STEINER 3-DESIGNS AS EXTENSIONS*

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Abstract. In this article, we construct a Steiner system with the parameters $S(3,6,42)$, settling one of the smallest open parameter sets of Steiner 3-designs. Furthermore, we establish the existence of rotational Steiner quadruple systems on 46 and 92 points.

Our construction method is based on extending Steiner 2-designs using prescribed extension groups. We also consider extensions to designs of higher strength. The article includes a table and a discussion of the status of all admissible parameters for Steiner 3-designs on at most 50 points.

1. Introduction

Steiner systems are among the most fascinating objects in Design Theory, if not all of Combinatorics. They have a long history [14] and many applications, as well as connections to other important combinatorial structures [6]. A *Steiner system* $\mathcal{D} = (X, \mathcal{B})$ with parameters $S(t,k,v)$ comprises a set X of v *points*, and a family $\mathcal{B}$ of k-element subsets of X called *blocks* such that every t-subset of points is contained in exactly one block. Alternatively, $\mathcal{D}$ is called a *Steiner t-design* and

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the parameters are written as t-$(v, k, 1)$. A necessary condition for the existence of Steiner systems is that

$$\binom{k-i}{t-i} \text{ divides } \binom{v-i}{t-i}, \quad \text{for all } i \in \{0, \ldots, t\}. \tag{1}$$

Given a point $P \in X$, the *derived design* $\mathrm{der}_P\,\mathcal{D}$ has $X \setminus \{P\}$ as points and $\{B \setminus \{P\} \mid B \in \mathcal{B},\, P \in B\}$ as blocks, and is a Steiner system $S(t-1, k-1, v-1)$ (for $t \geq 1$). We refer to [2] and [5] for other results and background material about combinatorial designs.

It was proved by probabilistic arguments that Steiner t-designs exist for all t [27, 17]. However, explicit examples are known only for $t \leq 5$, and only finitely many explicit examples are known for $t \geq 4$. Many known designs were constructed by prescribing groups of automorphisms and using computational methods, such as the Kramer-Mesner method [35]. In the recent paper [28], this method was fine-tuned for Steiner systems. Implementations of the different computational steps are available in the GAP [16] package *Prescribed Automorphism Groups* [36]. The paper [28] focuses on Steiner 2-designs, but the method can be used for arbitrary $S(t, k, v)$ designs, subject to time and memory limitations.

The purpose of this paper is to construct explicit examples of small Steiner 3-designs. There are 25 admissible parameter sets $S(3, k, v)$ with $v \leq 50$. Previously, designs were known for 20 of these parameter sets, nonexistence was established for one, and existence was an open problem for the remaining four. Our main result is the construction of a design with parameters $S(3, 6, 42)$, settling one of the open cases. Furthermore, we establish the existence of rotational Steiner quadruple systems for $v = 46$ and $v = 92$, and we construct many new Steiner designs with parameters for which existence was already known. We made all constructed designs available in GAP-readable format, compatible with the DESIGN [48] package. They can be accessed as a Zenodo data set [29] and through our web page

`https://web.math.hr/~krcko/results/steiner3.html`.

In Section 2, we present a table of admissible parameters $S(3, k, v)$ with $v \leq 50$, containing information on the number of non-isomorphic designs and extensibility. Examples of the newly established designs are given by listing the prescribed groups and base blocks (Theorems 2.1, 2.2, and 2.3).

In Section 3, we describe a procedure for extending Steiner systems with prescribed extension groups, based on the Kramer-Mesner method. The new $S(3, 6, 42)$ design was constructed in this way. We also tried constructing designs for the remaining three open parameter sets $S(3,6,46)$, $S(3,5,41)$, and $S(3,5,50)$. No such design could be found, but many new Steiner 2-designs for the derived parameters $S(2,5,45)$, $S(2,4,40)$, and $S(2,4,49)$ were constructed in this process (Theorems 3.1, 3.2, and 3.3).

