} = 10^d + 2^{p-m},$$

which can be rewritten as

$$2^{p-m}(2^{n-p} - 1) = 10^d = 2^d \cdot 5^d.$$

By the Fundamental Theorem of Arithmetic, we have

$$2^{p-m} = 2^d \quad \text{and} \quad 2^{n-p} - 1 = 5^d.$$

The second equation has no positive integer solutions by Lemma 2.2, thus Equation (1.1) has no solutions with $5 \leq n \leq k$. From now on, we assume $n > k$.

3.3 The case $n > k$

3.3.1 A bound for n in terms of k

If $n > k$, we write a SageMath code to search for values of $m, n, p,$ and k satisfying (1.1) in the range $0 \leq m, p \leq 500$ with $n \in (m+p-2, m+p+8)$ via (3.4) and $k < n$. This yields the solutions stated in Theorem 1.1. So from now on, we take $\max\{m, p\} > 500$. We now rewrite (1.1) using (2.7) as

$$f_k(\alpha)(2\alpha - 1)\alpha^{n-1} + e_k(n) = \big(f_k(\alpha)(2\alpha - 1)\alpha^{m-1} + e_k(m)\big) \cdot 10^d + L_p^{(k)},$$

and obtain

$$f_k(\alpha)(2\alpha-1)\alpha^{n-1} - f_k(\alpha)(2\alpha-1)\alpha^{m-1}\cdot 10^d = e_k(m)\cdot 10^d + L_p^{(k)} - e_k(n).$$

Dividing through by $f_k(\alpha)(2\alpha-1)\alpha^{n-1}$ and taking absolute values on both sides gives

$$\begin{aligned}
\left|1-\frac{10^d}{\alpha^{n-m}}\right| &\le \left|\frac{e_k(m)\cdot 10^d}{f_k(\alpha)(2\alpha-1)\alpha^{n-1}} + \frac{L_p^{(k)}}{f_k(\alpha)(2\alpha-1)\alpha^{n-1}} - \frac{e_k(n)}{f_k(\alpha)(2\alpha-1)\alpha^{n-1}}\right| \\
&\le \left|\frac{e_k(m)\cdot 10^d}{f_k(\alpha)(2\alpha-1)\alpha^{n-1}}\right| + \left|\frac{L_p^{(k)}}{f_k(\alpha)(2\alpha-1)\alpha^{n-1}}\right| + \left|\frac{e_k(n)}{f_k(\alpha)(2\alpha-1)\alpha^{n-1}}\right| \\
&< \left|\frac{30L_p^{(k)}}{(2\alpha-1)\alpha^{n-1}}\right| + \left|\frac{2L_p^{(k)}}{(2\alpha-1)\alpha^{n-1}}\right| + \left|\frac{3}{(2\alpha-1)\alpha^{n-1}}\right| \\
&< \frac{30(2\alpha^p)}{\alpha^n} + \frac{4\alpha^p}{\alpha^n} + \frac{3}{\alpha^n} < \frac{70}{\alpha^{n-p}}.
\end{aligned}$$

Hence, we let

$$|\Gamma_1| := \left|1-\frac{10^d}{\alpha^{n-m}}\right| < \frac{70}{\alpha^{n-p}}. \tag{3.6}$$

Notice that $\Gamma_1 \neq 0$ otherwise we would have $\alpha^{n-m} = 10^d$, a contradiction since α^{n-m} is a unit in $\mathbb{Q}(\alpha)$ and yet 10^d is not. The relevant algebraic number field is $\mathbb{K} = \mathbb{Q}(\alpha)$ with degree $D = k$. Setting $t := 2$, we define

$$\begin{aligned}
&\eta_1 := \alpha, \quad \eta_2 := 10,\\
&b_1 := -(n-m), \quad b_2 := d,
\end{aligned}$$

and $h(\eta_1) = (1/k)\log(\alpha), h(\eta_2) = \log 10$. Thus we take $A_1 = \log(\alpha), A_2 = k\log 10$. Note that from (3.2), $d < p+2$ and via (3.4), $p < n-m+4$ thus $d < n-m+6$ and hence we take $B := n-m+6$. Applying Theorem 2.1, we get,

$$\log|\Gamma_1| > -1.4\cdot 30^5\cdot 2^{4.5}\cdot k^2(1+\log k)(1+\log(n-m+6))\log\alpha\cdot k\log 10.$$

