A NOTE ON CONDUCTORS OF FREY REPRESENTATIONS AT 2

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Abstract. In 2000, Darmon introduced the notion of Frey representations within the framework of the modular method for studying the generalized Fermat equation. A central step in this program is the computation of their conductors, with the case at the prime 2 presenting particular challenges. In this article we study the conductor exponent at 2 for Frey representations of signatures (p,p,r), (r,r,p), $(2,r,p)$, and $(3,5,p)$, all of which have hyperelliptic realizations. In particular we are able to determine the conductor at 2 for even degree Frey representations of signature (p,p,r) and $(3,5,p)$ and all rational parameters.

1. Introduction

In [Dar00], Darmon introduced the notion of a Frey representation

$$\rho(t) : G_{K(t)} \to \mathrm{GL}_2(\bar{\mathbb{F}}_p),$$

which can be thought of as a family of two-dimensional residual Galois representations of G_K varying with a parameter t, where K is a totally real field, and $G_{K(t)}$ and G_K denote the absolute Galois group of $K(t)$ and K, respectively.

Let $\mathfrak{q}_2$ be a prime in K above 2. In this paper, we study the conductor exponent at $\mathfrak{q}_2$ of Frey representations constructed from different families of hyperelliptic curves parametrized by $t \in \mathbb{Q}$.

The methods rely on explicit computations of hyperelliptic models and the particular algebraic symmetries present in each hyperelliptic realization. In each case we cover, we obtain explicit hyperelliptic models over a suitable extension of $\mathbb{Q}_2$. As the hyperelliptic models are readily verified to be semistable, the examples in this paper may inform more sophisticated approaches to semistable reduction of hyperelliptic curves over $\mathbb{Q}_2$.

Frey representations have a natural application to the study of generalized Fermat equations, which are of the form

$$Ax^{n_1} + By^{n_2} = Cz^{n_3}, \tag{1.1}$$

where A, B, C are fixed non-zero integers and n_i are positive integers. The triple of exponents (n_1, n_2, n_3) is called the *signature* of (1.1) and if we fix it such that $\sum_{i=1}^{3} 1/n_i < 1$, it is known that (1.1) has finitely many *non-trivial primitive* solutions, i.e. solutions $(a,b,c) \in \mathbb{Z}^3$ where $abc \neq 0$ and $\gcd(a,b,c) = 1$ [DG95, Theorem 2].

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In [Dar00], Darmon introduced a novel approach to study (1.1). When following Darmon's program one specializes Frey representations to specific values of $t \in \mathbb{Q}$. More concretely, if (a, b, c) is a non-trivial primitive solution to (1.1), one specializes at $t = Aa^{n_1}/Cc^{n_3}$.

By adopting this framework, one needs to know the conductor of the Frey representations. The particular case where the representation can be constructed from a hyperelliptic curve is of special interest, since there are different tools to compute the conductor of this. In a series of recent papers (see [ACIK+24, Azo25, CV25]), the conductor exponent at odd primes has been computed for all signatures which are known so far to admit hyperelliptic realizations, namely $(n_1, n_2, n_3) \in \{(p, p, r), (r, r, p), (2, r, p)\}$.

The aim of the present article is to complement the current state of the art by providing the computation of the conductor exponent at $\mathfrak{q}_2$ for these cases. In addition, a central contribution of this paper is the introduction of a new Frey hyperelliptic curve with signature $(3, 5, p)$, together with a complete determination of its conductor exponent at $\mathfrak{q}_2$. So far, this signature has remained unexplored, and our work represents a fundamental step towards its study through Darmon's program.

It is worth mentioning that for signatures (p, p, r) and (r, r, p), the conductor exponent at $\mathfrak{q}_2$ was determined in [Azo25] for those values of t coming from a solution to (1.1). As a complement, in the present article we focus on an arbitrary parameter $t \in \mathbb{Q}$.

Subject to natural 2-adic conditions on t, in Table 1 we summarize our main results. For the prime 2, three cases arise: $v_2(t) > 0$, $v_2(1-t) > 0$, or $v_2(t) < 0$. Moreover, $v_2(t) < 0$ if and only if $v_2(1-t) < 0$, and in this situation we have $v_2(1-t) = v_2(t)$. Therefore, we will naturally split the analysis into these three cases.

Signature	Degree	$v_2(t)$	$v_2(1-t)$	Conductor exponent
(p, p, r)	even	$< 0, \equiv 0 \pmod r$	$< 0, \equiv 0 \pmod r$	0
		$< 0, \not\equiv 0 \pmod r$	$< 0, \not\equiv 0 \pmod r$	2
		> 0	0	1
		0	> 0	1
	odd	$\leq -4, \equiv -2 \pmod r$	$\leq -4, \equiv -2 \pmod r$	0
		$\leq -4, \not\equiv -2 \pmod r$	$\leq -4, \not\equiv -2 \pmod r$	2
(r, r, p)	odd	$\geq 4, \equiv 4 \pmod r$	0	0
		$\geq 4, \not\equiv 4 \pmod r$	0	2
		0	$\geq 4, \equiv 4 \pmod r$	0
		0	$\geq 4, \not\equiv 4 \pmod r$	2
$(2, r, p)$	odd	0	$\geq 6, \equiv 6 \pmod r$	0
		0	$\geq 6, \not\equiv 6 \pmod r$	2
$(3, 5, p)$	even	$> 0, \equiv 0 \pmod 3$	0	0
		$> 0, \not\equiv 0 \pmod 3$	0	2
		0	$> 0, \equiv 0 \pmod 5$	0
		0	$> 0, \not\equiv 0 \pmod 5$	2
		< 0	<0	1

