

FINITE ELEMENT COMPLEXES WITH TRACES STRUCTURES: A UNIFIED FRAMEWORK FOR COHOMOLOGY AND BOUNDED INTERPOLATION

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ABSTRACT. This paper considers the cohomology and bounded interpolation of nonstandard finite element complexes, e.g. Stokes, Hessian, Elasticity, divdiv. Compared to the standard finite element exterior calculus, the main challenge is the existence of extra smoothness. This paper provides a unified framework for finite element complexes with extra smoothness. The trace structure is introduced to derive the bubble complexes in different dimensions (vertices, edges, faces). It is shown that if the bubble complexes in different dimensions are all exact, then the finite element has the correct cohomology. Moreover, the L^2 bounded interpolation can be constructed.

1. INTRODUCTION

Constructing finite element spaces and finite element pairs with certain continuity and differential conditions is a central topic of finite element methods. In past decades, a common way to encode this condition is through the differential complexes. For solving PDEs and simulating physical systems, preserving the de Rham complex (and its cohomology) provides stability, convergence, and structure-preserving properties. This viewpoint has become central in the area of Finite Element Exterior Calculus (FEEC) [4, 1, 3, 30]. The huge success of the Finite element exterior calculus (FEEC) has been witnessed in both mathematics and engineering, see [3] and the references therein.

Recently, more and more discretizations are emerged and developed rapidly, especially in three dimensions. For example, non-standard discretizations of the Stokes complex in three dimensions [43, 40]. Moreover, many other complexes arising in various applications and relating to different differential equations, e.g., hessian complex [32], elasticity complex [16, 13, 33], divdiv complex [14, 34, 35, 33]. Some complexes with the numerical relativity and the Cotton-York operators are also considered [38]. Some generalizations to higher regularity in two and three dimensions are considered in [11, 12]. Unfortunately, in the present stage, no theory can provide a good guide for conforming discretization. Based on Bernstein-Gelfand-Gelfand discretizations, some attempts have been made, see [39, 37, 15]. However, most of discussions around the discrete complexes concentrate on the assumption that the domain Ω is topologically trivial, while a theory dealing with general topology is missed in the literature. An important question is: *are these*

discrete complexes preserving topology on nontrivial domains. For results on finite element de Rham cohomology without boundary conditions, see [4], and for those with homogeneous boundary conditions, see [20]. For results concerning the discrete de Rham complex with partial boundary conditions on domains with non-trivial topology, see [42]. The finite element systems (FES) [19] gives a systematic view relating discrete de Rham complexes to a discrete sheaf theory, yielding the cohomological results. Later, Christiansen and Hu generalized this idea to elasticity complexes in two dimensions [17]. There are two significant properties in the theory of FES: (1) FES requires lots of algebraic structures that fit perfectly at any dimension. (2) FES can only handle the finite elements that has no extra smoothness. As a consequence, the use of FES with non-standard finite element complexes can involve nontrivial difficulties.

This paper proposes a unified framework, called Finite Element Complex with Trace Structures (FECTS), which is established under more relaxed conditions. Specifically, we are interested in two types of applications.

Cohomology: The cohomology of the finite-element complex can be computed using the FECTS. These results signify that the some finite element complexes (those can be fit into this framework) preserve their structure even in the presence of non-trivial topologies, such as those with holes in various dimensions. Moreover, due to the correspondence between harmonic forms and geometric concepts (vertices, edges, and faces), the harmonic form can be readily calculated through simplicial (co)homology. This calculation lays the groundwork for further exploration of topics within the finite element methods, including the development of fast solvers and adaptive techniques. [24].

Locally bounded interpolation: Recently, Arnold and Guzmán provided a local L^2 bounded commuting projection from the Sobolev de Rham complexes to the FEEC complexes, based on the special structures. This projection is of great significance as it essentially encompasses the previous constructions, including the global constructions [45, 20] and $H\Lambda^k$ bounded [25, 27]. They play an important role in the analysis of nonstandard finite element methods [9, 10], especially in the optimal error estimates of the mixed formulation of Hodge-Laplacian [4] and the eigenvalue problem [7].

In this paper, we give a unified framework obtaining the cohomology of the discrete complexes (mainly coming from finite elements). In this framework, we introduce two key tools, which resolves the extra smoothness constraints in FES while addressing the cohomology.

Trace structures. (Section 2.1) We introduce the *trace structure* as a reformulation of the finite elements, which integrates these additional continuity conditions when considering traces. For example, in two dimensions trace structures can be designed to enforce specific data: C^1 data (values and first-order derivatives) on each edge, and C^2 data (including values, first-order derivatives, and second-order

derivatives) at each vertex. This modification, unfortunately, leads to the sacrifice of the traditional trace composition rule, such that $\text{tr} \circ \text{tr} \neq \text{tr}$ any longer. The underlying principle is that when focusing on points located on the edges, normal derivatives cannot be “seen”. Consequently, pure normal second-order derivatives cannot be captured through the composition. Instead, we stipulate that $\text{tr} \circ \text{tr} \subset \text{tr}$. In essence, when examining the composition, only partial trace information can be recorded. Based on the newly defined trace structure, the concept of geometric decomposition and finite element construction can be seamlessly transferred, resulting the finite element spaces with additional regularity.

Based on the trace structures, we propose the *finite element complexes with traces structures* (FECTS). The FECTS can be regarded as a diagram, where for each entry we assign a finite-dimensional space (usually polynomial in this paper). Each column contains a trace structure while each row forms a differential complex. The FECTS assumes some commuting properties between d and tr .

Generalized currents. (Section 2.3) To obtain finer algebraic results, we introduce the concept of the *currents*, which was introduced by de Rham. He constructed a set of $\mathbb{R}$ -valued distributions $\delta_\bullet$. These distributions are associated with geometric concepts in a triangulation in a way that the Stokes formula holds: $\delta_\sigma(dw) = \delta_{\partial\sigma}(w)$.

For various smooth differential complexes (such as the Hessian, elasticity, and divdiv complexes), we define the *generalized currents*. The generalized currents are a family of Hilbert space-valued distributions that satisfy the Stokes formula. For example, in the context of the Hessian complex, if w is a suitable differential form and σ is a simplex in the triangulation, the relationship $\delta_\sigma(dw) = \delta_{\partial\sigma}(w)$ holds for these generalized currents. This set of generalized currents has been used in constructing finite element-distribution complexes, as discussed in [39, 18], where the cohomological results for those complexes were first presented.

The paper is organized as follows. In Section 2 we propose the main results about the FECTS. In Sections 3 and 4, we show the examples in two and three dimensions, respectively. The proofs are shown in Section 5.

2. MAIN RESULTS

In this section, we present the main results regarding the discrete complexes. Throughout this paper, these complexes are based on finite elements, though the result is not restricted to just this case.

This section is organized as follows. First, in Section 2.1, we introduce the trace structure. Finite element spaces and finite element complexes equipped with the trace structure are also presented in this section. In Section 2.2, we show a basic recap on the complexes. Particularly, we introduce the cohomology of the de Rham complex. In Section 2.3, we introduce the definition of the generalized currents, we also give some examples of the generalized currents for some tensor-valued complexes in 2D and 3D. In Section 2.4, we give the definition of the finite

element complexes with trace structure (FECTS). Then the cohomology and the commuting bounded projections can be derived.

We begin by introducing notations, conventions, and assumptions that are frequently used in this paper.

Let $\mathcal{T}$ be a conforming triangulation of a domain $\Omega \subset \mathbb{R}^n$. Denote by $\mathcal{T}_s$ the collection of all s faces (sub-simplices) of $\mathcal{T}$, and $\mathcal{T}_{\leq s}$ the collection of all faces of $\mathcal{T}$ with dimension not greater than s . Generally, we will use Greek letters σ, τ, η to denote a simplex of $\mathcal{T}$. Particularly, denote by $\mathcal{V}, \mathcal{E}, \mathcal{F}$ and $\mathcal{K}$ the set of all vertices, edges, faces, and tetrahedron, respectively.

For each $\sigma \in \mathcal{T}$, let h_σ denote the diameter of $\sigma \in \mathcal{T}$, and denote $h := \max_{\sigma \in \mathcal{T}_n} h_\sigma$ as the mesh size. We say $\tau \trianglelefteq \eta$ if τ is a subsimplex of η , and say $\tau \trianglelefteq_1 \eta$ if τ is a subsimplex of η with codimension 1. For a simplex $\tau \in \mathcal{T}$, let $\text{star}(\tau) := \cup\{\sigma \in \mathcal{T}_n : \tau \trianglelefteq \sigma\}$ and $\text{star}^1(\tau) := \cup\{\sigma \in \mathcal{T}_n : \bar{\tau} \cap \bar{\sigma} \neq \emptyset\} = \cup_{v \trianglelefteq \tau} \text{star}(v)$ and $\text{star}^2(\tau) = \text{star}^1(\text{star}^1(\tau))$. In this paper, we suppose that each patch $\text{star}^1(\tau)$ for $\tau \in \mathcal{T}$ is simply connected, which is satisfied when h is sufficiently small. Note that $\text{star}(\tau)$ is always simply connected, by the fact that $\mathcal{T}$ is conforming.

Throughout this paper, we denote the space of all $n \times n$ matrices by $\mathbb{M}^n$, all symmetric $n \times n$ matrices by $\mathbb{S}^n$, and all trace-free $n \times n$ matrices by $\mathbb{T}^n$. We use the standard notation $L^2(\omega)$ and $H^m(\omega)$ to denote the Sobolev space on domain ω , and the L^2 -inner product and the L^2 -norm are defined by $(\cdot, \cdot)_\omega$ and $\|\cdot\|_{L^2(\omega)}$, respectively. Let $C^\infty(\omega)$ and $P_k(\omega)$ denote the smooth function space and the set of all polynomials of total degree not greater than k on ω , respectively. We also use $L^2(\omega) \otimes \mathbb{X}$, $C^\infty(\omega) \otimes \mathbb{X}$ and $P_k(\omega) \otimes \mathbb{X}$ to denote the tensor-valued spaces on ω , which taking values in the finite dimensional space $\mathbb{X}$. Here $\mathbb{X}$ could be $\mathbb{R}^n, \mathbb{M}^n, \mathbb{S}^n, \mathbb{T}^n$, etc.

Some conventions about continuity are made here. We say a piecewise smooth function u is in $C^1(\mathcal{V})$, if its value and first derivatives are single-valued at all vertices. The notation $C^k(\mathcal{V}), C^k(\mathcal{E})$ and $C^k(\mathcal{F})$, for a given nonnegative integer k , are understood in a similar manner.

2.1. Trace Structure: Algebraic tools for extra smoothness. For each simplex $\tau \in \mathcal{T}$, we define the finite element shape function space $A(\tau)$ as the finite dimensional subspace of $C^\infty(\tau) \otimes \mathbb{A}^{\dim \tau}$, where $\mathbb{A}^{\dim \tau}$ could be $\mathbb{R}^n$ (vectors), $\mathbb{M}^n$ (matrices), $\mathbb{S}^n$ (symmetric matrices), $\mathbb{T}^n$ (traceless matrices), etc. In this paper, all shape function spaces $A(\sigma), \sigma \in \mathcal{T}_n$ of discrete complexex are polynomial function spaces, and therefore a subspace of smooth functions. We now introduce the trace structure of the finite element space.

Definition 2.1 (Trace Structure). *For each $\sigma \in \mathcal{T}_n$ and $\tau \trianglelefteq \sigma$, assume that there exists a linear trace operator*

$$\text{tr}_{\sigma \rightarrow \tau} : A(\sigma) \rightarrow A(\tau).$$

Particularly, denote by $\text{tr}_{\sigma \rightarrow \sigma} = \text{Id}$. For each $\tau \trianglelefteq \sigma$ and $\eta \trianglelefteq \tau$, there also exists a linear trace operator $\text{tr}_{\tau \rightarrow \eta} : A(\tau) \rightarrow A(\eta)$, such that for any $u \in A(\sigma)$,

$$(2.1) \quad \text{tr}_{\sigma \rightarrow \eta} u = 0 \text{ implies } \text{tr}_{\tau \rightarrow \eta} \circ \text{tr}_{\sigma \rightarrow \tau} u = 0.$$

Also, denote by $\text{tr}_{\tau \rightarrow \tau} = \text{Id}$. Then we call (A, tr) a trace structure.

Remark 2.1. In this paper, the shape function space $A(\eta)$ could be the zero-space $\{0\}$. This means that the finite element has no trace structure on η . In this case, set the trace operator $\text{tr}_{\tau \rightarrow \eta} = 0$ for all $\eta \trianglelefteq \tau$.

Remark 2.2. One may expect the composition of traces is again a trace. This is not true in general for the trace systems, as the motivation of introducing the trace systems is exactly to break the composition law. Instead, we do require that the composition of the traces is “included” in the trace. One approach to understand condition (2.1) is shown as follows. Suppose that there exists a linear operator $\rho_{\tau, \eta}$ such that $\rho_{\tau, \eta} \circ \text{tr}_{\sigma \rightarrow \eta} = \text{tr}_{\tau \rightarrow \eta} \circ \text{tr}_{\sigma \rightarrow \tau}$. Then (2.1) holds. In practice, such ρ can be easily found, as differential operators or orthogonal projections.

Now we can define the global space.

Definition 2.2 (Global Space). *The global space with respect to a trace structure (A, tr) is defined as follows:*

$$\mathbf{A} = \{u \in L^2(\Omega) \otimes \mathbb{A}^n : u|_{\sigma} \in A(\sigma), \text{tr}_{\sigma \rightarrow \tau} u|_{\sigma} = \text{tr}_{\sigma' \rightarrow \tau} u|_{\sigma'}, \text{ for all } \tau \trianglelefteq \sigma, \sigma' \text{ and } \sigma, \sigma' \in \mathcal{T}_n\}.$$

The space $\mathbf{A}$ collects the discrete functions with single-valued traces, reflecting certain continuity. In the following text, we will always use bold letter $\mathbf{A}^\bullet$ to denote the global space. We use two examples to illustrate what the trace structure is, and how it reflects a certain finite element.

Example 2.1. Consider the following trace structure in two dimensions:

- $A(f) := P_k(f)$, $A(e) = P_k(e)$, and $A(x) = \mathbb{R} \oplus \mathbb{R}^2$ for all $f \in \mathcal{F}$, $e \in \mathcal{E}$ and $x \in \mathcal{V}$.
- $\text{tr}_{f \rightarrow e}(u) = u|_e$, $\text{tr}_{f \rightarrow x}(u) = u(x) \oplus \nabla u(x)$ for all edges e on f and vertices x of f .
- $\text{tr}_{e \rightarrow x}(q) = q(x) \oplus \frac{d}{dt} q(x) \mathbf{t}$ for all vertices x of e .

Here, $\mathbf{t}$ is the unit tangential vector of e . It is easy to see that for each $u \in \mathbf{A}$, $u \in C^1(\mathcal{V}) \cap C^0(\mathcal{E})$, which means that the global space is the space of the Hermite finite element. See Figure 1 for an illustration.

Example 2.2. Consider the following trace structure in two dimensions:

- $A(f) := P_k(f)$, $A(e) = P_k(e) \oplus P_{k-1}(e)$, $A(x) = \mathbb{R} \oplus \mathbb{R}^2 \oplus \mathbb{M}^2$ for all $f \in \mathcal{F}$, $e \in \mathcal{E}$ and $x \in \mathcal{V}$.
- $\text{tr}_{f \rightarrow e}(u) = u|_e \oplus (\frac{\partial u}{\partial \mathbf{n}})|_e$, $\text{tr}_{f \rightarrow x}(u) = u(x) \oplus \nabla u(x) \oplus \nabla^2 u(x)$ for all edges e on f and vertices x of f .

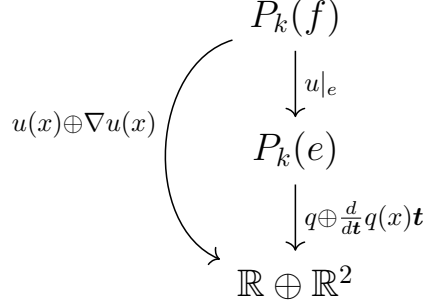


FIGURE 1. The trace structure of Hermite element.

- $\text{tr}_{e \rightarrow x}(u_1, u_2) = u_1(x) \oplus (u_2(x)\mathbf{n} + \frac{d}{dt}u_1(x)\mathbf{t}) \oplus (\frac{d}{dt}u_2(x)\mathbf{n}\mathbf{t}^T + \frac{d^2}{dt^2}u_1(x)\mathbf{t}\mathbf{t}^T)$ for all vertices x of e .

Here, $\mathbf{t}$ and $\mathbf{n}$ are the unit tangential vector and normal vector of e , respectively. The global space is the space of the Argyris finite element.

To give a better picture relating the classical constructions of finite element spaces, let us introduce the definition of local bubble functions (also known as trace-free functions). Given $\sigma \in \mathcal{T}_n$, then for any $\tau \trianglelefteq \sigma$, the local bubble functions on each $\tau \in \mathcal{T}$ collects the functions that contribute nothing to any subsimplices of τ .

Definition 2.3 (Local Bubble Functions). *Define the bubble space $B(\tau)$ as*

$$B(\tau) := \{a \in \text{tr}_{\sigma \rightarrow \tau} A(\sigma) : \text{tr}_{\tau \rightarrow \eta} a = 0 \text{ for all } \eta \trianglelefteq \tau \text{ and } \eta \neq \tau\}.$$

Note that for any $u \in \mathbf{A}$, we have $\text{tr}_{\sigma \rightarrow \tau} u|_{\sigma} = \text{tr}_{\sigma' \rightarrow \tau} u|_{\sigma'}$ for all $\tau \trianglelefteq \sigma, \sigma'$ and $\sigma, \sigma' \in \mathcal{T}_n$. So the definitions of bubble spaces here will not be ambiguous in the following text.

Example 2.3. The bubble space of the Hermite element is:

- $B(v) = \mathbb{R} \oplus \mathbb{R}^2$, corresponding to the value and the gradient at vertex v of the triangle f .
- $B(e) = \{p \in P_k(e) : p(x) = 0, \frac{\partial p}{\partial \mathbf{t}} = 0 \text{ for all end points } x \text{ of } e\} = \lambda_e^2 P_{k-4}(e)$ for the edge e of f .
- $B(f) = \{p \in P_k(f) : p|_{\partial f} = 0\} = \lambda_f P_{k-3}(f)$.

Example 2.4. The bubble space of the Argyris element is:

- $B(v) = \mathbb{R} \oplus \mathbb{R}^2 \oplus \mathbb{S}^2$ for the vertex v of the f .
- $B(e) = \lambda_e^3 P_{k-6}(e) \oplus \lambda_e^2 P_{k-5}(e)$ for the edge e of f .
- $B(f) = \lambda_f^2 P_{k-6}(f)$.

Here λ_e and λ_f denote the standard bubble polynomial function on edge e and face f , respectively.

In this paper, all the global finite element spaces will satisfy the following geometric decomposition:

Definition 2.4 (Geometric Decomposition). *We say the trace structure satisfies the geometric decomposition, if*

$$\sum_{\tau \trianglelefteq \sigma} \dim B(\tau) = \dim A(\sigma), \quad \forall \sigma \in \mathcal{T}_n.$$

The above definition leads to the following finite element construction.

