

Geodesic Deviation to All Orders via a Tangent Bundle Formalism

Joon-Hwi Kim¹

¹ *Walter Burke Institute for Theoretical Physics, California Institute of Technology, Pasadena, CA 91125*

We establish an *in-in* formalism for geodesic deviation as an alternative to Synge calculus, based on a covariant calculus of differential forms in tangent bundle. This derives the exact Lagrangian and equations governing the finite geodesic deviation between a free-falling test particle and an arbitrary observer, in terms of infinite sums whose coefficients are products of binomial coefficients. Explicit expressions are provided up to tenth order, finding agreements with the previous fourth-order result.

Introduction.—The geodesic deviation equation (GDE) is a foundational topic in general relativity, commonly covered in textbooks [1–7]. First derived in 1927 by Levi-Civita and Synge [8–12], this equation describes the evolution of the *infinitesimal* separation between two nearby free-falling test particles. Explicitly, it reads

$$\frac{D^2 y^\mu}{d\tau^2} = -R^\mu{}_{\nu\rho\sigma} u^\nu y^\rho u^\sigma, \quad (1)$$

where y^μ describes the infinitesimal separation as a vector and u^μ describes the unit-normalized four-velocity.

The generalization of the GDE to the case of *finite* separations has been a topic of research in the literature [13–29]. Here, the objective is to find corrections to the GDE perturbatively in the orders of the separation vector y , covariantly defined as a tangent to the geodesic segment joining the two particles. Namely, the right-hand side of Eq. (1) is augmented with terms involving nonlinear powers of y , coupled to the Riemann tensor $R^\mu{}_{\nu\rho\sigma}$ and its derivatives. Physically, this captures the higher-order tidal effects due to the non-infinitesimal size of y .

The higher-order extensions of the GDE have found fruitful physical applications [25, 29]: gravitational wave detectors, astrophysical jets, measurement of spacetime curvature, and an analytical treatment of eccentric relativistic orbits [17–24, 30–38]. For instance, Ref. [34] remarks that employing a higher-order version of the GDE could have improved the accuracy of LIGO’s original data analysis by 10% [25].

The explicit GDE valid up to $\mathcal{O}(y^3)$ and $\mathcal{O}(y^4)$ were obtained by works [13–16] in the 1970s and Vines [25] in 2014, respectively. Moreover, Vines [25] also provided formulae for the all-orders extension of the GDE and its Lagrangian formulation, though in terms of tensor expressions that are not fully expanded out in terms of Riemann curvature and its derivatives. To find the explicit equation or Lagrangian at an order, one has to solve a group of interrelated recursion relations in Appendix 3 of Ref. [25], which arise in the context of various and intricate identities of the Synge bitensor formalism [39–42].

In this paper, we intend to revisit this problem from a different framework. The achievements are the following. Firstly, we develop an alternative to the Synge formalism in which the relevant tensor expressions are computed from a covariant calculus of differential forms. Secondly, we provide the exact all-orders formula for the so-called Jacobi propagators [43–46], fully expanded out in terms

of Riemann tensor and its derivatives. Our exact expressions are infinite sums whose coefficients are products of binomial coefficients, showing agreements with Vines [25]. Consequently, we provide the explicit GDE and its Lagrangian up to $\mathcal{O}(y^{10})$ in the ancillary file `A11.nb` and Appendix B 2.

The key idea of our approach is to formulate geodesic deviation as an initial-value problem like in works [47–52]. Namely, we specify an *in-in* boundary condition for the geodesic: point x and a tangent vector y at x . This is to be contrasted with the Synge bitensor formalism [39–42] where a geodesic segment is characterized by its two endpoints x and z as an *in-out* boundary condition.

Geodesic Deviation in Tangent Bundle.—Let $(\mathcal{M}, g)$ be a d -dimensional real-analytic manifold with local coordinates x^μ and metric $g_{\mu\nu}$, which we call spacetime. Its tangent bundle, $T\mathcal{M}$, can be viewed as a $2d$ -dimensional manifold with local coordinates (x^μ, y^μ) , based on its local trivialization by the coordinate vector fields.

Coordinate transformations of $T\mathcal{M}$ are restricted in the form $(x^\mu, y^\mu) \mapsto (f^\mu(x), f^\mu{}_{,\nu}(x)y^\nu)$, where $f^\mu(x)$ describes a set of real-analytic functions in a local patch of $\mathcal{M}$. Importantly, the notion of covariance in $T\mathcal{M}$ is defined with respect to these coordinate transformations.

In particular, the following vector field in $T\mathcal{M}$ is *invariant* under such coordinate transformations:

$$N = y^\mu \frac{\partial}{\partial x^\mu} - \Gamma^\mu{}_{\rho\sigma}(x) y^\rho y^\sigma \frac{\partial}{\partial y^\mu}, \quad (2)$$

where $\Gamma^\mu{}_{\rho\sigma}(x)$ denote the Christoffel symbols. This can be easily seen by considering its interior products with a one-form basis in $T\mathcal{M}$ that transform covariantly:

$$\iota_N dx^\mu = y^\mu, \quad \iota_N Dy^\mu = 0. \quad (3)$$

Here, D denotes the covariant exterior derivative: $Dy^\mu = dy^\mu + \Gamma^\mu{}_{\nu\rho}(x) y^\nu dx^\rho$. In light of its invariance, N defines a structure characteristic of the tangent bundle. In the mathematical jargon, it describes a horizontal vector field due to the Ehresmann notion of a connection [53, 54].

Crucially, N holds the very significance as the *generator of geodesic deviation*. To see this, consider the first-order formulation of the geodesic equation:

$$\begin{aligned} \frac{d}{d\eta} X^\mu(\eta) &= Y^\mu(\eta), \\ \frac{d}{d\eta} Y^\mu(\eta) &= -\Gamma^\mu{}_{\rho\sigma}(X(\eta)) Y^\rho(\eta) Y^\sigma(\eta). \end{aligned} \quad (4)$$

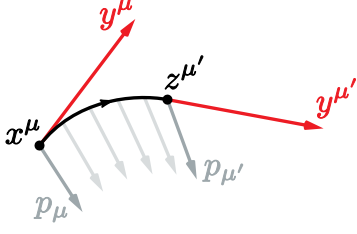


FIG. 1. A geodesic segment joins two spacetime points x (observer) and z (test particle). The tangent vectors are respectively denoted as y^μ and $y^{\mu'}$, which are normalized such that x and z are related by unit-time geodesic flows.

Evidently, the explicit components of N in Eq. (2) encode Eq. (4). This implies that the power series solution of Eq. (4) is given by $X^\mu(\eta) = e^{\eta N} x^\mu$ and $Y^\mu(\eta) = e^{\eta N} y^\mu$ for initial conditions $X^\mu(0) = x^\mu$ and $Y^\mu(0) = y^\mu$, where N is understood as a differential operator. Namely, the time- η flow of N solves the geodesic equation.

In particular, consider the unit-time flow:

$$X^\mu(1) = e^N x^\mu, \quad Y^\mu(1) = e^N y^\mu. \quad (5)$$

Borrowing the notation in the Synge bitensor formalism [39–42], we rephrase Eq. (5) as the following:

$$z^{\mu'} = \delta^{\mu'}_\mu (e^N x^\mu), \quad y^{\mu'} = \delta^{\mu'}_\mu (e^N y^\mu), \quad (6)$$

where $\delta^{\mu'}_\mu$ is Kronecker delta. Namely, we assign unprimed and primed indices to objects at the original (x) and deviated (z) points, respectively. The rationale might be that indices represent the behavior under coordinate transformations, and tensors at x and z transform with different Jacobian factors. Fig. 1 provides a space-time picture that visualizes Eq. (6).

The unit-time flow of N can be regarded as a diffeomorphism in $T\mathcal{M}$. By the very definition of the Lie derivative, the pullback of a tensor α in $T\mathcal{M}$ is given by $e^{\mathcal{L}_N} \alpha$. Practically, this pullback replaces every x with $e^N x$ and y with $e^N y$. For instance, the following identity holds:

$$dz^{\mu'} = \delta^{\mu'}_\mu (e^{\mathcal{L}_N} dx^\mu). \quad (7)$$

Notably, for differential forms, Lie derivatives are neatly computed by using the Cartan magic formula:

$$\mathcal{L}_N = d\iota_N + \iota_N d. \quad (8)$$

We leave it as an exercise to check Eq. (7) at each order in y by using Eq. (8).

Geodesic Deviation in A Direct Sum Bundle.—For pedagogical reasons, let us also consider a slight generalization of the above construction. Suppose the direct sum of the tangent and cotangent bundles: $\mathcal{P} = T\mathcal{M} \oplus T^*\mathcal{M}$, where local coordinates are (x^μ, y^μ, p_μ) . Eventually, this 3d-dimensional space will serve as our geometrical arena for a first-order formulation of GDE. In this larger bundle, the vector field N is uniquely defined by $\iota_N dx^\mu = y^\mu$, $\iota_N Dy^\mu = 0$, and $\iota_N Dp_\mu = 0$.

It is not difficult to see that this N is the generator of geodesic deviation and parallel transport. In particular, consider the analog of the first-order differential equations in Eq. (4) to find that $p_{\mu'} = (e^N p_\mu) \delta^{\mu'}_\mu$ is the covector p_μ parallel-transported to the point z along the geodesic; see Fig. 1. This finding is succinctly stated as

$$p_{\mu'} = p_\mu W^\mu_{\mu'}, \quad (9)$$

where $W^\mu_{\mu'}$ denotes the *parallel propagator*: the Wilson line of the Levi-Civita connection computed about the geodesic between x and z . Namely, $p_\mu \mapsto p_{\mu'} = p_\mu W^\mu_{\mu'}$ describes the isomorphism between the cotangent spaces at x and z facilitated by the geodesic parallel transport.

Again, $(x, y, p) \mapsto e^{\eta N}(x, y, p)$ describes the time- η flow in $\mathcal{P}$ generated by the vector field N . Due to this geometrical interpretation, the unit-time geodesic deviation and parallel transport is given by the exponentiated Lie derivative $e^{\mathcal{L}_N}$.

For a concrete example, consider a one-form $p_\mu dx^\mu$, i.e., the trivial extension of the cotangent bundle's tautological one-form to $\mathcal{P}$. Then we have

$$p_{\mu'} dz^{\mu'} = e^{\mathcal{L}_N} (p_\mu dx^\mu), \quad (10)$$

the right-hand side of which can be computed order-by-order by series-expanding $e^{\mathcal{L}_N}$. We leave it as an exercise to carry out this computation and check consistency between the left-hand and right-hand sides.

Covariant Lie Derivative and Dressing Identity.—The computations utilizing $\mathcal{L}_N$, however, are not very efficient. One encounters unoccupied connection coefficients arising from the components of N , so covariance at the original point x is not manifested in intermediate steps.

In this light, we introduce a shorthand notation,

$$\mathcal{L}_N^D = D\iota_N + \iota_N D, \quad (11)$$

defined on any tensor-valued differential form. This describes a covariantized analog of the Lie derivative; more information might be found in Refs. [55–57].