TABLE 1. Summary of conductors at $\mathfrak{q}_2$ up to quadratic twist

The paper is organized as follows. Section 2 provides a brief introduction to hyperelliptic curves, Jacobians, and Frey representations. In Section 3 we study Frey representations arising from hyperelliptic equations whose defining polynomial has even degree, while Section 4 is devoted to the odd degree case.

2. Preliminaries

2.1. **Hyperelliptic equations.** In this section, we summarize the basic theory of hyperelliptic equations. For a detailed study of the subject, the reader may consult [Liu96, Liu02, Loc94].

A hyperelliptic equation E over a field K is an equation of the form

$$y^2 + Q(x)y = P(x),$$

where $Q, P \in K[x]$, $\deg Q \le g + 1$, and $\deg P \le n = 2g + 2$ with

$$2g + 1 \le \max(2 \deg Q, \deg P) \le 2g + 2.$$

Let

$$R = 4P + Q^2,$$

and suppose κ is the leading coefficient of R. The discriminant of E is given by

$$\Delta_E := \begin{cases} 2^{-4(g+1)}\Delta(R) & \text{if } \deg R = 2g+2, \\ 2^{-4(g+1)}\kappa^2\Delta(R) & \text{if } \deg R = 2g+1, \end{cases}$$

where $\Delta(H)$ denotes the discriminant of a polynomial $H \in K[x]$. In particular, if P is monic, $\deg P = 2g + 1$, and $\deg Q \le g$, then

$$\Delta_E = 2^{4g}\Delta(P + Q^2/4),$$

using the fact that $\Delta(H)$ is homogeneous of degree $2n - 2$ in the coefficients of H.

A hyperelliptic equation E over a field K gives rise to a *hyperelliptic curve* C over K by gluing the following two affine schemes over K:

$$\operatorname{Spec} K[x,y]/(y^2 + Q(x)y - P(x)), \tag{2.1}$$

$$\operatorname{Spec} K[u,z]/(z^2 + T(u)z - S(u)), \tag{2.2}$$

along the open sets $x \neq 0$ and $u \neq 0$, respectively, by the relations

$$x = 1/u, \quad y = z/u^{g+1},$$

where

$$P(x) = u^{2g+2}S(u), \quad Q(x) = u^{g+1}T(u).$$

The curve C is non-singular if and only if $\Delta_E \neq 0$ and we refer to the affine schemes in (2.1)–(2.2) as affine patches of C.

If K has characteristic 0, we say that C has *odd* (resp. *even*) degree accordingly as the degree of R as above is odd (resp. even).

Lemma 2.3. *Let $E: y^2 + Q(x)y = P(x)$ and $\widetilde{E}: Y^2 + \widetilde{Q}(X)Y = \widetilde{P}(X)$ be two hyperelliptic equations describing the curve C/K. The coordinates (x,y) and (X,Y) are related by a change of variables*

$$x = \frac{aX+b}{cX+d}, \qquad \text{and} \qquad y = \frac{eY+R(X)}{(cX+d)^{g+1}},$$

with $a,b,c,d,e \in K$, $R(X) \in K[X]$, and $ad-bc, e \neq 0$. We have the equality between discriminants

$$\Delta_{\widetilde{E}} = e^{-4(2g+1)}(ad-bc)^{2(g+1)(2g+1)}\,\Delta_E.$$

2.2. **Jacobian criterion in characteristic 2.** Let C be the hyperelliptic curve of genus g given by

$$y^2 + yQ(x) = P(x), \tag{2.4}$$

over a field k of characteristic 2.

The following are useful criteria for the determining the semistability of C [BBW17].

Lemma 2.5. *Let (a,b) be in an affine patch of C. Then (a,b) is a singular point if and only if*

$$Q(a) = 0, \tag{2.6}$$

$$P'(a)^2 = Q'(a)^2P(a). \tag{2.7}$$

Proof. By the Jacobian criterion (a,b) is a singular point of C if and only if

$$b^2 + bQ(a) = P(a)$$
$$Q(a) = 0$$
$$bQ'(a) = P'(a).$$

Thus, if (a,b) is a singular point, then (2.6)–(2.7) hold.