Proposition 2.1 (Finite element from the trace structure). *Let (A, tr) be a trace structure satisfying geometric decomposition. For $\sigma \in \mathcal{T}_n$ with $\tau \trianglelefteq \sigma$, let $A(\sigma)$ denotes the shape function space over σ , $B(\tau)$ be the bubble spaces, $B(\tau)^\vee$ be the corresponding dual space defined by the trace structure. Then, for $u \in A(\sigma)$, the local degrees of freedom are as follows: $L(\text{tr}_{\sigma \rightarrow \tau} u)$ for any $L \in B(\tau)^\vee$. And the degrees of freedom are unisolvent for $A(\sigma)$.*

Proof. Note that

$$\sum_{\tau \trianglelefteq \sigma} \dim B(\tau)^\vee = \sum_{\tau \trianglelefteq \sigma} \dim B(\tau) = \dim A(\sigma), \quad \forall \sigma \in \mathcal{T}_n.$$

It suffices to prove that for any $u \in A(\sigma)$, if

$$L(\text{tr}_{\sigma \rightarrow \tau} u) = 0, \quad \forall \tau \trianglelefteq \sigma, L \in B(\tau)^\vee,$$

then $u = 0$.

We first show that for any $\tau \trianglelefteq \sigma$ and $\tau \neq \sigma$, it holds that $\text{tr}_{\sigma \rightarrow \tau} u = 0$. If $\tau \in \mathcal{T}_0$, it is easy to see that $\text{tr}_{\sigma \rightarrow \tau} u = 0$. Next we assume that for any $0 \leq k \leq n-2$, $\tau \in \mathcal{T}_k$, we have $\text{tr}_{\sigma \rightarrow \tau} u = 0$. Then for any $\tau \in \mathcal{T}_{k+1}$ with $\tau \trianglelefteq \sigma$, note that for any $\eta \trianglelefteq \tau$ and $\eta \neq \tau$, it holds that

$$\text{tr}_{\tau \rightarrow \eta} \circ \text{tr}_{\sigma \rightarrow \tau} u = 0.$$

Hence $\text{tr}_{\sigma \rightarrow \tau} u \in B(\tau)$, then from $L(\text{tr}_{\sigma \rightarrow \tau} u) = 0, \quad \forall L \in B(\tau)^\vee$, we can get $\text{tr}_{\sigma \rightarrow \tau} u = 0$. The argument above shows that $u \in B(\sigma)$. Finally, since $L(u) = 0$ for all $L \in B(\sigma)^\vee$, it holds that $u = 0$, which completes the proof. $\square$

2.2. Recap on Complexes: de Rham and beyond. This section gives a basic recap on the complexes in finite element methods, and the readers can refer to [3] for more details. A (smooth) differential complex has the form

$$0 \longrightarrow C^\infty(\Omega) \otimes \mathbb{X}^0 \xrightarrow{d} C^\infty(\Omega) \otimes \mathbb{X}^1 \xrightarrow{d} \cdots \xrightarrow{d} C^\infty(\Omega) \otimes \mathbb{X}^n \longrightarrow 0.$$

Here $\mathbb{X}^i$ is a linear space and can be $\mathbb{R}^n, \mathbb{M}^n, \mathbb{S}^n, \mathbb{T}^n$, etc. The operator d is the linear differential operator. Denote by M the maximum order of the differential operators. In the examples considered in this paper, d will be some first or second

order operator. The complex requires $d \circ d = 0$, and therefore the cohomology can be defined:

$$\mathcal{H}^k = \frac{\ker(d : C^\infty(\Omega) \otimes \mathbb{X}^k \rightarrow C^\infty(\Omega) \otimes \mathbb{X}^{k+1})}{\operatorname{im}(d : C^\infty(\Omega) \otimes \mathbb{X}^{k-1} \rightarrow C^\infty(\Omega) \otimes \mathbb{X}^k)}.$$

In this paper, we will consider L^2 bounded projections, which needs the following L^2 complex.

$$0 \longrightarrow L^2(\Omega) \otimes \mathbb{X}^0 \xrightarrow{d} L^2(\Omega) \otimes \mathbb{X}^1 \xrightarrow{d} \dots \xrightarrow{d} L^2(\Omega) \otimes \mathbb{X}^n \longrightarrow 0.$$

Note here all the differential operators are considered as unbounded operators.

We also introduce the adjoint complex, which takes the adjoint operator of each d in the L^2 norm.

$$0 \longrightarrow L^2(\Omega) \otimes \mathbb{X}^n \xrightarrow{d^*} L^2(\Omega) \otimes \mathbb{X}^{n-1} \xrightarrow{d^*} \dots \xrightarrow{d^*} L^2(\Omega) \otimes \mathbb{X}^0 \longrightarrow 0.$$

Specifically, the formal adjoint operator $d^* : L^2(\Omega) \otimes \mathbb{X}^k \rightarrow L^2(\Omega) \otimes \mathbb{X}^{k-1}$ is the unbounded operator satisfying that

$$(d^*a, b)_\Omega = (a, db)_\Omega, \quad \forall b \in \mathcal{D}(d)$$

here $\mathcal{D}(d)$ is the domain of d . Next we introduce smooth complexes considered in this paper. The most important complex is the de Rham complex, for clarity here we only consider the vector proxy version.

In two dimensions, the de Rham complex reads as

$$0 \longrightarrow C^\infty(\Omega) \xrightarrow{\operatorname{grad}} C^\infty(\Omega) \otimes \mathbb{R}^2 \xrightarrow{\operatorname{rot}} C^\infty(\Omega) \longrightarrow 0,$$

while in three dimensions, it reads as

$$0 \longrightarrow C^\infty(\Omega) \xrightarrow{\operatorname{grad}} C^\infty(\Omega) \otimes \mathbb{R}^3 \xrightarrow{\operatorname{curl}} C^\infty(\Omega) \otimes \mathbb{R}^3 \xrightarrow{\operatorname{div}} C^\infty(\Omega) \longrightarrow 0.$$

The cohomology determined by the de Rham complex is called de Rham cohomology, denoted as $\mathcal{H}_{dR}(\Omega)$. de Rham first proved that the de Rham cohomology is dual to the simplicial homology, where he proposed the idea of *currents* [23]. A current of a k simplex is defined as a k -form valued distribution sending the test function to the trace with respect to this simplex. Based on the standard vector proxies, we define the following currents in three dimensions by associating

- (1) each vertex a linear functional $\delta_x \in (C^\infty(\Omega))' : u \mapsto u(x)$;
- (2) each edge with the unit tangential vector $\mathbf{t}$ a linear functional $\delta_e \in (C^\infty(\Omega) \otimes \mathbb{R}^3)' : \mathbf{u} \mapsto \int_e \mathbf{u} \cdot \mathbf{t}$;
- (3) each face with the unit normal vector $\mathbf{n}$ a linear functional $\delta_f \in (C^\infty(\Omega) \otimes \mathbb{R}^3)' : \mathbf{u} \mapsto \int_f \mathbf{u} \cdot \mathbf{n}$;
- (4) each cell a linear functional $\delta_K \in (C^\infty(\Omega))' : u \mapsto \int_K u$.

By using the currents, de Rham established the duality between the simplicial homology and the de Rham cohomology, which will be also used in this paper. The key insight of his proof is based on the following Stokes' formula: for any $\sigma \in \mathcal{T}_m$ and $w \in C^\infty(\Omega) \otimes \mathbb{X}^{m-1}$, it holds that

$$\delta_\sigma(dw) = \delta_{\partial\sigma}(w) := \sum_{\tau \trianglelefteq_1 \sigma} \mathcal{O}(\tau, \sigma) \delta_\tau(w).$$

Here, the orientation is standardly defined as in [29]: for two oriented simplices τ and $\sigma = [v_0, \dots, v_s]$, $\mathcal{O}(\tau, \sigma) = (-1)^j$ if $\tau = [v_0, \dots, \widehat{v_j}, v_s]$ ($\widehat{v_j}$ means deleting v_j), and $\mathcal{O}(\tau, \sigma) = 0$ otherwise.

2.3. Generalized Currents. There are also some other complexes, whose cohomology can be related to de Rham cohomology. The procedure that produces such a complex is called Bernstein-Gelfand-Gelfand reduction, see [5] for more details. Specifically, in this paper, the following complexes will be considered as examples. The definitions of the differential operators can be found in Section 3 and 4. In two dimensions, we have the hessian complex:

$$0 \longrightarrow C^\infty(\Omega) \xrightarrow{\text{hess}} C^\infty(\Omega) \otimes \mathbb{S}^2 \xrightarrow{\text{rot}} C^\infty(\Omega) \otimes \mathbb{R}^2 \longrightarrow 0,$$

whose cohomology is isomorphic to $P_1(\Omega) \otimes \mathcal{H}_{dR}(\Omega)$, and the divdiv-complex

$$0 \longrightarrow C^\infty(\Omega) \otimes \mathbb{R}^2 \xrightarrow{\text{sym curl}} C^\infty(\Omega) \otimes \mathbb{S}^2 \xrightarrow{\text{div div}} C^\infty(\Omega) \longrightarrow 0,$$

whose cohomology is isomorphic to $\mathcal{RT}^2 \otimes \mathcal{H}_{dR}(\Omega)$. Here $\mathcal{RT}^2 := \{\mathbf{a} + b\mathbf{x}, \mathbf{a} \in \mathbb{R}^2, b \in \mathbb{R}\}$, whose dimension is 3.

In three dimensions, we have the hessian complex

$$0 \longrightarrow C^\infty(\Omega) \xrightarrow{\text{hess}} C^\infty(\Omega) \otimes \mathbb{S}^3 \xrightarrow{\text{curl}} C^\infty(\Omega) \otimes \mathbb{T}^3 \xrightarrow{\text{div}} C^\infty(\Omega) \otimes \mathbb{R}^3 \longrightarrow 0,$$

whose cohomology is isomorphic to $P_1(\Omega) \otimes \mathcal{H}_{dR}(\Omega)$, and the divdiv complex

$$0 \longrightarrow C^\infty(\Omega) \otimes \mathbb{R}^3 \xrightarrow{\text{dev grad}} C^\infty(\Omega) \otimes \mathbb{T}^3 \xrightarrow{\text{sym curl}} C^\infty(\Omega) \otimes \mathbb{S}^3 \xrightarrow{\text{div div}} C^\infty(\Omega) \longrightarrow 0,$$

whose cohomology is isomorphic to $\mathcal{RT}^3 \otimes \mathcal{H}_{dR}(\Omega)$, and the elasticity complex

$$0 \longrightarrow C^\infty(\Omega) \otimes \mathbb{R}^3 \xrightarrow{\text{sym grad}} C^\infty(\Omega) \otimes \mathbb{S}^3 \xrightarrow{\text{inc}} C^\infty(\Omega) \otimes \mathbb{S}^3 \xrightarrow{\text{div}} C^\infty(\Omega) \otimes \mathbb{R}^3 \longrightarrow 0,$$

whose cohomology is isomorphic to $\mathcal{RM}^3 \otimes \mathcal{H}_{dR}(\Omega)$. Here $\mathcal{RT}^3 := \{\mathbf{a} + b\mathbf{x}, \mathbf{a} \in \mathbb{R}^3, b \in \mathbb{R}\}$, whose dimension is 4. And $\mathcal{RM}^3 := \{\mathbf{a} + \mathbf{b} \times \mathbf{x}, \mathbf{a} \in \mathbb{R}^3, \mathbf{b} \in \mathbb{R}^3\}$, whose dimension is 6.

Definition 2.5 (Generalized Currents). *Given a finite dimensional Hilbert space $\mathcal{H}$. Suppose that for the complex*

(2.2)

$$0 \longrightarrow C^\infty(\Omega) \otimes \mathbb{X}^0 \xrightarrow{d} C^\infty(\Omega) \otimes \mathbb{X}^1 \cdots \xrightarrow{d} C^\infty(\Omega) \otimes \mathbb{X}^n \longrightarrow 0,$$

and each $\sigma \in \mathcal{T}_m$, there exists a linear surjective mapping $\Upsilon_\sigma : C^\infty(\Omega) \otimes \mathbb{X}^m \rightarrow \mathcal{H}$, such that the generalized Stokes' formula holds for $w \in C^\infty(\Omega) \otimes \mathbb{X}^{m-1}$:

$$\Upsilon_\sigma(dw) = \Upsilon_{\partial\sigma}(w) = \sum_{\tau \trianglelefteq_1 \sigma} \mathcal{O}(\tau, \sigma) \Upsilon_\tau(w).$$

We then call $(\Upsilon_\bullet, \mathcal{H})$ a family of generalized currents for (2.2).

Next, we introduce some examples of tensor complexes (e.g., hessian, elasticity, divdiv). Note that the cohomolgy of each complex is isomorphic to $\mathcal{Z} \otimes \mathcal{H}_{dR}(\Omega)$, where $\mathcal{Z}$ is a finite-dimensional Hilbert space on Ω with normalized orthogonal basis $\{z_i\}_{1 \leq i \leq \dim \mathcal{Z}}$.

2.3.1. Two-dimensional Results. First, we introduce the generalized currents for 2D Hessian complex. Note that now $\mathcal{Z} = P_1(\Omega)$.

Lemma 2.1 (Generalized currents for 2D Hessian). *Let ψ_1, ψ_2, ψ_3 be a basis of $\mathcal{RT}^2 = \{\mathbf{a} + b\mathbf{x} : \mathbf{a} \in \mathbb{R}^2, b \in \mathbb{R}\}$. Then set*

$$\Upsilon_x : C^\infty(\bar{\Omega}) \rightarrow \mathcal{Z}, \quad \Upsilon_x(u) = \sum_{i=1}^3 ((\psi_i \cdot \nabla u)(x) - \frac{1}{2}(\operatorname{div} \psi_i \cdot u)(x)) z_i,$$

$$\Upsilon_e : C^\infty(\bar{\Omega}) \otimes \mathbb{S}^2 \rightarrow \mathcal{Z}, \quad \Upsilon_e(\boldsymbol{\sigma}) = \left(\int_e (\boldsymbol{\sigma} \mathbf{t}) \cdot \psi_i \right) z_i,$$

and

$$\Upsilon_f : C^\infty(\bar{\Omega}) \otimes \mathbb{R}^2 \rightarrow \mathcal{Z}, \quad \Upsilon_f(\mathbf{p}) = \left(\int_f \mathbf{p} \cdot \psi_i \right) z_i,$$

for each $x \in \mathcal{V}, e \in \mathcal{E}$ and $f \in \mathcal{F}$. Then, $(\Upsilon_\bullet, \mathcal{Z})$ is the generalized current.

Proof. By direct integrations by parts. $\square$

Next, we introduce the generalized currents for 2D divdiv complex. Here $\mathcal{Z} = \mathcal{RT}^2 = \{\mathbf{a} + b\mathbf{x} : \mathbf{a} \in \mathbb{R}^2, b \in \mathbb{R}\}$, then we can introduce the following generalized currents.

Lemma 2.2 (Generalized Currents for 2D divdiv). *Let ψ_1, ψ_2, ψ_3 be a basis of $P_1(\Omega)$. Set*

$$\Upsilon_x : C^\infty(\bar{\Omega}) \otimes \mathbb{R}^2 \rightarrow \mathcal{Z}, \quad \Upsilon_x(\mathbf{u}) = \sum_{i=1}^3 \left(\frac{1}{2}(\operatorname{div} \mathbf{u} \psi_i)(x) - (\mathbf{u} \cdot \nabla \psi_i)(x) \right) z_i,$$

$$\Upsilon_e : C^\infty(\bar{\Omega}) \otimes \mathbb{S}^2 \rightarrow \mathcal{Z}, \quad \Upsilon_e(\boldsymbol{\sigma}) = \sum_{i=1}^3 \left(\int_e (\operatorname{div} \boldsymbol{\sigma} \cdot \mathbf{n}) \psi_i - (\boldsymbol{\sigma} \mathbf{n} \cdot \nabla \psi_i) \right) z_i,$$

$$\Upsilon_f : C^\infty(\bar{\Omega}) \rightarrow \mathcal{Z}, \quad \Upsilon_f(q) = \sum_{i=1}^3 \left(\int_f q \psi_i \right) z_i,$$

for each $x \in \mathcal{V}, e \in \mathcal{E}$ and $f \in \mathcal{F}$. Then, $(\Upsilon_\bullet, \mathcal{Z})$ is the generalized current.

Proof. See [39, Lemma 4.2 and 4.4]. $\square$

Remark 2.3. The generalized currents were used to construct distributional Hessian complex, as described in [39]. The construction comes from the duality of Hessian complex and divdiv complex.

2.3.2. Three-dimensional Results. Here, we introduce the generalized currents of tensor complexes in three dimensions. We first introduce the generalized currents for Hessian complex in three dimensions.

Lemma 2.3 (Generalized currents for 3D Hessian). *Note that $\mathcal{Z} = P_1(\Omega)$, let $\{\psi_i\}_{1 \leq i \leq 4}$ be a basis of $\mathcal{RT}^3 = \{\mathbf{a} + b\mathbf{x} : \mathbf{a} \in \mathbb{R}^3, b \in \mathbb{R}\}$. Set*

$$\Upsilon_x : C^\infty(\bar{\Omega}) \rightarrow \mathcal{Z}, \quad \Upsilon_x(u) = \sum_{i=1}^4 ((\psi_i \cdot \nabla u)(x) - \frac{1}{3}(\operatorname{div} \psi_i \cdot u)(x)) z_i,$$

$$\Upsilon_e : C^\infty(\bar{\Omega}) \otimes \mathbb{S}^3 \rightarrow \mathcal{Z}, \quad \Upsilon_e(\boldsymbol{\sigma}) = \sum_{i=1}^4 \left(\int_e (\boldsymbol{\sigma} \mathbf{t}) \cdot \psi_i \right) z_i,$$

$$\Upsilon_f : C^\infty(\bar{\Omega}) \otimes \mathbb{T}^3 \rightarrow \mathcal{Z}, \quad \Upsilon_f(\boldsymbol{\tau}) = \sum_{i=1}^4 \left(\int_f (\boldsymbol{\tau} \mathbf{n}) \cdot \psi_i \right) z_i,$$

$$\Upsilon_T : C^\infty(\bar{\Omega}) \otimes \mathbb{R}^3 \rightarrow \mathcal{Z}, \quad \Upsilon_T(\mathbf{q}) = \sum_{i=1}^4 \left(\int_T \mathbf{q} \cdot \psi_i \right) z_i.$$

for any $x \in \mathcal{V}, e \in \mathcal{E}, f \in \mathcal{F}$ and $T \in \mathcal{K}$. Then, $(\Upsilon_\bullet, \mathcal{Z})$ is the generalized currents.

Proof. Integration by parts. $\square$

Lemma 2.4 (Generalized currents for 3D divdiv). *Note that $\mathcal{Z} = \mathcal{RT}^3 = \{\mathbf{a} + b\mathbf{x}, \mathbf{a} \in \mathbb{R}^3, b \in \mathbb{R}\}$, and let $\{\psi_i\}_{1 \leq i \leq 4}$ be a basis of $P_1(\Omega)$. Set*

$$\Upsilon_x : C^\infty(\bar{\Omega}) \otimes \mathbb{R}^3 \rightarrow \mathcal{Z}, \quad \Upsilon_x(\mathbf{u}) = \sum_{i=1}^4 \left(\frac{1}{3}(\psi_i \cdot \operatorname{div} \mathbf{u})(x) - (\nabla \psi_i \cdot \mathbf{u})(x) \right) z_i,$$

$$\Upsilon_e : C^\infty(\bar{\Omega}) \otimes \mathbb{T}^3 \rightarrow \mathcal{Z}, \quad \Upsilon_e(\boldsymbol{\tau}) = \sum_{i=1}^4 \left(\int_e \frac{1}{2}(\operatorname{div} \boldsymbol{\tau} \cdot \mathbf{t}) \cdot \psi_i - (\boldsymbol{\tau} \mathbf{t}) \cdot \nabla \psi_i \right) z_i,$$

$$\Upsilon_f : C^\infty(\bar{\Omega}) \otimes \mathbb{S}^3 \rightarrow \mathcal{Z}, \quad \Upsilon_f(\boldsymbol{\sigma}) = \sum_{i=1}^4 \left(\int_f (\operatorname{div} \boldsymbol{\sigma} \cdot \mathbf{n}) \cdot \psi_i - (\boldsymbol{\sigma} \mathbf{n}) \cdot \nabla \psi_i \right) z_i,$$

$$\Upsilon_T : C^\infty(\bar{\Omega}) \rightarrow \mathcal{Z}, \quad \Upsilon_T(q) = \sum_{i=1}^4 \left(\int_T q \cdot \psi_i \right) z_i.$$

for any $x \in \mathcal{V}, e \in \mathcal{E}, f \in \mathcal{F}$ and $T \in \mathcal{K}$. Then, $(\Upsilon_\bullet, \mathcal{Z})$ is the generalized currents.

