

Spectrality of Prime Size Tiles

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Abstract

We prove that if a tile in \mathbb{Z}^d has prime size p , then it must be spectral. The proof is by contradiction, it is simply shown that the tiling complement of such a tile can not annihilate all p -subgroups. In addition, with a simple transformation we prove that any p points in general linear positions in \mathbb{Z}^d must be both tiling and spectral if $d \geq p - 1$.

Keywords: Fuglede conjecture; tiling sets; spectral sets.

2020 MSC: 42A99; 05B45

1 Introduction

Let G be a finitely generated Abelian group, a set A is *tiling* (or is a *tile*) in G if there exists some B such that $\{A + b\}_{b \in B}$ forms a partition of G , and is *spectral* if there exists a set of group characters of G that forms an orthogonal basis on $L^2(A)$. In such cases B (resp. S) is said to be a *tiling complement* (resp. *spectrum*) of A , while (A, B) (resp. (A, S)) is called a tiling pair (resp. spectral pair) in G , and we often write $G = A \oplus B$.

Fuglede conjectured that being spectral and being tiling are actually equivalent [7], this has been studied extensively and disproved in general when the dimension is at least three [6, 11, 12, 18, 22] but remains open for one dimensional and two dimensional cases. Researches over the Fuglede conjecture in finite Abelian groups as well as in p -adic fields have been active in the past years, see e.g., [4, 5, 10, 13, 14, 16, 17, 23] (the full list is of course too long to be enumerated here, this is only an attempt to mention some of the results in one dimensional or two dimensional cases, see perhaps references therein for more on the history and development of this problem).

It is well known that a tiling set of prime size in \mathbb{Z}^d must have at least one periodic tiling complement [21] (it may be worthing mentioning that being periodic in d dimensions

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means having d linearly independent periods, while having periods in one direction is called weakly periodic instead, readers may also find the periodic tiling conjecture and its positive and negative cases in e.g., [1, 8, 9, 15] relevant). The aim of this short note is to establish the following statements (Theorem 1 and 2 in the last section):

Theorem. *Let p be a prime number and $A \subset \mathbb{Z}^d$ with $|A| = p$, if A tiles \mathbb{Z}^d , then it is also spectral.*

Theorem. *Any p points in general linear positions in \mathbb{Z}^d must be both tiling and spectral if $d \geq p - 1$.*

Here p points being in *general linear positions* means that they are not contained in any hyperplane or affine hyperplane of dimension $p - 2$, alternatively this means if we fix any one of these p points, then the $p - 1$ vectors formed from the fixed point to the other $p - 1$ points are linearly independent in \mathbb{Z}^d . It is also worth mentioning that for $d \geq p$ the second theorem is actually already included in [18, Corollary 2.4].

The proof of the first theorem applies several techniques: First we reduce the problem to \mathbb{Z}_n^d by using periodicity, then we produce a size p spectrum from p -subgroups of \mathbb{Z}_n^d . The construction involves dissecting the group into equivalence classes formed by generators of cyclic subgroups, while the existence of such a set being included in the zero set of A is shown by contradiction. Projecting onto $\mathbb{Z}_{p^k}^d$ (where p^k is the largest power of p that divides n) and invoking the uncertainty principle, we are able to show that if the tiling complement annihilates all p -subgroups (which will be the case if A annihilates none), then its size must be divisible by p^{kd} (a statement that is similar to the Coven-Meyerowitz T1 property in [2]), thus leading to a contradiction if A does not admit such a spectrum.

The proof of the second theorem then reduces to proving any p points in general linear positions are tiling, for which we can simply show that they tile the sublattice they generate. This follows from the fact that the set formed by the origin and $(1, 0, \dots, 0)$, $(0, 2, 0, \dots, 0)$, \dots , $(0, \dots, 0, p - 1)$ is tiling (a tiling complement can be found explicitly by computation), then we can map this set to the p points we are looking at by a linear transform.