Conversely, if (2.6)–(2.7) hold, the result follows from the fact that (a,b) lies on C, together with the computation of the partial derivative of (2.4) with respect to x at (a,b). ◻

Lemma 2.8. *Let (a,b) be in an affine patch of C and suppose (a,b) is a singular point. Then (a,b) is an ordinary double point if and only if $Q'(a) \neq 0$.*

Proof. We recall the argument in [BBW17]. Assume (a,b) is a point in the affine part of C which is singular. The tangent cone at (a,b) is

$$(y+b)^2 + Q'(a)(y+b)(x+a) + c(x+a)^2,$$

where c is the coefficient of x^2 in the Taylor series expansion of P at $x = a$. As k has characteristic 2, the quadratic form is nondegenerate if and only if $Q'(a) \neq 0$. ◻

Lemma 2.9. *Let (a,b) be in an affine patch of C. Then (a,b) is a smooth or ordinary double point if and only if the following does not hold:*

$$Q(a) = Q'(a) = P'(a) = 0.$$

Proof. If (a,b) is smooth, then $Q'(a) \neq 0$ or $P'(a)^2 \neq Q'(a)^2P(a)$. In the latter case, we cannot have $Q'(a) = 0 = P'(a)$ so at least one of $Q'(a) = 0$ or $P'(a) = 0$. If (a,b) is ordinary double, then $Q'(a) \neq 0$.

If (a,b) is not smooth and not ordinary double, then $Q(a) = Q'(a) = 0$ and this forces $P'(a) = 0$. $\square$

A smooth or ordinary double point is called a *semistable* point.

Corollary 2.10. *Let (a,b) be in an affine patch of C and assume $Q(a) = Q'(a) = 0$. If $P'(a) = 0$ then (a,b) is a singular point. In particular, if a is a double root of $P(x)$ and C is given by $y^2 = P(x)$, then (a,b) is a singular point which is not semistable.*

2.3. **Frey representations and GL_2-type abelian varieties.** The following definition is a slight modification of Darmon's original one (see [Dar00, Definition 1.1]) adapted in [BCDF25, Definition 2.1]. Here K is a number field and $\bar{\mathbb{F}}_p$ is a fixed algebraic closure of the finite field with p elements. Recall that the absolute Galois group of $\bar{K}(t)$, denoted by $G_{\bar{K}(t)}$, sits inside $G_{K(t)}$, the absolute Galois group of $K(t)$, as a normal subgroup.

Definition 2.11. A Frey representation in characteristic p over $K(t)$ of signature $(n_i)_{i=1,\dots,m}$ with respect to the points $(t_i)_{i=1,\dots,m}$, where $n_i \in \mathbb{N}$ and $t_i \in \mathbb{P}^1(K)$, is a Galois representation

$$\rho : G_{K(t)} \to \mathrm{GL}_2(\bar{\mathbb{F}}_p)$$

satisfying:

(1) the restriction $\rho|_{G_{\bar{K}(t)}}$ has trivial determinant and is irreducible,
(2) the projectivization $\mathbb{P}\rho|_{G_{\bar{K}(t)}} : G_{\bar{K}(t)} \to \mathrm{PSL}_2(\bar{\mathbb{F}}_p)$ of $\rho|_{G_{\bar{K}(t)}}$ is unramified outside the t_i,
(3) the projectivization $\mathbb{P}\rho|_{G_{\bar{K}(t)}}$ maps the inertia subgroups at t_i to subgroups of $\mathrm{PSL}_2(\mathbb{F})$ generated by elements of order n_i, respectively, for $i = 1, \dots, m$.

As a natural source of Frey representations, we will work with abelian varieties of GL_2-type.

Definition 2.12. Let A be an abelian variety over a field L of characteristic 0. We say that A/L is of GL_2-type (or $\mathrm{GL}_2(K)$-type) if there is an embedding $K \hookrightarrow \mathrm{End}_L(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ where K is a number field with $[K : \mathbb{Q}] = \dim A$.

An abelian variety of GL_2-type is so called precisely because it induces a strictly compatible system of two-dimensional Galois representations [Shi67, Section 11.10]. In the present article, we work with Jacobians $\mathcal{J}(t)$ of different families of rational hyperelliptic curves $\mathcal{C}(t)$ that become of GL_2-type over a totally real number field K. Therefore, we will consider $\mathcal{J}(t)$ always as a variety over K.

If $\mathfrak{p}$ is a prime in K above p, then the residual representation of the $\mathfrak{p}$-member of the family of Galois representations associated to $\mathcal{J}(t)$ is the Frey representation that we want to consider. This will be denoted by $\bar{\rho}_{\mathcal{J}(t),\mathfrak{p}} : G_{K(t)} \to \mathrm{GL}_2(\bar{\mathbb{F}}_p)$.

Let $r \geq 3$ be a prime integer. Throughout this paper ζ_r will denote a fixed primitive r-th root of unity. For all $1 \leq j \leq r-1$, we let $\omega_j = \zeta_r^j + \zeta_r^{-j}$ and $K = \mathbb{Q}(\omega_1) = \mathbb{Q}(\zeta_r)^+$, the maximal totally real subextension of the cyclotomic field $\mathbb{Q}(\zeta_r)$. Let

$$h(x) = \prod_{j=1}^{\frac{r-1}{2}} (x - \omega_j)$$

be the minimal polynomial of ω_1 and, following Darmon's [Dar00], let

$$f(x) = (-1)^{\frac{r-1}{2}} x h(2 - x^2).$$

Let $J(s)$ be the Jacobian of

$$C(s) : y^2 = f(x) + s.$$

Proposition 2.13. *Let $\mathfrak{p}$ be a prime ideal in K above a prime p. Then the representation $\overline{\rho}_{J(s),\mathfrak{p}}$ is a Frey representation of signature (p,p,r) with respect to the points $(-2,2,\infty)$*

Proof. See [Dar00, p. 420]. □

2.3.1. *Signature (p,p,r).* If we take t another variable satisfying $s = 2 - 4t$ we get $J_r^-(t)$, which gives rise to a Frey representation of signature (p,p,r) with respect to the points $(0,1,\infty)$.