Proof. See [39, Proposition 5.3-5.5]. $\square$

Lemma 2.5 (Generalized currents for 3D Elasticity). *Note that $\mathcal{Z} = \mathcal{RM}^3 = \{\mathbf{a} + \mathbf{b} \times \mathbf{x}, \mathbf{a} \in \mathbb{R}^3, \mathbf{b} \in \mathbb{R}^3\}$, and let $\{\psi_i\}_{1 \leq i \leq 6}$ be a basis of $\mathcal{RM}^3$. Set*

$$\Upsilon_x : C^\infty(\bar{\Omega}) \otimes \mathbb{R}^3 \rightarrow \mathcal{Z}, \quad \Upsilon_x(\mathbf{u}) = \sum_{i=1}^6 \left(\frac{1}{2}(\psi_i \cdot \operatorname{curl} \mathbf{u})(x) + \frac{1}{2}(\operatorname{curl} \psi_i \cdot \mathbf{u})(x) \right) z_i,$$

$$\Upsilon_e : C^\infty(\bar{\Omega}) \otimes \mathbb{S}^3 \rightarrow \mathcal{Z}, \quad \Upsilon_e(\boldsymbol{\sigma}) = \sum_{i=1}^6 \left(\int_e \left(\frac{1}{2} \boldsymbol{\sigma} \mathbf{t} \right) \cdot \operatorname{curl} \psi_i + ((\operatorname{curl} \boldsymbol{\sigma}) \mathbf{t}) \cdot \psi_i \right) z_i,$$

$$\Upsilon_f : C^\infty(\bar{\Omega}) \otimes \mathbb{S}^3 \rightarrow \mathcal{Z}, \quad \Upsilon_f(\boldsymbol{\tau}) = \sum_{i=1}^6 \left(\int_f (\boldsymbol{\tau} \mathbf{n}) \cdot \psi_i \right) z_i,$$

$$\Upsilon_T : C^\infty(\bar{\Omega}) \otimes \mathbb{R}^3 \rightarrow \mathcal{Z}, \quad \Upsilon_T(\mathbf{q}) = \sum_{i=1}^6 \left(\int_T \mathbf{q} \cdot \psi_i \right) z_i.$$

for any $x \in \mathcal{V}, e \in \mathcal{E}, f \in \mathcal{F}$ and $T \in \mathcal{K}$. Then $(\Upsilon_\bullet, \mathcal{Z})$ is the generalized currents.

Proof. See [18, Section 5.1]. $\square$

2.4. FECTS: Cohomologies and Commuting Projections. This subsection is devoted to finer structures of the FECTS. Firstly, we give the precise definition of the finite element complexes with trace structure.

Definition 2.6. *We call a sequence of finite element spaces $\mathbf{A}^k$ with the trace structure $(A^k, \operatorname{tr}^k)$ for $0 \leq k \leq n$ form a conforming finite element subcomplex with the trace structure (FECTS) of*

$$0 \longrightarrow L^2(\Omega) \otimes \mathbb{X}^0 \xrightarrow{d^0} L^2(\Omega) \otimes \mathbb{X}^1 \xrightarrow{d^1} \dots \xrightarrow{d^{n-1}} L^2(\Omega) \otimes \mathbb{X}^n \longrightarrow 0.$$

if they satisfy that

- (1) $\mathbf{A}^k \subset L^2(\Omega) \otimes \mathbb{X}^k$ and for any $u \in \mathbf{A}^k$, we have that $d^k u \in \mathbf{A}^{k+1}$.
- (2) For any simplex $\tau \in \mathcal{T}$, there exists a differential operator $d_\tau^k : A^k(\tau) \rightarrow A^{k+1}(\tau)$ such that $d_\tau^{k+1} \circ d_\tau^k = 0$ and for any $\sigma \in \mathcal{T}_n$ and $\tau \trianglelefteq \sigma$, there holds the following commuting property

$$d_\tau^k \operatorname{tr}_{\sigma \rightarrow \tau}^k u = \operatorname{tr}_{\sigma \rightarrow \tau}^{k+1} d_\sigma^k u, \quad \forall u \in A^k(\sigma), 0 \leq k \leq n-1.$$

Moreover, we have that

$$d_\sigma^k = d^k|_\sigma, \quad 0 \leq k \leq n-1, \sigma \in \mathcal{T}_n.$$

Remark 2.4. In the following text, we will abbreviate d_τ^k as d and $\operatorname{tr}_{\sigma \rightarrow \tau}^k$ as $\operatorname{tr}_{\sigma \rightarrow \tau}$ without causing confusion.

Remark 2.5. Note that we only need d commutes with the trace operator tr of the top dimension. In contrast, FES [21] need commuting properties at all levels.

We suppose that there exists a finite dimensional Hilbert space $\mathcal{Z}$ on Ω with L^2 - inner product, such that for any contractible domain $\omega \subset \Omega$, the following complex is exact

$$\mathcal{Z}(\omega) \xrightarrow{\subset} C^\infty(\omega) \otimes \mathbb{X}^0 \xrightarrow{d^0} \dots \xrightarrow{d^{n-1}} C^\infty(\omega) \otimes \mathbb{X}^n \longrightarrow 0,$$

here $\mathcal{Z}(\omega)$ is the restriction of $\mathcal{Z}$ on ω . The exactness of the above-mentioned complexes (i.e., de Rham, hessian, divdiv, elasticity) can be found in [5]. Moreover, we assume that the following dual complex is also exact:

$$0 \xrightarrow{\subset} H_0^M(\omega) \otimes \mathbb{X}^n \xrightarrow{(d^{n-1})^*} \dots \xrightarrow{(d^0)^*} H_0^M(\omega) \otimes \mathbb{X}^0 \cap \mathcal{Z}(\omega)^\perp \longrightarrow 0,$$

recall that M is the maximum order of d^k .

Definition 2.7 (Compatibility). *We say that the FECTS (A, tr, d) is compatible with respect to the generalized currents $\{\Upsilon_\bullet, \mathcal{Z}\}$ if the following condition holds:*

- (1) *The generalized currents can be localized via the trace systems. That is, for each simplex τ , there exists $\tilde{\Upsilon}_\tau : A^{\dim \tau}(\tau) \rightarrow \mathcal{Z}$ such that*

$$\tilde{\Upsilon}_\tau(\text{tr}_{\sigma \rightarrow \tau} w) = \Upsilon_\tau(w) \text{ for any } w \in A^{\dim \tau}(\sigma), \sigma \in \mathcal{T}_n, \text{ and } \tau \trianglelefteq \sigma.$$

Recall that $A^k(\sigma)$ for any $\sigma \in \mathcal{T}_n, 0 \leq k \leq n$ is polynomial function space, therefore the right-hand side is well-defined.

- (2) *It holds that $\mathcal{Z} \subset \mathbf{A}^0$. And the following sequence of bubble space*

$$(2.3) \quad 0 \longrightarrow B^0(\tau) \xrightarrow{d} \dots \xrightarrow{d} B^{\dim(\tau)-1}(\tau) \xrightarrow{d} B^{\dim \tau}(\tau) \cap \ker(\tilde{\Upsilon}_\tau) \xrightarrow{d} \dots \xrightarrow{d} B^n(\tau) \longrightarrow 0$$

forms an exact complex for any $\tau \in \mathcal{T}$.

Remark 2.6. Since the commuting properties of d and tr at each level are not required, in general (2.3) does not form a complex.

Intuitively, the localized property reflects some requirements of the trace system in the sense that the generalized currents should be incorporated as a part of trace. The exactness condition shows that when removing the part contributed by generalized currents, the remaining parts have essentially no contributions on the cohomology.

Theorem 2.1 (Cohomology). *Suppose that the FECTS satisfies the geometric decomposition assumption, and is compatible with respect to the generalized currents $\{\Upsilon_\bullet, \mathcal{Z}\}$. Then the cohomology of the global discrete complex*

$$0 \longrightarrow \mathbf{A}^0 \xrightarrow{d} \dots \xrightarrow{d} \mathbf{A}^n \longrightarrow 0$$

is isomorphic to $\mathcal{Z} \otimes \mathcal{H}_{dR}(\Omega)$.

A direct result from Theorem 2.1 is that if the domain Ω is contractible and the FECTS satisfies the geometric decomposition assumption, and is compatible with respect to the generalized currents $\{\Upsilon_\bullet, \mathcal{Z}\}$, then the following discrete complex

$$\mathcal{Z} \xrightarrow{\subset} \mathbf{A}^0 \xrightarrow{d} \dots \xrightarrow{d} \mathbf{A}^n \longrightarrow 0$$

is exact.

In what follows, we will construct projections π^k from $L^2(\Omega) \otimes \mathbb{X}^k$ to $\mathbf{A}^k$, such that $d\pi^k w = \pi^{k+1} dw$ whenever $w \in L^2(\Omega) \otimes \mathbb{X}^k$ and $dw \in L^2(\Omega) \otimes \mathbb{X}^{k+1}$. Specifically, we will construct π^k as the following form:

Definition 2.8. *We say a function $\pi : L^2(\Omega) \otimes \mathbb{X} \rightarrow \mathbf{A}$ is of inner-product type, if*

$$\pi(w) := \sum_{i=1}^I (w, \Omega_i)_\Omega \Phi_i.$$

Here $\Omega_i \in L^2(\Omega) \otimes \mathbb{X}$ and $\Phi_i \in \mathbf{A}$. Moreover, we say that π is a local operator in $\mathcal{T}$, if the restriction of $\pi(w)$ on each $\sigma \in \mathcal{T}_n$ only depends on the restriction of w on $\text{star}^2 \sigma$.

Theorem 2.2 (Commuting projections). *Under the assumptions in Theorem 2.1, and assume that each element patch $\text{star}^1 \tau$ is simply connected for $\tau \in \mathcal{T}$.*

Then, there exists a projection $\pi^k : L^2(\Omega) \otimes \mathbb{X}^k \rightarrow \mathbf{A}^k$, such that each π^k is of local inner-product type. Furthermore, for any $u \in L^2(\Omega) \otimes \mathbb{X}^k$ with $du \in L^2(\Omega) \otimes \mathbb{X}^{k+1}$, it holds that $d\pi^k u = \pi^{k+1} du$.

Finally, we consider the boundedness of the interpolation operators. Denote by $\text{ord}(d_k)$ the order of d_k . Recall that $M := \max_k \text{ord}(d_k)$. For all the BGG complexes discussed in this paper, the generalized Bogovskii operators exists, namely, the following results hold [46, 22].

Proposition 2.2. *Let ω be a bounded, contractible, Lipschitz domain. Then for any $u \in H_0^M(\omega) \otimes \mathbb{X}^k$ with $(d^{k-1})^* u = 0$, and also $u \in \mathcal{Z}^\perp$ if $k = 0$, there exists $p \in H_0^{M+\text{ord}(d_k)}(\omega) \otimes \mathbb{X}^{k+1} \subset H_0^M(\omega) \otimes \mathbb{X}^{k+1}$, such that $(d^k)^* p = u$ and $\|p\|_{L^2(\omega)} \leq C(\omega) \text{diag}(\omega)^{\text{ord}(d^k)} \|u\|_{L^2(\omega)}$. Here the Poincaré constant $C(\omega)$ depends on ω .*

Moreover, we assume that the generalized currents are homogenous with respect to scaling.

Definition 2.9 (Homogeneity). *We say that the generalized currents $\{\Upsilon_\bullet\}$ are homogenous if for any $0 \leq k \leq n$, there exists a constant $\ell_k \in \mathbb{R}$ such that for any $\tau \in \mathcal{T}_k$ and $\hat{\mathbf{x}} = \varphi(\mathbf{x}) := (\mathbf{x} - x_0)/a$ with fixed vertex x_0 of τ and $a > 0$ which maps τ to $\hat{\tau} = \varphi(\tau)$, it holds that*

$$\Upsilon_\tau(\phi) = a^{\ell_k} \Upsilon_{\hat{\tau}}(\hat{\phi}), \quad \text{for any } \phi \in C^\infty(\Omega) \otimes \mathbb{X}^k$$

here $\hat{\phi} = (\varphi^{-1})^* \phi = \phi \circ \varphi^{-1}$.

Remark 2.7. If $\{\Upsilon_\bullet\}$ are homogeneity, then from the definition of $\{\Upsilon_\bullet\}$, it holds that

$$\ell_{k+1} = \ell_k + \text{ord}(d_k)$$

Theorem 2.3. *Suppose moreover that the triangulation is shape regular, the generalized currents $\{\Upsilon_\bullet\}$ are homogenous, and the Poincaré constant $C(\text{star}^1(\tau))$ in Proposition 2.2 is uniformly bounded for all $\tau \in \mathcal{T}$. Then the operator π^k constructed above is locally L^2 bounded, namely, there is a constant C , depends only on the regularity constant, such that*

$$\|\pi^k u\|_{L^2(\sigma)} \leq C \|u\|_{L^2(\text{star}^2(\sigma))}.$$

A direct consequence is that

$$\|\pi^k u\|_{L^2(\Omega)} \leq C' \|u\|_{L^2(\Omega)}.$$

Remark 2.8. We emphasize that in this paper, we do not require the exactness of the differential complex in Ω . Instead, we only need local exactness on the element patch. Therefore, our results are also applicable to non-trivial domains.

3. FINITE ELEMENT COMPLEXES WITH TRACE STRUCTURES: 2D CASE

This section explores finite-element complexes with trace structures in two-dimensional spaces. The resulting complexes have already been presented in the existing literature. Consequently, the primary emphasis of this section (and the subsequent section as well) is to point out that these constructions indeed possess finer structures.

Note that $\mathcal{Z}$ for each complex is finite dimensional Hilbert space, then there exists a normalized orthogonal basis $\{z_i\}_{1 \leq i \leq \dim \mathcal{Z}}$.

3.1. Notations. This subsection collects the notations frequently used in the sequel. Suppose that the face f is on the $x - y$ plane. In two dimensions, for a given edge e , let $\mathbf{t}$, $\mathbf{n}$ be its unit tangential and normal vectors, respectively. Given a scalar function u and vector function $\mathbf{q} = (q_1, q_2)^T$, we can then define the differential operator by

$$\begin{aligned} \text{grad } u &= \nabla u = (\partial_x u, \partial_y u)^T, & \text{curl } u &= (-\partial_y u, \partial_x u)^T, \\ \text{div } \mathbf{q} &= \partial_x q_1 + \partial_y q_2, & \text{rot } \mathbf{q} &= \partial_x q_2 - \partial_y q_1. \end{aligned}$$

For vector and tensor function, these differential operators act row-wise. We can also define the hessian operator

$$\text{hess } u = \nabla^2 u = \begin{pmatrix} \partial_{xx} u & \partial_{xy} u \\ \partial_{xy} u & \partial_{yy} u \end{pmatrix}.$$

Furthermore, given a matrix $\mathbf{M} \in \mathbb{M}^n$, define

$$\text{sym } \mathbf{M} = \frac{1}{2}(\mathbf{M} + \mathbf{M}^T), \quad \text{dev } \mathbf{M} = \mathbf{M} - \frac{1}{n} \text{Tr}(\mathbf{M}) I_{n \times n}$$

here $\text{Tr}(\mathbf{M})$ denotes the trace of the matrix $\mathbf{M}$.

Next, we introduce some bubble polynomial function spaces on edges and faces. For any nonnegative integer l we define the bubble space $\mathring{P}_k^{(l)}(e)$ on each edge $e \in \mathcal{E}$ by

$$\mathring{P}_k^{(l)}(e) := \{u \in P_k(e) : u, \partial_{\mathbf{t}}^m u \text{ vanish at the end point of } e \text{ with } m \leq l\}.$$

Similarly, for any nonnegative integer m, l we define the bubble space $\mathring{P}_k^{(m,l)}(f)$ on each face $f \in \mathcal{F}$ by

$$\begin{aligned} \mathring{P}_k^{(m,l)}(f) := \{u \in P_k(f) : D^\alpha u \text{ vanish at all vertices of } f \text{ with } |\alpha| \leq m, \\ D^\beta u \text{ vanish on all edges of } f \text{ with } |\beta| \leq l\}. \end{aligned}$$

Specially, we define the the bubble space $\mathring{P}_k^{(m,-1)}(f)$ on each face $f \in \mathcal{F}$ by

$$\mathring{P}_k^{(m,-1)}(f) := \{u \in P_k(f) : D^\alpha u \text{ vanish at all vertices of } f \text{ with } |\alpha| \leq m\}.$$

We introduce the jet at vertices, a concept that simplifies the expressions in the trace complexes (c.f. [44, 16]). For each integer m and $x \in \mathcal{V}$, we define $\mathbb{H}^m(x)$ to be the space of the constant m -tensor in n dimensions. It can be observed that $\mathbb{H}^0(x) \cong \mathbb{R}$, $\mathbb{H}^1(x) \cong \mathbb{R}^n$ and $\mathbb{H}^2(x) \cong \mathbb{R}^{n \times n}$. Then define the m jet as

$$\mathbb{J}^m(x) = \mathbb{H}^0(x) \oplus \mathbb{H}^1(x) \cdots \mathbb{H}^m(x).$$

For the non-scalar case, we use the tensor product-type space to denote the corresponding jets. For example, $\mathbb{J}^m(x) \otimes \mathbb{R}^n$ denotes an m jet with respect to the jet of vector-valued function. Then we can define the differential operators on jets. Here we present two examples.

Example 3.1. Given a 1-jet with respect to a two-dimensional vector function $\mathbf{v}$ at x as

$$\begin{pmatrix} v^1 \\ v^2 \end{pmatrix} \oplus \begin{pmatrix} v_1^1 & v_2^1 \\ v_1^2 & v_2^2 \end{pmatrix} \in \mathbb{J}^1(x) \otimes \mathbb{R}^2$$

where v^1 and v^2 are two components of $\mathbf{v}$ and v_j^i is the j -th derivative of the i -th components. Then the divergence of the 1-jet can be represented as

$$\text{div} \left(\begin{pmatrix} v^1 \\ v^2 \end{pmatrix} \oplus \begin{pmatrix} v_1^1 & v_2^1 \\ v_1^2 & v_2^2 \end{pmatrix} \right) = v_1^1 + v_2^2 \in \mathbb{J}^0(x)$$

Example 3.2. Given a 2-jet with respect to a scalar function v at x in 2D:

$$v \oplus \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \oplus \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix} \in \mathbb{J}^2(x)$$

where v_i is the first derivative of v and v_{ij} is the second derivative of v . Then the gradient of the 2-jet can be represented as

$$\text{grad} \left(v \oplus \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \oplus \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix} \right) = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \oplus \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix} \in \mathbb{J}^1(x) \otimes \mathbb{R}^2$$

Other differential operators (curl, hess, sym curl, div div, etc) can be represented in a similar way.