2 Preliminaries

Let A be a multiset (i.e., a set that allows repetitive elements) on \mathbb{Z}_n^d (i.e., each element of A is a member of \mathbb{Z}_n^d), the multiplicity of an element $a \in A$ (i.e., the number of copies of a in A) will be denoted by $\text{mul}(a)$, the characteristic function on such a multiset A will be defined and written as

$$\mathbf{1}_A(x) = \begin{cases} \text{mul}(x) & x \in A, \\ 0 & x \notin A. \end{cases}$$

The Fourier transform of $\mathbf{1}_A(x)$ on \mathbb{Z}_n^d is

$$\widehat{\mathbf{1}}_A(\xi) = \sum_{a \in A} \text{mul}(a) \cdot e^{2\pi i \langle a, \xi \rangle / n},$$

where $\langle \cdot, \cdot \rangle$ is the Euclidean inner product. It is straightforward to verify that

$$(A, S) \text{ is a spectral pair on } \mathbb{Z}_n^d \Leftrightarrow \begin{cases} |A| = |S|, \\ \Delta S \subseteq Z(\widehat{\mathbf{1}}_A), \end{cases} \quad (1)$$

$$(A, B) \text{ is a tiling pair on } \mathbb{Z}_n^d \Leftrightarrow \begin{cases} |A| \cdot |B| = n^2, \\ \Delta \mathbb{Z}_n^d \subseteq Z(\widehat{\mathbf{1}}_A) \cup Z(\widehat{\mathbf{1}}_B), \end{cases} \quad (2)$$

where the difference set ΔS of a usual set (i.e., a set with distinct elements only) S and the zero set $Z(f)$ of a function f is defined respectively as

$$\Delta S = \{s - s' : s, s' \in S, s \neq s'\}, \quad Z(f) = \{x : f(x) = 0\}.$$

One may also verify the *Poisson summation formula* below:

$$\widehat{\mathbf{1}}_H = |H| \cdot \mathbf{1}_{H^\perp}, \quad (3)$$

where $H \triangleleft \mathbb{Z}_n^d$ is a subgroup and

$$H^\perp = \{x \in \mathbb{Z}_n^d : \langle x, h \rangle = 0, \forall h \in H\},$$

is its *orthogonal group* in \mathbb{Z}_n^d . That (3) holds for cyclic subgroups is quite trivial, while non-cyclic subgroups can be viewed as products of cyclic ones, then its orthogonal set is simply the intersection over orthogonal sets of its cyclic factors.

Let the support of a function f in \mathbb{Z}_n^d be defined as

$$\text{supp}(f) = \mathbb{Z}_n^d \setminus Z(f),$$

then the *uncertainty principle* on finite Abelian groups [3, 20] asserts that

$$|\text{supp}(f)| \cdot |\text{supp}(\widehat{f})| \geq |\mathbb{Z}_n^d|, \quad (4)$$

holds for any non-zero function f defined on \mathbb{Z}_n^d .

Lemma 1. *Let A be a multiset on \mathbb{Z}_n^d , if $\Delta \mathbb{Z}_n^d \subseteq Z(\mathbf{1}_A)$, then $\mathbf{1}_A$ must be a scalar multiple of $\mathbf{1}_{\mathbb{Z}_n^d}$, i.e., $\mathbf{1}_A = m\mathbf{1}_{\mathbb{Z}_n^d}$ for some $m \in \mathbb{N}$.*

Proof. Assume the contrary that $\mathbf{1}_A$ is not a scalar multiple of $\mathbf{1}_{\mathbb{Z}_n^d}$, then let m be the smallest multiplicity of elements in A , i.e.,

$$m = \min\{\text{mul}(a) : a \in A\},$$

and consider

$$f = \mathbf{1}_A - m\mathbf{1}_{\mathbb{Z}_n^d}.$$

With this construction we would have $f(a) = 0$ for all a whose multiplicity in A is m and f is not identically 0 (since $\mathbf{1}_A$ is assumed not to be a scalar multiple of $\mathbf{1}_{\mathbb{Z}_n^d}$). Therefore we get

$$0 < |\text{supp}(f)| < |\mathbb{Z}_n^d|, \quad (5)$$

On the other hand, it is straightforward to check that

$$Z(\widehat{\mathbf{1}_{\mathbb{Z}_n^d}}) = \mathbb{Z}_n^d \setminus \{0\} = \Delta\mathbb{Z}_n^d,$$