Also from (3.6), we have

$$\log\Gamma_1 < \log 70 - (n-p)\log\alpha.$$

Also, since $\log\alpha < \log 2$, and using the inequality

$$1+\log k = \log k\left(1+\frac{1}{\log k}\right) < 2\log k,$$

and similarly,

$$1+\log(n-m+6) = \log(n-m+6)\left(1+\frac{1}{\log(n-m+6)}\right) < 2\log(n-m+6),$$

for $k \ge 3$ and $n \ge 5$, we derive the bound

$$n-p < 7.2\cdot 10^9 k^3\log k\log(n-m+6). \tag{3.7}$$

Next, we revisit (1.1) and by (2.7), we rewrite it as

$$f_k(\alpha)(2\alpha-1)\alpha^{n-1} + e_k(n) = L_m^{(k)}\cdot 10^d + f_k(\alpha)(2\alpha-1)\alpha^{p-1} + e_k(p),$$

we obtain

$$\begin{aligned}
f_k(\alpha)(2\alpha-1)\alpha^{n-1} - f_k(\alpha)(2\alpha-1)\alpha^{p-1} - L_m^{(k)}\cdot 10^d &= -e_k(n) + e_k(p),\\
f_k(\alpha)(2\alpha-1)\alpha^{n-1}(1-\alpha^{p-n}) - L_m^{(k)}\cdot 10^d &= -e_k(n) + e_k(p).
\end{aligned}$$

Dividing through by $f_k(\alpha)(2\alpha-1)\alpha^{n-1}(1-\alpha^{p-n})$ and taking absolutes on both sides yields

$$\begin{aligned}\left|1-\frac{L_m^{(k)}\cdot 10^d}{f_k(\alpha)(2\alpha-1)\alpha^{n-1}(1-\alpha^{p-n})}\right| &\le \left|\frac{-e_k(n)}{f_k(\alpha)(2\alpha-1)\alpha^{n-1}(1-\alpha^{p-n})}+\frac{e_k(p)}{f_k(\alpha)(2\alpha-1)\alpha^{n-1}(1-\alpha^{p-n})}\right|\\ &\le \left|\frac{e_k(n)}{f_k(\alpha)(2\alpha-1)\alpha^{n-1}(1-\alpha^{p-n})}\right|+\left|\frac{e_k(p)}{f_k(\alpha)(2\alpha-1)\alpha^{n-1}(1-\alpha^{p-n})}\right|\\ &< \frac{18}{\alpha^{n-1}},\end{aligned}$$

where we have used the estimates $\frac{1}{f_k(\alpha)}<2$ and $\frac{1}{1-\alpha^{p-n}}<3$ for all $k\ge 3$ and $n-p\ge 1$. Fix

$$|\Gamma_2| := \left|1-\frac{L_m^{(k)}\cdot 10^d}{f_k(\alpha)(2\alpha-1)\alpha^{n-1}(1-\alpha^{p-n})}\right| < \frac{18}{\alpha^{n-1}}, \tag{3.8}$$

Suppose $\Gamma_2=0$; then we would have

$$L_m^{(k)}\cdot 10^d = f_k(\alpha)(2\alpha-1)\alpha^{n-1}(1-\alpha^{p-n}).$$

Conjugating both sides by an automorphism $\rho_i:\alpha\mapsto\alpha_i$, for $i\ge 2$, and taking absolute values, we apply (2.5) to get

$$10<\left|L_m^{(k)}\cdot 10^d\right| = |f_k(\alpha)|\cdot|2\alpha-1|\cdot\left|\alpha^{n-1}\right|\cdot\left|1-\alpha^{p-n}\right|<6,$$

which is clearly false. This contradiction implies that $\Gamma_2\neq 0$, so we may now proceed to apply Theorem 2.1 to (3.8). The number field remains $\mathbb{K}=\mathbb{Q}(\alpha)$ with degree $D=k$. and $t:=3$,

$$\begin{aligned}&\eta_1:=\alpha,\quad \eta_2:=10,\quad \eta_3:=\frac{L_m^{(k)}}{f_k(\alpha)(2\alpha-1)(1-\alpha^{p-n})},\\ &b_1:=-(n-1),\quad b_2:=d,\quad b_3:=1.\end{aligned}$$