Particularly, denote the symmetric m - tensor by $\tilde{\mathbb{S}}^m(x)$. It is obvious that the symmetric m -tensor is the range of the m -th order full derivative of a smooth functions, and therefore isomorphic to the homogenous polynomials with degree $= m$. Therefore, we can define the bubble space at vertex $x \in \mathcal{V}$

$$\mathbb{J}_B^m(x) = \tilde{\mathbb{S}}^0(x) \oplus \cdots \oplus \tilde{\mathbb{S}}^m(x)$$

Then there exists a natural isomorphism between $\mathbb{J}_B^m(x)$ and the polynomial space $P_m(\Omega)$.

We first consider the smooth de Rham complex in 2D:

$$(3.1) \quad 0 \longrightarrow C^\infty(\Omega) \otimes \mathbb{R} \xrightarrow{\text{grad}} C^\infty(\Omega) \otimes \mathbb{R}^2 \xrightarrow{\text{rot}} C^\infty(\Omega) \otimes \mathbb{R} \longrightarrow 0.$$

Note that now $\mathcal{Z} = \mathbb{R}$. Then we can choose $\Upsilon_\bullet$ as follows: $\Upsilon_f : u \mapsto \int_f u$, $\Upsilon_e : \mathbf{v} \mapsto \int_e \mathbf{v} \cdot \mathbf{t}$ and $\Upsilon_x : u \mapsto u(x)$. Therefore, it holds that $(\Upsilon_\bullet, \mathbb{R})$ is a generalized currents of (3.1).

Note that for the complexes without additional smoothness, the results have been extensively studied in [3], or in the concept of finite element system (FES) [19]. In this regard, the results of this paper can be considered as an improvement of the existing results in within the FES framework.

3.2. Hermite-Stenberg de Rham Complexes. Consider the following FECTS with $k \geq 3$.

$$(3.2) \quad \begin{array}{ccc} P_k(f) & \xrightarrow{\text{grad}} & P_{k-1}(f) \otimes \mathbb{R}^2 \xrightarrow{\text{rot}} P_{k-2}(f) \\ \downarrow u|_e & & \downarrow \mathbf{v}|_e \cdot \mathbf{t} \\ P_k(e) & \xrightarrow{\partial_t} & P_{k-1}(e) \\ \downarrow u(x) \oplus \frac{d}{dt} u(x) \mathbf{t} & & \downarrow v(x) \mathbf{t} \\ \mathbb{J}^1(x) & \xrightarrow{\text{grad}} & \mathbb{J}^0(x) \otimes \mathbb{R}^2 \end{array}$$

Here, we specify the trace structure from faces to vertices. At the first column, the mapping $P_k(f) \rightarrow \mathbb{J}^1(x)$ is defined as $u \mapsto u(x) \oplus \nabla u(x)$; at the second column, the mapping $P_{k-1}(f) \otimes \mathbb{R}^2 \rightarrow \mathbb{J}^0(x) \otimes \mathbb{R}^2$ is defined as $\mathbf{v} \mapsto v(x)$.

Remark 3.1. Note that here the choice of the trace structure from edges to vertices is not unique. For example, denote by $\mathbf{e}_i, 1 \leq i \leq 2$, the standard basis of $\mathbb{R}^2$. Fixed i , then an alternative choice of the trace structure from edges to vertices is that: At the first column, the mapping $P_k(e) \rightarrow \mathbb{J}^1(x)$ is defined as $u \mapsto u(x) \oplus \frac{d}{dt} u(x) \mathbf{e}_i$; at the second column, the mapping $P_{k-1}(e) \rightarrow \mathbb{J}^0(x) \otimes \mathbb{R}^2$ is defined as $v \mapsto v(x) \mathbf{e}_i$. The reason we can do this is that we no longer require the

composition of traces to be a trace again, and we do not need the trace structure from edges to vertices commutes with differential operators.

Proposition 3.1. *The FECTS (3.2) is compatible with respect to the generalized currents for de Rham (3.1).*

Proof. Set $\tilde{\Upsilon}_x : a_0 \oplus a_1 \mapsto a_0$ for all vertex $x \in \mathcal{V}$, $\tilde{\Upsilon}_e : v \mapsto \int_e v$ for all edge $e \in \mathcal{E}$ and $\tilde{\Upsilon}_f : q \mapsto \int_f q$ for all face $f \in \mathcal{F}$. Then it is easy to see that $\Upsilon_\bullet$ can be localized. Note that the face bubble complex is

$$0 \longrightarrow \mathring{P}_k^{(1,0)}(f) \xrightarrow{\text{grad}} \mathring{P}_{k-1}^{(0,-1)}(\text{rot}, f; \mathbb{R}^2) \xrightarrow{\text{rot}} P_{k-2}(f)/\mathbb{R} \longrightarrow 0.$$

here the bubble space $\mathring{P}_{k-1}^{(0,-1)}(\text{rot}, f; \mathbb{R}^2) := \{\mathbf{u} \in \mathring{P}_{k-1}^{(0,-1)}(f) \otimes \mathbb{R}^2 : \mathbf{u} \cdot \mathbf{t} = 0 \text{ on each edge } e \subset \partial f\}$, which is exact (cf. [16]). The bubble complex on each edge $e \in \mathcal{E}$ is

$$0 \longrightarrow \mathring{P}_k^{(0)}(e) \xrightarrow{\partial_t} P_{k-1}(e)/\mathbb{R} \longrightarrow 0,$$

and the bubble complex on each vertex $x \in \mathcal{V}$ is isomorphic to

$$0 \longrightarrow P_1(\Omega)/\mathbb{R} \xrightarrow{\text{grad}} P_0(\Omega) \otimes \mathbb{R}^2 \longrightarrow 0,$$

which are also exact. □

The corresponding global space forms the following complex [16]:

$$(3.3) \quad 0 \longrightarrow \mathbf{Hm}_k \xrightarrow{\text{grad}} \mathbf{St}_{k-1} \xrightarrow{\text{rot}} \mathbf{DG}_{k-2} \longrightarrow 0.$$

Here, $\mathbf{Hm}_k$ is the space of Hermite element: $\mathbf{Hm}_k = \{u : u|_f \in P_k(f), u \in C^1(\mathcal{V}), u \in C^0(\mathcal{E})\}$. $\mathbf{St}_{k-1}$ is the space of rotational Stenberg element: $\mathbf{St}_{k-1} = \{\mathbf{u} : \mathbf{u}|_f \in P_{k-1}(f) \otimes \mathbb{R}^2, \mathbf{u} \in C^1(\mathcal{V}), \mathbf{u} \cdot \mathbf{t} \in C^0(\mathcal{E})\}$. $\mathbf{DG}_{k-2}$ is the discontinuous piecewise P_{k-2} space. As a consequence of Theorem 2.1, the cohomology of (3.3) is isomorphic to $\mathcal{H}_{dR}(\Omega)$.

3.3. Falk-Neilan Stokes Complexes. Consider the following FECTS with $k \geq 5$, which will lead to a finite element de Rham complex with higher regularity. Such complexes have been applied in the computation of the Stokes problem, see [26].

$$(3.4) \quad \begin{array}{ccccc} P_k(f) & \xrightarrow{\text{grad}} & P_{k-1}(f) \otimes \mathbb{R}^2 & \xrightarrow{\text{rot}} & P_{k-2}(f) \\ \downarrow (u|_e, (\partial_n u)|_e) & & \downarrow (v|_e \cdot \mathbf{t}, v|_e \cdot \mathbf{n}) & & \downarrow w(x) \\ \left(\begin{array}{c} P_k(e) \\ P_{k-1}(e) \end{array} \right) & \xrightarrow{\mathbf{d}(\partial_t, id)} & \left(\begin{array}{c} P_{k-1}(e) \\ P_{k-1}(e) \end{array} \right) & & \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{J}^2(x) & \xrightarrow{\text{grad}} & \mathbb{J}^1(x) \otimes \mathbb{R}^2 & \xrightarrow{\text{rot}} & \mathbb{J}^0(x) \end{array}$$

Here, $\mathbf{d}(\partial_t, \text{id})$ is an abbreviation of $\begin{pmatrix} \partial_t & 0 \\ 0 & \text{id} \end{pmatrix}$. Hereafter, we shall adopt this conventional notation to shorten the expression. We first specify the trace structures from faces to vertices: At the first column, the mapping $P_k(f) \rightarrow \mathbb{J}^2(x)$ is defined as $u \mapsto u(x) \oplus \nabla u(x) \oplus \nabla^2 u(x)$; at the second column, the mapping $P_{k-1}(f) \otimes \mathbb{R}^2 \rightarrow \mathbb{J}^1(x) \otimes \mathbb{R}^2$ is defined as $\mathbf{v} \mapsto \mathbf{v}(x) \oplus \nabla \mathbf{v}(x)$.

Then we specify the trace structures from edges to vertices: The mapping at the first column is $\begin{pmatrix} P_k(e) \\ P_{k-1}(e) \end{pmatrix} \rightarrow \mathbb{J}^2(x)$ is $(a_0, a_1) \mapsto a_0(x) \oplus \left(\frac{d}{dt}a_0(x)\mathbf{t} + a_1(x)\mathbf{n}\right) \oplus \left(\frac{d^2}{dt^2}a_0(x)\mathbf{t}\mathbf{t}^T + \frac{d}{dt}a_1(x)\mathbf{n}\mathbf{t}^T\right)$; at the second column, the mapping $\begin{pmatrix} P_{k-1}(e) \\ P_{k-1}(e) \end{pmatrix} \rightarrow \mathbb{J}^1(x) \otimes \mathbb{R}^2$ is $(a_0, a_1) \mapsto (a_0(x)\mathbf{t} + a_1(x)\mathbf{n}) \oplus \left(\frac{d}{dt}a_0(x)\mathbf{t}\mathbf{t}^T + \frac{d}{dt}a_1(x)\mathbf{n}\mathbf{t}^T\right)$.

Proposition 3.2. *The FECTS (3.4) is compatible with respect to the generalized currents for de Rham (3.1).*

Proof. Set $\tilde{\Upsilon}_x : a_0 \oplus a_1 \oplus a_2 \mapsto a_0$ for all vertex $x \in \mathcal{V}$, $\tilde{\Upsilon}_e : (v_0, v_1) \mapsto \int_e v_0$ for all edge $e \in \mathcal{E}$ and $\tilde{\Upsilon}_f : q \mapsto \int_f q$ for all face $f \in \mathcal{F}$. Then it is easy to see that $\Upsilon_\bullet$ can be localized. The bubble complex on each face $f \in \mathcal{F}$ is

$$0 \longrightarrow \mathring{P}_k^{(2,1)}(f) \xrightarrow{\text{grad}} \mathring{P}_{k-1}^{(1,0)}(f) \otimes \mathbb{R}^2 \xrightarrow{\text{rot}} \mathring{P}_{k-2}^{(0,-1)}(f)/\mathbb{R} \longrightarrow 0.$$

which is exact (cf. [26]). The bubble complex on each edge $e \in \mathcal{E}$ is

$$0 \longrightarrow \begin{pmatrix} \mathring{P}_k^{(2)}(e) \\ \mathring{P}_{k-1}^{(1)}(e) \end{pmatrix} \xrightarrow{\mathbf{d}(\partial_t, \text{id})} \begin{pmatrix} \mathring{P}_{k-1}^{(1)}(e) \\ \mathring{P}_{k-1}^{(1)}(e) \end{pmatrix} / \begin{pmatrix} \mathbb{R} \\ 0 \end{pmatrix} \longrightarrow 0,$$

and the bubble complex at each vertex $x \in \mathcal{V}$ is isomorphic to

$$0 \longrightarrow P_2(\Omega)/\mathbb{R} \xrightarrow{\text{grad}} P_1(\Omega) \otimes \mathbb{R}^2 \xrightarrow{\text{rot}} P_0(\Omega) \longrightarrow 0,$$

which are exact. □

The corresponding global space complex is the Argyris–Falk–Neilan:

$$(3.5) \quad 0 \longrightarrow \mathbf{Ar}_k \longrightarrow \mathbf{Hm}_{k-1} \otimes \mathbb{R}^2 \longrightarrow \mathbf{DG}_{k-2}^{(0)} \longrightarrow 0.$$

Here, $\mathbf{Ar}_k$ is the space of Argyris element: $\mathbf{Ar}_k = \{u : u|_f \in P_k(f), u \in C^2(\mathcal{V}), u \in C^1(\mathcal{E})\}$. The last space $\mathbf{DG}_{k-2}^{(0)}$ is the space of piecewise P_{k-2} function u such that u is continuous at vertices. As a consequence of Theorem 2.1, the cohomology of (3.5) is isomorphic to $\mathcal{H}_{dR}(\Omega)$.

3.4. Hu-Zhang Hessian Complexes. Now we move our focus beyond the de Rham complexes. Consider the following Hessian complexes in two dimensions:

$$(3.6) \quad 0 \longrightarrow C^\infty(\Omega) \xrightarrow{\text{hess}} C^\infty(\Omega) \otimes \mathbb{S}^2 \xrightarrow{\text{rot}} C^\infty(\Omega) \otimes \mathbb{R}^2 \longrightarrow 0.$$

The generalized currents are listed in Lemma 2.1.

Now we consider the following FECTS with for Hessian complex with $k \geq 5$:

$$(3.7) \quad \begin{array}{ccc} P_k(f) & \xrightarrow{\text{hess}} & P_{k-2}(f) \otimes \mathbb{S}^2 \xrightarrow{\text{rot}} P_{k-3}(f) \otimes \mathbb{R}^2 \\ \downarrow (u|_e, (\partial_{\mathbf{n}} u)|_e) & & \downarrow (t^T \sigma|_e, \mathbf{n}^T \sigma|_e t) \\ \left(\begin{array}{c} P_k(e) \\ P_{k-1}(e) \end{array} \right) & \xrightarrow{\mathbf{d}(\partial_{\mathbf{t}\mathbf{t}}, \partial_{\mathbf{t}})} & \left(\begin{array}{c} P_{k-2}(e) \\ P_{k-2}(e) \end{array} \right) \\ \downarrow & & \downarrow \\ \mathbb{J}^2(x) & \xrightarrow{\text{hess}} & \mathbb{J}^0(x) \otimes \mathbb{M}^2 \end{array}$$

We first specify the trace structures from faces to vertices: At the first column, the mapping $P_k(f) \rightarrow \mathbb{J}^2(x)$ is defined as $u \mapsto u(x) \oplus \nabla u(x) \oplus \nabla^2 u(x)$; at the second column, the mapping $P_{k-2}(f) \otimes \mathbb{S}^2 \rightarrow \mathbb{J}^0(x) \otimes \mathbb{M}^2$ is defined as $\mathbf{v} \mapsto \mathbf{v}(x)$. Then we specify the trace structures from edges to vertices: The mapping at the first column $\left(\begin{array}{c} P_k(e) \\ P_{k-1}(e) \end{array} \right) \rightarrow \mathbb{J}^2(x)$ is $(a_0, a_1) \mapsto a_0(x) \oplus \left(\frac{d}{dt} a_0(x) \mathbf{t} + a_1(x) \mathbf{n} \right) \oplus \left(\frac{d^2}{dt^2} a_0(x) \mathbf{t} \mathbf{t}^T + \frac{d}{dt} a_1(x) \mathbf{n} \mathbf{t}^T \right)$; and the mapping at the second column $\left(\begin{array}{c} P_{k-2}(e) \\ P_{k-2}(e) \end{array} \right) \rightarrow \mathbb{J}^0(x) \otimes \mathbb{M}^2$ is $(a_0, a_1) \mapsto a_0(x) \mathbf{t} \mathbf{t}^T + a_1(x) \mathbf{n} \mathbf{t}^T$.

Proposition 3.3. *The FECTS (3.7) is compatible with respect to the generalized currents for Hessian (3.6).*

Proof. We first show that $\Upsilon_\bullet$ can be localized. Let

$$\tilde{\Upsilon}_f : P_{k-3} \otimes \mathbb{R}^2 \rightarrow \mathcal{Z}, \quad \mathbf{v} \mapsto \sum_{i=1}^3 \left(\int_f \mathbf{v} \cdot \psi_i \right) z_i,$$

$$\tilde{\Upsilon}_e : \left(\begin{array}{c} P_{k-2}(e) \\ P_{k-2}(e) \end{array} \right) \rightarrow \mathcal{Z}, \quad (a_0, a_1) \mapsto \sum_{i=1}^3 \left(\int_e a_0(\psi_i \cdot \mathbf{t}) + a_1(\psi_i \cdot \mathbf{n}) \right) z_i,$$

and

$$\tilde{\Upsilon}_x : \mathbb{J}^2(x) \rightarrow \mathcal{Z}, \quad a_0 \oplus a_1 \oplus a_2 \mapsto \sum_{i=1}^3 \left((\psi_i(x) \cdot a_1) - \frac{1}{2} \operatorname{div} \psi_i(x) \cdot a_0 \right) z_i.$$

for each $x \in \mathcal{V}$, $e \in \mathcal{E}$ and $f \in \mathcal{F}$. Therefore, the generalized currents can be localized. The bubble complex on each face $f \in \mathcal{F}$ is

$$0 \longrightarrow \mathring{P}_k^{(2,1)}(f) \xrightarrow{\text{hess}} \mathring{P}_{k-2}^{(0,-1)}(\operatorname{rot}, f; \mathbb{S}^2) \xrightarrow{\text{rot}} P_{k-3}(f) \otimes \mathbb{R}^2 / \mathcal{RT}^2(f) \longrightarrow 0,$$

which is exact (cf. [16]). Here the bubble space $P_{k-2}^{(0,-1)}(\text{rot}, f; \mathbb{S}^2) := \{\mathbf{u} \in \mathring{P}_{k-2}^{(0,-1)}(f) \otimes \mathbb{S}^2 : \mathbf{u} \cdot \mathbf{t} = 0 \text{ on each edge } e \subset \partial f\}$. The bubble complex on edge $e \in \mathcal{E}$ is

$$0 \longrightarrow \begin{pmatrix} \mathring{P}_k^{(2)}(e) \\ \mathring{P}_{k-1}^{(1)}(e) \end{pmatrix} \xrightarrow{\mathbf{d}(\partial \mathbf{t} \mathbf{t}, \partial \mathbf{t})} \begin{pmatrix} \mathring{P}_{k-2}^{(0)}(e) \\ \mathring{P}_{k-2}^{(0)}(e) \end{pmatrix} / \begin{pmatrix} P_1(e) \\ \mathbb{R} \end{pmatrix} \longrightarrow 0,$$

and the bubble complex at each vertex $x \in \mathcal{V}$ is isomorphic to

$$0 \longrightarrow P_2(\Omega)/P_1(\Omega) \xrightarrow{\text{hess}} P_0(\Omega) \otimes \mathbb{S}^2 \longrightarrow 0.$$

which are exact. $\square$

The global space complex is:

$$(3.8) \quad 0 \longrightarrow \mathbf{A} \mathbf{r}_k \xrightarrow{\text{hess}} \mathbf{H} \mathbf{Z}_{k-2} \xrightarrow{\text{rot}} \mathbf{D} \mathbf{G}_{k-3} \otimes \mathbb{R}^2 \longrightarrow 0.$$

Here, $\mathbf{H} \mathbf{Z}_{k-2}$ is the space of Hu-Zhang element [31]:

$$\mathbf{H} \mathbf{Z}_{k-2} = \{\boldsymbol{\sigma} : \boldsymbol{\sigma}|_f \in P_k(f) \otimes \mathbb{S}^2, \boldsymbol{\sigma} \in C^0(\mathcal{V}), \boldsymbol{\sigma} \mathbf{n} \in C^0(\mathcal{E})\}.$$

As a consequence of Theorem 2.1, the cohomology of (3.8) is isomorphic to $P_1(\Omega) \otimes \mathcal{H}_{dR}(\Omega)$.