i.e., $\Delta\mathbb{Z}_n^d \subseteq Z(\mathbf{1}_{\mathbb{Z}_n^d})$, thus combined with the assumption that $\Delta\mathbb{Z}_n^d \subseteq Z(\mathbf{1}_A)$ we get

$$\Delta\mathbb{Z}_n^d \subseteq Z(\widehat{f}),$$

i.e.,

$$\text{supp}(\widehat{f}) = \{0\}.$$

Together with (5) this leads to

$$|\text{supp}(f)| \cdot |\text{supp}(\widehat{f})| = |\text{supp}(f)| \cdot 1 = |\text{supp}(f)| < |\mathbb{Z}_n^d|,$$

which contradicts the uncertainty principle in (4). \square

For any $x = (x_1, \dots, x_d) \in \mathbb{Z}^d$, set

$$\pi_n(x) = (x_1 \bmod n, \dots, x_d \bmod n),$$

so that π_n is the projection from \mathbb{Z}^d to \mathbb{Z}_n^d (as a map between sets). Given a set $A \subseteq \mathbb{Z}^d$, $\pi_n(A)$ will denote the multiset obtained by applying π_n on every element of A . Clearly $\pi_n(A)$ will only be a usual set if (and only if) the map $A \mapsto \pi_n(A)$ is injective.

Lemma 2 ([21]). *Let p be a prime number and $A \subset \mathbb{Z}^d$ with $|A| = p$, if $A \oplus B = \mathbb{Z}^d$, then there is some $n \in \mathbb{N}$ and $B' \in \mathbb{Z}_n^d$ such that $A \mapsto \pi_n(A)$ is injective and $\pi_n(A) \oplus B' = \mathbb{Z}_n^d$.*

The way that Lemma 2 is stated here is slightly different from its original version in [21, Theorem 17], but it is clear that if $\{(t_1, 0, \dots, 0), \dots, (0, \dots, 0, t_d)\}$ is a set of periods of B , then the least common multiple of t_1, \dots, t_d can be taken as n , and B' can be obtained by removing all duplications from $\pi_n(B)$.

Lemma 3. *Let $n = p^k m$ with p being a prime number that does not divide m , and set $A' = \pi_{p^k}(A)$. Let both f, g be characteristic functions on A' but with the domain of f being the entire \mathbb{Z}_n^d while the domain of g being $\mathbb{Z}_{p^k}^d$ only. If $x \in \mathbb{Z}_n^d$ is an element of order p^t for some $1 \leq t \leq k$, then we have*

$$\widehat{\mathbf{1}_A}(x) = \widehat{f}(x) = \widehat{g}(x'),$$

where $x' \in \mathbb{Z}_{p^k}^d$ is the element that satisfies $x = mx'$, and the last Fourier transform \widehat{g} is performed in $\mathbb{Z}_{p^k}^d$ only (the first two Fourier transforms $\widehat{\mathbf{1}_A}$ and \widehat{f} take place in \mathbb{Z}_n^d).

Proof. Every $a = (a_1, \dots, a_d) \in A$ can be written as

$$a = p^k q_a + r_a,$$

where q_a, r_a respectively collect quotients and remainders of a_1, \dots, a_d divided by p^k . In particular we have

$$A' = \pi_{p^k}(A) = \{r_a : a \in A\}.$$

The support set of f, g coincide, but they are defined on different subgroups, thus we have

$$\widehat{f}(\xi) = \sum_{a \in A} e^{2\pi i \langle r_a, \xi \rangle / n}, \quad \widehat{g}(\xi) = \sum_{a \in A} e^{2\pi i \langle r_a, \xi \rangle / p^k}$$

in which Fourier transforms in \mathbb{Z}_n^d and in $\mathbb{Z}_{p^k}^d$ are applied respectively.