2.3.2. *Signature (r,r,p).* In [BCDF25, Section 2.4] it has been proved that if we take t to be another variable satisfying

$$\frac{1}{s^2 - 4} = t(t-1)$$

and we take α to be the square-root of $t(t-1)$ then, the twist by α of the base change of $C(s)$ to $K(s,t)$ has a model over $K(t)$, namely the curve we denote by $C_{r,r}^-(t)$. Its Jacobian $J_{r,r}^-(t)$ gives rise to a Frey representation of signature (r,r,p) with respect to the points $(0,1,\infty)$ [BCDF25, Theorem 2.8].

2.3.3. *Signature $(2,r,p)$.* Let $J_{2,r}^-(t)$ be the Jacobian of the hyperelliptic curve

$$H_{2,r}^-(t) : y^2 = (-t(t-1))^{\frac{r-1}{2}} x h\Big(2 - \frac{x^2}{t(t-1)}\Big) + 2(t-1)^{\frac{r-1}{2}} t^{\frac{r+1}{2}},$$

which has discriminant

$$\Delta(H_{2,r}^-(t)) = (-1)^{\frac{r-1}{2}} 2^{3(r-1)} r^r t^{\frac{r(r-1)}{2}} (t-1)^{\frac{(r-1)^2}{2}}. \tag{2.14}$$

The argument to prove that $J_{2,r}^-(t)$ is of $\mathrm{GL}_2(K)$-type follows the above strategy. Let $K(s,t)$ be the function field where s and t are related by

$$\frac{1}{s^2 - 4} = \frac{t-1}{4}.$$

We can see both $K(s)$ and $K(t)$ as subfields of $K(s,t)$. Define α as the square-root of $t(t-1)$. Then, by twisting by α the base change of $C(s)$ to $K(s,t)$ we get the model $H_{2,r}^-(t)$.

Replicating the arguments in [BCDF25, Theorem 2.8], we have that there is an embedding $K \hookrightarrow \mathrm{End}_{K(t)}(J_{2,r}^-(t)) \otimes \mathbb{Q}$ and that for every specialization of t to $t_0 \in \mathbb{P}^1(K) - \{0,1,\infty\}$ it is well-defined.

The ramification set $\{\pm 2, \infty\}$ of $C(s)$ corresponds to the ramification set $\{\infty, 1\}$ for $H^-_{2,r}(t)$. In addition, from (2.14) we see that 0 is a point of ramification for $H^-_{2,r}(t)$. Since $K(s,t)/K(s)$ is a degree two extension, and $t = 0$ corresponds to $s = 0$ (only one point), the projective order of inertia at $t = 0$ is 2. The orders at $t = 1$ and $t = \infty$ are r and p, respectively, from the correspondence of points. Hence, $J^-_{2,r}(t)$ gives rise to a Frey representation of signature $(2, r, p)$ with respect to the points $(0, 1, \infty)$.

2.4. **Hyperelliptic curves with potentially good reduction at 2.** Let K be a number field and $\mathfrak{q}_2$ be a prime in K above 2. Let C/K be a hyperelliptic curve and J be its Jacobian.

Over this section, assume that $C/K_{\mathfrak{q}_2}$ acquires good reduction over a finite extension with ramification index a prime number $r \geq 3$ and that there is no unramified extension of $K_{\mathfrak{q}_2}$ where C or J attains good reduction.

Assume that J/K is of $\mathrm{GL}_2(K)$-type and denote by $\{\rho_{J,\lambda}\}$ its associated strictly compatible system of two-dimensional Galois representations of G_K, the absolute Galois group of K. Let $I_{\mathfrak{q}_2}$ be the inertia subgroup of G_K at $\mathfrak{q}_2$.

Proposition 2.15. *The inertial type of J/K at $\mathfrak{q}_2$ is a principal series if $r \mid \#\mathbb{F}^\times_{\mathfrak{q}_2}$ and supercuspidal otherwise. Moreover, for all $\lambda \nmid 2$ in K, the image of inertia $\rho_{J,\lambda}(I_{\mathfrak{q}_2})$ is cyclic of order r.*

Proof. This follows exactly as in [BCDF25, Proposition 5.3], but we include the proof here for convenience.

We know that $J/K_{\mathfrak{q}_2}$ obtains good reduction over any extension $L/K_{\mathfrak{q}_2}$ with ramification degree r. In particular, this shows that $\rho_{J,\lambda}|_{D_{\mathfrak{q}_2}}$ becomes unramified over L, hence the inertial type of J/K is not Steinberg at $\mathfrak{q}_2$. Moreover, the inertial type of J/K at $\mathfrak{q}_2$ is a principal series if and only if there is an abelian extension $L/K_{\mathfrak{q}_2}$ with ramification degree r and supercuspidal otherwise. By local class field theory, such an extension exists if and only if r divides $\#\mathbb{F}^\times_{\mathfrak{q}_2}$. This proves the first statement.