3.5. Hu-Ma-Zhang divdiv+ complexes. Let us finally consider a finite element divdiv complex with a slightly higher regularity by [36]. The example is specially chosen to show that the proposed framework is also capable to handle some additional smoothness, which actually shares the similar philosophy of the Stokes complex discussed before.

Consider the two-dimensional divdiv complex:

$$(3.9) \quad 0 \longrightarrow C^\infty(\Omega) \otimes \mathbb{R}^2 \xrightarrow{\text{sym curl}^1} C^\infty(\Omega) \otimes \mathbb{S}^2 \xrightarrow{\text{div div}} C^\infty(\Omega) \longrightarrow 0.$$

The generalized currents of the complex is discussed in Lemma 2.2.

We now introduce the following FECTS with $k \geq 4$:

$$(3.10) \quad \begin{array}{ccc} P_k(f) \otimes \mathbb{R}^2 & \xrightarrow{\text{sym curl}^1} & P_{k-1}(f) \otimes \mathbb{S}^2 \xrightarrow{\text{div div}} P_{k-3}(f) \\ \downarrow (\mathbf{u}|_e \cdot \mathbf{n}, \mathbf{u}|_e \cdot \mathbf{t}, (\text{div } \mathbf{u})|_e) & & \downarrow (\mathbf{n}^T \boldsymbol{\sigma}|_e \mathbf{n}, \mathbf{t}^T \boldsymbol{\sigma}|_e \mathbf{n}, (\text{div } \boldsymbol{\sigma})|_e \cdot \mathbf{n}) \\ \begin{pmatrix} P_k(e) \\ P_k(e) \\ P_{k-1}(e) \end{pmatrix} & \xrightarrow{\begin{pmatrix} \partial_t & 0 & 0 \\ 0 & \partial_t & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} \partial_t \end{pmatrix}} & \begin{pmatrix} P_{k-1}(e) \\ P_{k-1}(e) \\ P_{k-2}(e) \end{pmatrix} \\ \downarrow & & \downarrow \\ \mathbb{J}^1(x) \otimes \mathbb{R}^2 & \xrightarrow{\text{sym curl}^1} & \mathbb{J}^0(x) \otimes \mathbb{M}^2 \end{array}$$

We first specify the trace structures from faces to vertices: At the first column, the mapping $P_k(f) \otimes \mathbb{R}^2 \rightarrow \mathbb{J}^1(x) \otimes \mathbb{R}^2$ is defined as $\mathbf{u} \mapsto \mathbf{u}(x) \oplus \nabla \mathbf{u}(x)$; at the second column, the mapping $P_{k-1}(f) \otimes \mathbb{S}^2 \rightarrow \mathbb{J}^0(x) \otimes \mathbb{M}^2$ is defined as $\mathbf{v} \mapsto \mathbf{v}(x)$. Then we specify the trace structures from the edges to vertices:

The mapping at the first column $\begin{pmatrix} P_k(e) \\ P_k(e) \\ P_{k-1}(e) \end{pmatrix} \rightarrow \mathbb{J}^1(x) \otimes \mathbb{R}^2$ is $(a_0, a_1, a_2) \mapsto (a_0(x)\mathbf{n} + a_1(x)\mathbf{t}) \oplus (\frac{d}{dt}a_0(x)\mathbf{n}\mathbf{n}^T + \frac{d}{dt}a_1(x)\mathbf{t}\mathbf{t}^T + a_2(x)\mathbf{n}\mathbf{t}^T)$; the mapping at the second column $\begin{pmatrix} P_{k-1}(e) \\ P_{k-1}(e) \\ P_{k-2}(e) \end{pmatrix} \rightarrow \mathbb{J}^0(x) \otimes \mathbb{M}^2$ is $(a_0, a_1, a_2) \mapsto a_0(x)\mathbf{n}\mathbf{n}^T + a_1(x)\mathbf{n}\mathbf{t}^T$.

The operators on the edge are defined according to the following lemma, cf. [36].

Lemma 3.1. *Suppose that $\boldsymbol{\sigma} = \text{sym curl } \mathbf{u}$, then it holds that*

$$\begin{aligned} \mathbf{n}^T \boldsymbol{\sigma} \mathbf{n} &= \partial_t(\mathbf{u} \cdot \mathbf{n}), \\ \mathbf{t}^T \boldsymbol{\sigma} \mathbf{n} &= \partial_t(\mathbf{u} \cdot \mathbf{t}) - \frac{1}{2} \text{div } \mathbf{u}, \\ \text{div } \boldsymbol{\sigma} \cdot \mathbf{n} &= \frac{1}{2} \partial_t(\text{div } \mathbf{u}). \end{aligned}$$

Proposition 3.4. *The FECTS (3.10) is compatible with respect to the generalized currents with respect to divdiv complex (3.9).*

Proof. We first show that $\Upsilon_\bullet$ can be localized. Let

$$\begin{aligned} \tilde{\Upsilon}_x : (a_0, a_1) &\mapsto \sum_{i=1}^3 ((a_0 \cdot \nabla \psi_i)(x) - \frac{1}{2}(\text{Tr}(a_1)\psi_i)(x))z_i, \\ \tilde{\Upsilon}_e : \begin{pmatrix} P_{k-1}(e) \\ P_{k-1}(e) \\ P_{k-2}(e) \end{pmatrix} &\rightarrow \mathbb{R}^3, \quad (a_0, a_1, a_2) \mapsto \sum_{i=1}^3 \left(\int_e a_0(\nabla \psi_i \cdot \mathbf{n}) + a_1(\nabla \psi_i \cdot \mathbf{t}) - a_2 \psi_i \right) z_i, \\ \tilde{\Upsilon}_f : P_{k-3}(f) &\rightarrow \mathcal{Z}, \quad \tilde{\Upsilon}_f(q) = \sum_{i=1}^3 \left(\int_f q \psi_i \right) z_i \end{aligned}$$

for each $x \in \mathcal{V}$, $e \in \mathcal{E}$ and $f \in \mathcal{F}$. Thus, the generalized currents can be localized.

The bubble complex in face f is

$$0 \longrightarrow \mathring{P}_k^{(1,0) \cap \text{div}}(f; \mathbb{R}^2) \xrightarrow{\text{sym curl}} \mathring{P}_{k-1}^{(0,-1) \cap \text{div}}(\text{divdiv}, f; \mathbb{S}^2) \xrightarrow{\text{divdiv}} P_{k-3}(f)/P_1(f) \longrightarrow 0.$$

which is exact (cf. [36]). Here the bubble space $\mathring{P}_k^{(1,0) \cap \text{div}}(f; \mathbb{R}^2) := \{\mathbf{u} \in P_k^{(1,0)}(f) \otimes \mathbb{R}^2 : \text{div } \mathbf{u} \in P_{k-1}^{(0,0)}(f)\}$ and the space $\mathring{P}_{k-1}^{(0,-1) \cap \text{div}}(\text{divdiv}, f; \mathbb{S}^2) := \{\boldsymbol{\sigma} \in P_{k-1}^{(0,-1)} \otimes$

$\mathbb{S}^2 : \boldsymbol{\sigma} \mathbf{n} = 0, \operatorname{div} \boldsymbol{\sigma} \cdot \mathbf{n} = 0$ on ∂f }. The bubble space on edge $e \in \mathcal{E}$ is

$$0 \longrightarrow \begin{pmatrix} \dot{P}_k^{(1)}(e) \\ \dot{P}_k^{(1)}(e) \\ \dot{P}_{k-1}^{(0)}(e) \end{pmatrix} \xrightarrow{\begin{pmatrix} \partial_t & 0 & 0 \\ 0 & \partial_t & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2}\partial_t \end{pmatrix}} \begin{pmatrix} \dot{P}_{k-1}^{(0)}(e) \\ \dot{P}_{k-1}^{(0)}(e) \\ P_{k-2}(e) \end{pmatrix} / \left\{ \begin{pmatrix} \mathbb{R} \\ \partial_t b \\ -b \end{pmatrix} : b \in P_1(e) \right\} \longrightarrow 0.$$

The quotient space here is defined as

$$\begin{pmatrix} \dot{P}_{k-1}^{(0)}(e) \\ \dot{P}_{k-1}^{(0)}(e) \\ P_{k-2}(e) \end{pmatrix} / \left\{ \begin{pmatrix} \mathbb{R} \\ \partial_t b \\ -b \end{pmatrix} : b \in P_1(e) \right\} = \{(u, v, w)^T \in (\dot{P}_{k-1}^{(0)}(e), \dot{P}_{k-1}^{(0)}(e), P_{k-2}(e))^T \\ : \int_e (ua + v\partial_t b - wb)dx = 0, \text{ for all } b \in P_1(e), a \in P_0(e)\}.$$

To verify that the sequence forms a complex, it is sufficient to perform integration by parts. When combined with dimension counting, it can be checked that the edge bubble complex is exact. Finally, the bubble complex at vertex $x \in \mathcal{V}$ is isomorphic to

$$0 \longrightarrow P_1(\Omega) \otimes \mathbb{R}^2 / \mathcal{RT}^2 \xrightarrow{\operatorname{sym curl}} P_0(\Omega) \otimes \mathbb{S}^2 \longrightarrow 0,$$

which is also exact. $\square$

The corresponding global space complex is

$$(3.12) \quad 0 \longrightarrow \mathbf{Hm}_k^{\operatorname{div}} \xrightarrow{\operatorname{sym curl}} \mathbf{HMZ}_{k-1} \xrightarrow{\operatorname{div div}} \mathbf{DG}_{k-3} \longrightarrow 0.$$

Here, $\mathbf{Hm}_k^{\operatorname{div}} = \{\mathbf{u} : \mathbf{u} \in P_k(\Omega) \otimes \mathbb{R}^2, \mathbf{u} \in C^1(\mathcal{V}), \mathbf{u}, \operatorname{div} \mathbf{u} \in C^0(\mathcal{E})\}$ has $H^1(\operatorname{div})$ global continuity, and $\mathbf{HMZ}_{k-1}$ is the space of Hu-Ma-Zhang $H(\operatorname{div}) \cap H(\operatorname{div div})$ element:

$$\mathbf{HMZ}_{k-1} := \{\boldsymbol{\sigma} : \boldsymbol{\sigma} \in P_{k-1}(\Omega) \otimes \mathbb{S}^2 : \boldsymbol{\sigma} \in C^0(\mathcal{V}), \boldsymbol{\sigma} \mathbf{n}, \operatorname{div} \boldsymbol{\sigma} \cdot \mathbf{n} \in C^0(\mathcal{E})\}$$

As a consequence of Theorem 2.1, the cohomology of (3.12) is isomorphic to $\mathcal{RT}^2 \otimes \mathcal{H}_{dR}(\Omega)$.

4. FINITE ELEMENT COMPLEXES WITH TRACE STRUCTURE: 3D CASE

This subsection deals with complexes in three-dimensional spaces. Note that for the examples presented herein, the exactness of all cell bubble complexes had been proven at the time when the finite element complexes were proposed. Consequently, we will always omit the discussion of the cell complexes hereinafter. Moreover, some face bubbles were introduced as an intermediate step of the construction. However, some of them are not suitable for deriving compatibility conditions. Therefore, we present the bubble complexes that can be produced by

the compatibility conditions, without providing detailed proofs. The edge bubble complex appears to be rarely discussed in the literature. Thus, we introduce them in detail, and their exactness can be easily verified through dimension counting.

We first introduce some standard differential operators: Given a scalar function u and vector function $\mathbf{q} = (q_1, q_2, q_3)^T$, the differential operators are defined by

$$\begin{aligned}\text{grad } u &= \nabla u = (\partial_x u, \partial_y u, \partial_z u)^T, \\ \text{curl } \mathbf{q} &= \nabla \times \mathbf{q} = (\partial_y q_3 - \partial_z q_2, \partial_z q_1 - \partial_x q_3, \partial_x q_2 - \partial_y q_1)^T, \\ \text{div } \mathbf{q} &= \nabla \cdot \mathbf{q} = \partial_x q_1 + \partial_y q_2 + \partial_z q_3.\end{aligned}$$

For vector and tensor function, these differential operators act row-wise. We can also define the hessian operator

$$\text{hess } u = \nabla^2 u = \begin{pmatrix} \partial_{xx} u & \partial_{xy} u & \partial_{xz} u \\ \partial_{xy} u & \partial_{yy} u & \partial_{yz} u \\ \partial_{xz} u & \partial_{yz} u & \partial_{zz} u \end{pmatrix}.$$

In this subsection, we need to discuss differential operators on surfaces: for a given face f , we use $\mathbf{t}_1, \mathbf{t}_2$ as its two orthogonal unit tangential vectors, let $\mathbf{n}$ be its unit normal vectors such that $\mathbf{t}_1 \times \mathbf{t}_2 = \mathbf{n}$. Let $\Pi_f(\mathbf{u}) = \begin{pmatrix} \mathbf{u} \cdot \mathbf{t}_1 \\ \mathbf{u} \cdot \mathbf{t}_2 \end{pmatrix}$ be the projection to the face tangential space. Given a scalar function u and vector function $\mathbf{q} = (q_1, q_2)^T$, we can then define the surface differential operator on face f by

$$\begin{aligned}\text{grad}_f u &= (\partial_{\mathbf{t}_1} u, \partial_{\mathbf{t}_2} u)^T, & \text{curl}_f u &= (-\partial_{\mathbf{t}_2} u, \partial_{\mathbf{t}_1} u)^T \\ \text{div}_f \mathbf{q} &= \partial_{\mathbf{t}_1} q_1 + \partial_{\mathbf{t}_2} q_2, & \text{rot}_f \mathbf{q} &= \partial_{\mathbf{t}_1} q_2 - \partial_{\mathbf{t}_2} q_1.\end{aligned}$$

and

$$\text{hess}_f u = \nabla_f^2 u = \begin{pmatrix} \partial_{\mathbf{t}_1 \mathbf{t}_1} u & \partial_{\mathbf{t}_1 \mathbf{t}_2} u \\ \partial_{\mathbf{t}_1 \mathbf{t}_2} u & \partial_{\mathbf{t}_2 \mathbf{t}_2} u \end{pmatrix}.$$

For vector and tensor function, the surface differential operators act row-wise.

For a given edge, we use $\mathbf{t}$ to denote its unit tangential vector, $\mathbf{n}_1, \mathbf{n}_2$ to denote its two orthogonal unit normal vectors, such that $\mathbf{n}_1 \times \mathbf{n}_2 = \mathbf{t}$. Let $N_e(\mathbf{u}) = \begin{pmatrix} \mathbf{u} \cdot \mathbf{n}_1 \\ \mathbf{u} \cdot \mathbf{n}_2 \end{pmatrix}$ be the projection to the edge normal space. For the matrix case, Π_f, N_e on the left (right, resp.) denotes the projection rowwise (columnwise, resp.). When considering a face f and one of its edge e , we still use $\mathbf{t}$ to denote the tangential vector of e and $\mathbf{n}_{f,e} = \mathbf{t} \times \mathbf{n}$ to denote the edge-face normal vector.

Finally, denote by $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ the standard basis of $\mathbb{R}^3$. Then given a two-dimensional vector $\mathbf{v} = (v_1, v_2)^T$, we can define the extension operator $E_3(\mathbf{v}) = v_1 \mathbf{e}_1 + v_2 \mathbf{e}_2$, the definition of E_3 can be extended naturally to high order tensor on two-dimensional face. Namely, E_3 induces a mapping from $(\mathbb{R}^2)^{\otimes n}$ to $(\mathbb{R}^3)^{\otimes n}$.

4.1. Hessian Complexes. This subsection focuses on the hessian complex in three dimensions:

(4.1)

$$0 \longrightarrow C^\infty(\Omega) \xrightarrow{\text{hess}} C^\infty(\Omega) \otimes \mathbb{S}^3 \xrightarrow{\text{curl}} C^\infty(\Omega) \otimes \mathbb{T}^3 \xrightarrow{\text{div}} C^\infty(\Omega) \otimes \mathbb{R}^3 \longrightarrow 0.$$

The generalized currents of this complex is discussed in Lemma 2.3.

We now introduce the following FECTS with $k \geq 9$, which is modified from [32].

(4.2)

$$\begin{array}{ccccccc}
 P_k(T) & \xrightarrow{\text{hess}} & P_{k-2}(T) \otimes \mathbb{S}^3 & \xrightarrow{\text{curl}} & P_{k-3}(T) \otimes \mathbb{T}^3 & \xrightarrow{\text{div}} & P_{k-4}(T) \otimes \mathbb{R}^3 \\
 \downarrow (u|_f, (\partial_{\mathbf{n}} u)|_f) & & \downarrow (\Pi_f \boldsymbol{\sigma}|_f \Pi_f, \mathbf{n}^T (\Pi_f \boldsymbol{\sigma}|_f)) & & \downarrow (\Pi_f(\boldsymbol{\tau}|_f \mathbf{n}), \mathbf{n}^T \boldsymbol{\tau}|_f \mathbf{n}) & & \downarrow \mathbf{v}(x) \\
 \begin{pmatrix} P_k(f) \\ P_{k-1}(f) \end{pmatrix} & \xrightarrow{\mathbf{d}(\text{hess}_f, \text{grad}_f)} & \begin{pmatrix} P_{k-2}(f) \otimes \mathbb{S}^2 \\ P_{k-2}(f) \otimes \mathbb{R}^2 \end{pmatrix} & \xrightarrow{\mathbf{d}(\text{rot}_f, \text{rot}_f)} & \begin{pmatrix} P_{k-3}(f) \otimes \mathbb{R}^2 \\ P_{k-3}(f) \otimes \mathbb{R} \end{pmatrix} & & \\
 \downarrow & & \downarrow & & \downarrow & & \\
 \begin{pmatrix} P_k(e) \\ P_{k-1}(e) \otimes \mathbb{R}^2 \\ P_{k-2}(e) \otimes \mathbb{M}^2 \end{pmatrix} & \xrightarrow{\mathbf{d}(\partial_{tt}, \partial_t, \text{id})} & \begin{pmatrix} P_{k-2}(e) \\ P_{k-2}(e) \otimes \mathbb{R}^2 \\ P_{k-2}(e) \otimes \mathbb{M}^2 \end{pmatrix} & & & & \\
 \downarrow & & \downarrow & & \downarrow & & \\
 \mathbb{J}^4(x) & \xrightarrow{\text{hess}} & \mathbb{J}^2(x) \otimes \mathbb{M}^3 & \xrightarrow{\text{curl}} & \mathbb{J}^1(x) \otimes \mathbb{M}^3 & \xrightarrow{\text{div}} & \mathbb{J}^0(x) \otimes \mathbb{R}^3
 \end{array}$$

We specify these mappings as follows. For convenience, we use (I, J) to denote the entry with I -th row and J -th column.