Finally, in Section 4, we discuss extensions of $S(3,k,v)$ designs to t-designs of strength $t > 3$. Using the procedure from the previous section, we prove that the new $S(3,6,42)$ design does not extend to a 4-design if any nontrivial extension group is assumed (Theorem 4.1). We also prove that the only known $S(5,7,28)$ design does not extend to a 6-design (Theorem 4.2).

2. A table of admissible parameters

Table 1 contains all admissible parameters $S(3,k,v)$ with $v \leq 50$. Admissibility means that the divisibility conditions (1) and Fisher's inequality for the derived designs are satisfied. The latter condition eliminates only the parameters $S(3,7,22)$ in our range. The first two columns contain v and k, followed by the number b of blocks. The fourth column labeled 'Nd' contains the number of $S(3,k,v)$ designs up to isomorphism, or a lower bound if the exact number is not known. Let $t_{\max}$ be the largest integer t such that a $S(t,k+t-3,v+t-3)$ design exists. Note that, by taking derived designs, all $S(t,k+t-3,v+t-3)$ with $t \leq t_{\max}$ do exist. The fifth and the sixth columns contain the sharpest known lower and upper bounds on $t_{\max}$, respectively. The last two columns contain references and comments.

We first discuss the parameters from Table 1 belonging to known infinite families of Steiner 3-designs. The most widely studied family are Steiner quadruple systems $S(3,4,v)$, abbreviated as $\mathrm{SQS}(v)$. Two comprehensive surveys are [42] and [24]. $\mathrm{SQS}(v)$ are known to exist for all admissible v, i.e. for $v \equiv 2,4 \pmod 6$, $v \geq 8$. This was first proved in [21], and simplifications were given in [22, 23, 40, 53]. A subfamily with parameters $S(2,4,2^n)$, $n \geq 3$ are the n-dimensional affine spaces $\mathrm{AG}_2(n,2)$ over the binary field, with planes as blocks. This family was already described by T. P. Kirkman [31]. Another subfamily are the spherical geometries of order $q = 3$ with parameters $S(3,4,3^n+1)$, $n \geq 2$, discussed below.

The uniqueness of SQS(8) and SQS(10) was proved in [1] and [52]. In [44], it was computationally proved that there are exactly four SQS(14) up to isomorphism. A computer enumeration of SQS(16) was performed in [26], requiring about 12 years of CPU time. We repeated the calculation using different software and confirmed 1 054 163 as the

v	k	b	Nd	$t_{\max}\geq$	$t_{\max}\leq$	References	Comments
8	4	14	1	3	3	[1, 52]	$AG_2(3,2)$
10	4	30	1	5	5	[1, 52, 51]	Möb(3)
14	4	91	4	3	3	[44]	
16	4	140	1 054 163	3	3	[26, 46]	$AG_2(4,2)$
17	5	68	1	3	3	[52]	Möb(4)
20	4	285	≥60 077	3	15		
22	4	385	≥52 240	5	17	[12]	
22	6	77	1	5	5	[52]	
26	4	650	≥51 145	3	21		
26	5	260	≥1	5	8	[12, 19]	
26	6	130	1	3	3	[4, 10, 51]	Möb(5)
28	4	819	≥12 149	3	3		Spherical
32	4	1240	≥1516	3	27		$AG_2(5,2)$
34	4	1496	≥1569	5	29	[3]	
37	7	222	0	1	1		Möb(6)
38	4	2109	≥1547	3	3		
40	4	2470	≥1557	3	35		
41	5	1066	?	2	16		
42	6	574	≥1	3	9		Theorem 2.3
44	4	3311	≥1504	3	39		
46	4	3795	≥1559	5	41	[18]	
46	6	759	?	2	3		
50	4	4900	≥1535	3	45		
50	5	1960	?	2	20		
50	8	350	1	3	3	[11, 51]	Möb(7)

Table 1. Steiner 3-designs with at most 50 points.

number of SQS(16) up to isomorphism; see Section 3 for more details. In [41], it was established that there are at least 10^{17} non-isomorphic SQS(20), making a computer enumeration infeasible. Asymptotic estimates for the number of non-isomorphic SQS(v) for sufficiently large admissible v are known [13, 39].