We compute the logarithmic heights:

$$h(\eta_1)=h(\alpha)=\frac{1}{k}\log\alpha,\qquad h(\eta_2)=\log 10,\qquad A_1=\log\alpha,\qquad A_2=k\log 10.$$

To bound $h(\eta_3)$, we use:

$$\begin{aligned}h(\eta_3)&=h\left(\frac{L_m^{(k)}}{f_k(\alpha)(2\alpha-1)(1-\alpha^{p-n})}\right)\\ &\le h(2\alpha^m)+h(f_k(\alpha))+h(2\alpha-1)+h(1-\alpha^{p-n})\\ &\le \frac{m}{k}\log\alpha+2\log k+\frac{1}{k}\log\alpha+\frac{|p-n|}{k}\log\alpha+4\log 2.\\ &=(m+1+|p-n|)\frac{\log\alpha}{k}+\log k\left(2+\frac{4\log 2}{\log k}\right).\end{aligned}$$

Since $m+1<n-p+3$, then

$$h(\eta_3)<(2|p-n|+3)\frac{\log\alpha}{k}+5\log k,$$

thus

$$A_3=(2|p-n|+3)\log\alpha+5k\log k.$$

Applying Theorem 2.1 to (3.8), we obtain the lower bound

$$\log\Gamma_2>-1.4\cdot 30^6\cdot 3^{4.5}\cdot k^2(1+\log k)(1+\log(n-1))(\log\alpha)(k\log 10)((2|p-n|+3)\log\alpha+5k\log k).$$

Also, from (3.8), we have

$$\log\Gamma_2<\log 18-(n-1)\log\alpha.$$

Comparing both bounds yields

$$(n-1) < 1.4 \times 10^{12} k^3 \log k \log(n-1)((2|p-n|+3)\log\alpha + 5k\log k). \tag{3.9}$$

Now, we examine the cases $p \le m$ and $m < p$ separately.
Case 1: When $p \le m$, then $n-m \le n-p$ and hence we rewrite (3.7) as

$$\begin{aligned} n-p &< 7.2 \cdot 10^9 k^3 \log k \log(n-p+6) \\ &< 7.2 \cdot 10^9 k^3 \log k \log 7(n-p) \\ &< 2.9 \cdot 10^{10} k^3 \log k \log(n-p) \end{aligned}$$

To proceed, we cite this important result, [[23], Lemma 7].

Lemma 3.1. *If $e \ge 1$ and $H > (4e^2)^e$, then*

$$\frac{f}{(\log(f))^e} < H \Rightarrow f < 2^e H(\log(H))^e.$$

We take $e := 1$ and $H := 2.9 \cdot 10^{10} k^3 \log k, f = n-p$ hence by Lemma 3.1 we obtain

$$(n-p) < 1.7 \cdot 10^{12} k^3 (\log k)^2. \tag{3.10}$$

From (3.4), $n < m+p+8 < 2m+8$ and $m < n-p+2$, we find that

$$n < 2(1.7 \cdot 10^{12} k^3 (\log k)^2) + 12,$$

and thus

$$n < 3.5 \cdot 10^{12} k^3 (\log k)^2. \tag{3.11}$$

Case 2: When $m < p$, since $\alpha < k$, we have that

$$(2|p-n|+3)\log\alpha + 5k\log k < 5k\log k + (2|p-n|+3)\log k.$$

From $2|p-n|+3 < 5|p-n|$ for $|p-n| > 0$ and

$$5k\log k + 5|p-n|\log k = |p-n|k\log k\left(\frac{5}{|p-n|} + \frac{5}{k}\right) < 7|p-n|k\log k,$$