Here the other mappings in the first column are:

- (1,1) to (3,1) : from $P_k(T)$ to $\begin{pmatrix} P_k(e) \\ P_{k-1}(e) \otimes \mathbb{R}^2 \\ P_{k-2}(e) \otimes \mathbb{M}^2 \end{pmatrix}$ is $u \mapsto \begin{pmatrix} u|_e \\ [(\partial_{\mathbf{n}_i} u)|_e]_i \\ [(\partial_{\mathbf{n}_i} \mathbf{n}_j u)|_e]_{ij} \end{pmatrix}$
with $i, j = 1, 2$;
- (1,1) to (4,1): from $P_k(T)$ to $\mathbb{J}^4(x)$ is $u \mapsto u(x) \oplus \nabla u(x) \oplus \nabla^2 u(x) \oplus \nabla^3 u(x) \oplus \nabla^4 u(x)$;
- (2,1) to (3,1): from $\begin{pmatrix} P_k(f) \\ P_{k-1}(f) \end{pmatrix}$ to $\begin{pmatrix} P_k(e) \\ P_{k-1}(e) \otimes \mathbb{R}^2 \\ P_{k-2}(e) \otimes \mathbb{M}^2 \end{pmatrix}$ is

$$(a_0, a_1) \mapsto \begin{pmatrix} a_0|_e \\ \frac{\partial a_0}{\partial \mathbf{n}_{f,e}}|_e \Pi_f(\mathbf{t}) + a_1|_e \Pi_f(\mathbf{n}_{f,e}) \\ \frac{\partial^2 a_0}{\partial \mathbf{n}_{f,e}^2}|_e \Pi_f(\mathbf{t}) \Pi_f(\mathbf{n}_{f,e})^T + \frac{\partial a_1}{\partial \mathbf{n}_{f,e}}|_e \Pi_f(\mathbf{n}_{f,e}) \Pi_f(\mathbf{n}_{f,e})^T \end{pmatrix};$$

- (2,1) to (4,1): from $\begin{pmatrix} P_k(f) \\ P_{k-1}(f) \end{pmatrix}$ to $\mathbb{J}^4(x)$ is $(a_0, a_1) \mapsto a_0(x) \oplus (E_3 \nabla_f a_0(x) + a_1(x) \mathbf{e}_3) \oplus (E_3 \nabla_f^2 a_0(x) + E_3 \nabla_f a_1(x) \otimes \mathbf{e}_3) \oplus (E_3 \nabla_f^3 a_0(x) + E_3 \nabla_f^2 a_1(x) \otimes \mathbf{e}_3) \oplus (E_3 \nabla_f^4 a_0(x) + E_3 \nabla_f^3 a_1(x) \otimes \mathbf{e}_3)$;
- (3,1) to (4,1): from $\begin{pmatrix} P_k(e) \\ P_{k-1}(e) \otimes \mathbb{R}^2 \\ P_{k-2}(e) \otimes \mathbb{M}^2 \end{pmatrix}$ to $\mathbb{J}^4(x)$ is $(a_0, a_1, a_2) \mapsto a_0(x) \oplus (\frac{d}{dt} a_0(x) \mathbf{e}_3 + E_3 a_1(x)) \oplus (\frac{d^2}{dt^2} a_0(x) \mathbf{e}_3^{\otimes 2} + E_3 \frac{d}{dt} a_1(x) \otimes \mathbf{e}_3 + E_3 a_2(x)) \oplus (\frac{d^3}{dt^3} a_0(x) \mathbf{e}_3^{\otimes 3} + E_3 \frac{d^2}{dt^2} a_1(x) \otimes \mathbf{e}_3^{\otimes 2} + E_3 \frac{d}{dt} a_2(x) \otimes \mathbf{e}_3) \oplus (\frac{d^4}{dt^4} a_0(x) \mathbf{e}_3^{\otimes 4} + E_3 \frac{d^3}{dt^3} a_1(x) \otimes \mathbf{e}_3^{\otimes 3} + E_3 \frac{d^2}{dt^2} a_2(x) \otimes \mathbf{e}_3^{\otimes 2})$, here $\mathbf{e}_3^{\otimes k}$ represents the k -th tensor power of the vector $\mathbf{e}_3$.

The other mappings in the second column are:

- (1,2) to (3,2): from $P_{k-2}(T) \otimes \mathbb{S}^3$ to $\begin{pmatrix} P_{k-2}(e) \\ P_{k-2}(e) \otimes \mathbb{R}^2 \\ P_{k-2}(e) \otimes \mathbb{M}^2 \end{pmatrix}$ is $\boldsymbol{\sigma} \mapsto \begin{pmatrix} \mathbf{t}^T \boldsymbol{\sigma} |_e \mathbf{t} \\ [\mathbf{n}_i^T \boldsymbol{\sigma} |_e \mathbf{t}]_i \\ [\mathbf{n}_i^T \boldsymbol{\sigma} |_e \mathbf{n}_j]_{ij} \end{pmatrix}$ with $i, j = 1, 2$;
- (1,2) to (4,2): from $P_{k-2}(T) \otimes \mathbb{S}^3$ to $\mathbb{J}^2(x) \otimes \mathbb{M}^3$ is $\boldsymbol{\sigma} \mapsto \boldsymbol{\sigma}(x) \oplus \nabla \boldsymbol{\sigma}(x) \oplus \nabla^2 \boldsymbol{\sigma}(x)$;
- (2,2) to (3,2): from $\begin{pmatrix} P_{k-2}(f) \otimes \mathbb{S}^2 \\ P_{k-2}(f) \otimes \mathbb{R}^2 \end{pmatrix}$ to $\begin{pmatrix} P_{k-2}(e) \\ P_{k-2}(e) \otimes \mathbb{R}^2 \\ P_{k-2}(e) \otimes \mathbb{M}^2 \end{pmatrix}$ is $(a_0, a_1) \mapsto \begin{pmatrix} \Pi_f(\mathbf{t})^T a_0 |_e \Pi_f(\mathbf{t}) \\ (\Pi_f(\mathbf{t})^T a_0 |_e \Pi_f(\mathbf{n}_{f,e})) \Pi_f(\mathbf{t}) + (\Pi_f(\mathbf{t})^T a_1 |_e) \Pi_f(\mathbf{n}_{f,e}) \\ (\Pi_f(\mathbf{n}_{f,e})^T a_0 |_e \Pi_f(\mathbf{n}_{f,e})) \Pi_f(\mathbf{t}) \Pi_f(\mathbf{n}_{f,e})^T + (\Pi_f(\mathbf{n}_{f,e})^T a_1 |_e) \Pi_f(\mathbf{n}_{f,e}) \Pi_f(\mathbf{n}_{f,e})^T \end{pmatrix}$;
- (2,2) to (4,2): from $\begin{pmatrix} P_{k-2}(f) \otimes \mathbb{S}^2 \\ P_{k-2}(f) \otimes \mathbb{R}^2 \end{pmatrix}$ to $\mathbb{J}^2(x) \otimes \mathbb{M}^3$ is $(a_0, a_1) \mapsto (E_3 a_0(x) + E_3 a_1(x) \otimes \mathbf{e}_3) \oplus (E_3 \nabla_f a_0(x) + E_3 \nabla_f a_1(x) \otimes \mathbf{e}_3) \oplus (E_3 \nabla_f^2 a_0(x) + E_3 \nabla_f^2 a_1(x) \otimes \mathbf{e}_3)$;
- (3,2) to (4,2): from $\begin{pmatrix} P_{k-2}(e) \\ P_{k-2}(e) \otimes \mathbb{R}^2 \\ P_{k-2}(e) \otimes \mathbb{M}^2 \end{pmatrix}$ to $\mathbb{J}^2(x) \otimes \mathbb{M}^3$ is $(a_0, a_1, a_2) \mapsto (a_0(x) \mathbf{e}_3^{\otimes 2} + E_3 a_1(x) \otimes \mathbf{e}_3 + E_3 a_2(x)) \oplus (\frac{d}{dt} a_0(x) \mathbf{e}_3^{\otimes 3} + E_3 \frac{d}{dt} a_1(x) \otimes \mathbf{e}_3^{\otimes 2} + E_3 \frac{d}{dt} a_2(x) \otimes \mathbf{e}_3) \oplus (\frac{d^2}{dt^2} a_0(x) \mathbf{e}_3^{\otimes 4} + E_3 \frac{d^2}{dt^2} a_1(x) \otimes \mathbf{e}_3^{\otimes 3} + E_3 \frac{d^2}{dt^2} a_2(x) \otimes \mathbf{e}_3^{\otimes 2})$

The hidden mappings in the third column are

- (1,3) to (4,3): from $P_{k-3}(T) \otimes \mathbb{T}^3$ to $\mathbb{J}^1(x) \otimes \mathbb{M}^3$ is $\boldsymbol{\tau} \mapsto \boldsymbol{\tau}(x) \oplus \nabla \boldsymbol{\tau}(x)$;
- (2,3) to (4,3): from $\begin{pmatrix} P_{k-3}(f) \otimes \mathbb{R}^2 \\ P_{k-3}(f) \otimes \mathbb{R} \end{pmatrix}$ to $\mathbb{J}^1(x) \otimes \mathbb{M}^3$ is $(a_0, a_1) \mapsto (E_3 a_0(x) \otimes \mathbf{e}_3 + a_1(x) \mathbf{e}_3^{\otimes 2}) \oplus (E_3 \nabla_f a_0(x) \otimes \mathbf{e}_3 + E_3 \nabla_f a_1(x) \otimes \mathbf{e}_3^{\otimes 2})$.

Proposition 4.1. *The FECTS (4.2) is compatible with respect to the generalized currents for Hessian complex (4.1).*

Proof. First, we show the localizable property. Let

$$\tilde{\Upsilon}_T : P_{k-4}(T) \otimes \mathbb{R}^3 \rightarrow \mathcal{Z} : \mathbf{q} \mapsto \sum_{i=1}^4 \left(\int_T \mathbf{q} \cdot \boldsymbol{\psi}_i \right) z_i$$

and

$$\tilde{\Upsilon}_f : \begin{pmatrix} P_{k-3}(f) \otimes \mathbb{R}^2 \\ P_{k-3}(f) \otimes \mathbb{R} \end{pmatrix} \rightarrow \mathcal{Z} : (a_0, a_1) = \sum_{i=1}^4 \left(\int_f a_0 \cdot (\Pi_f \boldsymbol{\psi}_i) + a_1 (\boldsymbol{\psi}_i \cdot \mathbf{n}_f) \right) z_i,$$

and

$$\tilde{\Upsilon}_e : \begin{pmatrix} P_{k-2}(e) \\ P_{k-2}(e) \otimes \mathbb{R}^2 \\ P_{k-2}(e) \otimes \mathbb{M}^2 \end{pmatrix} \rightarrow \mathcal{Z} : (a_0, a_1, a_2) \mapsto \sum_{i=1}^4 \left(\int_e a_0 (\boldsymbol{\psi}_i \cdot \mathbf{t}) + a_1 \cdot (N_e \boldsymbol{\psi}_i) \right) z_i,$$

and

$$\tilde{\Upsilon}_x : \mathbb{J}^4 \rightarrow \mathcal{Z} : a_0 \oplus a_1 \oplus a_2 \oplus a_3 \oplus a_4 \mapsto \sum_{i=1}^4 \left((\boldsymbol{\psi}_i \cdot a_1)(x) - \frac{1}{3}(\operatorname{div} \boldsymbol{\psi}_i \cdot a_0)(x) \right) z_i$$

for any $T \in \mathcal{K}, f \in \mathcal{F}, e \in \mathcal{E}$ and $x \in \mathcal{V}$. Then the generalized currents can be localized.

The bubble complex in cell is shown in [32]. The bubble complex on faces are:

$$0 \longrightarrow \begin{pmatrix} \mathring{P}_k^{(4,2)}(f) \\ \mathring{P}_{k-1}^{(3,1)}(f) \end{pmatrix} \xrightarrow{\mathbf{d}(\operatorname{hess}_f, \operatorname{grad}_f)} \begin{pmatrix} \mathring{P}_{k-2}^{(2,0)}(f) \otimes \mathbb{S}^2 \\ \mathring{P}_{k-2}^{(2,0)}(f) \otimes \mathbb{R}^2 \end{pmatrix} \xrightarrow{\mathbf{d}(\operatorname{rot}_f, \operatorname{rot}_f)} \begin{pmatrix} \mathring{P}_{k-3}^{(1,-1)}(f) \otimes \mathbb{R}^2 \\ \mathring{P}_{k-3}^{(1,-1)}(f) \otimes \mathbb{R} \end{pmatrix} / \begin{pmatrix} RT(f) \\ \mathbb{R} \end{pmatrix} \longrightarrow 0,$$

here $RT(f) := \{a + \mathbf{b} \cdot \Pi_f \mathbf{x}, a \in \mathbb{R}, b \in \mathbb{R}^2\}$. The bubble complex on edges are:

$$0 \longrightarrow \begin{pmatrix} \mathring{P}_k^{(4)}(e) \\ \mathring{P}_{k-1}^{(3)}(e) \otimes \mathbb{R}^2 \\ \mathring{P}_{k-2}^{(2)}(e) \otimes \mathbb{S}^2 \end{pmatrix} \xrightarrow{\mathbf{d}(\partial_{\mathbf{t}\mathbf{t}}, \partial_{\mathbf{t}}, \operatorname{id})} \begin{pmatrix} \mathring{P}_k^{(2)}(e) \\ \mathring{P}_{k-1}^{(2)}(e) \otimes \mathbb{R}^2 \\ \mathring{P}_{k-2}^{(2)}(e) \otimes \mathbb{S}^2 \end{pmatrix} / \begin{pmatrix} P_1(e) \\ \mathbb{R}^2 \\ 0 \end{pmatrix} \longrightarrow 0.$$

The bubble complex at vertex is isomorphic to

$$0 \longrightarrow P_4(\Omega)/P_1(\Omega) \xrightarrow{\operatorname{hess}} P_2(\Omega) \otimes \mathbb{S}^3 \xrightarrow{\operatorname{curl}} P_1(\Omega) \otimes \mathbb{T}^3 \xrightarrow{\operatorname{div}} P_0(\Omega) \otimes \mathbb{R}^3 \longrightarrow 0.$$

It can be readily checked that the above complexes are exact. $\square$

The global space of the finite element system is

$$(4.3) \quad 0 \longrightarrow \mathbf{Z}_k \xrightarrow{\operatorname{hess}} \mathbf{HL}_{k-2}^{\operatorname{curl}} \xrightarrow{\operatorname{curl}} \mathbf{HL}_{k-3}^{\operatorname{div}} \xrightarrow{\operatorname{div}} \mathbf{DG}_{k-4}^{(0)} \otimes \mathbb{R}^3 \longrightarrow 0.$$

Here, each space in this complex is specified as follows: $\mathbf{Z}_k$ is the space of Zenisek–Zhang C^1 element in three dimensions:

$$\mathbf{Z}_k := \{u : u|_K \in P_k(K), u \in C^4(\mathcal{V}) \cap C^2(\mathcal{E}) \cap C^1(\mathcal{F})\}.$$

$\mathbf{HL}_{k-2}^{\text{curl}}$ is the space of Hu–Liang $H(\text{curl}; \mathbb{S}^3)$ element:

$$\mathbf{HL}_{k-2}^{\text{curl}} := \{\boldsymbol{\sigma} : \boldsymbol{\sigma}|_K \in P_{k-2}(K) \otimes \mathbb{S}^3, \boldsymbol{\sigma} \in C^2(\mathcal{V}) \cap C^0(\mathcal{E}), \Pi_f \boldsymbol{\sigma} \in C^0(\mathcal{F})\}.$$

$\mathbf{HL}_{k-3}^{\text{div}}$ is the space of Hu–Liang $H(\text{div}; \mathbb{T}^3)$ element:

$$\mathbf{HL}_{k-3}^{\text{div}} := \{\boldsymbol{\tau} : \boldsymbol{\tau}|_K \in P_{k-3}(K) \otimes \mathbb{T}^3, \boldsymbol{\tau} \in C^1(\mathcal{V}), \boldsymbol{\tau} \mathbf{n} \in C^0(\mathcal{F})\}.$$

$\mathbf{DG}_{k-4}^{(0)} \otimes \mathbb{R}^3$ is defined by

$$\mathbf{DG}_{k-4}^{(0)} \otimes \mathbb{R}^3 := \{\mathbf{q} : \mathbf{q}|_K \in P_{k-4}(K) \otimes \mathbb{R}^3, \mathbf{q} \in C^0(\mathcal{V})\}.$$

As a consequence, the cohomology of (4.3) is isomorphic to $P_1(\Omega) \otimes \mathcal{H}_{dR}(\Omega)$.

5. PROOFS

This section gives the proofs of the Theorem 2.1 and 2.2. The proof of Theorem 2.3 comes from scaling argument, and we omit the details here.

5.1. Skeletal degrees of freedom, harmonic forms and cohomologies. In this subsection, we will prove the cohomology of the finite element complex in Theorem 2.1, provided that the FECTS is compatible with respect to the generalized currents. The goal of this section is to show a finer structure of the global finite element space and identify the harmonic form. This is stronger than the cohomological result we obtained in the previous section.

We first introduce the harmonic inner product, which have been intensively studied and used in [4, 2].

Definition 5.1 (Harmonic inner product). *Suppose that $X' \xrightarrow{d'} X \xrightarrow{d} X''$ is an exact sequence of Hilbert spaces. Then, the following bilinear form:*

$$\langle u, v \rangle_X = (\mathcal{P}_{d'(X')} u, \mathcal{P}_{d'(X')} v)_X + (du, dv)_{X''},$$

is an inner product. Here $\mathcal{P}_{d'(X')}$ is the projection operator onto $d'(X')$.

For each τ , we denote by

$$\tilde{B}^k(\tau) = \begin{cases} B^k(\tau) \cap \ker(\tilde{\Upsilon}_\tau), & k = \dim \tau, \\ B^k(\tau), & \text{otherwise.} \end{cases}$$

Then from (2.3), it can be shown that

$$(5.1) \quad 0 \rightarrow \tilde{B}^0(\tau) \xrightarrow{d} \tilde{B}^1(\tau) \xrightarrow{d} \cdots \xrightarrow{d} \tilde{B}^n(\tau) \rightarrow 0$$

is an exact complex. Let $\mathcal{N}_{\tau,0}$ be the dimension of the bubble space $\tilde{B}^0(\tau)$. Moreover, we set $\mathcal{N}_{\tau,-1} = 0$. Note that each bubble space $\tilde{B}^k(\tau)$ can be seen as a

Hilbert space with L^2 -norm on τ . Then we can show a normalized orthogonal basis of $\tilde{B}^k(\tau)$ with respect to the harmonic inner products.