Also, from Proposition 4.3, there is no unramified extension of $K_{\mathfrak{q}_2}$ over which J has good reduction, hence $\#\rho_{J,\lambda}(I_{\mathfrak{q}_2}) \neq 1$ and divides r. The conclusion follows. ∎

Let $\mathfrak{p}$ be an odd prime in K. Consider $\bar{\rho}_{J,\mathfrak{p}}$, the residual representation of the $\mathfrak{p}$-member of the strictly compatible system.

Proposition 2.16. *The conductor exponent of $\bar{\rho}_{J,\mathfrak{p}}$ at $\mathfrak{q}_2$ equals* 2.

Proof. From Corollary 2.15 we have that the inertial type at $\mathfrak{q}_2$ of $\rho_{J,\mathfrak{p}}$ is either principal series or supercuspidal, and that $\rho_{J,\mathfrak{p}}(I_{\mathfrak{q}_2})$ is cyclic of order r. Then, the proof follows as in [BCDF25, Theorem 5.16]. ∎

3. Even degree cases

In this section, we study Frey representations of signature (p, p, r) and $(3, 5, p)$. The abelian varieties giving rise to these representations are Jacobians $\mathcal{J}(t)$ of a family of rational hyperelliptic curves parametrized by $t \in \mathbb{Q}$, and are of $\mathrm{GL}(K)$-type, where $K = \mathbb{Q}(\zeta_r)^+$ and

$K = \mathbb{Q}(\zeta_5)^+$, respectively. Let $\mathfrak{p}$ and $\mathfrak{q}_2$ be primes in K above p and 2, respectively. In this section we compute the conductor of the Frey representation $\bar{\rho}_{\mathcal{J}(t),\mathfrak{p}}$ at $\mathfrak{q}_2$.

Recall for the prime 2, three cases arise: $v_2(t) > 0$, $v_2(1-t) > 0$, or $v_2(t) < 0$. Moreover, $v_2(t) < 0$ if and only if $v_2(1-t) < 0$, and in this situation we have $v_2(1-t) = v_2(t)$. Therefore, we will naturally split the analysis into these three cases.

3.1. **The Jacobian $J_r^+(t)$.** Keeping the notation of Section 2.3, let $J_r^+(t)$ be the Jacobian of the hyperelliptic curve

$$C_r^+(t) : y^2 = (x+2)(f(x) + 2 - 4t), \tag{3.1}$$

introduced in [Dar00]. By Theorem [Dar00, Theorem 1.10] it gives rise to a Frey representation of signature (p,p,r). The discriminant of $C_r^+(t)$ equals $\Delta(C_r^+(t)) = 2^{2(r+1)} r^r t^{\frac{r+3}{2}} (1-t)^{\frac{r-1}{2}}$.

In [Azo25], the author studies the conductor exponent at $\mathfrak{q}_2$ using the sophisticated new theory developed in [Geh25]. In particular, in the case where $J_r^+(t)$ has potentially toric reduction at $\mathfrak{q}_2$, certain restrictions are imposed on the possible values of t (see Hypothesis 3 loc. cit.). In contrast, using only elementary arguments, we will provide a complete description for all possible parameter values.

Using the identity $f(x) + 2 = (x+2)h(-x)^2$ (see [Dar00, p. 11]) we get

$$y^2 = (x+2)^2 h(-x)^2 - 4t(x+2).$$

Making the substitution $y \to 2y + (x+2)h(-x)$ we obtain the model

$$y^2 + (x+2)h(-x)y = -t(x+2), \tag{3.2}$$

whose discriminant, by Lemma 2.3, equals

$$\Delta = r^r t^{\frac{r+3}{2}} (1-t)^{\frac{r-1}{2}}. \tag{3.3}$$

3.1.1. *Case $v_2(t) < 0$.* Let us prove that the curve acquires good reduction over $K_{\mathfrak{q}_2}((1/t)^{\frac{1}{r}})$. To simplify notation, let $u = (1/t)^{\frac{1}{r}}$, and so $t = 1/u^r$. Making the substitution $x \to x/u$, $y \to y/u^{\frac{r+1}{2}}$ in (3.2) we get the integral model

$$y^2 + (x+2u)u^{\frac{r-1}{2}} h(-x/u)y = -(x+2u). \tag{3.4}$$

From (3.3) and Lemma 2.3, its discriminant is a unit, so the above is a model with good reduction.

3.1.2. *Case $v_2(t) > 0$.* From Lemma 2.5, the model (3.2) has the singular points $(-2,0)$ and $(-\omega_j, 0)$ for all $1 \le j \le (r-1)/2$, and by Lemma 2.9 it has semistable reduction at $\mathfrak{q}_2$.

3.1.3. *Case $v_2(1-t) > 0$.* Similarly as in the previous analysis, we can use the identity $f(x) - 2 = (x-2)h(-x)^2$ and get the model

$$y^2 = (x^2 - 4)h(-x)^2 + 4(1-t)(x+2).$$

Making the substitution $y \to 2y + xh(-x)$ we obtain the model

$$y^2 + xh(-x)y = -h(-x)^2 + (1-t)(x+2). \tag{3.5}$$

Once again, by Lemma 2.5, we have that $(-\omega_j, 0)$ are singular points of the model (3.5), which has semistable reduction at $\mathfrak{q}_2$, by Lemma 2.9.