Then direct computation shows that

$$\begin{aligned} \widehat{\mathbf{1}}_A(x) &= \sum_{a \in A} e^{2\pi i \langle a, x \rangle / n} = \sum_{a \in A} e^{2\pi i \langle p^k q_a + r_a, mx' \rangle / n} = \sum_{a \in A} e^{2\pi i \langle r_a, mx' \rangle / n}, \\ &= \sum_{a \in A} e^{2\pi i \langle r_a, x \rangle / n} (= \widehat{f}(x)), \\ &= \sum_{a \in A} e^{2\pi i \langle r_a, x' \rangle / p^k} (= \widehat{g}(x')), \end{aligned}$$

which is the desired result. □

Given a finite Abelian group G , let \sim be the equivalence relation on G so that $g \sim g'$ if g and g' generate the same cyclic subgroup in G . If E is an equivalence class under \sim , then we define the order of E , denoted by $\text{ord}(E)$, to be the order of any element (as a member of the group G) in E (by definition all elements in the same class will have the same order).

Lemma 4. *Let E be an equivalence class under \sim in \mathbb{Z}_n^d and $A \subseteq \mathbb{Z}_n^d$. If A annihilates one element of E , then it annihilates entire E .*

Proof. Set $m = \text{ord}(E)$ and $r = n/m$. Let x be an arbitrary element of E , then $x = rx'$ for some $x' \in \mathbb{Z}_m^d$, consequently

$$\widehat{\mathbf{1}}_A(x) = \sum_{a \in A} e^{2\pi i \langle a, x \rangle / n} = \sum_{a \in A} e^{2\pi i \langle a, x' \rangle / m}.$$

We shall view the above expression as the polynomial

$$P(z) = \sum_{a \in A} z^{\langle a, x' \rangle},$$

evaluated at the m -th root of unity $\omega = e^{2\pi i/m}$, i.e.,

$$\widehat{1}_A(x) = P(\omega).$$

If $x \in Z(\widehat{1}_A)$, then $P(\omega) = 0$, which means $P(z)$ is divisible by the m -th cyclotomic polynomial $\Phi_m(z)$. On the other hand, if y is another element of E , then by the definition of E we must have $y = \lambda x$ for some λ that is coprime to m , thus

$$\widehat{1}_A(y) = \sum_{a \in A} e^{2\pi i \langle a, \lambda x \rangle / n} = \sum_{a \in A} e^{2\pi i \lambda \langle a, x \rangle / n} = \sum_{a \in A} e^{2\pi i \lambda \langle a, x' \rangle / m} = P(\omega^\lambda) = 0,$$

where the last equality holds since λ is coprime to m , and so that ω^λ is also a root of $\Phi_m(z)$. This finishes the proof since x, y are chosen arbitrarily from E . \square

A set E' will be called a *derived set* of E if

$$E' \subseteq E \cup \{0\}, \quad \Delta E' \subseteq E.$$

Lemma 5. *Let E be an equivalence class under \sim in some finite Abelian group. If p is the smallest prime divisor of $\text{ord}(E)$, then E admits a derived set of size p .*

Proof. Let h be an arbitrary element of E , then we claim that

$$E' = \{0, h, 2h, \dots, (p-1)h\},$$

is a desired derived set. Indeed, that $|E'| = p$ is obvious, to see $\Delta E' \subseteq E$, set $n = \text{ord}(E)$ and simply consider the group isomorphism from the cyclic group generated by h to \mathbb{Z}_n given by $\varphi : h \mapsto 1$. Let F be the equivalence class formed by taking all generators of \mathbb{Z}_n , then F consists of all numbers that are coprime to n in \mathbb{Z}_n and by this construction we have $\varphi(E) = F$. Now set $F' = \{0, 1, \dots, p-1\}$, then $\Delta F' = \{\pm 1, \dots, \pm(p-1)\}$, clearly every member of $\Delta F'$ is coprime to n since p is the smallest divisor of n . Therefore $\Delta F' \subseteq F$, apply φ^{-1} at both sides we get $\Delta E' \subseteq E$. \square

In particular, Lemma 5 holds for $\text{ord}(E)$ being a power of p , which is how it will be applied later in this article.

3 Main results

Theorem 1. *Let p be a prime number and $A \subset \mathbb{Z}^d$ with $|A| = p$, if A tiles \mathbb{Z}^d , then it is also spectral.*

Proof. Let n and B' be as asserted in Lemma 2, and set $A' = \pi_n(A)$, so that we have $A' \oplus B' = \mathbb{Z}_n^d$. It suffices to show that A' is spectral in \mathbb{Z}_n^d .