In Table 1, we give rough lower bounds on the number of SQS(v) for $20 \leq v \leq 50$, based on explicitly constructed examples. It is clear that the actual numbers 'Nd' are many orders of magnitude larger. Our lower bounds are primarily due to memory considerations, since we are making the designs available in [29]. For $v \geq 32$, we stopped the search when the number of constructed designs exceeded 1500. The designs were constructed by prescribing groups of automorphisms and using the implementation of the Kramer-Mesner method from [36], described in [28]. *Nauty / Traces* [43] was used for isomorphism tests and to compute the full automorphism groups.

We aimed to choose the prescribed groups large enough to make the computation feasible, while still producing many non-isomorphic designs. For some parameters, there is a "classical" design $\mathcal{D}$ with a large full automorphism group. For example, one SQS(28) is the spherical geometry for $q = n = 3$, with full automorphism group $\mathrm{Aut}(\mathcal{D}) \cong \mathrm{PSL}(2, 27) \rtimes C_6$ of order 58 968. One example of a SQS(32) is $\mathrm{AG}_2(5, 2)$, with $\mathrm{Aut}(\mathcal{D}) \cong \mathrm{AGL}(5, 2)$ of order 319 979 520. We can take subgroups $G \leq \mathrm{Aut}(\mathcal{D})$ and construct designs with G as prescribed group, possibly getting other designs besides $\mathcal{D}$. When there is no such "classical" design, we prescribed extensions of regular permutation representations of groups of orders v and $v - 1$, utilizing the GAP [16] library of small groups.

An SQS(v) is called *cyclic* and denoted by CSQS(v) if it has a cyclic automorphism of order v. It is called *rotational* and denoted by RoSQS(v) if it has a cyclic automorphism of order $v - 1$ with a fixed point ∞. Designs of type CSQS(v) and RoSQS(v) have been extensively studied, but the sets of all values v for which they exist have not been fully determined. For CSQS(v), there is only one unknown value below 100, namely $v = 94$ [15, Theorem 4.6]. We were not able to construct a CSQS(94) with our method, nor to rule it out. For RoSQS(v), there were seven open values below 100: $v = 46$, 56, 70, 82, 86, 92, and 98 [25, Table I]. We were able to construct a RoSQS(46) and a RoSQS(92). Examples can be downloaded from [29], and here we describe one of each by giving base blocks.

Theorem 2.1. *There exists a* RoSQS(46).

Proof. Let G be the group $\mathbb{Z}_{45} = \{0, \ldots, 44\}$ with addition modulo 45. Then the union of the G-orbits of the 4-subsets listed below forms a RoSQS(46):

$\{0,1,2,21\}$	$\{0,1,3,24\}$	$\{0,1,4,22\}$	$\{0,1,5,42\}$	$\{0,1,6,41\}$
$\{0,1,7,31\}$	$\{0,1,8,27\}$	$\{0,1,9,19\}$	$\{0,1,10,13\}$	$\{0,1,11,\infty\}$
$\{0,1,12,37\}$	$\{0,1,14,35\}$	$\{0,1,15,25\}$	$\{0,1,16,40\}$	$\{0,1,17,26\}$
$\{0,1,18,28\}$	$\{0,1,23,33\}$	$\{0,1,29,38\}$	$\{0,1,30,34\}$	$\{0,1,32,36\}$
$\{0,1,39,43\}$	$\{0,2,4,35\}$	$\{0,2,5,29\}$	$\{0,2,6,26\}$	$\{0,2,7,28\}$
$\{0,2,8,14\}$	$\{0,2,9,36\}$	$\{0,2,10,17\}$	$\{0,2,11,40\}$	$\{0,2,12,22\}$