(3.9) becomes

$$\begin{aligned} n-1 &< 9.9 \cdot 10^{12} k^4 (\log k)^2 \log(n-1) \cdot |p-n| \\ &< 9.9 \cdot 10^{12} k^4 (\log k)^2 \log(n-1) \cdot 7.2 \cdot 10^9 k^3 \log k \log(n-m+6). \end{aligned}$$

Note that $n-m < n-p < n-1$ and thus we get

$$n-1 < 1.5 \cdot 10^{23} k^7 (\log k)^3 (\log(n-1))^2, \tag{3.12}$$

where we have used the fact that

$$n-1+6 = (n-1)\left(1 + \frac{6}{n-1}\right) < 3(n-1),$$

and

$$\log 3(n-1)) = \log 3 + \log(n-1) = \log(n-1)\left(1 + \frac{\log 3}{\log(n-1)}\right) < 2\log(n-1).$$

Applying Lemma 3.1 to Equation (3.12) yields $n-1 < 2.1 \cdot 10^{27} k^7 (\log k)^5$ and thus

$$n < 2.2 \cdot 10^{27} k^7 (\log k)^5. \tag{3.13}$$

Thus in both cases we can see that the bound in (3.13) is valid. We now analyze the two cases based on the value of k.

3.3.2 The case $k \leq 500$

When $k \leq 500$, it follows from (3.13) that

$$n < 2.2 \cdot 10^{27} k^7 (\log k)^5.$$

We proceed to reduce the upper bound for n. First, we recall (3.6) as,

$$\Gamma_1 = \frac{10^d}{\alpha^{n-m}} - 1 = e^{\Lambda_1} - 1. \tag{3.14}$$

We already showed that $\Gamma_1 \neq 0$, thus $\Lambda_1 \neq 0$. If $(n-p) \geq 6$, we have $|e^{\Lambda_1} - 1| = |\Gamma_1| < 0.5$, which yields $e^{|\Lambda_1|} \leq 1 + |\Gamma_1| < 1.5$. Thus,

$$|\Gamma_1| := \left| 1 - \frac{10^d}{\alpha^{n-m}} \right| < \frac{70}{\alpha^{n-p}}.$$

We have;

$$|d \log 10 - (n-m) \log \alpha| < \frac{106}{\alpha^{n-p}},$$

and hence

$$\left| \frac{\log \alpha}{\log 10} - \frac{d}{n-m} \right| < \frac{106}{\alpha^{n-p} \cdot (n-m) \log 10}.$$

If

$$\frac{106}{(n-m)\alpha^{n-p} \log 10} < \frac{1}{2(n-m)^2},$$

then $\frac{d}{n-m}$ is a convergent of the continued fraction expansion of the irrational number $\frac{\log \alpha}{\log 10}$, say. $\frac{p_i}{q_i}$. Since $q_i \leq n - m \leq n - 1 \leq 2.2 \cdot 10^{27} k^7 (\log k)^5$, for each k we find an upper bound of indices $i = i_0$ and therefore the maximum value of $a_{\max} := \max[a_0, a_1, a_2, \cdots, a_{i_0}]$ of the continued fraction of $\log \alpha / \log 10$. By using a property of continued fractions, we write

$$\frac{1}{(a_{\max}+2)(n-m)^2} \leq \frac{1}{(a_i+2)(n-m)^2} < \left| \frac{\log \alpha}{\log 10} - \frac{d}{n-m} \right| < \frac{106}{\alpha^{n-p} \cdot (n-m) \cdot \log 10}.$$

Thus, we get the inequality

$$\alpha^{n-p} < \frac{106 \cdot (a_{\max}+2) \cdot 2.2 \cdot 10^{27} k^7 (\log k)^5}{\log 10},$$

which gives an upper bound for $(n-p)$ and none of them is greater than 181. If

$$\frac{106}{(n-m)\alpha^{n-p} \log 10} \geq \frac{1}{2(n-m)^2},$$

then we find a more strict bound for $(n-p)$ as

$$\alpha^{n-p} < \frac{106 \cdot 2.2 \cdot 10^{27} k^7 (\log k)^5}{\log 10}.$$

We apply this procedure for each $k \in [3, 500]$ and we find that $(n-p) < 181$. From (3.4), we also find a bound for m as $m < 183$. Since $\max\{m, p\} > 500$, we may assume $m < p$. Fix