Crucially, since $p_\mu dx^\mu$ is a scalar-valued one-form carrying no free indices, using D instead of d makes no difference. Hence we can equivalently use $\exp(\mathcal{L}_N^D)$ in Eq. (10). It is a privilege of exponentiated operators that they are distributable as

$$e^{\mathcal{L}_N^D} (p_\mu dx^\mu) = (e^{\mathcal{L}_N^D} p_\mu) (e^{\mathcal{L}_N^D} dx^\mu) = p_\mu (e^{\mathcal{L}_N^D} dx^\mu), \quad (12)$$

where the last equality follows from $\mathcal{L}_N^D p_\mu = \iota_N Dp_\mu = 0$. Therefore, from Eqs. (9), (10), and (12), it follows that

$$W^\mu_{\mu'} dz^{\mu'} = e^{\mathcal{L}_N^D} dx^\mu. \quad (13)$$

In the same way, it can be shown that the $\exp(\mathcal{L}_N^D)$ of a tensor-valued differential form computes its value at the deviated point, followed by the parallel-transportation back to the original point via dressing by the Wilson lines. This fact will be referred to as the *dressing identity*.

$$\begin{array}{ccccc}
dx^\mu & \xrightarrow{D\iota_N} & Dy^\mu & \longrightarrow & 0 \\
\iota_N D \downarrow & & \downarrow & & \\
0 & & (\iota_N R^\mu{}_\nu) y^\nu & \longrightarrow & 0 \\
& & \downarrow & & \\
& & (\iota_N D \iota_N R^\mu{}_\nu) y^\nu & \longrightarrow & 0 \\
& & \downarrow & & \\
& & \vdots & &
\end{array}$$

FIG. 2. A sequence of one-forms originating from dx^μ via the covariant Lie derivative $\mathcal{L}_N^D = D\iota_N + \iota_N D$. Note that $(D\iota_N)(\iota_N D) = 0$, while Ddx^μ vanishes for zero torsion.

Recursion for Jacobi Propagators.—Having set up the foundations of our formalism, we now concern explicit evaluations. In particular, it follows from Eq. (11) that the right-hand side of Eq. (13) evaluates as

$$dx^\mu + Dy^\mu + \sum_{\ell=2}^{\infty} \frac{1}{\ell!} ((\iota_N D)^{\ell-2} \iota_N R^\mu{}_\nu) y^\nu, \quad (14)$$

where $R^\mu{}_\nu$ denotes the Riemann curvature two-form such that $D^2 y^\mu = R^\mu{}_\nu y^\nu$. This computation is illustrated in

Fig. 2 as a curved deformation of the sequence of differential forms due to the Cartan magic formula. As a tensor at the point x , Eq. (14) eventually boils down to the following form:

$$W^\mu{}_{\mu'} dz^{\mu'} = e^{\mathcal{L}_N^D} dx^\mu = X^\mu{}_\sigma dx^\sigma + Y^\mu{}_\sigma Dy^\sigma. \quad (15)$$

The objective now is to find the tensors $X^\mu{}_\sigma$ and $Y^\mu{}_\sigma$ explicitly. Note that $W^{\mu'}{}_\mu X^\mu{}_\sigma$ and $W^{\mu'}{}_\mu Y^\mu{}_\sigma$ are exactly what are known as Jacobi propagators (denoted as $K^\mu{}_\sigma$ and $H^\mu{}_\sigma$ in Refs. [25, 58]), where $W^{\mu'}{}_\mu$ is the inverse of $W^\mu{}_{\mu'}$. We find it preferable to peel off the Wilson line, for which the path-ordered exponential formula is well-known and amenable.

To this end, one needs to compute $(\iota_N D)^{\ell-2} \iota_N R^\mu{}_\nu$ for $\ell \geq 2$. For $\ell=3$, one finds $\iota_N D \iota_N R^\mu{}_\nu = y^\kappa R^\mu{}_{\nu\rho\sigma;\kappa}(x) y^\rho dx^\sigma + R^\mu{}_{\nu\rho\sigma}(x) y^\rho Dy^\sigma$. When one hits this with a yet another $\iota_N D$, the covariant exterior derivative D can act on Dy^σ to generate another Riemann tensor. As a result, one finds two single-Riemann terms and one double-Riemann term at $\ell=4$ (see Appendix A). In the same fashion, higher concatenations of Riemann tensors arise at higher orders.

A recursive structure can be identified in these calculations. First of all, it follows that

$$(\iota_N D)^{\ell-2} \iota_N R^\mu{}_\nu y^\nu = \sum_{p=1}^{\lfloor \ell/2 \rfloor} \sum_{\alpha \in \Omega_p(\ell)} c^{\alpha_1, \dots, \alpha_p} (Q_{\alpha_1} \cdots Q_{\alpha_p})^\mu{}_\sigma dx^\sigma + \sum_{p=1}^{\lfloor \frac{\ell-1}{2} \rfloor} \sum_{\alpha \in \Omega_p(\ell-1)} c'^{\alpha_1, \dots, \alpha_p} (Q_{\alpha_1} \cdots Q_{\alpha_p})^\mu{}_\sigma Dy^\sigma, \quad (16)$$

where α runs over ordered partitions such that

$$\alpha = (\alpha_1, \alpha_2, \dots, \alpha_p) \in \Omega_p(\ell) \iff \alpha_1 + \alpha_2 + \dots + \alpha_p = \ell, \quad \alpha_i \geq 2. \quad (17)$$

Note that the total number of such partitions at each ℓ , i.e., $\sum_{p=1}^{\lfloor \ell/2 \rfloor} |\Omega_p(\ell)|$, is the $(\ell-1)$ th Fibonacci number. In Eq. (16), the “ Q -tensors” are defined as

$$(Q_\ell)^\mu{}_\sigma := y^{\kappa_1} \cdots y^{\kappa_\ell} R^\mu{}_{\kappa_1 \kappa_2 \sigma; \kappa_3; \dots; \kappa_\ell}(x) \implies (Q_\ell)_{\mu\sigma} = (Q_\ell)_{\sigma\mu}, \quad (Q_\ell)^\mu{}_\sigma y^\sigma = 0. \quad (18)$$

The recursion relations for the coefficients in Eq. (16) are easily found from identifying the following action of $\iota_N D$:

$$\iota_N D : (Q_{\alpha_1} \cdots Q_{\alpha_p})^\mu{}_\sigma dx^\sigma \mapsto \left[\sum_{i=1}^p (Q_{\alpha_1} \cdots Q_{\alpha_{i-1}} Q_{\alpha_i+1} Q_{\alpha_{i+1}} \cdots Q_{\alpha_p})^\mu{}_\sigma dx^\sigma \right] + (Q_{\alpha_1} \cdots Q_{\alpha_p})^\mu{}_\sigma Dy^\sigma, \quad (19a)$$

$$(Q_{\alpha_1} \cdots Q_{\alpha_p})^\mu{}_\sigma Dy^\sigma \mapsto \left[\sum_{i=1}^p (Q_{\alpha_1} \cdots Q_{\alpha_{i-1}} Q_{\alpha_i+1} Q_{\alpha_{i+1}} \cdots Q_{\alpha_p})^\mu{}_\sigma Dy^\sigma \right] + (Q_{\alpha_1} \cdots Q_{\alpha_p} Q_2)^\mu{}_\sigma dx^\sigma. \quad (19b)$$

With the proper identification of the boundary conditions [59], the solution is determined as [60]

$$c^{\alpha_1, \dots, \alpha_p} = \prod_{i=1}^p \binom{(\sum_{j=i}^p \alpha_j) - 2}{\alpha_i - 2}, \quad c'^{\alpha_1, \dots, \alpha_p} = \prod_{i=1}^p \binom{(\sum_{j=i}^p \alpha_j) - 1}{\alpha_i - 2}, \quad (20)$$

which describes products of binomial coefficients. Finally, plugging in Eq. (20) to Eq. (16), we arrive at the following formula for $e^{\mathcal{L}_N^D} dx^\mu$ from which the tensors X and Y in Eq. (15) are readily read off:

$$e^{\mathcal{L}_N^D} dx^\mu = dx^\mu + Dy^\mu + \sum_{\ell=2}^{\infty} \frac{1}{\ell!} \sum_{p=1}^{\lfloor \ell/2 \rfloor} \sum_{\alpha \in \Omega_p(\ell)} \left[\prod_{i=1}^p \binom{(\sum_{j=i}^p \alpha_j) - 2}{\alpha_i - 2} \right] \left((Q_{\alpha_1} \cdots Q_{\alpha_p})^\mu{}_\sigma dx^\sigma + (\alpha_p - 2) (Q_{\alpha_1} \cdots Q_{\alpha_{p-1}} Q_{\alpha_p-1})^\mu{}_\sigma Dy^\sigma \right). \quad (21)$$

See Appendix B 1 for explicit enumerations up to $\mathcal{O}(y^{10})$.

The explicit solution for the Jacobi propagators in Eq. (21) has not been spelled out in the literature to our best knowledge, though Appendix 3 of Vines [25] has identified a set of relevant recursion relations by building upon Ottewill and Wardell [61] and Dixon [58].

The Lagrangians.—We are now ready to derive the all-orders GDE and its Lagrangian. When described with the primed variables, the first-order action of a free-falling test particle is

$$\int d\sigma \left[p_{\mu'} \frac{dz^{\mu'}}{d\sigma} - \frac{e}{2} \left(g^{\mu'\nu'}(z) p_{\mu'} p_{\nu'} + m^2 \right) \right], \quad (22)$$

where e is the einbein (a Lagrange multiplier), and m is the rest mass. To describe this particle from an observer's worldline, $\sigma \mapsto x^\mu(\sigma)$, we identify that the first term in Eq. (22) originates from the one-form $p_{\mu'} dz^{\mu'} = p_\mu (\exp(\mathcal{L}_N^D) dx^\mu)$. Consequently, Eq. (22) boils down to

$$\int d\sigma \left[p_\mu \left(X^\mu{}_\nu \frac{dx^\nu}{d\sigma} + Y^\mu{}_\nu \frac{Dy^\nu}{d\sigma} \right) - \frac{e}{2} (p^2 + m^2) \right], \quad (23)$$

where $p^2 = g^{\mu\nu}(x) p_\mu p_\nu$ since Wilson lines due to the Levi-Civita connection respect the metric. By integrating out p_μ , we also find a second-order Lagrangian:

$$\int d\sigma \left[\frac{1}{2e} \left(X \frac{dx}{d\sigma} + Y \frac{Dy}{d\sigma} \right)^2 - \frac{m^2 e}{2} \right]. \quad (24)$$

Adopting an invariant measure of time $d\tau = m e d\sigma$ for $m \neq 0$ reproduces Vines [25]'s action for affinely parameterized worldlines (isochronous correspondence [16, 25]):

$$m \int d\tau \frac{1}{2} \left((X u + Y v)^2 - 1 \right). \quad (25)$$

Here, we have denoted $u^\mu := dx^\mu/d\tau$ and $v^\mu := Dy^\mu/d\tau$.