$\{0,2,13,31\}$	$\{0,2,15,19\}$	$\{0,2,16,\infty\}$	$\{0,2,18,38\}$	$\{0,2,20,24\}$
$\{0,2,25,39\}$	$\{0,2,27,34\}$	$\{0,2,30,42\}$	$\{0,2,32,37\}$	$\{0,3,6,13\}$
$\{0,3,7,\infty\}$	$\{0,3,9,30\}$	$\{0,3,11,32\}$	$\{0,3,12,16\}$	$\{0,3,14,38\}$
$\{0,3,15,37\}$	$\{0,3,17,20\}$	$\{0,3,18,39\}$	$\{0,3,19,22\}$	$\{0,3,23,34\}$
$\{0,3,25,40\}$	$\{0,4,9,26\}$	$\{0,4,11,28\}$	$\{0,4,12,21\}$	$\{0,4,17,31\}$
$\{0,4,18,37\}$	$\{0,4,19,39\}$	$\{0,4,20,34\}$	$\{0,4,23,38\}$	$\{0,4,29,40\}$
$\{0,5,11,19\}$	$\{0,5,12,28\}$	$\{0,5,14,36\}$	$\{0,5,17,32\}$	$\{0,5,18,25\}$
$\{0,5,20,37\}$	$\{0,5,23,\infty\}$	$\{0,5,24,35\}$	$\{0,5,27,39\}$	$\{0,5,31,38\}$
$\{0,6,15,38\}$	$\{0,6,16,22\}$	$\{0,6,17,23\}$	$\{0,6,18,36\}$	$\{0,6,19,\infty\}$
$\{0,6,20,32\}$	$\{0,7,15,29\}$	$\{0,7,17,33\}$	$\{0,8,16,32\}$	$\{0,8,19,35\}$
$\{0,8,21,33\}$	$\{0,8,25,\infty\}$	$\{0,9,21,\infty\}$	$\{0,9,22,35\}$	$\{0,15,30,\infty\}$

$\square$

Theorem 2.2. *There exists a* RoSQS(92).

Proof. Let G be the subgroup of the symmetric group on $\mathbb{Z}_{91}$ generated by $x \mapsto x+1$ and $x \mapsto 4x$. Then, G is of order 546 and isomorphic to a semidirect product $C_{91} \rtimes C_6$. The union of the G-orbits of the 4-subsets listed below forms a RoSQS(92):

$\{0,1,2,13\}$	$\{0,1,3,52\}$	$\{0,1,4,29\}$	$\{0,1,5,31\}$	$\{0,1,6,17\}$
$\{0,1,7,59\}$	$\{0,1,8,26\}$	$\{0,1,9,48\}$	$\{0,1,10,66\}$	$\{0,1,11,\infty\}$
$\{0,1,14,39\}$	$\{0,1,15,27\}$	$\{0,1,16,43\}$	$\{0,1,18,76\}$	$\{0,1,19,58\}$
$\{0,1,20,85\}$	$\{0,1,21,40\}$	$\{0,1,22,61\}$	$\{0,1,24,36\}$	$\{0,1,25,60\}$
$\{0,1,28,79\}$	$\{0,1,32,84\}$	$\{0,1,33,63\}$	$\{0,1,34,45\}$	$\{0,1,35,83\}$
$\{0,1,37,55\}$	$\{0,1,38,71\}$	$\{0,1,41,47\}$	$\{0,1,44,70\}$	$\{0,1,46,67\}$
$\{0,1,50,77\}$	$\{0,1,54,80\}$	$\{0,1,56,89\}$	$\{0,1,57,62\}$	$\{0,1,64,78\}$
$\{0,1,72,82\}$	$\{0,1,73,81\}$	$\{0,1,75,86\}$	$\{0,2,5,71\}$	$\{0,2,7,44\}$
$\{0,2,8,30\}$	$\{0,2,9,33\}$	$\{0,2,10,67\}$	$\{0,2,11,14\}$	$\{0,2,12,34\}$
$\{0,2,20,42\}$	$\{0,2,24,48\}$	$\{0,2,32,57\}$	$\{0,2,35,72\}$	$\{0,2,36,61\}$
$\{0,2,37,\infty\}$	$\{0,2,41,56\}$	$\{0,2,43,63\}$	$\{0,2,55,60\}$	$\{0,2,59,81\}$
$\{0,3,9,21\}$	$\{0,3,18,51\}$	$\{0,3,31,53\}$	$\{0,3,36,72\}$	$\{0,3,43,76\}$