$$\Lambda_2 := d \log 10 - (n-1) \log \alpha + \log \left(\frac{L_m^{(k)}}{f_k(\alpha)(2\alpha - 1)(1 - \alpha^{p-n})} \right),$$

for which we already established that $\Gamma_2 \neq 0$ and hence $\Lambda_2 \neq 0$. Assume that $(n-1) > 6$, this implies that

$$\left| (n-1) \log \alpha - d \log 10 + \log \left(\frac{f_k(\alpha)(2\alpha - 1)(1 - \alpha^{p-n})}{L_m^{(k)}} \right) \right| < \frac{28}{\alpha^{n-1}}.$$

For every $k \in [3, 500]$, $(n-p) \in [1, 181]$, $m \in [1, 183]$, we apply the LLL-algorithm to estimate a lower bound for the smallest nonzero value of the linear form, where the integer coefficients are bounded in absolute value by

$$n < 2.2 \cdot 10^{27} k^7 (\log k)^5.$$

To this end, we employ the approximation lattice

$$\mathcal{A}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \lfloor C\log\alpha\rfloor & \lfloor C\log(1/10)\rfloor & \left\lfloor C\log\left(\left(f_k(\alpha)(2\alpha-1)(1-\alpha^{p-n})\right)/L_m^{(k)}\right)\right\rfloor \end{pmatrix},$$

with $C := 5\cdot 10^{150}$ and $y := (0,0,0)$. Applying Lemma 2.3, we obtain

$$l(\mathcal{L}, y) = |\Lambda| > c_1 = 10^{-52} \quad \text{and} \quad \delta = 3.4\cdot 10^{50}.$$

Via Lemma 2.4, we have that $S = 5.1\cdot 10^{100}$ and $T = 2.4\cdot 10^{50}$. Since $\delta^2 \geq T^2 + S$, choosing $c_3 := 28$ and $c_4 := \log\alpha$, we get $n-1 \leq 343$ hence $n \leq 344$. To conclude this case, we conducted a computational search using SageMath, checking all values of $L_n^{(k)}$ for $k \in [3, 500]$, $0 \leq m \leq n-p+4$, and $k+1 \leq n \leq n(k)$, taking into account inequality (3.2) and ensuring that (1.1) is satisfied. We found only the results stated in Theorem 1.1. We now proceed to examine the case $k > 500$.

3.3.3 The case $k > 500$

If $k > 500$, then

$$n < 2.2\cdot 10^{27} k^7 (\log k)^5 < 2^{k/2}. \tag{3.15}$$

We prove the following result.

Lemma 3.2. *Let (p, m, n, k) be a solution to the Diophantine equation (1.1) with $n \geq 5$, $k > 500$ and $n \geq k+1$, then*

$$k < 4.9\cdot 10^{27} \quad \text{and} \quad n < 1.6\cdot 10^{230}$$

Proof. Since $k > 500$, then (3.15) holds and we can use the sharper estimate in (2.8) together with (3.5) and write

$$\left|3\cdot 2^{m-2}\cdot 10^d + 3\cdot 2^{p-2} - 3\cdot 2^{n-2}\right| < 3\cdot 2^{n-2}\cdot \frac{36}{2^{k/2}}.$$

Thus

$$\left|3\cdot 2^{m-2}\cdot 10^d - 3\cdot 2^{n-2}\right| < 3\cdot 2^{n-2}\cdot \frac{36}{2^{k/2}} + \left|3\cdot 2^{p-2}\right|,$$

and hence

$$\left|1 - \frac{10^d}{2^{n-m}}\right| < \frac{36}{2^{k/2}} + 2^{p-n} < \frac{1}{2^{\frac{k}{6}-6}} + \frac{1}{2^{n-p}} < \frac{1}{2^{\min\{(k/2)-7,\, n-p-1\}}}.$$

Therefore,

$$\left|1 - \frac{10^d}{2^{n-m}}\right| < \frac{1}{2^{\min\{(k/2)-7, n-p-1\}}}. \tag{3.16}$$