Eqs. (23)-(25) provide exact first-order and second-order Lagrangian formulations of the all-orders geodesic deviation, where the deviation y^μ is defined as a vector attached to the observer's worldline. We clarify that the observer's worldline is introduced as a nondynamical reference [25], while y^μ and p_μ are dynamical variables [62].

The Lagrangian in Eq. (25) is explicitly given up to $\mathcal{O}(y^{10})$ in Appendix B 2, showing perfect agreement with the previous $\mathcal{O}(y^5)$ result due to Vines [25].

Note that the second-order actions in Eqs. (24) and (25) could have been directly obtained by implementing our formalism simply in the tangent bundle $T\mathcal{M}$, instead of employing the extended bundle $\mathcal{P}$.

The All-Orders GDE.—The all-orders GDE follows by varying the above Lagrangians. Otherwise, it can also be derived from our formalism in the following way.

Consider the first-order formulation of the free-falling equations of motion, associated with Eq. (22):

$$\frac{dz^{\mu'}}{d\sigma} = e p^{\mu'}, \quad \frac{Dp^{\mu'}}{d\sigma} = 0. \quad (26)$$

The idea is to re-covariantize Eq. (26) at the observer's position, x^μ , via the geodesic Wilson line dressing. Earlier, we have obtained $W^\mu{}_{\mu'} dz^{\mu'} = X^\mu{}_\nu dx^\nu + Y^\mu{}_\nu Dy^\nu$ in Eq. (15). This applies to the first equation in Eq. (26). Taking a similar approach for the second equation as well, we obtain a first-order formulation of the all-orders GDE:

$$e p^\mu = X^\mu{}_\nu \frac{dx^\nu}{d\sigma} + Y^\mu{}_\nu \frac{Dy^\nu}{d\sigma}, \quad (27a)$$

$$\frac{Dp^\mu}{d\sigma} = - \left(\dot{X}^\mu{}_{\nu\sigma} \frac{dx^\sigma}{d\sigma} + \dot{Y}^\mu{}_{\nu\sigma} \frac{Dy^\sigma}{d\sigma} \right) p^\nu. \quad (27b)$$

For obtaining Eq. (27b), we have applied the dressing identity to the one-form Dp^μ :

$$W^\mu{}_{\mu'} Dp^{\mu'} = e^{\mathcal{L}_N^D} Dp^\mu = Dp^\mu + \dot{X}^\mu{}_{\nu\sigma} p^\nu dx^\sigma + \dot{Y}^\mu{}_{\nu\sigma} p^\nu Dy^\sigma. \quad (28)$$

The tensors $\dot{X}^\mu{}_{\nu\sigma}$ and $\dot{Y}^\mu{}_{\nu\sigma}$ are given as

$$\dot{X}^\mu{}_{\nu\sigma} = \sum_{\ell=2}^{\infty} \frac{1}{(\ell-1)!} \sum_{p=1}^{[\ell/2]} \sum_{\alpha \in \Omega_p(\ell)} \left[\prod_{i=1}^p \left(\binom{\sum_{j=i}^p \alpha_j}{\alpha_i - 2} \right) \right] (\dot{Q}_{\alpha_1})^\mu{}_{\nu\kappa} (Q_{\alpha_2} \cdots Q_{\alpha_p})^\kappa{}_\sigma, \quad (29a)$$

$$\dot{Y}^\mu{}_{\nu\sigma} = \sum_{\ell=2}^{\infty} \frac{1}{(\ell-1)!} \sum_{p=1}^{[\ell/2]} \sum_{\alpha \in \Omega_p(\ell)} \left[\prod_{i=1}^p \left(\binom{\sum_{j=i}^p \alpha_j}{\alpha_i - 2} \right) \right] (\alpha_p - 2) (\dot{Q}_{\alpha_1})^\mu{}_{\nu\kappa} (Q_{\alpha_2} \cdots Q_{\alpha_{p-1}} Q_{\alpha_p - 1})^\kappa{}_\sigma, \quad (29b)$$

which simply replaces $\ell!$ in Eq. (21) to $(\ell-1)!$ regarding the combinatorial factors. Here, we have defined, for $\ell \geq 2$,

$$(\dot{Q}_\ell)^\mu{}_{\nu\sigma} := y^{\kappa_1} \cdots y^{\kappa_{\ell-1}} R^\mu{}_{\nu\kappa_1\sigma; \kappa_2 \cdots \kappa_{\ell-1}}(x) \implies \begin{cases} (\dot{Q}_\ell)_{\mu\nu\sigma} = -(\dot{Q}_\ell)_{\nu\mu\sigma}, & (\dot{Q}_\ell)_{\mu\nu\sigma} y^\sigma = 0, \\ (\dot{Q}_\ell)_{[\mu\nu\sigma]} = 0, & (\dot{Q}_\ell)^\mu{}_{\nu\sigma} y^\nu = (Q_\ell)^\mu{}_\sigma. \end{cases} \quad (30)$$

Combining Eqs. (27a) and (27b), the second-order formulation of the all-orders GDE in the isochronous correspondence is found as

$$-Y^\mu{}_\nu \frac{Dv^\nu}{d\tau} = \left(\nabla_\rho X^\mu{}_\sigma + \dot{X}^\mu{}_{\nu\rho} X^\nu{}_\sigma \right) u^\rho u^\sigma + 2 \left(\nabla_\rho Y^\mu{}_\sigma + \dot{Y}^\mu{}_{\nu\rho} Y^\nu{}_\sigma \right) u^\rho v^\sigma + \left(\frac{\partial}{\partial y^\rho} Y^\mu{}_\sigma + \dot{Y}^\mu{}_{\nu\rho} Y^\nu{}_\sigma \right) v^\rho v^\sigma, \quad (31)$$

where it should be understood that ∇_ρ will be acted only on the Riemann tensors inside the Q -tensors.

By inverting the matrix $Y^\mu{}_\nu$ on the left-hand side, the explicit GDE in the form $-Dv^\mu/d\tau = \dots$ is obtained from Eq. (31) and shown up to $\mathcal{O}(y^{10})$ in the ancillary file `A11.nb`. See Appendix B3 for the specifics of this computation. Moreover, we have also verified that the GDE obtained in this way is exactly reproduced from varying the second-order Lagrangian in Eq. (25), up to $\mathcal{O}(y^5)$ in the ancillary file `Low.nb`. Here, we spell out the explicit GDE up to $\mathcal{O}(y^5)$ [63]:

$$\begin{aligned}
-\frac{Dv}{d\tau} = & \dot{Q}_2(u, u) + \frac{1}{2!} \left[\left(\dot{Q}_3(u, u) + \nabla_u(Q_2 u) \right) + 4\dot{Q}_2(v, u) \right] \\
& + \frac{1}{3!} \left[\left(\dot{Q}_4(u, u) + \nabla_u(Q_3 u) + \dot{Q}_2(u, Q_2 u) + 3\dot{Q}_2(Q_2 u, u) - \textcolor{blue}{Q_2 \dot{Q}_2(u, u)} \right) + \left(2\nabla_v(Q_2 u) + 6\dot{Q}_3(v, u) \right) + 4\dot{Q}_2(v, v) \right] \\
& + \frac{1}{4!} \left[\left(\dot{Q}_5(u, u) + \nabla_u(Q_4 u) + (\nabla_u Q_2)(Q_2 u) + Q_2 \nabla_u(Q_2 u) - \textcolor{blue}{2Q_2 \nabla_u(Q_2 u)} \right. \right. \\
& \quad \left. \left. + 6\dot{Q}_3(Q_2 u, u) + 3\dot{Q}_3(u, Q_2 u) + \dot{Q}_2(u, Q_3 u) + 4\dot{Q}_2(Q_3 u, u) - \textcolor{blue}{2Q_2 \dot{Q}_3(u, u)} - \textcolor{blue}{2Q_3 \dot{Q}_2(u, u)} \right) \right. \\
& \quad \left. + \left(4\nabla_u(Q_3 v) + 8\dot{Q}_4(v, u) + 8\dot{Q}_2(v, Q_2 u) + 8\dot{Q}_2(Q_2 v, u) - \textcolor{blue}{8Q_2 \dot{Q}_2(v, u)} \right) \right. \\
& \quad \left. + \left(2\nabla_v(Q_2 v) + 10\dot{Q}_3(v, v) \right) \right] \\
& + \frac{1}{5!} \left[\left(\dot{Q}_6(u, u) + \nabla_u(Q_5 u) + 10\dot{Q}_4(Q_2 u, u) + 10\dot{Q}_3(Q_3 u, u) + 5\dot{Q}_2(Q_4 u, u) + 5\dot{Q}_2(Q_2 Q_2 u, u) \right. \right. \\
& \quad \left. \left. + Q_2 \nabla_u(Q_3 u) + 3Q_3 \nabla_u(Q_2 u) + (\nabla_u Q_2) Q_3 + 3(\nabla_u Q_3) Q_2 - \frac{10}{3} Q_2 \nabla_u(Q_3 u) - 5Q_3 \nabla_u(Q_2 u) \right. \right. \\
& \quad \left. \left. + \dot{Q}_2(u, Q_4 u) + 4\dot{Q}_3(u, Q_3 u) + 6\dot{Q}_4(u, Q_2 u) + 10\dot{Q}_2(Q_2 u, Q_2 u) + \dot{Q}_2(u, Q_2 Q_2 u) - \frac{10}{3} Q_2 \dot{Q}_4(u, u) \right. \right. \\
& \quad \left. \left. - \textcolor{blue}{10Q_2 \dot{Q}_2(Q_2 u, u)} - \textcolor{blue}{5Q_3 \dot{Q}_3(u, u)} - \textcolor{blue}{3Q_4 \dot{Q}_2(u, u)} + \frac{7}{3} Q_2 Q_2 \dot{Q}_2(u, u) - \frac{10}{3} Q_2 \dot{Q}_2(u, Q_2 u) \right) \right. \\
& \quad \left. + \left(10\dot{Q}_5(v, u) + 6\nabla_u(Q_4 v) + 2Q_2 \nabla_u(Q_2 v) + 2(\nabla_u Q_2)(Q_2 v) - \frac{20}{3} Q_2 \nabla_u(Q_2 v) + 20\dot{Q}_3(Q_2 v, u) \right) \right. \\
& \quad \left. + \left(20\dot{Q}_2(Q_3 v, u) + 10\dot{Q}_2(v, Q_3 u) + 30\dot{Q}_3(v, Q_2 u) - \textcolor{blue}{20Q_2 \dot{Q}_3(v, u)} - \textcolor{blue}{20Q_3 \dot{Q}_2(v, u)} \right) \right. \\
& \quad \left. + \left(6\nabla_v(Q_3 v) + 18\dot{Q}_4(v, v) + 6\dot{Q}_2(Q_2 v, v) - 2Q_2 \dot{Q}_2(v, v) + 10\dot{Q}_2(v, Q_2 v) - \frac{40}{3} Q_2 \dot{Q}_2(v, v) \right) \right] \\
& + \mathcal{O}(y^6).
\end{aligned} \tag{32}$$

We have color-coded terms due to Y^{-1} and adopted a condensed notation: $Dv/d\tau \rightarrow Dv^\mu/d\tau$, $\dot{Q}_3(v, u) \rightarrow (\dot{Q}_3)^\mu{}_{\rho\sigma} v^\rho u^\sigma$, $Q_2 \dot{Q}_2(u, u) \rightarrow (Q_2)^\mu{}_\nu (\dot{Q}_2)^\nu{}_{\rho\sigma} u^\rho u^\sigma$, $\nabla_v(Q_2 u) \rightarrow v^\rho (\nabla_\rho(Q_2)^\mu{}_\sigma) u^\sigma$, $(\nabla_u Q_2)(Q_2 u) \rightarrow u^\rho (\nabla_\rho(Q_2)^\mu{}_\sigma) (Q_2)^\sigma{}_\kappa u^\kappa$, etc. The sources of the minus signs on the right-hand side are either Y^{-1} or the Riemann tensor identities used for simplifying the quadratic-in- v part.