Proposition 3.6. *The conductor exponent of* $\bar{\rho}_{J_r^+(t),\mathfrak{p}}$ *at* $\mathfrak{q}_2$ *equals*

$$\begin{cases} 0 & \text{if } v_2(t) < 0 \text{ and } v_2(t) \equiv 0 \pmod{r}, \\ 2 & \text{if } v_2(t) < 0 \text{ and } v_2(t) \not\equiv 0 \pmod{r}, \\ 1 & \text{if } v_2(t(1-t)) > 0. \end{cases}$$

Proof. The first and the third case follow directly from the previous analysis. In the second case, the curve acquires good reduction over an extension with ramification index r and then the result follows from Proposition 2.16. □

3.2. **The Jacobian $J_{3,5}^+(t)$.** Let $J_{3,5}^+(t)$ be the Jacobian of the hyperelliptic curve

$$H_{3,5}^+(t) : y^2 + y(x^3 + t(1-t)^2) = 2t(1-t)^2x^3 + 3t^2(1-t)^3x + t^2(1-t)^4,$$

whose discriminant equals $\Delta(H_{3,5}^+(t)) = 3^6 5^5 t^{10}(t-1)^{18}$.

Setting $t = Aa^3/Cc^p$ and $1 - t = Bb^5/Cc^p$ gives rise to an isomorphism to a Frey hyperelliptic curve for signature $(3, 5, p)$, namely

$$y^2 + y(x^3 + AB^2b) = 2AB^2bx^3 + 3A^2B^3ax + A^2B^4b^2,$$

having discriminant $\Delta = 3^6 5^5 A^{10} B^{18}(Aa^3 + Bb^5)^2$. As far as we know, this is the first time that a Frey hyperelliptic curve for signature $(3, 5, p)$ has been considered. The forthcoming work [PV] explains how to construct the curve $H_{3,5}^+(t)$ from the theory of hypergeometric motives, a framework that was recently incorporated into the modular method to address generalized Fermat equations [GP24].

3.2.1. *Case $v_2(t) > 0$.* Let us prove that the curve acquires good reduction over $K_{\mathfrak{q}_2}(t^{\frac{1}{3}})$. To simplify notation, set $u = t^{\frac{1}{3}}$. Then, the initial model is

$$y^2 + y(x^3 + u^3(1-u^3)^2) = 2u^3(1-u^3)^2x^3 + 3u^6(1-u^3)^3x + u^6(1-u^3)^4.$$

Applying the change of variables $x \to ux$ and $y \to u^3y$ we get the model

$$y^2 + y(x^3 + (1-u^3)^2) = 2(1-u^3)^2x^3 + 3u(1-u^3)^3x + (1-u^3)^4,$$

whose discriminant is a unit, by Lemma 2.3.

3.2.2. *Case $v_2(1-t) > 0$.* Let us prove that the curve acquires good reduction over $K_{\mathfrak{q}_2}((1-t)^{\frac{1}{5}})$. To simplify notation, set $u = (1-t)^{\frac{1}{5}}$. Then, the initial model is

$$y^2 + y(x^3 + (1-u^5)u^{10}) = 2u(1-u^5)u^{10}x^3 + 3(1-u^5)^2u^{15}x + u^2(1-u^5)^2u^{20}.$$

Applying the change of variables $x \to u^3x$ and $y \to u^9y$ we get the model

$$y^2 + y(x^3 + (1-u^5)u) = 2(1-u^5)ux^3 + 3(1-u^5)^2x + (1-u^5)^2u^2,$$

whose discriminant is a unit, by Lemma 2.3.

3.2.3. *Case $v_2(t) < 0$.* Recall that in this case we have $v_2(1-t) = v_2(t)$. Under this assumption, we can see that the curve has toric reduction over $K_{\mathfrak{q}_2}(1/t)$.

Set $u = (1-t)/t$, which is a unit. By applying the change of variables $x \to tx$ and $y \to t^3 y$ we get the model

$$y^2 + y(x^3 + u^2) = 2u^2x^3 + 3u^3x + u^4$$

which a semistable model by Lemma 2.9, whose special fiber is given by

$$y^2 + y(x^3 - 1) = x + 1,$$

with singular points at $x = \zeta_3$, by Lemma 2.5.

For the purposes of this work, we assume that $J^+_{3,5}(t)$ is of $\mathrm{GL}_2(K)$-type for $K = \mathbb{Q}(\zeta_5)^+$ (this is also proved in [PV]). Under this assumption, we can follow the strategy in the proof of [Dar00, Theorem 1.10] to use the above arguments to show that the attached Galois representation $\bar{\rho}_{J^+_{3,5}(t),\mathfrak{p}} : G_{K(t)} \to \mathrm{GL}_2(\bar{\mathbb{F}}_p)$ is a Frey representation of signature $(3,5,p)$ with respect to the points $(0,1,\infty)$.