A' being a tile in \mathbb{Z}_n^d implies that p divides n^d . Since p is prime this also implies that p divides n . Let m be the number so that $n = p^k m$ and $\gcd(p, m) = 1$.

Consider all equivalence classes in \mathbb{Z}_n^d whose orders are powers of p , by Lemma 4, each equivalence class must be either completely inside $Z(\widehat{\mathbf{1}}_A)$ or completely disjoint with $Z(\widehat{\mathbf{1}}_A)$.

By Lemma 5, each of these equivalence classes has a derived set of size p , thus if any of these classes is in $Z(\widehat{\mathbf{1}}_{A'})$, then the corresponding derived set would be a spectrum of A' and we are done.

To see this must be the case, assume the contrary that all equivalence classes whose orders are powers of p are disjoint with $Z(\widehat{\mathbf{1}}_{A'})$, then by (2) they must be annihilated by $\widehat{\mathbf{1}}_{B'}$. By Lemma 3 this further implies

$$\Delta\mathbb{Z}_{p^k}^d \subseteq Z(\widehat{\mathbf{1}}_{\pi_{p^k}(B')}).$$
 (6)

Indeed, every element $x' \in \Delta\mathbb{Z}_{p^k}^d$ can be lifted to an element of the same order in \mathbb{Z}_n^d through the map $x' \mapsto mx'$, and Lemma 3 asserts that

$$\widehat{\mathbf{1}}_{\pi_{p^k}(B')}(x') = \widehat{\mathbf{1}}_{B'}(mx').$$
 (7)

The order of x' in $\mathbb{Z}_{p^k}^d$ is obviously a power of p , thus the order of mx' in \mathbb{Z}_n^d is also a power of p (since it equals the order of x' in $\mathbb{Z}_{p^k}^d$), which means by assumption the right-hand side of (7) must vanish. Repeating the argument on every member of $\Delta\mathbb{Z}_{p^k}^d$ leads to (6).

Now as $\pi_{p^k}(B')$ is a multiset on $\mathbb{Z}_{p^k}^d$, Lemma 1 indicates that $\widehat{\mathbf{1}}_{\pi_{p^k}(B')}$ is just a scalar multiple of $\widehat{\mathbf{1}}_{\mathbb{Z}_{p^k}^d}$, which means $|\mathbb{Z}_{p^k}^d|$ divides $|\pi_{p^k}(B')|$. Since $|\pi_{p^k}(B')| = |B'|$, this further indicates that p^{kd} divides $|B'|$, consequently we would have $p^{kd}m^d = |\mathbb{Z}_n^d| = |A| \cdot |B|$ is divisible by p^{kd+1} , which is a contradiction. \square

Lemma 6. *Let e_i be the i -th standard Euclidean basis, then the set*

$$A = \{0, e_1, 2e_2, \dots, (p-1)e_{p-1}\},$$

is both tiling and spectral in \mathbb{Z}_p^{p-1} .

Proof. Let $x = (1, \dots, 1)$, and set $\omega = e^{2\pi i/p}$, then it is easy to verify that

$$\widehat{\mathbf{1}}_A(x) = 1 + \omega + \dots + \omega^{p-1} = 0.$$

Therefore the cyclic group generated by x (denoted by $\langle x \rangle$) is a spectrum of A . To see that A is tiling, we will construct a set B so that $|A| \cdot |B| = |\mathbb{Z}_p^{p-1}|$ and

$$Z(\widehat{\mathbf{1}}_B) = \mathbb{Z}_p^{p-1} \setminus \langle x \rangle,$$
 (8)

then it will follow from (2) that (A, B) forms a tiling pair.

Since p is a prime number, each of $1, \dots, p-1$ is a $(p-1)$ -st root of unity in the finite field \mathbb{F}_p , hence all entries of the $(p-1) \times (p-1)$ Fourier matrix (i.e., the Vandermonde matrix generated by these roots of unity $1, \dots, p-1$) are in \mathbb{Z}_p , and the set of its columns, denoted by u_1, \dots, u_{p-1} , is a basis (the Fourier basis) on \mathbb{Z}_p^{p-1} (understood as a vector space over \mathbb{F}_p). In particular, since x is a column of it we may without loss of generality order the columns in the way so that $u_1 = x$.