Lemma 5.1 (Orthogonal basis of bubble space). *For each $\tau \in \mathcal{T}$, there exists a normalized orthogonal basis $\{\phi_{\tau,0}^i\}_{1 \leq i \leq \mathcal{N}_{\tau,0}}$ of bubble space $\tilde{B}^0(\tau)$ such that*

$$\langle \phi_{\tau,0}^i, \phi_{\tau,0}^j \rangle_{\tilde{B}^0(\tau)} = (d\phi_{\tau,0}^i, d\phi_{\tau,0}^j)_\tau = \delta_{ij}, \quad \forall 1 \leq i, j \leq \mathcal{N}_{\tau,0}.$$

For each $1 \leq k \leq n$, there exists a set $\{\phi_{\tau,k}^i\}_{1 \leq i \leq \mathcal{N}_{\tau,k}}$ such that $\{d\phi_{\tau,k-1}^i\}_{1 \leq i \leq \mathcal{N}_{\tau,k-1}} \cup \{\phi_{\tau,k}^i\}_{1 \leq i \leq \mathcal{N}_{\tau,k}}$ form a basis of bubble space $\tilde{B}^k(\tau)$, and satisfies that

$$\begin{aligned} \langle d\phi_{\tau,k-1}^i, \phi_{\tau,k}^j \rangle_{\tilde{B}^k(\tau)} &= (d\phi_{\tau,k-1}^i, \phi_{\tau,k}^j)_\tau = 0, \quad \forall 1 \leq i \leq \mathcal{N}_{\tau,k-1}, 1 \leq j \leq \mathcal{N}_{\tau,k}, \\ \langle d\phi_{\tau,k-1}^i, d\phi_{\tau,k-1}^j \rangle_{\tilde{B}^k(\tau)} &= (d\phi_{\tau,k-1}^i, d\phi_{\tau,k-1}^j)_\tau = \delta_{ij}, \quad \forall 1 \leq i, j \leq \mathcal{N}_{\tau,k-1}, \\ \langle \phi_{\tau,k}^i, \phi_{\tau,k}^j \rangle_{\tilde{B}^k(\tau)} &= (d\phi_{\tau,k}^i, d\phi_{\tau,k}^j)_\tau = \delta_{ij}, \quad \forall 1 \leq i, j \leq \mathcal{N}_{\tau,k}. \end{aligned}$$

Proof. It can be deduced from induction and the exactness of the bubble complex. $\square$

Lemma 5.2 (Local degrees of freedom). *For each $\tau \in \mathcal{T}$ and $0 \leq k \leq n$, a basis of the dual space of $B^k(\tau)$ can be given as*

$$B^k(\tau)^\vee = \begin{cases} \{\langle \cdot, \phi \rangle_{\tilde{B}^k(\tau)}\}, & \phi \in \{d\phi_{\tau,k-1}^i\}_i \cup \{\phi_{\tau,k}^i\}_i, \quad \text{if } k \neq \dim \tau, \\ \{\langle \cdot, \phi \rangle_{\tilde{B}^k(\tau)}\} \cup \tilde{\Upsilon}_\tau(\cdot), & \phi \in \{d\phi_{\tau,k-1}^i\}_i \cup \{\phi_{\tau,k}^i\}_i, \quad \text{if } k = \dim \tau. \end{cases}$$

Then from the geometric decomposition, for any $w \in A^k(\sigma)$, $\sigma \in \mathcal{T}_n$, it can be uniquely determined by

- $\langle \text{tr}_{\sigma \rightarrow \tau} w, \phi \rangle_{\tilde{B}^k(\tau)}$ for any $\phi \in \{d\phi_{\tau,k-1}^i\}_i \cup \{\phi_{\tau,k}^i\}_i$ and $\tau \trianglelefteq \sigma$,
- $\Upsilon_\tau(w)$ for any $\tau \trianglelefteq \sigma$ with $\dim \tau = k$.

Proof. This is directly from the orthogonal basis of bubble spaces and $\tilde{\Upsilon}_\tau(\text{tr}_{\sigma \rightarrow \tau} w) = \Upsilon_\tau(w)$. $\square$

The degrees of freedom provide a direct sum decomposition of the global finite element space

$$\mathbf{A}^k = \mathbf{S}^k(\mathcal{T}) \oplus \bigoplus_{\tau \in \mathcal{T}} \mathbb{B}^k(\tau).$$

Here, the space $\mathbb{B}^k(\tau)$ consists of all functions $w \in \mathbf{A}^k$ such that

- $\langle \text{tr}_{\sigma \rightarrow \eta} w, \phi \rangle_{\tilde{B}^k(\eta)} = 0$ for any $\phi \in \{d\phi_{\eta,k-1}^i\}_i \cup \{\phi_{\eta,k}^i\}_i$ and $\eta \in \mathcal{T}, \eta \neq \tau$;
- $\Upsilon_\eta(w) = 0$ for any $\eta \in \mathcal{T}_k$.

And the space $\mathbf{S}^k(\mathcal{T})$ consists of all functions $w \in \mathbf{A}^k$ such that

- $\langle \text{tr}_{\sigma \rightarrow \eta} w, \phi \rangle_{\tilde{B}^k(\eta)} = 0$ for any $\phi \in \{d\phi_{\eta,k-1}^i\}_i \cup \{\phi_{\eta,k}^i\}_i$ and $\eta \in \mathcal{T}$.

Remark 5.1. Note that from the definition of $\mathbf{A}^k$, it holds that for any $w \in \mathbf{A}^k$, $\text{tr}_{\sigma \rightarrow \tau} w|_\sigma = \text{tr}_{\sigma' \rightarrow \tau} w|_{\sigma'}$ for all $\tau \trianglelefteq \sigma, \sigma'$ and $\sigma, \sigma' \in \mathcal{T}_n$. Therefore there is no confusion to use $\text{tr}_{\sigma \rightarrow \eta} w$.

Lemma 5.3 (Closedness). *For any $\tau \in \mathcal{T}$, $0 \leq k \leq n-1$ and $w \in \mathbb{B}^k(\tau)$, it holds that $dw \in \mathbb{B}^{k+1}(\tau)$; for any $v \in \mathbf{S}^k(\mathcal{T})$, it holds that $dv \in \mathbf{S}^{k+1}(\mathcal{T})$.*

Proof. For any $w \in \mathbb{B}^k(\tau)$, since $dw \in \mathbf{A}^{k+1}$ and

$$\Upsilon_\theta(dw) = \Upsilon_{\partial\theta}(w) = \sum_{\eta \leq_1 \theta} \mathcal{O}(\eta, \theta) \Upsilon_\eta(w) = 0, \quad \text{for all } \theta \in \mathcal{T}_{k+1}.$$

It suffices to prove that $\langle \text{tr}_{\sigma \rightarrow \eta} dw, \phi \rangle_{\tilde{B}^{k+1}(\eta)} = 0$ for any $\phi \in \{d\phi_{\eta,k}^i\}_i \cup \{\phi_{\eta,k+1}^i\}_i$ and $\eta \in \mathcal{T}, \eta \neq \tau$. Note that

$$\begin{aligned} \langle \text{tr}_{\sigma \rightarrow \eta} dw, \phi \rangle_{\tilde{B}^{k+1}(\eta)} &= \langle d \text{tr}_{\sigma \rightarrow \eta} w, \phi \rangle_{\tilde{B}^{k+1}(\eta)} = (d \text{tr}_{\sigma \rightarrow \eta} w, \mathcal{P}_{d(\tilde{B}^k(\eta))} \phi)_\eta = 0, \\ &\quad \text{for all } \phi \in \{d\phi_{\eta,k}^i\}_i \cup \{\phi_{\eta,k+1}^i\}_i, \end{aligned}$$

where we use the fact $d \circ \text{tr}_{\sigma \rightarrow \eta} = \text{tr}_{\sigma \rightarrow \eta} \circ d$. Hence we can derive that $dw \in \mathbb{B}^{k+1}(\tau)$.

For any $v \in \mathbf{S}^k(\mathcal{T})$, it is obvious that $dv \in \mathbf{A}^{k+1}$, and from a similar argument, we can get that

$$\langle \text{tr}_{\sigma \rightarrow \eta} dv, \phi \rangle_{\tilde{B}^{k+1}(\eta)} = 0, \quad \text{for all } \phi \in \{d\phi_{\eta,k}^i\}_i \cup \{\phi_{\eta,k+1}^i\}_i, \eta \in \mathcal{T},$$

hence $dv \in \mathbf{S}^{k+1}(\mathcal{T})$, this completes the proof. $\square$

From the above lemma, we can get the decomposition of the finite element complex.

Proposition 5.1 (Direct sum decomposition of finite element complexes). *The complex*

$$(5.2) \quad 0 \longrightarrow \mathbf{A}^0 \xrightarrow{d} \mathbf{A}^1 \xrightarrow{d} \cdots \xrightarrow{d} \mathbf{A}^n \longrightarrow 0$$

is a direct sum of

$$(5.3) \quad 0 \longrightarrow \mathbf{S}^0(\mathcal{T}) \xrightarrow{d} \mathbf{S}^1(\mathcal{T}) \xrightarrow{d} \cdots \xrightarrow{d} \mathbf{S}^n(\mathcal{T}) \longrightarrow 0$$

and

$$(5.4) \quad 0 \longrightarrow \mathbb{B}^0(\tau) \xrightarrow{d} \mathbb{B}^1(\tau) \xrightarrow{d} \cdots \xrightarrow{d} \mathbb{B}^n(\tau) \longrightarrow 0$$

with all $\tau \in \mathcal{T}$.

In the remaining part of this subsection, we will discuss the cohomology of the skeletal complex (5.3) and the bubble complex (5.4). First, we introduce an extension operator.

Definition 5.2 (Extension of bubble functions). *For any $\tau \in \mathcal{T}$ and $0 \leq k \leq n$, we define an extension operator $\mathbf{E}_\tau^k : \tilde{B}^k(\tau) \rightarrow \mathbb{B}^k(\tau) \subset \mathbf{A}^k$ by: For any $b \in \tilde{B}^k(\tau)$, set the degrees of freedom of $\mathbf{E}_\tau^k b$ as*

- $\langle \text{tr}_{\sigma \rightarrow \tau} \mathbf{E}_\tau^k b, \phi \rangle_{\tilde{B}^k(\tau)} = \langle b, \phi \rangle_{\tilde{B}^k(\tau)}$ for any $\phi \in \{d\phi_{\tau,k-1}^i\}_i \cup \{\phi_{\tau,k}^i\}_i$;
- $\langle \text{tr}_{\sigma \rightarrow \eta} \mathbf{E}_\tau^k b, \phi \rangle_{\tilde{B}^k(\eta)} = 0$ for any $\phi \in \{d\phi_{\eta,k-1}^i\}_i \cup \{\phi_{\eta,k}^i\}_i$ and $\eta \in \mathcal{T}, \eta \neq \tau$.
- $\Upsilon_\eta(\mathbf{E}_\tau^k b) = 0$ for any $\eta \in \mathcal{T}_k$.

From the definition of E_τ^k , we can readily obtain that

$$\text{tr}_{\sigma \rightarrow \eta} E_\tau^k b = 0, \quad \forall b \in \tilde{B}^k(\tau) \text{ and } \eta \trianglelefteq \tau, \eta \neq \tau.$$

This leads to

$$\text{tr}_{\tau \rightarrow \eta} \circ \text{tr}_{\sigma \rightarrow \tau} E_\tau^k b = 0,$$

and $\Upsilon_\eta(E_\tau^k b) = 0$ for any $\eta \in \mathcal{T}_k$, which implies that

$$\text{tr}_{\sigma \rightarrow \tau} E_\tau^k b \in \tilde{B}^k(\tau), \quad \forall b \in \tilde{B}^k(\tau).$$

Hence, from the definition of E_τ^k , it leads to that

$$\text{tr}_{\sigma \rightarrow \tau} E_\tau^k b = b, \quad \forall b \in \tilde{B}^k(\tau).$$

Then $\text{tr}_{\sigma \rightarrow \tau}$ gives an isomorphism from $\mathbb{B}^k(\tau)$ to $\tilde{B}^k(\tau)$ with the inverse operator E_τ^k .

Then we have the following result, telling us that the bubble complex $\mathbb{B}^\bullet$ contributes a trivial cohomology.

Proposition 5.2 (Exactness of bubble functions). *For any $\tau \in \mathcal{T}$ and $0 \leq k \leq n$, the trace operator $\text{tr}_{\sigma \rightarrow \tau}$ gives an isomorphism from $\mathbb{B}^k(\tau)$ to $\tilde{B}^k(\tau)$. The inverse operator is given by E_τ^k . Moreover, it holds that the differential operator d commutes with E_τ^k . Then we get that the bubble sequence (5.4)*

$$0 \longrightarrow \mathbb{B}^0(\tau) \xrightarrow{d} \mathbb{B}^1(\tau) \xrightarrow{d} \cdots \xrightarrow{d} \mathbb{B}^n(\tau) \longrightarrow 0$$

is exact.

Proof. Note that the commuting property is implied by the commuting property of tr . Hence E_τ^k induces an isomorphism from the cohomology of bubble complex (5.1) to the cohomology of bubble sequence (5.4). Since the bubble complex (5.1) is exact. Then we get the exactness of bubble complex (5.1). $\square$

In what follows, we show that the skeletal complex gives the nontrivial cohomology.

Definition 5.3 (Extension of the skeletal functions). *We define the extension operator for the skeletal complex: $E_S^k : \mathcal{Z} \otimes \mathcal{T}_k \rightarrow \mathcal{S}^k(\mathcal{T})$ such that*

$$\Upsilon_\tau(E_S^k(z \otimes \tau)) = \delta_{\tau\tau'} z.$$

Clearly, E_S^k is an isomorphism, and therefore invertible. The inverse operator L_S^k can be given as follows:

$$L_S^k(w) = \sum_{\tau \in \mathcal{T}_k} \Upsilon_\tau(w) \otimes \tau.$$

Proposition 5.3. *The skeletal lifting operator $E_S^\bullet$ gives an isomorphism from the simplicial cochain complex*

$$0 \rightarrow \mathcal{Z} \otimes \mathcal{T}_0 \xrightarrow{\partial^\vee} \mathcal{Z} \otimes \mathcal{T}_1 \xrightarrow{\partial^\vee} \cdots \xrightarrow{\partial^\vee} \mathcal{Z} \otimes \mathcal{T}_n \rightarrow 0.$$

to the skeletal complex.

Proof. It can be readily shown that $\mathbf{L}_5^{k+1} \circ d \circ \mathbf{E}_5^k = \partial^\vee$. Therefore, we conclude the result. $\square$

Now we can conclude the result in this subsection.

Theorem 5.1. *The cohomology of $\mathbf{A}^\bullet$ is isomorphic to $\mathcal{Z} \otimes \mathcal{H}_{dR}(\Omega)$. Moreover, the isomorphism is induced by $\mathbf{L}_5^\bullet$.*

5.2. Locally bounded interpolation operators. In this subsection, we introduce the locally bounded interpolation operators in Theorem 2.2. The most part of the idea of this part originates from [2].

By the degrees of freedom of finite elements, for each $w \in \mathbf{A}^k$, it has an decomposition as:

$$(5.5) \quad \begin{aligned} w = & \sum_{\tau \in \mathcal{T}_k} \sum_{i=1}^{\dim \mathcal{Z}} (\Upsilon_\tau(w), z_i)_\mathcal{Z} \mathbf{E}_S^k(z_i \otimes \tau) + \sum_{\tau \in \mathcal{T}} \sum_{i=1}^{\mathcal{N}_{\tau,k}} \langle \text{tr}_{\sigma \rightarrow \tau} w, \phi_{\tau,k}^i \rangle_{\tilde{B}^k(\tau)} \mathbf{E}_\tau^k(\phi_{\tau,k}^i) \\ & + \sum_{\tau \in \mathcal{T}} \sum_{i=1}^{\mathcal{N}_{\tau,k-1}} \langle \text{tr}_{\sigma \rightarrow \tau} w, d\phi_{\tau,k-1}^i \rangle_{\tilde{B}^k(\tau)} \mathbf{E}_\tau^k(d\phi_{\tau,k-1}^i). \end{aligned}$$

here $\{z_i\}_i$ is a normalized orthogonal basis of $\mathcal{Z}$ and $(\cdot, \cdot)_\mathcal{Z}$ denotes the inner product of $\mathcal{Z}$. The aim of the construction is to consider the following form

$$(5.6) \quad \pi^k(w) = \pi_S^k(w) + \pi_B^k(w),$$

such that

$$(5.7) \quad \pi_S^k(w) = \sum_{\tau \in \mathcal{T}_k} \sum_{i=1}^{\dim \mathcal{Z}} (w, \Xi_\tau^i)_{\text{star}^1(\tau)} \mathbf{E}_S^k(z_i \otimes \tau),$$

and

$$(5.8) \quad \pi_B^k(w) = \sum_{\tau \in \mathcal{T}} \sum_{i=1}^{\mathcal{N}_{\tau,k}} (w, d^* \Omega_{\tau,k+1}^i)_{\text{star}(\tau)} \mathbf{E}_\tau^k(\phi_{\tau,k}^i) + \sum_{\tau \in \mathcal{T}} \sum_{i=1}^{\mathcal{N}_{\tau,k-1}} (w, \Omega_{\tau,k}^i)_{\text{star}(\tau)} \mathbf{E}_\tau^k(d\phi_{\tau,k-1}^i).$$

with some $\Xi_\tau^i \in H_0^M(\text{star}^1(\tau)) \otimes \mathbb{X}^{\dim \tau}$ and $\Omega_{\tau,k}^i \in C_0^{2M}(\text{star}(\tau)) \otimes \mathbb{X}^k$. In what follows, we consider the properties of π_S and π_B respectively. One useful result is that the above construction (5.8) satisfies the commuting property automatically.

Proposition 5.4. *It holds that $\pi_B^k(dw) = d\pi_B^{k-1}(w)$ for $w \in L^2(\Omega) \otimes \mathbb{X}^{k-1}$ with $dw \in L^2(\Omega) \otimes \mathbb{X}^k$.*

Proof. By definition, the left-hand side is

$$\begin{aligned}
\pi_{\mathbf{B}}^k(dw) &= \sum_{\tau \in \mathcal{T}} \sum_i (dw, d^* \Omega_{\tau, k+1}^i)_{\text{star}(\tau)} \mathbf{E}_{\tau}^k(\phi_{\tau, k}^i) + \sum_{\tau \in \mathcal{T}} \sum_i (dw, \Omega_{\tau, k}^i)_{\text{star}(\tau)} \mathbf{E}_{\tau}^k(d\phi_{\tau, k-1}^i) \\
&= \sum_{\tau \in \mathcal{T}} \sum_i (w, d^* d^* \Omega_{\tau, k+1}^i)_{\text{star}(\tau)} \mathbf{E}_{\tau}^k(\phi_{\tau, k}^i) + \sum_{\tau \in \mathcal{T}} \sum_i (w, d^* \Omega_{\tau, k}^i)_{\text{star}(\tau)} \mathbf{E}_{\tau}^k(d\phi_{\tau, k-1}^i) \\
&= \sum_{\tau \in \mathcal{T}} \sum_i (w, d^* \Omega_{\tau, k}^i)_{\text{star}(\tau)} \mathbf{E}_{\tau}^k(d\phi_{\tau, k-1}^i).
\end{aligned}$$

The right-hand side is

$$\begin{aligned}
d\pi_{\mathbf{B}}^{k-1}(w) &= \sum_{\tau \in \mathcal{T}} \sum_i (w, d^* \Omega_{\tau, k}^i)_{\text{star}(\tau)} d\mathbf{E}_{\tau}^{k-1}(\phi_{\tau, k-1}^i) + \sum_{\tau \in \mathcal{T}} \sum_i (w, \Omega_{\tau, k-1}^i)_{\text{star}(\tau)} d\mathbf{E}_{\tau}^{k-1}(d\phi_{\tau, k-2}^i) \\
&= \sum_{\tau \in \mathcal{T}} \sum_i (w, d^* \Omega_{\tau, k}^i)_{\text{star}(\tau)} \mathbf{E}_{\tau}^k(d\phi_{\tau, k-1}^i) + \sum_{\tau \in \mathcal{T}} \sum_i (w, \Omega_{\tau, k-1}^i)_{\text{star}(\tau)} \mathbf{E}_{\tau}^k(d \circ d\phi_{\tau, k-2}^i) \\
&= \sum_{\tau \in \mathcal{T}} \sum_i (w, d^* \Omega_{\tau, k}^i)_{\text{star}(\tau)} \mathbf{E}_{\tau}^k(d\phi_{\tau, k-1}^i),
\end{aligned}$$

where we use the fact that d commutes with $\mathbf{E}_{\tau}^{\bullet}$. This completes the proof. $\square$

The above result shows that the commuting property is automatically valid for such a type of $\pi_{\mathbf{B}}^k$. It suffices to make sure that $\pi_{\mathbf{B}}^k$ is a projection. Therefore, we can choose

$$(w, d^* \Omega_{\tau, k+1}^i)_{\text{star}(\tau)} = \langle \text{tr}_{\sigma \rightarrow \tau} w, \phi_{\tau, k}^i \rangle_{\tilde{B}^k(\tau)}$$

and

$$(w, \Omega_{\tau, k}^i)_{\text{star}(\tau)} = \langle \text{tr}_{\sigma \rightarrow \tau} w, d\phi_{\tau, k-1}^i \rangle_{\tilde{B}^k(\tau)}$$

for all $w \in \mathbf{A}^k(\text{star}(\tau))$. We show that, the latter equation implies the former one.