Proposition 3.7. *Assume that $J^+_{3,5}(t)$ is of $\mathrm{GL}_2(\mathbb{Q}(\zeta_5^+))$-type. The conductor exponent of $\bar{\rho}_{J^+_{3,5}(t),\mathfrak{p}}$ at $\mathfrak{q}_2$ equals*

$$\begin{cases} 0 & \text{if } v_2(t(1-t)) > 0,\ v_2(t) \equiv 0 \pmod 3 \text{ and } v_2(1-t) \equiv 0 \pmod 5, \\ 2 & \text{if } v_2(t(1-t)) > 0 \text{ and } v_2(t) \not\equiv 0 \pmod 3 \text{ or } v_2(1-t) \not\equiv 0 \pmod 5, \\ 1 & \text{if } v_2(t) < 0. \end{cases}$$

Proof. The first and the third case follow directly from the previous analysis. In the second case, the curve acquires good reduction over an extension with ramification index 3 if $v_2(t) \not\equiv 0 \pmod 3$ and 5 if $v_2(1-t) \not\equiv 0 \pmod 5$. Then the result follows from Proposition 2.16. □

4. Odd degree cases

In this section we study Frey representations of signature (p,p,r), (r,r,p) and $(2,r,p)$, where $r \geq 3$ is a fixed prime and $p \neq r$ is a prime variable. The abelian varieties giving rise to these representations are Jacobians $\mathcal{J}(t)$ of a family of rational hyperelliptic curves parametrized by $t \in \mathbb{Q}$, and are of $\mathrm{GL}_2(K)$-type, where $K = \mathbb{Q}(\zeta_r)^+$. Let $\mathfrak{p}$ and $\mathfrak{q}_2$ be a primes in K above p and 2, respectively. We focus on computing the conductor exponent of the Frey representation $\bar{\rho}_{\mathcal{J}(t),\mathfrak{p}}$ at $\mathfrak{q}_2$ for the cases where $\mathcal{J}(t)$ has potentially good reduction at $\mathfrak{q}_2$. The case where $\mathcal{J}(t)$ has (potentially) multiplicative reduction at $\mathfrak{q}_2$ remains outside the scope of this paper for these Frey representations.

We will carry out this aim by following the framework introduced in [CV25].

4.1. **A common framework.** Keeping the notation of Section 2.3, consider the hyperelliptic curve

$$C = C(z,s) : y^2 = (-z)^{\frac{r-1}{2}} x h\left(2 - \frac{x^2}{z}\right) + s,$$

where $z, s \in \mathbb{Q}$. C is a rational curve, but we will consider its base change over K. Let $J = J(z,s)/K$ be its Jacobian. The discriminant of C (see [CV25, Lemma 3.1]) equals

$$\Delta_C = (-1)^{\frac{r-1}{2}} 2^{2(r-1)} r^r (s^2 - 4z^r)^{\frac{r-1}{2}}. \tag{4.1}$$

Lemma 4.2. *$C(\delta^2 z, \delta^r s)$ is isomorphic to the quadratic twist by δ of $C(z,s)$.*

Over the completion $K_{\mathfrak{q}_2}$ we will set the notation π_K to be a uniformizer, $\mathcal{O}_K$ the ring of integers and v_K the corresponding valuation.

Proposition 4.3. *Suppose $v_K(z^r) \geq v_K(s^2) + 4$. Let $L/K_{\mathfrak{q}_2}$ be a finite extension with ramification index r. Then, up to a quadratic twist by $\pm\pi_K$, C has good reduction over L.*

Proof. Let π_L be the uniformizer of the extension, $\mathcal{O}_L$ its ring of integers and v_L the corresponding valuation. It follows from Lemma 4.2 that, up to a quadratic twist by $\pm\pi_K$, we may assume $v_K(s)$ is even and that $s/\pi_K^{v_K(s)} \equiv 1 \pmod{\pi_K^2}$.

Applying the substitution $x \to \pi_L^{v_K(s)} x$, $y \to \pi_L^{r\frac{v_K(s)}{2}} y$ and dividing out by $\pi_L^{rv_K(s)}$ we get the model

$$y^2 = x^r + \sum_{k=1}^{\frac{r-1}{2}} c_k \left(\frac{z}{\pi_L^{v_K(s^2)}}\right)^k x^{r-2k} + \frac{s}{\pi_L^{rv_K(s)}}.$$

Now, making the substitution $x \to \pi_L x$, $y \to \pi_L^r y + 1$ and dividing out by π_L^{2r} we get the model

$$y^2 + \frac{2}{\pi_L^r} y = x^r + \sum_{k=1}^{\frac{r-1}{2}} c_k \left(\frac{z}{\pi_L^{v_K(s^2)+4}}\right)^k x^{r-2k} + \frac{\frac{s}{\pi_L^{rv_K(s)}} - 1}{\pi_L^{2r}}, \tag{4.4}$$

where $v_L(z) = v_K(z^r) \geq v_K(s^2)+4$ by hypothesis and $v_L(2/\pi_L^{rv_K(s)}-1) = rv_K(2/\pi_K^{v_K(s)}-1) \geq 2r$. Therefore (4.4) is defined over $\mathcal{O}_L$. Moreover, from (4.1) and Lemma 2.3, its discriminant equals

$$(-1)^{\frac{r-1}{2}} \frac{2^{2(r-1)}}{\pi_L^{2r(r-1)}} \cdot r^r \cdot \frac{(s^2 - 4z^r)^{\frac{r-1}{2}}}{\pi_L^{r(r-1)v_K(s)}},$$

which is a unit in $\mathcal{O}_L$. ☐

Remark 4.5. Notice that, if $v_K(s^2) + 4 \equiv 0 \pmod{r}$, the model (4.4) is defined over $K_{\mathfrak{q}_2}$. Otherwise, there is no unramified extension where C or J attains good reduction.