Zero-Torsion Identities.—One may notice that the behavior of $\dot{X}^\mu{}_{\nu\rho}$ and $\dot{Y}^\mu{}_{\nu\rho}$ in Eq. (31) is suggestive of connection coefficients. In fact, they arise from the conjugation $e^{\mathcal{L}_N^D} D e^{-\mathcal{L}_N^D} = e^{[\mathcal{L}_N^D, \cdot]} D$ of the covariant exterior derivative. In turn, it can be seen that they encode the covariant derivative at the deviated point: see Eq. (C9).

On a related note, the torsion-free condition for the Levi-Civita connection, as $Ddx^\mu = \Gamma^\mu{}_{\rho\sigma} dx^\rho \wedge dx^\sigma = 0$, implies $0 = (e^{\mathcal{L}_N^D} D e^{-\mathcal{L}_N^D})(e^{\mathcal{L}_N^D} dx^\mu) = D(X^\mu{}_\sigma dx^\sigma + Y^\mu{}_\sigma Dy^\sigma) + (\dot{X}^\mu{}_{\nu\rho} dx^\rho + \dot{Y}^\mu{}_{\nu\rho} Dy^\rho) \wedge (X^\nu{}_\sigma dx^\sigma + Y^\nu{}_\sigma Dy^\sigma)$, which unpacks into three identities:

$$\nabla_{[\rho} X^\mu{}_{\sigma]} + \dot{X}^\mu{}_{\nu[\rho} X^\nu{}_{\sigma]} = -\frac{1}{2} Y^\mu{}_\lambda R^\lambda{}_{\nu\rho\sigma}, \tag{33a}$$

$$\nabla_\rho Y^\mu{}_\sigma + \dot{X}^\mu{}_{\nu\rho} Y^\nu{}_\sigma = \frac{\partial}{\partial y^\rho} X^\mu{}_\sigma + \dot{Y}^\mu{}_{\nu\rho} X^\nu{}_\sigma, \tag{33b}$$

$$\frac{\partial}{\partial y^\rho} Y^\mu{}_\sigma + \dot{Y}^\mu{}_{\nu[\rho} Y^\nu{}_{\sigma]} = 0. \tag{33c}$$

Especially, we have made a use of Eq. (33b) in Eq. (31) to simplify the computation of the term linear in both u and v . The ancillary file `Low.nb` provides an explicit check of Eqs. (33a)-(33c) up to $\mathcal{O}(y^5)$ or $\mathcal{O}(y^4)$, which exploits various identities about the Riemann tensor. Also,

note that more identities follow in a similar fashion by conjugating $D^2 = R$, $[D, R] = 0$, etc.

Summary and Outlook.—In this paper, we revisited the problem of finding the all-orders-exact GDE for finite separations by formulating geodesic deviation and transport as a flow along a vector field in tangent bundle. The technique of covariant Lie derivative then systematically defines and generates various bitensors with the parallel propagators peeled off, directly producing manifestly covariantized expressions at the original point. This achieves an in-in formalism for geodesic deviation that serves as an alternative to the Synge calculus.

The explicit outcomes are the all-orders formula for the Jacobi propagators in Eq. (21) as well as the second-order GDE and its Lagrangian given up to $\mathcal{O}(y^{10})$.

Our framework is versatile and could find further applications in the context of quantum field theory: manifestly covariant perturbation theories for gauge and gravitational interactions, worldline formalism [64–68], or field space geometry [69, 70] and sigma models [71–73]. Appendix C implements our formalism in nonabelian gauge theories, which describes gauge-covariant translations.

Moreover, we realize that our innocuous attempt to re-

master the fundamental subject of all-orders-in-deviation GDE surprisingly connects to a persistent and seemingly disparate problem in the modern times: finding the all-orders-in-spin equations of motion for the Kerr black hole in its effective point-particle description [74–81]. By implementing a probe counterpart of the Newman-Janis algorithm [82], we have found that the all-orders GDE for an imaginary deviation can deduce a part of the black hole’s all-orders-in-spin equations of motion, yield-

ing a nonlinear completion of the Mathisson-Papapetrou-Dixon [43, 83, 84] equations. More details will be presented in a follow-up article [85].

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Appendix A: The “Organic Chemistry” of Covariant Lie Derivative Calculus

In this appendix, we devise a slight extension of the Penrose graphical notation [87–89] with the following rules.

$$\begin{aligned}
 \mu \text{---} \circ &= dx^\mu \\
 \mu \text{---} \bullet &= y^\mu \\
 \mu \text{---} \odot &= \mathcal{D}y^\mu
 \end{aligned}
 \quad
 \begin{aligned}
 \begin{array}{c} \mu \\ \diagup \\ \text{---} \times \text{---} \\ \diagdown \end{array} &= \begin{array}{c} \nu \quad \rho \\ \diagup \quad \diagdown \\ \text{---} \times \text{---} \\ \diagdown \quad \diagup \end{array} = R^\mu{}_{\nu\rho\sigma}(x)
 \end{aligned}
 \tag{A1}$$

The higher covariant derivatives of the Riemann tensor will be denoted as the following.

$$\begin{array}{c} \mu \\ \diagup \\ \text{---} \times \text{---} \\ \diagdown \end{array} \alpha_k = \begin{array}{c} \mu \\ \diagup \\ \text{---} \times \text{---} \\ \diagdown \end{array} \begin{array}{c} \nu \quad \rho \\ \diagup \quad \diagdown \\ \text{---} \times \text{---} \\ \diagdown \quad \diagup \end{array} \alpha_k = R^\mu{}_{\nu\rho\sigma;\kappa_1 \dots \kappa_k}(x) y^\nu y^\rho y^{\kappa_1} \dots y^{\kappa_k}
 \tag{A2}$$

The relevant “molecular backbones” are the “carbon” (as Riemann tensor) chains of the following form.

$$\begin{array}{c} \alpha_i \\ \diagup \\ \text{---} \times \text{---} \\ \diagdown \end{array} \alpha_j \dots \begin{array}{c} \alpha_{p-1} \\ \diagup \\ \text{---} \times \text{---} \\ \diagdown \end{array} \alpha_p
 \tag{A3}$$

The computation of $\iota_N D(\iota_N R^\mu{}_\nu)$ and $(\iota_N D)^2(\iota_N R^\mu{}_\nu)$ proceeds as the following.

$$\begin{array}{c}
 \begin{array}{c} \mu \\ \diagup \\ \text{---} \times \text{---} \\ \diagdown \end{array} \\
 \downarrow \mathcal{D} \\
 \begin{array}{c} \mu \\ \diagup \\ \text{---} \times \text{---} \\ \diagdown \end{array} \odot \quad \begin{array}{c} \mu \\ \diagup \\ \text{---} \times \text{---} \\ \diagdown \end{array} \bullet \quad \begin{array}{c} \mu \\ \diagup \\ \text{---} \times \text{---} \\ \diagdown \end{array} \odot \quad \begin{array}{c} \mu \\ \diagup \\ \text{---} \times \text{---} \\ \diagdown \end{array} \odot \\
 \downarrow \iota_N \\
 \begin{array}{c} \mu \\ \diagup \\ \text{---} \times \text{---} \\ \diagdown \end{array} \odot \quad 0 \quad - \begin{array}{c} \mu \\ \diagup \\ \text{---} \times \text{---} \\ \diagdown \end{array} \odot \quad - \begin{array}{c} \mu \\ \diagup \\ \text{---} \times \text{---} \\ \diagdown \end{array} \odot \\
 \downarrow \\
 \begin{array}{c} \mu \\ \diagup \\ \text{---} \times \text{---} \\ \diagdown \end{array} \odot \quad \downarrow 0 \\
 \downarrow \\
 \begin{array}{c} \mu \\ \diagup \\ \text{---} \times \text{---} \\ \diagdown \end{array} \odot \quad \begin{array}{c} \mu \\ \diagup \\ \text{---} \times \text{---} \\ \diagdown \end{array} \odot \quad \begin{array}{c} \mu \\ \diagup \\ \text{---} \times \text{---} \\ \diagdown \end{array} \odot \quad \frac{1}{2} \begin{array}{c} \mu \\ \diagup \\ \text{---} \times \text{---} \\ \diagdown \end{array} \odot \\
 \downarrow \\
 \begin{array}{c} \mu \\ \diagup \\ \text{---} \times \text{---} \\ \diagdown \end{array} \odot \quad 0 \quad 0 \quad \begin{array}{c} \mu \\ \diagup \\ \text{---} \times \text{---} \\ \diagdown \end{array} \odot
 \end{array}
 \tag{A4}$$

The ordering for the differential forms should be clear from the orientations designated in Eq.(A1). The interior product ι_N is a “reduction reaction” that substitutes a “hydroxyl group” (as dx) with a “hydrogen” (as y). Observe the mechanism for the “carbon polymerization” (as Q -tensor concatenation).