Let B be the span (over \mathbb{F}_p) of u_2, \dots, u_{p-1} , then $|B| = p^{p-2}$ is of the correct size, and we simply notice that B and $\langle x \rangle$ are orthogonal groups of each other, thus (8) follows immediately from the Poisson summation formula (3). \square

Corollary 1. *The set*

$$A = \{0, e_1, \dots, e_{p-1}\},$$

is both tiling and spectral in \mathbb{Z}^{p-1} .

Proof. It suffices to show that A is tiling in \mathbb{Z}_p^{p-1} since if (A, B) is a tiling pair in \mathbb{Z}_p^{p-1} , then $(A, B \oplus (p\mathbb{Z})^{p-1})$ becomes a tiling pair in \mathbb{Z}^{p-1} .

Let

$$S = \{0, e_1, 2e_2, \dots, (p-1)e_{p-1}\},$$

be the set defined in Lemma 6, and set $D = \text{diag}(1, 2, \dots, p-1)$. Then D is a group automorphism on \mathbb{Z}_p^{p-1} , its inverse is simply $D^{-1} = \text{diag}(1, 2^{-1}, \dots, (p-1)^{-1})$ where for each $k \in \{1, \dots, p-1\}$, k^{-1} is the multiplicative inverse of k in \mathbb{Z}_p , each k^{-1} exists since p is a prime number.

By Lemma 6, S is tiling in \mathbb{Z}_p^{p-1} , therefore $A = D^{-1}S$ is also tiling in $D^{-1}\mathbb{Z}_p^{p-1}$, but $D^{-1}\mathbb{Z}_p^{p-1}$ equals \mathbb{Z}_p^{p-1} , thus A is indeed tiling in \mathbb{Z}_p^{p-1} . Its spectrality then follows from Theorem 1. \square

Theorem 2. *Any p points in general linear positions in \mathbb{Z}^d must be both tiling and spectral if $d \geq p-1$.*

Proof. In light of Theorem 1, it suffices to show that these points are tiling in \mathbb{Z}^d .

Without loss of generality we may assume that one of these points is the origin, and we denote the other $p-1$ points by v_1, \dots, v_{p-1} . Let V be an arbitrary linear transform that maps e_i to v_i for $i \leq p-1$, such V exists since $0, v_1, \dots, v_{p-1}$ are in general linear positions (which means that v_1, \dots, v_{p-1} are linearly independent). Further set

$$A = \{0, v_1, \dots, v_{p-1}\}, \quad E = \{0, e_1, \dots, e_{p-1}\}.$$

By Corollary 1, E is tiling in \mathbb{Z}^{p-1} , thus $A = VE$ is also tiling in the space obtained by applying V to $\text{span}(E)$, which is a sublattice since $\text{span}(E) = \mathbb{Z}^{p-1}$. This implies that A is also tiling in \mathbb{Z}^d . Its tiling complement can be produced explicitly: If E' is a tiling complement of E in \mathbb{Z}^{p-1} , and F is the set formed by taking one representative from each coset of $V\mathbb{Z}^{p-1}$ (viewed as an additive subgroup) in \mathbb{Z}^d , then $B = (VE') \oplus F$ is a tiling complement of A . \square

Theorem 2 produces some interesting consequences, for example we can assert that any 3 points must be tiling as long as they are not on a line. On the other hand, this also shows that the condition that p points have to be in general linear positions is tight, since 3 colinear points indeed need not be tiling, e.g., $\{0, 3, 4\}$ does not tile \mathbb{Z} (corresponds to $p = 3$, and $d = 1 < 2 = 3 - 1$, see also [19] for a characterization of prime power size tiles in \mathbb{Z}).

Acknowledgement

The author would like to thank Tao Zhang (XDU) for the hospitality in Xi'an, and the Chern Institute of Mathematics (Tianjin) for hosting while preparing for this paper.

The author thanks Jia Zheng (WHU) for mentioning the phenomenon described in Theorem 2 at $p = 3$ to him, as well as Shilei Fan (CCNU) and Gergely Kiss (ARIM) for short communications that improved Lemma 5.