Fix

$$\Gamma_3 := \left|1 - \frac{10^d}{2^{n-m}}\right| < \frac{1}{2^{\min\{(k/2)-7, n-p-1\}}}.$$

Clearly $\Gamma_3 \neq 0$ otherwise we would have $2^{n-m} = 10^d$. This is a contradiction because the left hand side is divisible by 5 and yet the right hand side is not. We take the algebraic number field $\mathbb{K} := \mathbb{Q}$ with $D = 1, t = 2$,

$$\eta_1 := 10, \quad \eta_2 := 2,$$
$$b_1 := d, \quad b_2 := -(n-m).$$

Since $h(\eta_1) := \log 10$ and $h(\eta_2) := \log 2$, we get $A_1 := \log 10$, $\quad A_2 := \log 2$. Also $B := n-m+6$. Applying Theorem 2.1, we get

$$\log|\Gamma_3| > -1.4\cdot 30^5\cdot 2^{4.5}\cdot 1^2\cdot (1+\log 1)(1+\log(n-m+6))\cdot \log 10\cdot \log 2.$$

Also

$$\log|\Gamma_3| < \log 2^{\min\{(k/2)-7, n-p-1\}},$$

thus

$$\min\{(k/2)-7, n-p-1\}\log 2 < 1.24\cdot 10^{9}(1+\log(n-m+6)).$$

Since $n-m<n$, we get

$$\min\{(k/2)-7, n-p-1\}\log 2 < 3.8\cdot 10^{10}\log k,$$

where we have used the bound in Equation (3.13).
This yields two cases:

(a) If $\min\{k/2-7, n-p-1\} := (k/2)-7$, then $(k/2)-7 < 3.8\cdot 10^{10}\log k$, hence $k < 7.7\cdot 10^{10}\log k$, and by Lemma 3.1, $k < 3.9\cdot 10^{12}$.

(b) If $\min\{(k/2)-7, n-p-1\} := n-p-1$, then $n-p < 3.9\cdot 10^{10}\log k$.

We proceed as in (3.16), beginning with

$$\left|3\cdot 2^{m-2}\cdot 10^{d} + 3\cdot 2^{p-2} - 3\cdot 2^{n-2}\right| < 3\cdot 2^{n-2}\cdot \frac{36}{2^{(k/2)}},$$

which implies

$$\left|3\cdot 2^{n-2}(1-2^{p-n}) - 3\cdot 2^{m-2}\cdot 10^{d}\right| < 3\cdot 2^{n-2}\cdot \frac{36}{2^{(k/2)}},$$

and consequently

$$\left|(1-2^{p-n}) - 2^{m-n}\cdot 10^{d}\right| < \frac{36}{2^{(k/2)}}.$$

This leads to

$$\left|1-\frac{10^{d}}{2^{n-m}\cdot(1-2^{p-n})}\right| < \frac{36}{2^{(k/2)}\cdot(1-2^{p-n})} < \frac{72}{2^{(k/2)}},$$

where we have used the estimate

$$\frac{1}{2^{(k/2)}\cdot(1-2^{p-n})} < \frac{1}{1-\frac{1}{2}} = 2.$$

Thus, we define

$$|\Gamma_4| := \left|1-\frac{10^{d}}{2^{n-m}\cdot(1-2^{p-n})}\right| < \frac{72}{2^{(k/2)}}. \tag{3.17}$$

Exactly the same argument in subsection 3.2 shows that $\Gamma_4 \neq 0$. Fix $\mathbb{K} := \mathbb{Q}$, we have $D := 1, t := 3$,

$$\begin{array}{lll} \eta_1 := 10, & \eta_2 := 2, & \eta_3 := (1-2^{p-n}),\\ b_1 := d, & b_2 := -(n-m), & b_3 := -1. \end{array}$$