Appendix B: Explicit Results up to Tenth Order

1. The Jacobi Propagators

The Jacobi propagators with the parallel propagator peeled off are given in Eq. (21). It is then trivial to enumerate them explicitly up to any desired order, say $\mathcal{O}(y^{10})$. With the definition of the Q -tensors in Eq. (18), they are

$$Dy^\mu = Dy^\mu, \quad (B1)$$

$$(\iota_N R^\mu{}_\nu) y^\nu = (Q_2)^\mu{}_\sigma dx^\sigma, \quad (B2)$$

$$(\iota_N D \iota_N R^\mu{}_\nu) y^\nu = (Q_3)^\mu{}_\sigma dx^\sigma + (Q_2)^\mu{}_\sigma Dy^\sigma, \quad (B3)$$

$$((\iota_N D)^2 \iota_N R^\mu{}_\nu) y^\nu = (Q_4 + Q_2 Q_2)^\mu{}_\sigma dx^\sigma + (2Q_3)^\mu{}_\sigma Dy^\sigma, \quad (B4)$$

$$((\iota_N D)^3 \iota_N R^\mu{}_\nu) y^\nu = (Q_5 + 3Q_3 Q_2 + Q_2 Q_3)^\mu{}_\sigma dx^\sigma + (3Q_4 + Q_2 Q_2)^\mu{}_\sigma Dy^\sigma, \quad (B5)$$

$$\begin{aligned} ((\iota_N D)^4 \iota_N R^\mu{}_\nu) y^\nu = & (Q_6 + 6Q_4 Q_2 + 4Q_3 Q_3 + Q_2 Q_4)^\mu{}_\sigma dx^\sigma \\ & + (4Q_5 + 4Q_3 Q_2 + 2Q_2 Q_3)^\mu{}_\sigma Dy^\sigma \\ & + (Q_2 Q_2 Q_2)^\mu{}_\sigma dx^\sigma, \end{aligned} \quad (B6)$$

$$\begin{aligned} ((\iota_N D)^5 \iota_N R^\mu{}_\nu) y^\nu = & (Q_7 + 10Q_5 Q_2 + 10Q_4 Q_3 + 5Q_3 Q_4 + Q_2 Q_5)^\mu{}_\sigma dx^\sigma \\ & + (5Q_6 + 10Q_4 Q_2 + 10Q_3 Q_3 + 3Q_2 Q_4)^\mu{}_\sigma Dy^\sigma \\ & + (5Q_3 Q_2 Q_2 + 3Q_2 Q_3 Q_2 + Q_2 Q_2 Q_3)^\mu{}_\sigma dx^\sigma \\ & + (Q_2 Q_2 Q_2)^\mu{}_\sigma Dy^\sigma, \end{aligned} \quad (B7)$$

$$\begin{aligned} ((\iota_N D)^6 \iota_N R^\mu{}_\nu) y^\nu = & (Q_8 + 15Q_6 Q_2 + 20Q_5 Q_3 + 15Q_4 Q_4 + 6Q_3 Q_5 + Q_2 Q_6)^\mu{}_\sigma dx^\sigma \\ & + (6Q_7 + 20Q_5 Q_2 + 30Q_4 Q_3 + 18Q_3 Q_4 + 4Q_2 Q_5)^\mu{}_\sigma Dy^\sigma \\ & + \left(\begin{array}{l} 15Q_4 Q_2 Q_2 + 18Q_3 Q_3 Q_2 + 6Q_2 Q_4 Q_2 \\ + 6Q_3 Q_2 Q_3 + 4Q_2 Q_3 Q_3 + Q_2 Q_2 Q_4 \end{array} \right)^\mu{}_\sigma dx^\sigma \\ & + (6Q_3 Q_2 Q_2 + 4Q_2 Q_3 Q_2 + 2Q_2 Q_2 Q_3)^\mu{}_\sigma Dy^\sigma \\ & + (Q_2 Q_2 Q_2 Q_2)^\mu{}_\sigma dx^\sigma, \end{aligned} \quad (B8)$$

$$\begin{aligned} ((\iota_N D)^7 \iota_N R^\mu{}_\nu) y^\nu = & (Q_9 + 21Q_7 Q_2 + 35Q_6 Q_3 + 35Q_5 Q_4 + 21Q_4 Q_5 + 7Q_3 Q_6 + Q_2 Q_7)^\mu{}_\sigma dx^\sigma \\ & + (7Q_8 + 35Q_6 Q_2 + 70Q_5 Q_3 + 63Q_4 Q_4 + 28Q_3 Q_5 + 5Q_2 Q_6)^\mu{}_\sigma Dy^\sigma \\ & + \left(\begin{array}{l} 35Q_5 Q_2 Q_2 + 63Q_4 Q_3 Q_2 + 42Q_3 Q_4 Q_2 + 10Q_2 Q_5 Q_2 \\ + 21Q_4 Q_2 Q_3 + 28Q_3 Q_3 Q_3 + 10Q_2 Q_4 Q_3 \\ + 7Q_3 Q_2 Q_4 + 5Q_2 Q_3 Q_4 + Q_2 Q_2 Q_5 \end{array} \right)^\mu{}_\sigma dx^\sigma \\ & + \left(\begin{array}{l} 21Q_4 Q_2 Q_2 + 28Q_3 Q_3 Q_2 + 10Q_2 Q_4 Q_2 \\ + 14Q_3 Q_2 Q_3 + 10Q_2 Q_3 Q_3 + 3Q_2 Q_2 Q_4 \end{array} \right)^\mu{}_\sigma Dy^\sigma \\ & + (7Q_3 Q_2 Q_2 Q_2 + 5Q_2 Q_3 Q_2 Q_2 + 3Q_2 Q_2 Q_3 Q_2 + Q_2 Q_2 Q_2 Q_3)^\mu{}_\sigma dx^\sigma \\ & + (Q_2 Q_2 Q_2 Q_2)^\mu{}_\sigma Dy^\sigma \end{aligned} \quad (B9)$$

$$\begin{aligned} ((\iota_N D)^8 \iota_N R^\mu{}_\nu) y^\nu = & (Q_{10} + 28Q_8 Q_2 + 56Q_7 Q_3 + 70Q_6 Q_4 + 56Q_5 Q_5 + 28Q_4 Q_6 + 8Q_3 Q_7 + Q_2 Q_8)^\mu{}_\sigma dx^\sigma \\ & + (8Q_9 + 56Q_7 Q_2 + 140Q_6 Q_3 + 168Q_5 Q_4 + 112Q_4 Q_5 + 40Q_3 Q_6 + 6Q_2 Q_7)^\mu{}_\sigma Dy^\sigma \\ & + \left(\begin{array}{l} 70Q_6 Q_2 Q_2 + 168Q_5 Q_3 Q_2 + 168Q_4 Q_4 Q_2 + 80Q_3 Q_5 Q_2 + 15Q_2 Q_6 Q_2 \\ + 56Q_5 Q_2 Q_3 + 112Q_4 Q_3 Q_3 + 80Q_3 Q_4 Q_3 + 20Q_2 Q_5 Q_3 \\ + 28Q_4 Q_2 Q_4 + 40Q_3 Q_3 Q_4 + 15Q_2 Q_4 Q_4 + 8Q_3 Q_2 Q_5 + 6Q_2 Q_3 Q_5 + Q_2 Q_2 Q_6 \end{array} \right)^\mu{}_\sigma dx^\sigma \\ & + \left(\begin{array}{l} 56Q_5 Q_2 Q_2 + 112Q_4 Q_3 Q_2 + 80Q_3 Q_4 Q_2 + 20Q_2 Q_5 Q_2 \\ + 56Q_4 Q_2 Q_3 + 80Q_3 Q_3 Q_3 + 30Q_2 Q_4 Q_3 \\ + 24Q_3 Q_2 Q_4 + 18Q_2 Q_3 Q_4 + 4Q_2 Q_2 Q_5 \end{array} \right)^\mu{}_\sigma Dy^\sigma \\ & + \left(\begin{array}{l} 28Q_4 Q_2 Q_2 Q_2 + 40Q_3 Q_3 Q_2 Q_2 + 24Q_3 Q_2 Q_3 Q_2 + 18Q_2 Q_3 Q_3 Q_2 + 15Q_2 Q_4 Q_2 Q_2 \\ + 6Q_2 Q_2 Q_4 Q_2 + 8Q_3 Q_2 Q_2 Q_3 + 6Q_2 Q_3 Q_2 Q_3 + 4Q_2 Q_2 Q_3 Q_3 + Q_2 Q_2 Q_2 Q_4 \end{array} \right)^\mu{}_\sigma dx^\sigma \\ & + (8Q_3 Q_2 Q_2 Q_2 + 6Q_2 Q_3 Q_2 Q_2 + 4Q_2 Q_2 Q_3 Q_2 + 2Q_2 Q_2 Q_2 Q_3)^\mu{}_\sigma Dy^\sigma \\ & + (Q_2 Q_2 Q_2 Q_2 Q_2)^\mu{}_\sigma dx^\sigma. \end{aligned} \quad (B10)$$

In the attached file `Q.nb`, we have verified this result up to $\mathcal{O}(y^{10})$ by direct computations of $((\iota_N D)^{\ell-1} \iota_N R^\mu{}_\nu) y^\nu$ from the covariant calculus of differential forms as in Eq. (A4). Eqs. (B1)-(B5) agree exactly with Eq. (83) of Ref. [25].

2. The Lagrangian

The Lagrangian for the all-orders GDE in isochronous correspondence is given in Eq. (25):

$$L = \frac{m}{2}(Xu + Yv)^2 - \frac{m}{2} = m \left(\frac{u^2 - 1}{2} + u \cdot v \right) + \frac{1}{2}mv^2 + m \sum_{\ell=2}^{\infty} \frac{1}{\ell!} \mathcal{L}_{\ell}. \quad (\text{B11})$$

In the last expression, the bracketed terms can be discarded as a constant plus a total derivative. The term $\frac{1}{2}mv^2$, on the other hand, is the standard kinetic energy. Hence it remains to spell out the “interaction Lagrangian” $\mathcal{L}_{\ell}$ at each order ℓ , which follows from the explicit Jacobi propagators by straightforward algebra:

$$\mathcal{L}_2 = uQ_2u, \quad (\text{B12})$$

$$\mathcal{L}_3 = uQ_3u + v(4Q_2)u, \quad (\text{B13})$$

$$\mathcal{L}_4 = u(Q_4 + 4Q_2Q_2)u + v(6Q_3)u + v(4Q_2)v, \quad (\text{B14})$$

$$\mathcal{L}_5 = u(Q_5 + 14Q_3Q_2)u + v(8Q_4 + 16Q_2Q_2)u + v(10Q_3)v, \quad (\text{B15})$$

$$\begin{aligned} \mathcal{L}_6 = & u(Q_6 + 22Q_4Q_2 + 14Q_3Q_3 + 16Q_2Q_2Q_2)u \\ & + v(10Q_5 + 50Q_3Q_2 + 30Q_2Q_3)u + v(18Q_4 + 16Q_2Q_2)v, \end{aligned} \quad (\text{B16})$$

$$\begin{aligned} \mathcal{L}_7 = & u(Q_7 + 32Q_5Q_2 + 50Q_4Q_3 + 62Q_3Q_2Q_2 + 66Q_2Q_3Q_2)u \\ & + v(12Q_6 + 108Q_4Q_2 + 108Q_3Q_3 + 52Q_2Q_4 + 64Q_2Q_2Q_2)u \\ & + v(28Q_5 + 112Q_3Q_2)v, \end{aligned} \quad (\text{B17})$$

$$\begin{aligned} \mathcal{L}_8 = & u \left(Q_8 + 44Q_6Q_2 + 82Q_5Q_3 + 50Q_4Q_4 \right. \\ & \left. + 114Q_4Q_2Q_2 + 302Q_3Q_3Q_2 + 62Q_3Q_2Q_3 + 174Q_2Q_4Q_2 + 64Q_2Q_2Q_2Q_2 \right)u \\ & + v \left(14Q_7 + 196Q_5Q_2 + 266Q_4Q_3 + 210Q_3Q_4 + 84Q_2Q_5 \right. \\ & \left. + 238Q_3Q_2Q_2 + 308Q_2Q_3Q_2 + 126Q_2Q_2Q_3 \right)u \\ & + v(40Q_6 + 220Q_3Q_3 + 272Q_4Q_2 + 64Q_2Q_2Q_2)v, \end{aligned} \quad (\text{B18})$$