References

- [1] S.Bhattacharya: Periodicity and decidability of tilings of \mathbb{Z}^2 . American Journal of Mathematics, 142, 255-266, 2020.
- [2] E.Coven, A.Meyerowitz: Tiling the integers with translates of one finite set. Journal of Algebra, 212, 161–174, 1999.
- [3] D.Donoho, P.Stark: Uncertainty principle and signal processing. SIAM Journal on Applied Mathematics, 49(3), 906-931, 1989.
- [4] T.Fallon, G.Kiss, G.Somlai: Spectral sets and tiles in $\mathbb{Z}_p^2 \times \mathbb{Z}_q^2$. Journal of Functional Analysis, 282(12), 109472, 2022.
- [5] A.Fan, S.Fan, R.Shi: Compact open spectral sets in \mathbb{Q}_p . Journal of Functional Analysis, 271(12), 3628-3661, 2016.
- [6] B.Farkas, M.Matolcsi, P.Móra: On Fuglede's conjecture and the existence of universal spectra. Journal of Fourier Analysis and Applications, 12(5), 483-494. 2006.
- [7] B.Fuglede: Commuting self-adjoint partial differential operators and a group theoretic problem. Journal of Functional Analysis, 16, 101-121, 1974.
- [8] R.Greenfeld, T.Tao: The structure of translational tilings in \mathbb{Z}^d . Discrete Analysis, 16, 2021.
- [9] R.Greenfeld, T.Tao: A counterexample to the periodic tiling conjecture. Annals of Mathematics, 200, 301-363, 2024.

- [10] A.Iosevich, A.Mayeli, J.Pakianathan: The Fuglede conjecture holds in $\mathbb{Z}_p \times \mathbb{Z}_p$. *Analysis & PDE*, 10(4), 757-764, 2017.
- [11] M.Kolountzakis, M.Matolcsi: Complex Hadamard matrices and the spectral set conjecture. *Collectanea Mathematica*, 57, 281-291, 2006.
- [12] M.Kolountzakis, M.Matolcsi: Tiles with no spectra. *Forum Mathematicum*, 18(3), 519-528, 2006.
- [13] I.Laba, I.Londner: The Coven–Meyerowitz tiling conditions for 3 odd prime factors. *Inventiones Mathematicae*, 232, 365-470, 2022.
- [14] I.Laba, I.Londner: The Coven-Meyerowitz tiling conditions for 3 prime factors: the even case. *Research in the Mathematical Sciences*, 12, 43, 2025.
- [15] J.Lagarias, Y.Wang: Tiling the line with translates of one tile. *Inventiones Mathematicae*, 124, 341–365, 1996.
- [16] J.Lagarias, Y.Wang: Spectral sets and factorizations of finite Abelian groups. *Journal of Functional Analysis*, 145, 73-98, 1997.
- [17] R.Malikiosis: On the structure of spectral and tiling subsets of cyclic groups. *Forum of Mathematics Sigma*, 10, 1-42, 2022.
- [18] M.Matolcsi: Fuglede’s conjecture fails in dimension 4. *Proceedings of the American Mathematical Society*, 133(10), 3021-3026, 2005.
- [19] D.Newman: Tessellation of integers. *Journal of Number Theory*, 9, 107-111, 1977.
- [20] K.Smith: The uncertainty principle on groups. *SIAM Journal on Applied Mathematics*, 50(2), 876-882, 1990.
- [21] M. Szegedy, Algorithms to tile the infinite grid with finite clusters. *Proceedings of the 39th Annual Symposium on Foundations of Computer Science (FOCS’98)*, IEEE Computer Society, Los Alamitos, CA, 137-145, 1998.
- [22] T.Tao: Fuglede’s conjecture is false in 5 and higher dimensions. *Mathematical Research Letters*, 11(2-3), 251-258, 2004.
- [23] T.Zhang: Fuglede’s conjecture holds in $\mathbb{Z}_p \times \mathbb{Z}_{p^n}$. *SIAM Journal on Discrete Mathematics*, 37(2), 1180-1197, 2023.