4.2. **The Jacobian $J_r^-(t)$.** Let $J_r^-(t)$ be the Jacobian of the hyperelliptic curve

$$\begin{aligned} C_r^-(t) : y^2 &= f(x) + 2 - 4t \\ &= (-1)^{\frac{r-1}{2}} x h(2 - x^2) + 2 - 4t, \end{aligned}$$

introduced in [Dar00]. As mentioned in Section 2.3, it gives rise to a Frey representation of signature (p,p,r).

Corollary 4.6. *Suppose $v_2(t) \leq -4$. Then, up to a quadratic twist of the curve, the conductor exponent of $\bar\rho_{J_r^-(t),\mathfrak{p}}$ at $\mathfrak{q}_2$ equals*

$$\begin{cases} 0 & \text{if } v(t) \equiv -2 \pmod{r}, \\ 2 & \text{if } v(t) \not\equiv -2 \pmod{r}. \end{cases}$$

Proof. In this case we have $z = 1$ and $s = 2 - 4t$. Since

$$v_2(z^r) = 0 = -4 + 4 \geq v(s^2) + 4,$$

the result follows from Proposition 2.16 and Remark 4.5. ☐

4.3. **The Jacobian $J^-_{r,r}(t)$.** Let $J^-_{r,r}(t)$ be the Jacobian of the hyperelliptic curve

$$H^-_{r,r}(t) : y^2 = (-t(t-1))^{\frac{r-1}{2}} xh\Big(2 - \frac{x^2}{t(t-1)}\Big) + ((t-1)t)^{\frac{r-1}{2}}(2t-1),$$

considered in [BCDF25]. As mentioned in Section 2.3, it gives rise to a Frey representation of signature (r,r,p). The curve $H^-_{r,r}(t)$ has discriminant

$$\Delta_{H^-_{r,r}(t)} = (-1)^{\frac{r-1}{2}} 2^{2(r-1)} r^r (t(t-1))^{\frac{(r-1)^2}{2}}.$$

Corollary 4.7. *Suppose $v_2(t(t-1)) \geq 4$. Then, up to a quadratic twist of the curve, the conductor exponent of $\bar{\rho}_{J^-_{r,r}(t),\mathfrak{p}}$ at $\mathfrak{q}_2$ equals*

$$\begin{cases} 0 & \textit{if } v(t(t-1)) \equiv 4 \pmod r, \\ 2 & \textit{if } v(t(t-1)) \not\equiv 4 \pmod r. \end{cases}$$

Proof. In this case we have $z = t(t-1)$ and $s = ((t-1)t)^{\frac{r-1}{2}}(2t-1)$. Since

$$v_2(z^r) = rv(t(t-1)) = (r-1)v(t(t-1)) + v(t(t-1)) \geq (r-1)v(t-1) + 4 = v(s^2) + 4,$$

the result follows from Proposition 2.16 and Remark 4.5. □

4.4. **The Jacobian $J^-_{2,r}(t)$.** Let $J^-_{2,r}(t)$ be the Jacobian of the hyperelliptic curve

$$H^-_{2,r}(t) : y^2 = (-t(t-1))^{\frac{r-1}{2}} xh\Big(2 - \frac{x^2}{t(t-1)}\Big) + 2(t-1)^{\frac{r-1}{2}} t^{\frac{r+1}{2}},$$

with discriminant given in (2.14). Setting $t = Aa^2/Cc^p$ we recover the Frey hyperelliptic curve associated to equation (1.1) for signature $(2,r,p)$ up to a twist (see e.g. [CV25, Section 6.3]). Indeed, as mentioned in Section 2.3, $J^-_{2,r}(t)$ gives rise to a Frey representation of signature $(2,r,p)$.

Corollary 4.8. *Suppose $v_2(t-1) \geq 6$. Then, up to a quadratic twist of the curve, the conductor exponent of $\bar{\rho}_{J^-_{2,r}(t),\mathfrak{p}}$ at $\mathfrak{q}_2$ equals*

$$\begin{cases} 0 & \textit{if } v(t-1) \equiv 6 \pmod r, \\ 2 & \textit{if } v(t-1) \not\equiv 6 \pmod r. \end{cases}$$

Proof. Suppose $v_2(t-1) \geq 6$ and so $v_2(t) = 0$. In this case we have $z = t(t-1)$ and $s = 2(t-1)^{\frac{r-1}{2}} t^{\frac{r+1}{2}}$. Since

$$v_2(z^r) = rv(t-1) = (r-1)v(t-1) + v(t-1) \geq (r-1)v(t-1) + 6 = v(s^2) + 4,$$

the result follows from Proposition 2.16 and Remark 4.5. □

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