$$\begin{aligned} \mathcal{L}_9 = & u \left(Q_9 + 58Q_7Q_2 + 126Q_6Q_3 + 182Q_5Q_4 \right. \\ & \left. + 198Q_5Q_2Q_2 + 626Q_4Q_3Q_2 + 238Q_4Q_2Q_3 + 916Q_3Q_4Q_2 + 364Q_3Q_3Q_3 + 370Q_2Q_5Q_2 \right. \\ & \left. + 254Q_3Q_2Q_2Q_2 + 674Q_2Q_3Q_2Q_2 \right)u \\ & + v \left(16Q_8 + 320Q_6Q_2 + 544Q_5Q_3 + 576Q_4Q_4 + 376Q_3Q_5 + 128Q_2Q_6 \right. \\ & \left. + 624Q_4Q_2Q_2 + 1288Q_3Q_3Q_2 + 488Q_3Q_2Q_3 + 928Q_2Q_4Q_2 + 736Q_2Q_3Q_3 + 240Q_2Q_2Q_4 \right. \\ & \left. + 256Q_2Q_2Q_2Q_2 \right)u \\ & + v(54Q_7 + 552Q_5Q_2 + 1188Q_4Q_3 + 492Q_3Q_2Q_2 + 372Q_2Q_3Q_2)v, \end{aligned} \quad (\text{B19})$$

$$\begin{aligned} \mathcal{L}_{10} = & u \left(Q_{10} + 74Q_8Q_2 + 184Q_7Q_3 + 308Q_6Q_4 + 182Q_5Q_5 \right. \\ & \left. + 326Q_6Q_2Q_2 + 1200Q_5Q_3Q_2 + 436Q_5Q_2Q_3 + 2118Q_4Q_4Q_2 + 1592Q_4Q_3Q_3 + 238Q_4Q_2Q_4 \right. \\ & \left. + 2200Q_3Q_5Q_2 + 1280Q_3Q_4Q_3 + 690Q_2Q_6Q_2 + 494Q_4Q_2Q_2Q_2 + 1540Q_3Q_3Q_2Q_2 \right. \\ & \left. + 1540Q_3Q_2Q_3Q_2 + 254Q_3Q_2Q_2Q_3 + 2226Q_2Q_4Q_2Q_2 + 1962Q_2Q_3Q_3Q_2 + 256Q_2Q_2Q_2Q_2Q_2 \right)u \\ & + v \left(18Q_9 + 486Q_7Q_2 + 990Q_6Q_3 + 1302Q_5Q_4 + 1134Q_4Q_5 + 630Q_3Q_6 + 186Q_2Q_7 + 1374Q_5Q_2Q_2 \right. \\ & \left. + 3726Q_4Q_3Q_2 + 1350Q_4Q_2Q_3 + 4320Q_3Q_4Q_2 + 3240Q_3Q_3Q_3 + 966Q_3Q_2Q_4 + 2220Q_2Q_5Q_2 \right. \\ & \left. + 2580Q_2Q_4Q_3 + 1602Q_2Q_3Q_4 + 438Q_2Q_2Q_5 + 1002Q_3Q_2Q_2Q_2 + 1674Q_2Q_3Q_2Q_2 \right. \\ & \left. + 1422Q_2Q_2Q_3Q_2 + 510Q_2Q_2Q_2Q_3 \right)u \\ & + v \left(70Q_8 + 1000Q_6Q_2 + 2660Q_5Q_3 + 1764Q_4Q_4 + 1356Q_4Q_2Q_2 + 3260Q_3Q_3Q_2 + 980Q_3Q_2Q_3 \right. \\ & \left. + 1300Q_2Q_4Q_2 + 256Q_2Q_2Q_2Q_2 \right)v. \end{aligned} \quad (\text{B20})$$

Eqs. (B12)-(B15) agree flawlessly with Eq. (5) in Vines [25].

3. The GDE

The GDE, in the second-order formulation, is given in Eq. (31). The definition of X^μ_σ and Y^μ_σ is given in Eq. (21). The definition of $\dot{X}^\mu_{\nu\rho}$ and $\dot{Y}^\mu_{\nu\rho}$ is given in Eq. (29). For the reader's sake, we explicitly spell them out at low orders:

$$X^\mu_\sigma = \delta^\mu_\sigma + \frac{1}{2!}(Q_2)^\mu_\sigma + \frac{1}{3!}(Q_3)^\mu_\sigma + \frac{1}{4!}(Q_4 + Q_2 Q_2)^\mu_\sigma + \frac{1}{5!}(Q_5 + 3Q_3 Q_2 + Q_2 Q_3)^\mu_\sigma + \mathcal{O}(y^6), \quad (\text{B21a})$$

$$Y^\mu_\sigma = \delta^\mu_\sigma + \frac{1}{3!}(Q_2)^\mu_\sigma + \frac{1}{4!}(2Q_3)^\mu_\sigma + \frac{1}{5!}(3Q_4 + Q_2 Q_2)^\mu_\sigma + \frac{1}{6!}(4Q_5 + 4Q_3 Q_2 + 2Q_2 Q_3)^\mu_\sigma + \mathcal{O}(y^6), \quad (\text{B21b})$$

$$\begin{aligned} \dot{X}^\mu_{\nu\sigma} &= \frac{1}{1!}(\dot{Q}_2)^\mu_{\nu\sigma} + \frac{1}{2!}(\dot{Q}_3)^\mu_{\nu\sigma} + \frac{1}{3!}\left((\dot{Q}_4)^\mu_{\nu\sigma} + (\dot{Q}_2)^\mu_{\nu\lambda}(Q_2)^\lambda_\sigma\right) \\ &\quad + \frac{1}{4!}\left((\dot{Q}_5)^\mu_{\nu\sigma} + 3(\dot{Q}_3)^\mu_{\nu\lambda}(Q_2)^\lambda_\sigma + (\dot{Q}_2)^\mu_{\nu\lambda}(Q_3)^\lambda_\sigma\right) + \mathcal{O}(y^5), \end{aligned} \quad (\text{B22a})$$

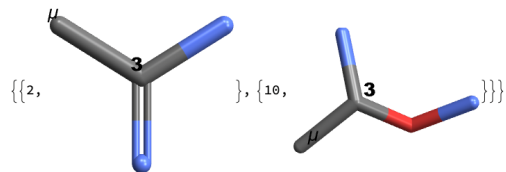
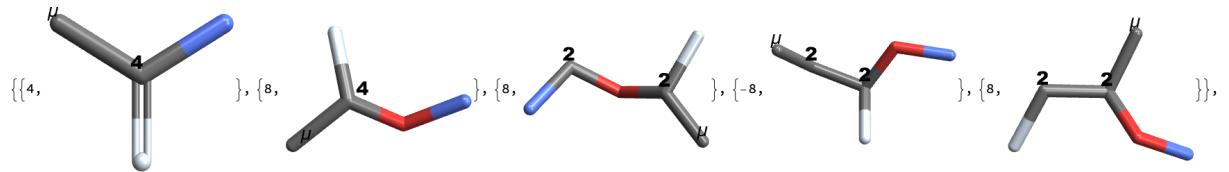
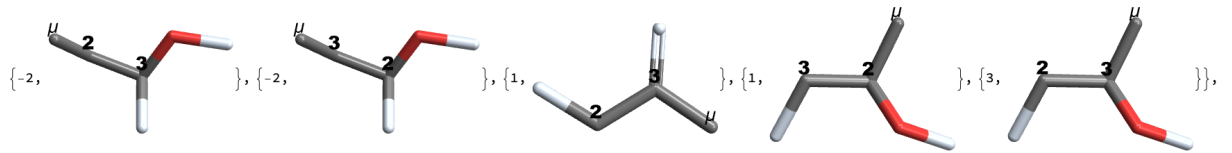
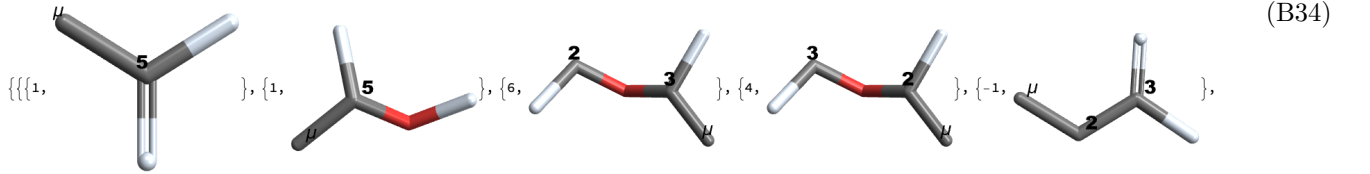
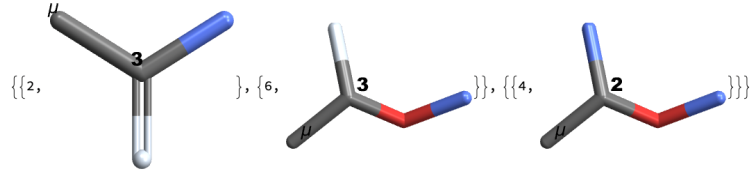
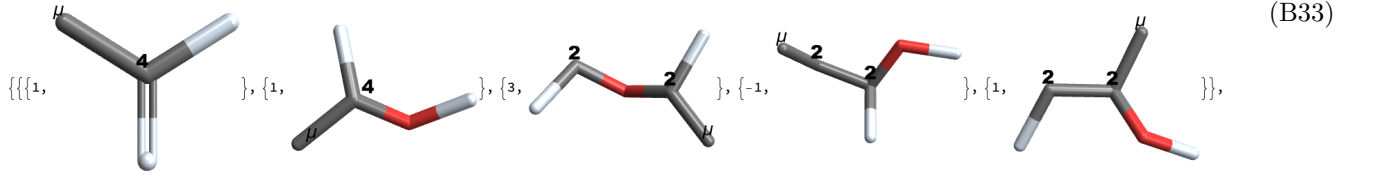
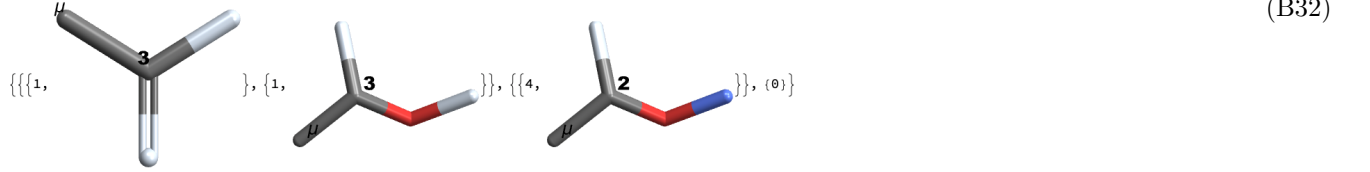
$$\begin{aligned} \dot{Y}^\mu_{\nu\sigma} &= \frac{1}{2!}(\dot{Q}_2)^\mu_{\nu\sigma} + \frac{1}{3!}(2\dot{Q}_3)^\mu_{\nu\sigma} + \frac{1}{4!}\left(3(\dot{Q}_4)^\mu_{\nu\sigma} + (\dot{Q}_2)^\mu_{\nu\lambda}(Q_2)^\lambda_\sigma\right) \\ &\quad + \frac{1}{5!}\left(4(\dot{Q}_5)^\mu_{\nu\sigma} + 4(\dot{Q}_3)^\mu_{\nu\lambda}(Q_2)^\lambda_\sigma + 2(\dot{Q}_2)^\mu_{\nu\lambda}(Q_3)^\lambda_\sigma\right) + \mathcal{O}(y^5). \end{aligned} \quad (\text{B22b})$$