Proposition 5.5. *If $(w, \Omega_{\tau, k}^i)_{\text{star}(\tau)} = \langle \text{tr}_{\sigma \rightarrow \tau} w, d\phi_{\tau, k-1}^i \rangle_{\tilde{B}^k(\tau)}$ for all $w \in \mathbf{A}^k(\text{star}(\tau))$, $1 \leq k \leq n$. Then, $(w', d^* \Omega_{\tau, k}^i)_{\text{star}(\tau)} = \langle \text{tr}_{\sigma \rightarrow \tau} w', \phi_{\tau, k-1}^i \rangle_{\tilde{B}^{k-1}(\tau)}$ for all $w' \in \mathbf{A}^{k-1}(\text{star}(\tau))$.*

Proof. Note that

$$\begin{aligned}
(w', d^* \Omega_{\tau, k}^i)_{\text{star}(\tau)} &= (dw', \Omega_{\tau, k}^i)_{\text{star}(\tau)} \\
&= \langle \text{tr}_{\sigma \rightarrow \tau} dw', d\phi_{\tau, k-1}^i \rangle_{\tilde{B}^k(\tau)} \\
&= (d \text{tr}_{\sigma \rightarrow \tau} w', d\phi_{\tau, k-1}^i)_{\tau} \\
&= \langle \text{tr}_{\sigma \rightarrow \tau} w', \phi_{\tau, k-1}^i \rangle_{\tilde{B}^{k-1}(\tau)}.
\end{aligned}$$

This concludes the result. $\square$

There are many ways to determine $\Omega_{\tau, k}^i$, here we introduce one method by Riesz representation theorem. Recall that M is the maximum number of the orders of the differential operators d^k . For each $\tau \in \mathcal{T}$ and $1 \leq k \leq n$, we define by $\mathbf{b}_{\tau}$ the bubble function on $\text{star}(\tau)$ such that

$$\mathbf{b}_{\tau} \in C^0(\text{star}(\tau)), \quad \mathbf{b}_{\tau}|_{\partial\sigma} = 0, \text{ for all } \sigma \subset \text{star}(\tau), \sigma \in \mathcal{T}_n.$$

Note that $(\cdot, (\mathbf{b}_\tau)^{2M} \cdot)_{\text{star}(\tau)}$ provides an L^2 inner product of $\mathbf{A}^k(\text{star}(\tau))$. Then from the Riesz representation theorem, for each $\phi_{\tau, k-1}^i, 1 \leq i \leq \mathcal{N}_{\tau, k-1}$, there exists $\tilde{\Omega}_{\tau, k}^i \in \mathbf{A}^k(\text{star}(\tau))$ such that

$$(w, (\mathbf{b}_\tau)^{2M} \tilde{\Omega}_{\tau, k}^i)_{\text{star}(\tau)} = \langle \text{tr}_{\sigma \rightarrow \tau} w, d\phi_{\tau, k-1}^i \rangle_{\tilde{B}^k(\tau)} \text{ for all } w \in \mathbf{A}^k(\text{star}(\tau)).$$

Next we choose $\Omega_{\tau, k}^i = (\mathbf{b}_\tau)^{2M} \tilde{\Omega}_{\tau, k}^i$ that $\Omega_{\tau, k}^i \in C_0^{2M}(\text{star}(\tau)) \otimes \mathbb{X}^k$, since $\tilde{\Omega}_{\tau, k}^i \in \mathbf{A}^k(\text{star}(\tau))$ is piecewise smooth.

Next, we consider the interpolation operator to the skeletal complex. We first consider a fixed element $z \in \mathcal{Z}$.

Proposition 5.6. *For any $\tau \in \mathcal{T}$, there exists $\Xi_\tau \in H_0^{M+\dim \tau}(\text{star}^1(\tau)) \otimes \mathbb{X}^{\dim \tau}$ such that*

$$(5.9) \quad (w, \Xi_\tau)_{\text{star}^1(\tau)} = (\Upsilon_\tau(w), z)_{\mathcal{Z}}, \quad \text{for any } w \in \mathbf{A}^{\dim \tau}(\text{star}^1(\tau))$$

and

$$(5.10) \quad d^* \Xi_\tau = \Xi_{\partial\tau} := \sum_{\eta \trianglelefteq_1 \tau} O(\eta, \tau) \Xi_\eta, \quad \text{for any } \tau \in \mathcal{T}_{\geq 1}.$$

Note that $\Xi_\tau = \Xi_\tau[z]$ depends on the choice of z .

Proof. We prove by an inductive construction on $\dim \tau$ that (5.9) and (5.10) hold. The construction boils down to two steps.

First step. For any $\tau \in \mathcal{T}$ with $\dim \tau = 0$, i.e., $\tau \in \mathcal{V}$ is vertex of the mesh. Follow the similar argument as above, since $(\cdot, (\mathbf{b}_\tau^1)^M \cdot)_{\text{star}^1(\tau)}$ provides an L^2 inner product of $\mathbf{A}^0(\text{star}^1(\tau))$, here $\mathbf{b}_\tau^1$ is bubble function on $\text{star}^1(\tau)$ with similar definition as $\mathbf{b}_\tau$. Then from the Riesz representation theorem, there exists $\tilde{\Xi}_\tau \in \mathbf{A}^0(\text{star}^1(\tau))$ such that

$$(w, (\mathbf{b}_\tau^1)^M \tilde{\Xi}_\tau)_{\text{star}^1(\tau)} = (\Upsilon_\tau(w), z)_{\mathcal{Z}}, \quad \text{for any } w \in \mathbf{A}^0(\text{star}^1(\tau))$$

Then we can choose $\Xi_\tau = (\mathbf{b}_\tau^1)^M \tilde{\Xi}_\tau$ that $\Xi_\tau \in C_0^M(\text{star}^1(\tau)) \otimes \mathbb{X}^0$, since $\tilde{\Xi}_\tau$ is piecewise smooth.

Second step. Assume that for any $\tau \in \mathcal{T}$ with $0 \leq \dim \tau \leq k-1 \leq n-1$, the (5.9) and (5.10) hold. Then for any $\tau \in \mathcal{T}$ with $\dim \tau = k$, consider $\Xi_{\partial\tau} = \sum_{\eta \trianglelefteq_1 \tau} O(\eta, \tau) \Xi_\eta$. Note that $\text{supp } \Xi_\eta \subset \text{star}^1(\eta) \subset \text{star}^1(\tau)$, then we get that $\Xi_{\partial\tau} \in H_0^{M+k-1}(\text{star}^1(\tau)) \otimes \mathbb{X}^{k-1}$. If $k \geq 2$, by induction we have

$$d^* \Xi_{\partial\tau} = \sum_{\eta \trianglelefteq_1 \tau} O(\eta, \tau) \Xi_{\partial\eta} = 0.$$

Otherwise, if $k = 1$, for any $z^0 \in \mathcal{Z}$, we have that

$$\begin{aligned} (z^0, \Xi_{\partial\tau})_{\text{star}^1(\tau)} &= \sum_{\eta \trianglelefteq_1 \tau} O(\eta, \tau)(z^0, \Xi_\eta)_{\text{star}^1(\tau)} \\ &= \sum_{\eta \trianglelefteq_1 \tau} O(\eta, \tau)(\Upsilon_\eta(z^0), z)_{\mathcal{Z}} \\ &= (\Upsilon_{\partial\tau}(z^0), z)_{\mathcal{Z}} = (\Upsilon_\tau(dz^0), z)_{\mathcal{Z}} = 0 \end{aligned}$$

which implies that $\Xi_{\partial\tau} \in H_0^M(\text{star}^1(\tau)) \otimes \mathbb{X}^0/\mathcal{Z}$. Since the element patch $\text{star}^1(\tau)$ is contractible, then from Proposition 2.2, there exists $\tilde{\Xi}_\tau \in H_0^{M+k}(\text{star}^1(\tau)) \otimes \mathbb{X}^k$ such that $d^*\tilde{\Xi}_\tau = \Xi_{\partial\tau}$ and

$$\|\tilde{\Xi}_\tau\|_{L^2(\text{star}^1(\tau))} \leq Ch^l \|\Xi_{\partial\tau}\|_{L^2(\text{star}^1(\tau))}$$

here l is the order of the differential operator d^{k-1} .

Next, we consider the modification of the following form: $\Xi_\tau = \tilde{\Xi}_\tau + d^*\Theta_\tau$. Clearly, in this case (5.10) always holds. Note that from the Theorem 2.1, the discrete complex on $\text{star}^1(\tau)$

$$\mathcal{Z} \xrightarrow{\subset} \mathbf{A}^0(\text{star}^1(\tau)) \xrightarrow{d} \cdots \xrightarrow{d} \mathbf{A}^n(\text{star}^1(\tau)) \longrightarrow 0$$

is exact. Denote by $\mathcal{N}(d)$ the kernel space of d , then there exists $\tilde{\Theta}_\tau \in \mathbf{A}^k(\text{star}^1(\tau))/\mathcal{N}(d)$ such that

$$(du, (\mathbf{b}_\tau^1)^{2M+k} d\tilde{\Theta}_\tau)_{\text{star}^1(\tau)} = (\Upsilon_\tau(u), z)_{\mathcal{Z}} - (u, \tilde{\Xi}_\tau)_{\text{star}^1(\tau)}, \quad \text{for all } u \in \mathbf{A}^k(\text{star}^1(\tau))/\mathcal{N}(d)$$

Now choose $\Theta_\tau = (\mathbf{b}_\tau^1)^{2M+k} d\tilde{\Theta}_\tau$, we then prove the identity (5.9) for $w \in \mathbf{A}^k(\text{star}^1(\tau))$. If $w \in \mathbf{A}^k(\text{star}^1(\tau))/\mathcal{N}(d)$, it follows from the definition of Θ_τ . Otherwise, if $dw = 0$, from the exactness of the discrete complex, there exists $u' \in \mathbf{A}^{k-1}(\text{star}^1(\tau))$ such that $w = du'$, then it holds that

$$(du', \tilde{\Xi}_\tau + d^*\Theta_\tau)_{\text{star}^1(\tau)} = (u', d^*\tilde{\Xi}_\tau)_{\text{star}^1(\tau)} = (u', \Xi_{\partial\tau})_{\text{star}^1(\tau)} = (\Upsilon_{\partial\tau}(u'), z)_{\mathcal{Z}} = (\Upsilon_\tau(du'), z)_{\mathcal{Z}}$$

which completes the proof $\square$

Now we choose $\Xi_\tau^i = \Xi_\tau[z_i]$, then the construction (5.7) satisfies the commuting property.

Proposition 5.7. *It holds that $\pi_{\mathcal{S}}^k(dw) = d\pi_{\mathcal{S}}^{k-1}(w)$ for $w \in L^2(\Omega) \otimes \mathbb{X}^{k-1}$ with $dw \in L^2(\Omega) \otimes \mathbb{X}^k$.*

Proof. By definition, it holds that

$$\begin{aligned}
\pi_{\mathbf{S}}^k(dw) &= \sum_{\tau \in \mathcal{T}_k} \sum_{i=1}^{\dim \mathcal{Z}} (dw, \Xi_{\tau}^i)_{\text{star}^1(\tau)} \mathbf{E}_{\mathbf{S}}^k(z_i \otimes \tau) \\
&= \sum_{\tau \in \mathcal{T}_k} \sum_{i=1}^{\dim \mathcal{Z}} (w, d^* \Xi_{\tau}^i)_{\text{star}^1(\tau)} \mathbf{E}_{\mathbf{S}}^k(z_i \otimes \tau) \\
&= \sum_{i=1}^{\dim \mathcal{Z}} \sum_{\tau \in \mathcal{T}_k} \sum_{\eta \leq_1 \tau} O(\eta, \tau)(w, \Xi_{\eta}^i)_{\text{star}^1(\eta)} \mathbf{E}_{\mathbf{S}}^k(z_i \otimes \tau) \\
&= \sum_{i=1}^{\dim \mathcal{Z}} \sum_{\eta \in \mathcal{T}_{k-1}} (w, \Xi_{\eta}^i)_{\text{star}^1(\eta)} \sum_{\eta \leq_1 \tau} O(\eta, \tau) \mathbf{E}_{\mathbf{S}}^k(z_i \otimes \tau) \\
&= \sum_{i=1}^{\dim \mathcal{Z}} \sum_{\eta \in \mathcal{T}_{k-1}} (w, \Xi_{\eta}^i)_{\text{star}^1(\eta)} d\mathbf{E}_{\mathbf{S}}^{k-1}(z_i \otimes \eta) = d\pi_{\mathbf{S}}^{k-1}(w)
\end{aligned}$$

here the last equation comes from the property of $\Upsilon_{\bullet}$ and the definition of $\mathbf{E}_{\mathbf{S}}^{k-1}(z_i \otimes \eta)$, which completes the proof. $\square$

Now we can conclude the result in this subsection.

Theorem 5.2. *Suppose that the conditions in Theorem 2.2 hold true, then the operators π^k constructed in (5.6) are commuting projections. Furthermore, suppose that the conditions in Theorem 2.3 hold true, then these projections π_k are locally L^2 bounded, namely, there is a constant C , depends only on the regularity constant, and*

$$\|\pi^k u\|_{L^2(\sigma)} \leq C \|u\|_{L^2(\text{star}^2(\sigma))}, \quad \text{for all } u \in L^2(\Omega) \otimes \mathbb{X}^k, \sigma \in \mathcal{T}_n$$

A direct consequence is that

$$\|\pi^k u\|_{L^2(\Omega)} \leq C' \|u\|_{L^2(\Omega)}, \quad \text{for all } u \in L^2(\Omega) \otimes \mathbb{X}^k$$

Proof. The proof is similar as [2, Theo. 3.1], which comes from scaling argument and compactness argument and Lemma A.1. $\square$

6. CONCLUSIONS AND PERSPECTIVES

In this paper, we establish the framework to elaborate on the construction of finite element complexes. Specifically, the trace structure allows us to consider the finite elements with additional smoothness. Based on the concept of generalized currents, we can decompose the finite element complexes into the skeletal part (contributed by the generalized currents) and the bubble part (coming from the remaining degrees of freedom). Based on this decomposition, we can readily derive the cohomology and the L^2 -bounded interpolation of the finite element

complexes. This will serve as a building block for the whole analysis of solving Hodge-Laplacian problems and their associated eigenvalue problems.

As previously mentioned, the examples presented above lie in the center of the finite element de Rham complexes and tensor complexes, where the latter essentially can be derived from the Bernstein-Gelfand-Gelfand construction [5]. To detail all the finite element complexes in this paper seems impractical. Especially, the construction of the divdiv complexes and the elasticity complexes involves numerous technicalities. Therefore, some constructions of the finite element complexes are not presented here, though the construction can be modified into FECTS to derive its cohomology and interpolation operators, e.g. [38, 28]

Nonetheless, we believe that the current examples are far from exhausting the capacity of using the FECTS framework. We hereby list some potential directions for further studies, which are left as future works.

- (1) Boundary treatment: In this paper, we only consider the free boundary case. We believe that both the free boundary case and the zero boundary case can be readily handled by the current framework. However, for mixed boundary conditions and essential (but non-zero) conditions, the framework requires some additional techniques. For the de Rham cases, this can be categorized by the “relative de Rham complexes”, cf. [8]. As far as the authors are aware, few results regarding the cohomology of the complicated boundary case are discussed in the discrete case. On the other hand, preserving the boundary value in such boundary cases seems nontrivial, see [41] and the references therein.
- (2) p version finite element complexes. One advantage is that the framework of FECTS does not require each cell to have the same space. Therefore, it is also possible to consider the case when the FECTS contains heterogeneous data, for example, the p finite elements. Some of the results for stress complexes in two dimensions be found in [6]. Meanwhile, a systematic framework for interpolation and approximation is expected.

APPENDIX A. L^2 BOUNDEDNESS OF THE INTERPOLATION OPERATOR

We first prove the following lemma

Lemma A.1. *For any $\phi_{\tau,k}^i$, assume the differential operator d is l -order, then it holds that*

$$\frac{1}{h_\tau^l} \|\mathbf{E}_\tau^k(\phi_{\tau,k}^i)\|_{L^2(\text{star}^1(\tau))} + \|\mathbf{E}_\tau^{k+1}(d\phi_{\tau,k}^i)\|_{L^2(\text{star}^1(\tau))} \leq Ch_\tau^{(n-\dim \tau)/2}$$

Proof. Follow the scaling argument in [4], let x_0 denote the first vertex of τ , and define $\varphi(x) = (x - x_0)/h_\tau$ which maps τ to $\hat{\tau} = \varphi(\tau)$ and $\text{star}(\tau)$ to $\text{star}(\hat{\tau}) = \varphi(\text{star}(\tau))$. Define $\hat{\phi}_{\tau,k}^i = \varphi^{-1*}\phi_{\tau,k}^i$ and it is easy to check that $\mathbf{E}_{\hat{\tau}}^k(\hat{\phi}_{\tau,k}^i) = \varphi^{-1*}\mathbf{E}_\tau^k(\phi_{\tau,k}^i)$, then

$$\|\mathbf{E}_\tau^k(\phi_{\tau,k}^i)\|_{L^2(\text{star}^1(\tau))} = h_\tau^{n/2} \|\mathbf{E}_{\hat{\tau}}^k(\hat{\phi}_{\tau,k}^i)\|_{L^2(\text{star}^1(\hat{\tau}))}.$$

Note that

$$\begin{aligned} \langle \hat{\phi}_{\tau,k}^i, \hat{\phi}_{\tau,k}^i \rangle_{\tilde{B}^k(\hat{\tau})} &= (d\hat{\phi}_{\tau,k}^i, d\hat{\phi}_{\tau,k}^i)_{L^2(\hat{\tau})} \\ &= h_\tau^{2l-\dim \tau} (d\phi_{\tau,k}^i, d\phi_{\tau,k}^i)_{L^2(\tau)} = h_\tau^{2l-\dim \tau}. \end{aligned}$$

Hence

$$\|\mathbf{E}_{\hat{\tau}}^k(\hat{\phi}_{\tau,k}^i)\|_{L^2(\text{star}^1(\hat{\tau}))} \leq C(\text{star}^1(\hat{\tau})) h_\tau^{l-\dim \tau/2}$$

here $C(\text{star}^1(\hat{\tau}))$ is a continuous function of $\text{star}^1(\hat{\tau})$, then the mesh regularity assumption implies that there exists constant C independent of τ and h_τ such that $C(\text{star}^1(\hat{\tau})) \leq C$. Then we get that

$$\|\mathbf{E}_\tau^k(\phi_{\tau,k}^i)\|_{L^2(\text{star}^1(\tau))} \leq Ch_\tau^{l+(n-\dim \tau)/2}.$$

Similarly

$$\|\mathbf{E}_\tau^{k+1}(d\phi_{\tau,k}^i)\|_{L^2(\text{star}^1(\tau))} \leq Ch_\tau^{(n-\dim \tau)/2}.$$

□

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