The logarithmic heights are given by $h(\eta_1) := \log 10$, $h(\eta_2) := \log 2$, and $h(\eta_3) \le 2.7\cdot 10^{10}\log k$, where we have used the fact that $h(\eta_3) \le |p-n|\log 2 + \log 2 = (|p-n|+1)\log 2 < (3.9\cdot 10^{10}+1)\log 2 < 2.7\cdot 10^{10}\log k$. Furthermore, we define $A_1 := \log 10$, $A_2 := \log 2$, $A_3 := 2.7\cdot 10^{10}\log k$, and $B := (b-m+6) < 7(n-m)$. Applying theorem 2.1 to (3.17), we get

$$\log|\Gamma_4| > -1.4\cdot 30^{6}\cdot 3^{4.5}\cdot 1^{2}\cdot(1+\log 1)(1+\log 7(n-m))\cdot\log 10\cdot\log 2\cdot 2.7\cdot 10^{10}\log k.$$

Also from 3.17

$$\log|\Gamma_4| < \log 72 - (k/2)\log 2.$$

Comparing both bounds yields $(k/2)\log 2 - \log 72 < 1.4\cdot 10^{23}(\log k)^2$, thus $k < 4.1\cdot 10^{23}(\log k)^2$. So by lemma 3.1, we get

$$k < 4.9\cdot 10^{27},$$

and by (3.13) we get

$$n < 1.6\cdot 10^{230}.$$

This completes the proof. □

We now proceed to reduce the bounds obtained in lemma 3.2. To do this, we revisit 3.16 and recall that

$$\Gamma_3 := \frac{10^d}{2^{n-m}} - 1.$$

We already showed that $\Gamma_3 \neq 0$, thus $\Lambda_3 \neq 0$. Also since $\min\{(k/2)-7, n-p-1\} > 2$, $|e^{\Lambda_3}-1| = |\Gamma_3| < 0.5$, which leads to $e^{|\Gamma_3|} \leq 1 + |\Gamma_3| < 1.5$. Therefore

$$|d \log 10 - (n-m)\log 2| < \frac{2}{2^{\min\{(k/2)-7, n-p-1\}}},$$

and hence

$$\left|\frac{\log 2}{\log 10} - \frac{d}{n-m}\right| < \frac{2}{(n-m)\cdot 2^{\min\{(k/2)-7, n-p-1\}}\log 10}.$$

First assume that

$$\frac{2}{(n-m)\cdot 2^{\min\{(k/2)-7, n-p-1\}}\log 10} < \frac{1}{2(n-m)^2},$$

then this means $\frac{d}{n-m}$ is a convergent of continued fractions of $\frac{\log 2}{\log 10}$, say $\frac{p_i}{q_i}$. Since $\gcd(p_i, q_i) = 1$, we deduce that $q_i \leq n-m \leq n-1 < 1.6 \cdot 10^{230}$. A quick calculation shows that $i < 136$. Let $[a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, \cdots] = [0, 3, 3, 9, 2, 2, 4, 6, 2, 1, \cdots]$ be the continued fraction expansion of $\log 2/\log 10$, then $\max\{a_i\} = 5393 \quad \text{for} \quad i = 0, 1, 2, \cdots, 136$. So we write

$$\frac{1}{5393\cdot(n-m)^2} \leq \frac{1}{(a_i+2)(n-m)^2} < \left|\frac{\log 2}{\log 10} - \frac{d}{n-m}\right| < \frac{2}{(n-m)\cdot 2^{\min((k/2)-7, n-p-1)}\cdot \log 10}.$$

Thus we have

$$2^{\min\{(k/2)-7, n-p-1\}} < \frac{2\cdot 5393\cdot 1.6\cdot 10^{230}}{\log 10} < 7.5\cdot 10^{233},$$

that is $\min\{(k/2)-7,\, n-p-1\} < 427$. On the other hand, the inequality

$$\frac{1}{2(n-m)^2} \leq \frac{2}{2^{\min((k/2)-7,\, n-p-1)}\cdot (n-m)\log 10}$$

implies

$$2^{\min\{(k/2)-7,\, n-p-1\}} < \frac{4(n-m)}{\log 10} < \frac{4\cdot 1.6\cdot 10^{230}}{\log 10} < 2.8\cdot 10^{230}.$$

So we see that $\min\{(k/2)-7,\, n-p-1\} < 427$ holds in this case too. We proceed in two ways.