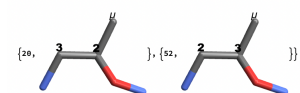
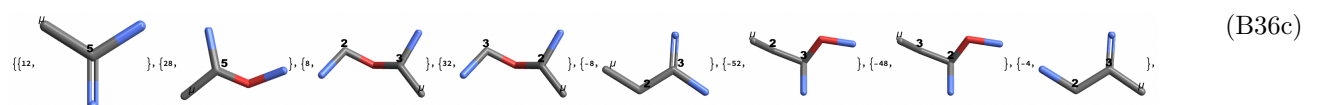
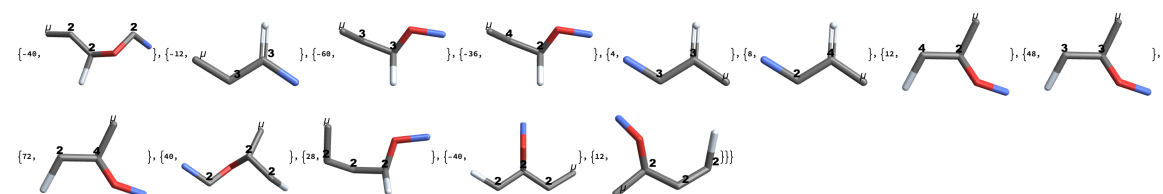
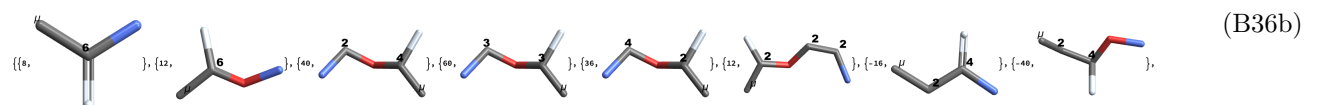
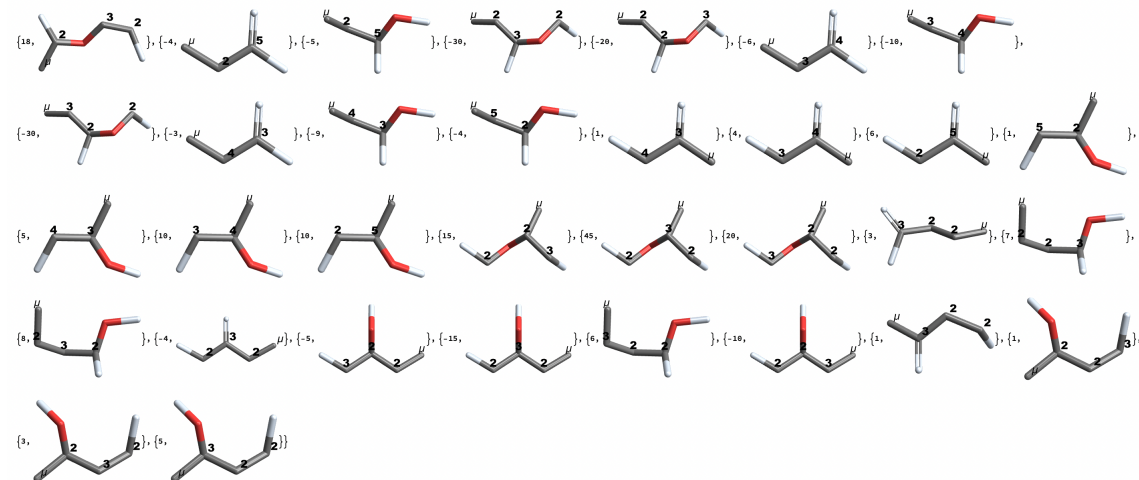
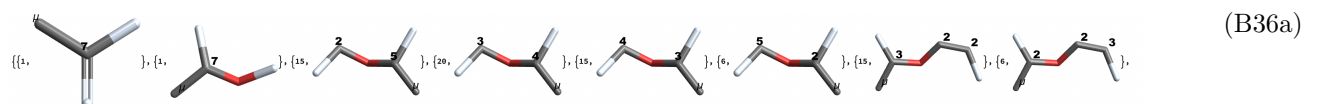
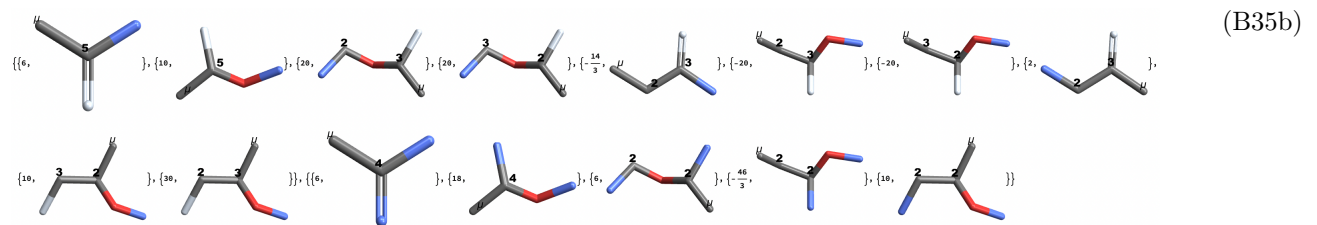
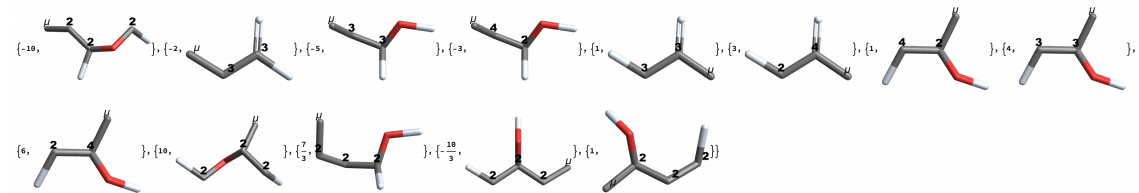
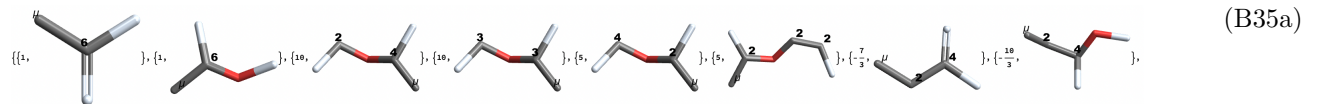
Next, we need to compute the inverse $(Y^{-1})^\mu_\nu$ of the Jacobi propagator Y^μ_ν in Eq. (B21b), which is viable by geometric series expansion around δ^μ_ν :

$$\begin{aligned} Y^{-1} &= \mathbb{1} - \frac{1}{3!}\left(Q_2\right) - \frac{1}{4!}\left(2Q_3\right) - \frac{1}{5!}\left(3Q_4 - \frac{7}{3}Q_2 Q_2\right) - \frac{1}{6!}\left(4Q_5 - 8Q_2 Q_3 - 6Q_3 Q_2\right) \\ &\quad - \frac{1}{7!}\left(5Q_6 - 11Q_4 Q_2 - 25Q_3 Q_3 - 18Q_2 Q_4 + \frac{31}{3}Q_2 Q_2 Q_2\right) \\ &\quad - \frac{1}{8!}\left(6Q_7 - \frac{52}{3}Q_5 Q_2 - 54Q_4 Q_3 - 66Q_3 Q_4 - \frac{100}{3}Q_2 Q_5 + 34Q_3 Q_2 Q_2 + \frac{124}{3}Q_2 Q_3 Q_2 + \frac{146}{3}Q_2 Q_2 Q_3\right) \\ &\quad - \frac{1}{9!}\left(7Q_8 - 25Q_6 Q_2 - 98Q_5 Q_3 - \frac{819}{5}Q_4 Q_4 - 140Q_3 Q_5 - 55Q_2 Q_6 + \frac{387}{5}Q_4 Q_2 Q_2 + 160Q_3 Q_3 Q_2\right. \\ &\quad \left.+ 182Q_3 Q_2 Q_3 + 106Q_2 Q_4 Q_2 + 226Q_2 Q_3 Q_3 + \frac{717}{5}Q_2 Q_2 Q_4 - \frac{381}{5}Q_2 Q_2 Q_2 Q_2\right) \\ &\quad - \frac{1}{10!}\left(8Q_9 - 34Q_7 Q_2 - 160Q_6 Q_3 - 336Q_5 Q_4 - 392Q_4 Q_5 - 260Q_3 Q_6 - 84Q_2 Q_7\right. \\ &\quad \left.+ 148Q_5 Q_2 Q_2 + 418Q_4 Q_3 Q_2 + 464Q_4 Q_2 Q_3 + 470Q_3 Q_4 Q_2 + 980Q_3 Q_3 Q_3 + 600Q_3 Q_2 Q_4\right. \\ &\quad \left.+ 220Q_2 Q_5 Q_2 + 660Q_2 Q_4 Q_3 + 756Q_2 Q_3 Q_4 + 336Q_2 Q_2 Q_5\right. \\ &\quad \left.- 310Q_3 Q_2 Q_2 Q_2 - 368Q_2 Q_3 Q_2 Q_2 - 394Q_2 Q_2 Q_3 Q_2 - 452Q_2 Q_2 Q_2 Q_3\right) \\ &\quad - \frac{1}{11!}\left(9Q_{10} - \frac{133}{3}Q_8 Q_2 - 243Q_7 Q_3 - 612Q_6 Q_4 - 896Q_5 Q_5 - 810Q_4 Q_6 - 441Q_3 Q_7 - \frac{364}{3}Q_2 Q_8\right. \\ &\quad \left.+ \frac{763}{3}Q_6 Q_2 Q_2 + \frac{2702}{3}Q_5 Q_3 Q_2 + 982Q_5 Q_2 Q_3 + 1383Q_4 Q_4 Q_2 + 2835Q_4 Q_3 Q_3\right. \\ &\quad \left.+ 1692Q_4 Q_2 Q_4 + \frac{3290}{3}Q_3 Q_5 Q_2 + 3240Q_3 Q_4 Q_3 + 3627Q_3 Q_3 Q_4 + 1554Q_3 Q_2 Q_5\right. \\ &\quad \left.+ \frac{1205}{3}Q_2 Q_6 Q_2 + \frac{4610}{3}Q_2 Q_5 Q_3 + 2472Q_2 Q_4 Q_4 + \frac{5936}{3}Q_2 Q_3 Q_5 + \frac{2050}{3}Q_2 Q_2 Q_6\right. \\ &\quad \left.- 855Q_4 Q_2 Q_2 Q_2 - \frac{5173}{3}Q_3 Q_3 Q_2 Q_2 - \frac{5348}{3}Q_3 Q_2 Q_3 Q_2 - 2028Q_3 Q_2 Q_2 Q_3\right. \\ &\quad \left.- \frac{3358}{3}Q_2 Q_4 Q_2 Q_2 - \frac{6494}{3}Q_2 Q_3 Q_3 Q_2 - \frac{7262}{3}Q_2 Q_3 Q_2 Q_3 - \frac{3787}{3}Q_2 Q_2 Q_4 Q_2\right. \\ &\quad \left.- \frac{7945}{3}Q_2 Q_2 Q_3 Q_3 - 1636Q_2 Q_2 Q_2 Q_4 + \frac{2555}{3}Q_2 Q_2 Q_2 Q_2 Q_2\right). \end{aligned} \quad (\text{B23})$$