$\{0,3,50,56\}$	$\{0,3,55,84\}$	$\{0,5,11,66\}$	$\{0,5,14,61\}$	$\{0,5,30,85\}$
$\{0,5,33,77\}$	$\{0,5,41,82\}$	$\{0,5,47,65\}$	$\{0,5,49,70\}$	$\{0,7,21,63\}$
$\{0,7,28,42\}$	$\{0,7,36,\infty\}$	$\{0,13,26,52\}$	$\{0,13,65,\infty\}$	$\{0,15,73,\infty\}$

□

Steiner systems $S(3,q+1,q^n+1)$ are known as *spherical geometries* or *inversive spaces*. They exist for all $n \geq 2$ and prime powers q [52]. The most studied case is $n=2$, called *inversive* or *Möbius planes* and denoted by $\text{Möb}(q)$ in Table 1. Classical inversive planes are called *Miquelian* because they are characterized by Miquel's theorem [9, Theorem 6.1.5]. The only known non-Miquelian examples have orders $q = 2^{2e+1}$, $e \geq 1$, and are associated with the ovoids of Tits [50].

The uniqueness of $\text{Möb}(q)$ was proved for $q=4$ in [52], for $q=5$ in [4] and [10], and for $q=7$ in [11]. For $q=3,5,7$, uniqueness also follows from a theorem of Thas [51] and the uniqueness of affine planes $\text{AG}(2,q)$. For similar results about larger orders q, see [47] and the references therein. Designs $\text{Möb}(6)$ do not exist because their derived designs would be affine planes of order 6.

There are three "sporadic" designs in Table 1, not belonging to the infinite families discussed above. The first is the twice derived Witt 5-design with parameters $S(3,6,22)$, also known to be unique by [52]. The second has parameters $S(3,5,26)$ and was investigated in [19], where the question was raised whether this design is in fact unique. In Section 3, we explore the possibility of a computational approach to this question, which remains open. The third sporadic example is the following new $S(3,6,42)$ design.

Theorem 2.3. *There exists an $S(3,6,42)$ design.*

Proof. Let $G = \langle \alpha, \beta \rangle \leq S_{42}$ be the permutation group generated by

$$\begin{aligned}\alpha = {} & (1,2,4)(3,9,7)(5,6,8)(10,11,13)(12,18,16)(14,15,17)(19,38,34)\\ & (20,39,36)(21,37,35)(22,42,32)(23,40,31)(24,41,33)(28,30,29),\end{aligned}$$

$$\begin{aligned}\beta = {} & (1,10)(2,11)(3,12)(4,16)(5,17)(6,18)(7,13)(8,14)(9,15)(19,23)\\ & (20,24)(21,22)(25,35)(26,36)(27,34)(28,33)(29,31)(30,32)\\ & (38,39)(41,42).\end{aligned}$$

This is a group of order 432 isomorphic to $\text{AGL}(2,3)$. Then the union of the G-orbits of the 6-subsets listed below is a $S(3,6,42)$:

$\{1,2,3,10,11,12\}$	$\{1,2,4,20,36,39\}$	$\{1,2,13,17,31,34\}$
$\{1,10,19,22,25,30\}$	$\{1,11,20,21,29,34\}$	$\{19,20,21,22,23,24\}$

$\{19, 22, 31, 36, 38, 42\}$

□

The group G from the proof of Theorem 2.3 is the full automorphism group of the design $\mathcal{D}$. It partitions the point set into two orbits, one of size 18 and the other of size 24. The unusual action of $\mathrm{Aut}(\mathcal{D})$ might be the reason why $\mathcal{D}$ remained undiscovered for so long. Another Steiner design constructed from a non-transitive group is the $S(5, 6, 36)$ from [3], where a permutation group comprising two copies of the natural action of $\mathrm{PGL}(2, 17)$ on 18 points was used. Our new design has $\mathrm{Aut}(\mathcal{D}) \cong \mathrm{AGL}(2, 3)$, suggesting a connection with the affine plane of order 3.