(a) If $\min((k/2)-7,\, n-p-1) = (k/2)-7$, then $k < 869$.

(b) If $\min((k/2)-7,\, n-p-1) = n-p-1$, then $n-p < 428$.

Applying the bound obtained on k to inequality (3.15), we get $n < 1.2\cdot 10^{52}$. We revisit (3.17) and recall that

$$\Gamma_4 := \frac{10^d}{2^{n-m}\cdot(1-2^{p-n})} - 1.$$

Since we have already shown that $\Gamma_4 \neq 0$, it follows that $\Lambda_4 \neq 0$. Moreover, since $k > 500$, we obtain the inequality $|e^{|\Lambda_4|} - 1| = |\Gamma_4| < 0.5$. Hence, we arrive at

$$|(n-p)\log(2) - d\log 10 + (n-m)\log 2| < \frac{108}{2^{k/2}}.$$

We now apply the LLL-algorithm, restricting the absolute values of the integer coefficients to be less than $1.6\cdot 10^{230}$. To do this, we consider the lattice

$$\mathcal{A}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \lfloor C\log(2)\rfloor & \lfloor C\log(1/10)\rfloor & \lfloor C\log 2\rfloor \end{pmatrix},$$

where we set $C := 4.1\cdot 10^{690}$ and take $y := (0,0,0)$ as before. Let $\delta := 3.3\cdot 10^{230}$, $S := 5.12\cdot 10^{460}$, and $T := 2.4\cdot 10^{230}$. We choose $c_3 := 108$ and $c_4 := \log 2$. This yields the inequality $(k/2) < 1538$, and therefore $k < 3076$, hence from (3.13), $n < 2.0\cdot 10^{56}$. We repeat the same procedure starting with this sub-subsection and find that $\min((k/2)-7,\, n-p-1) < 194$.Thus if $\min((k/2)-7,\, n-p-1) =$

$(k/2)-7$, then $k<402$, a contradiction. Also, if $\min((k/2)-7,\, n-p-1)=n-p-1$, then $n-p<195$. We recall

$$\Gamma_4 := \frac{10^d}{2^{n-m}\cdot(1-2^{p-n})}-1.$$

for which we already showed that $\Gamma_4 \neq 0$ and hence we write

$$\left| d\frac{\log 10}{\log 2} - (n-m) - \frac{1-2^{p-n}}{\log 2} \right| < \frac{108}{2^{(k/2)}\log 2}.$$

Applying 2.5, we take

$$\tau := \frac{\log 10}{\log 2}, \mu_{n-p} := \frac{1-2^{p-n}}{\log 2}, A := \frac{\log 108}{\log 2}, B := 2.$$

Let $\tau=[a_0,a_1,\cdots]=[3;3,9,2,2,4,6,2,2]$ be the continued fraction of τ, Let $M:=2\cdot 10^{56}$ which is such that $M>n-1\geq n-m+6>d$. With the help of SageMath, it is found that the convergent

$$\frac{p}{q}=\frac{p_{124}}{q_{124}}=\frac{597091836462299030175097337281893146251396203286949173405470}{179742552941244445968718032243953335920387528504165692302870},$$

is such that $q=q_{124}>6M$. Furthermore, it gives $\epsilon<0.49693$ and thus,

$$\frac{k}{2}\leq\frac{\log(108/\log 2)q/\epsilon}{\log 2}<213.$$

From this, we get $k<426$ which contradicts our initial assumption that $k>500$. Therefore, we conclude that the Diophantine equation (1.1) has no positive integer solutions for $k>500$, which completes the proof.

Conclusion

In this paper, we have completely solved the Diophantine equation $L_n^{(k)}=L_m^{(k)}\cdot 10^d+L_p^{(k)}$ involving concatenations of k-Lucas numbers for all $k\geq 3$. Our results show that such equations admit only finitely many solutions, confirming the rarity of concatenation-type structures in generalized Lucas numbers. This work extends earlier results on classical Lucas numbers and opens new directions for similar investigations in other recurrence sequences such as k-Pell numbers.

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