Finally, the right-hand side of Eq. (31) should be computed. The algebraic terms, $\dot{X}^\mu_{\nu\rho} X^\nu_\sigma$, $\dot{X}^\mu_{\nu\rho} Y^\nu_\sigma$, $\dot{Y}^\mu_{\nu\rho} X^\nu_\sigma$, and $\dot{Y}^\mu_{\nu\rho} Y^\nu_\sigma$, are straightforward to evaluate from Eq. (B21). The evaluation of the differential terms, however, is relatively less trivial. The issue in particular is the evaluation of the y -derivative. When a y -derivative hits a Q -tensor, it inserts v in all possible positions in the string of y -vectors. Such terms can be gathered and simplified by commuting covariant derivatives and using Bianchi identities of the Riemann tensor in various ways.

Finally, below, we enumerate the GDE (as the right-hand side of $-Dy^\mu/d\tau = \dots$) at each order in the chemical notation, up to $\mathcal{O}(y^6)$.





Appendix C: Gauge-Covariant Translations in Nonabelian Gauge Theory

It should be remarked that our framework applies to not only gravity but also nonabelian gauge theories. Suppose a nonabelian gauge theory in a d -dimensional flat spacetime $\mathbb{M}$ with gauge group G . Let $A^i_j = A^i_{j\mu}(x)dx^\mu$ be the gauge connection and let $F^i_j = dA^i_j + A^i_k \wedge A^k_j$ be its curvature, where $i, j, \dots = 1, 2, \dots, N$ are the fundamental indices. For example, suppose a vector bundle E over $\mathbb{M}$ whose typical fiber is $\mathbb{C}^N$. This bundle is coordinatized by x^μ and ψ^i , associated respectively with the base and fiber.

Our tangent bundle formalism then considers the larger bundle $\mathcal{P} = T\mathbb{M} \oplus E$, which is locally isomorphic to $\mathbb{R}^{2d} \times \mathbb{C}^N$. Local trivialization equips $\mathcal{P}$ with coordinates x^μ , y^μ , and ψ^i . In $\mathcal{P}$, the generator of gauge-covariant translations is given by the vector field

$$N = y^\rho \left(\frac{\partial}{\partial x^\rho} - A^i_{j\rho}(x) \psi^j \frac{\partial}{\partial \psi^i} \right). \quad (\text{C1})$$

Again, this describes a horizontal vector field due to the Ehresmann [53] notion of the connection A . Recalling Eq. (4), one derives the set of first-order differential equations it encodes, which leads to the conclusion that it generates gauge-covariant translations: $x^\mu \mapsto x^\mu + y^\mu$, $y^\mu \mapsto y^\mu$, and $\psi^i \mapsto \psi^{i'} = W^{i'}_i \psi^i$. Here, the Wilson line $W^{i'}_i$ is given by the path-ordered exponential,

$$\text{P exp} \left(- \int_0^1 ds A_\rho(x + ys) y^\rho \right), \quad (\text{C2})$$

which describes the parallel transport along the straight path from x to $x + y$. Under gauge transformations, Eq. (C2) transforms bilocally as $W^{i'}_i \mapsto \Lambda^{i'}_{j'}(z) W^{j'}_j (\Lambda^{-1}(x))^j_i$.

Now consider the “ $\mathcal{L}_N^D$ sequence” of the one-form $D\psi^i = d\psi^i + A^i_{j\rho}(x) \psi^j dx^\rho$.

$$\begin{array}{ccc} D\psi^i & \xrightarrow{D\iota_N} & 0 \\ \iota_N D \downarrow & & \\ \iota_N F^i_j \psi^j & \longrightarrow & 0 \\ \downarrow & & \\ \iota_N D \iota_N F^i_j \psi^j & \longrightarrow & 0 \\ \vdots & & \end{array} \quad (\text{C3})$$

This derives

$$e^{\mathcal{L}_N^D} D\psi^i = D\psi^i + \sum_{\ell=1}^{\infty} \frac{1}{\ell!} (\iota_N D)^{\ell-1} \iota_N F^i_j \psi^j, \quad (\text{C4a})$$

$$= D\psi^i + \sum_{\ell=1}^{\infty} \frac{1}{\ell!} \left((P_\ell)^i_{j\sigma} dx^\sigma + (\ell-1) (P_{\ell-1})^i_{j\sigma} dy^\sigma \right) \psi^j, \quad (\text{C4b})$$

where the “ P -tensors” are defined for $\ell \geq 1$ as

$$(P_\ell)^i_{j\sigma} := y^{\kappa_1} \dots y^{\kappa_\ell} F^i_{j\kappa_1\sigma; \kappa_2; \dots; \kappa_\ell}(x) \implies (P_\ell)^i_{j\sigma} y^\sigma = 0. \quad (\text{C5})$$

The ancillary file `P.nb` verifies Eq. (C4b) in a direct fashion up to $\mathcal{O}(y^{10})$. For a physical application, Eq. (C4) can be used for deriving the Wong equations [90] of a color-charged particle as seen by an arbitrary observer.

To be further concrete, the dressing identity $\delta^{i'}_i e^{\mathcal{L}_N}(D\psi^i) = W^{i'}_i (e^{\mathcal{L}_N^D} D\psi^i)$ unpacks the content of Eq. (C4b) as

$$d\psi^{i'} + A^{i'}_{j'\rho}(z) \psi^{j'} dz^\rho = W^{i'}_i \left(d\psi^i + A^i_{j\rho}(x) \psi^j dx^\rho \right) + W^{i'}_i \sum_{\ell=1}^{\infty} \frac{1}{\ell!} \left((P_\ell)^i_{j\sigma} dx^\sigma + (\ell-1) (P_{\ell-1})^i_{j\sigma} dy^\sigma \right) \psi^j, \quad (\text{C6})$$

where the deviated coordinates z^μ is understood as $x^\mu + y^\mu$. This reveals that Eq. (C4b) encodes

$$W^{i'}_{i'} A^{i'}_{j'\rho}(z) W^{j'}_j + W^{i'}_{i'} \frac{\partial}{\partial x^\rho} W^{i'}_j = A^i_{j\rho}(x) + \sum_{\ell=1}^{\infty} \frac{1}{\ell!} (P_\ell)^i_{j\rho}, \quad (\text{C7a})$$

$$W^{i'}_{i'} A^{i'}_{j'\rho}(z) W^{j'}_j + W^{i'}_{i'} \frac{\partial}{\partial y^\rho} W^{i'}_j = \sum_{\ell=2}^{\infty} \frac{1}{\ell!} (\ell-1) (P_{\ell-1})^i_{j\rho}. \quad (\text{C7b})$$

Especially, Eq. (C7a) describes a “gauge transformation” of the connection $A_{j\rho}^{i'}(z)$ at the deviated point z , via the Wilson line to the original point x . In this sense, its right-hand side describes an avatar of the gauge connection. In fact, the sum of P -tensors in Eqs. (C7a) and (C7b), which computes the difference between the connection at z dragged back to x and the connection at x , can be used for studying the Fock-Schwinger gauge [91].

Note that all variables in Eqs. (C7a) and (C7b) are understood as functions of x and y , by the very construction of our formalism. Also, Eqs. (C7a) and (C7b) can be reproduced from the following formula that describes generic variations of the Wilson line $W(s_2, s_1) = \text{P exp} \left(-\int_{s_1}^{s_2} ds A_\rho(\gamma(s)) \dot{\gamma}^\rho(s) \right)$ about an arbitrary contour $s \mapsto \gamma^\mu(s)$:

$$W(0, 1) \delta W(1, 0) = A_\sigma(\gamma(0)) \delta \gamma^\sigma(0) - W(0, 1) A_\sigma(\gamma(1)) \delta \gamma^\sigma(1) W(1, 0) + \int_0^1 ds \dot{\gamma}^\rho(s) W(0, s) F_{\rho\sigma}(\gamma(s)) W(s, 0) \delta \gamma^\sigma(s). \quad (\text{C8})$$

Note that the identities in Eq. (C7) are the consequences of the conjugation $e^{\mathcal{L}_N^D} D e^{-\mathcal{L}_N^D}$; recall the discussion around Eq. (33). In the same fashion, more identities follow by conjugating $D^2 = F$, $[D, F] = 0$, etc. by $e^{\mathcal{L}_N^D}$.

Lastly, the identities analogous to Eq. (C7) in Riemannian geometry are

$$W^\mu_{\mu'} \Gamma^{\mu'}_{\nu\rho'}(z) W^{\nu'}_{\nu} W^{\rho'}_{\kappa} X^\kappa_{\rho} + W^\mu_{\mu'} \left(\frac{\partial}{\partial x^\rho} - \Gamma^\kappa_{\lambda\rho}(x) y^\lambda \frac{\partial}{\partial y^\kappa} \right) W^{\mu'}_{\nu} = \Gamma^\mu_{\nu\rho}(x) + \dot{X}^\mu_{\nu\rho}, \quad (\text{C9a})$$

$$W^\mu_{\mu'} \Gamma^{\mu'}_{\nu\rho'}(z) W^{\nu'}_{\nu} W^{\rho'}_{\kappa} Y^\kappa_{\rho} + W^\mu_{\mu'} \frac{\partial}{\partial y^\rho} W^{\mu'}_{\nu} = \dot{Y}^\mu_{\nu\rho}, \quad (\text{C9b})$$

which can be deduced from Eq. (28). Notice the presence of the Jacobi propagators X^κ_{ρ} and Y^κ_{ρ} on the left-hand sides, which reflects the fact that the curve for the Wilson lines depends on the gravitational fields unlike as in gauge theory.

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