Three parameter sets in Table 1 remain open with regards to existence: $S(3, 5, 41)$, $S(3, 6, 46)$, and $S(3, 5, 50)$. In the next section, we describe the computational approach that led to the discovery of the $S(3, 5, 42)$ design and our attempts to construct designs for the remaining open parameter sets.

3. Extending designs with prescribed groups

Let $\mathcal{E}$ be a $S(t, k, v)$ design with a group G of automorphisms fixing a point ∞. Then the derived design $\mathcal{D} = \mathrm{der}_\infty \mathcal{E}$ is also G-invariant. In this situation, we refer to $\mathcal{E}$ as an *extension* of $\mathcal{D}$ and to G as an *extension group*. Clearly, G must be a subgroup of both $\mathrm{Aut}\,\mathcal{D}$ and $\mathrm{Aut}\,\mathcal{E}$. For example, a rotational $\mathrm{SQS}(v)$ is the extension of a cyclic Steiner triple system $\mathrm{STS}(v-1)$ with C_{v-1} as extension group. We use the customary abbreviation $\mathrm{STS}(v)$ for $S(2, 3, v)$ designs, known as Steiner triple systems.

Our strategy for constructing Steiner 3-designs is to extend Steiner 2-designs with prescribed extension groups. Typically, we first classify $S(2, k-1, v-1)$ designs $\mathcal{D}$ with a fixed prescribed group G. Depending on the parameters, this is a significantly smaller computation than classifying the corresponding 3-designs directly. Then, for each such design $\mathcal{D}$, we add ∞ to all blocks of $\mathcal{D}$ and then compute all extensions to a G-invariant $S(3, k, v)$ design $\mathcal{E}$. The G-invariance of $\mathcal{E}$ precisely means that the extension consists of complete G-orbits of k-subsets that do not contain ∞. For a fixed $\mathcal{D}$, this also requires less computation than a full classification of G-invariant designs $\mathcal{E}$, because the blocks through ∞ are already given. The k-subset orbits covering any 3-subset already covered by $\mathcal{D}$ can be discarded, reducing the size of the Kramer-Mesner system. The extension process is a modification of the standard Kramer-Mesner algorithm, available in the package [36] as a command `ExtendSteinerDesign`.

In [44] and [26], SQS(14) and SQS(16) were classified by extending STS(13) and STS(15), respectively. To get a full classification, all non-isomorphic STS must be extended with the trivial extension group. The 80 non-isomorphic STS(15) were determined in [20]. In [26], extending each STS(15) was set up as an instance of the exact cover problem, and D. E. Knuth's *dancing links* algorithm [32] was used to solve thes problems. We repeated the calculation using the *ssxcc* [33] implementation of the more recent *dancing cells* algorithm [34]. The obtained systems SQS(16) need to be checked for isomorphism, but this requires less CPU time than the extension phase. We used the `BlockDesignFilter` command from [36], relying on *nauty / Traces* [43]. We got the same total number of non-isomorphic SQS(16) and distribution by full automorphism group orders as in [26, Table 1].

One could attempt to prove the uniqueness of the $S(3, 5, 26)$ design in this manner. The derived designs have parameters $S(2, 4, 25)$ and were classified in [49]; there are 18 examples up to isomorphism. However, in this case the extension phase seems infeasible, because each $S(2, 4, 25)$ gives rise to a very large exact cover problem. In Knuth's terminology [32], each problem has 2100 items and 16 380 options, out of which 210 are to be chosen. We couldn't solve these exact cover problems with the resources available to us. For comparison, the 80 exact cover problems in the classification of SQS(16) have 420 items and 945 options, out of which 105 are to be chosen.

When a complete classification is out of reach, prescribing a nontrivial extension group G may reduce the computation to manageable size. An added benefit is that we have a natural way of choosing G: It must be a subgroup of $\mathrm{Aut}(\mathcal{D})$. For example, 15 non-isomorphic $S(2, 5, 41)$

$\lvert\mathrm{Aut}(\mathcal{D})\rvert$	Nd
205	1
120	2
24	2
20	1
18	4
12	1
9	1
6	2
2	1

TABLE 2. Distribution of the known $S(2, 5, 41)$ designs by full automorphism group orders.

designs are known, with full automorphism group orders given in Table 2. Among them are all designs with automorphisms of order 3, classified in [37]. The known $S(2,5,41)$ designs are included in the Zenodo data set [29]. Two of the designs, with $|\mathrm{Aut}(\mathcal{D})| = 24$ and $|\mathrm{Aut}(\mathcal{D})| = 18$, extend to a $S(3,6,42)$ design $\mathcal{E}$. For both designs $\mathcal{D}$, the full automorphism group can be used as extension group. The group of order 24 is large enough to construct the $S(3,6,42)$ design $\mathcal{E}$ even without prescribing the derived subdesign, but it would be difficult to guess its action on 42 points without the knowledge of the derived design. The group is isomorphic to $\mathrm{SL}(2,3)$ and acts in five orbits of sizes 1, 1, 8, 8, and 24. Instead of guessing, we get the extension groups from the prescribed derived subdesigns $\mathcal{D}$. We examined all known $S(2,5,41)$ designs with $G \leq \mathrm{Aut}(\mathcal{D})$, $|G| \geq 10$ as extension groups, and found only one $S(3,6,42)$ design up to isomorphism, given in Theorem 2.3.

We tried to construct designs for the remaining three open parameter sets from Table 1 by following the same procedure, but no designs were found. Derived designs of a hypothetical $S(3,6,46)$ have parameters $S(2,5,45)$. Previously, 65 non-isomorphic $S(2,5,45)$ designs were known [8]; see also [7] and references therein. We managed to construct over 1000 new examples by prescribing various groups of orders 16 and 8, and by applying so-called paramodifications [45].

Theorem 3.1. *There exist at least* 1072 *non-isomorphic* $S(2,5,45)$ *designs. Their distribution by orders of full automorphism groups is given in Table 3.*

These designs, as well as the ones from Theorems 3.2 and 3.3, are also available in [29]. We systematically ran the extension algorithm with the known examples and extension groups of orders $|G| > 10$, and even with most groups of orders 8 and 10 barring a few hard cases, but we didn't get any $S(3,6,46)$ designs. The search would take too long for smaller extension groups.

Derived designs of $S(3,5,41)$ have parameters $S(2,4,40)$. The classical example is $\mathrm{PG}(3,3)$; we managed to construct many more examples by prescribing various groups of automorphisms and using the Kramer-Mesner method.

Theorem 3.2. *There exist at least* 10 791 *non-isomorphic* $S(2,4,40)$ *designs. Their distribution by orders of full automorphism groups is given in Table 4.*

While trying to extend these designs to $S(3,5,41)$, we could eliminate all extension groups of orders greater than 40. Some groups of

$\lvert\mathrm{Aut}(\mathcal{D})\rvert$	Nd
360	1
160	1
72	6
40	1
32	3
24	3
16	16
8	142
6	2
4	221
2	648
1	28

TABLE 3. Distribution of the known $S(2,5,45)$ designs by full automorphism group orders.

$\lvert\mathrm{Aut}(\mathcal{D})\rvert$	Nd	$\lvert\mathrm{Aut}(\mathcal{D})\rvert$	Nd	$\lvert\mathrm{Aut}(\mathcal{D})\rvert$	Nd
12 130 560	1	486	13	96	22
151 632	1	480	1	81	43
15 552	1	468	1	80	3
10 368	1	432	13	78	13
7776	1	384	1	72	467
5184	4	324	153	64	1
3888	5	320	1	60	1
2916	1	288	12	54	11
2592	5	256	3	48	163
2106	3	216	21	40	25
1944	2	192	3	39	194
1458	1	162	19	36	5346
1296	18	160	3	32	29
972	11	156	3	27	6
864	2	144	82	26	176
648	57	120	1	24	2782
576	2	108	169	20	894

TABLE 4. Distribution of the known $S(2,4,40)$ designs by full automorphism group orders.

order 40 require a lot of computation and we couldn't examine them exhaustively. This is also true for groups of orders less than 40. To construct $S(3,5,50)$ designs by the same method, $S(2,4,49)$ designs are required. Again, we first applied the usual Kramer-Mesner approach to the parameters $S(2,4,49)$ with various prescribed groups.

Theorem 3.3. *There exist at least* 22 080 *non-isomorphic* $S(2,4,49)$ *designs. Their distribution by orders of full automorphism groups is given in Table 5.*

$\lvert\mathrm{Aut}(\mathcal{D})\rvert$	Nd
147	14
144	2
72	2
49	222
48	332
45	288
42	479
36	1672
30	1498
27	7432
20	9879
14	260

Table 5. Distribution of the known $S(2,4,49)$ designs by full automorphism group orders.

The cut-off order for the extension group is now 72. We could eliminate all extensions of the designs from Theorem 3.3 with $|G| \geq 72$, but as one can see from Table 5, these groups act on only 18 non-isomorphic $S(2,4,49)$ designs. There is no classical example with these parameters, and we did not find many designs $\mathcal{D}$ with large enough $\mathrm{Aut}(\mathcal{D})$ so that the extension process is computationally feasible.

4. Extending to higher strengths

In Table 1, the sixth column labeled '$t_{\max} \leq$' contains the largest strength t for which a $S(3,k,v)$ design could possibly be extended to an $S(t,k',v')$, according to the current knowledge of nonexistence results. Most entries follow from the divisibility conditions (1) on $S(t,k',v')$, with three exceptions. In [44] and in [46], it was proved by exhaustive computer searches that $S(4,5,15)$ and $S(4,5,17)$ designs do not exist.

Designs $S(4,6,18)$ were ruled out in [52, Satz 6] based on a case-by-case analysis. Hence, for $S(3,4,14)$, $S(3,4,16)$, and $S(3,5,17)$, the bound in the sixth column is $t_{\max} \leq 3$. The fifth column contains strengths of the largest known extensions. As it happens, all these (nontrivial) extensions are 5-designs, with parameters $S(5,6,12)$, $S(5,6,24)$, $S(5,8,24)$, $S(5,7,28)$, $S(5,6,36)$, and $S(5,6,48)$. Known designs with these parameters are also included in [29]. Not a single Steiner 4-design was discovered directly; all known examples are actually derived 5-designs.

We remark that the existence table in [6, Section 5.3] is organized differently: Admissible parameters $S(t,k,v)$ are grouped according to $k-t$ and $n = v-t$. There, all entries in the column labeled '$t \leq$' follow from necessary conditions on the parameters, while (non)existence results are given in separate columns labeled '$\exists$' and '$\nexists$'. The table was published in 2007, but only two entries need to be updated with new results. The first is the nonexistence of $S(4,5,17)$ from [46], and the second is the new $S(3,6,42)$ design from Theorem 2.3.

In view of the remark about Steiner 4-designs above, it would be noteworthy to extend some of the 3-designs from Table 1. The procedure of extending designs with prescribed groups, used in the previous section for $t = 3$, can be used without change for higher strengths. In the first place, we searched for extensions of the new $S(3,6,42)$ design $\mathcal{D}$ using subgroups $G \leq \mathrm{Aut}(\mathcal{D})$ as extension groups. We eliminated all nontrivial subgroups, thereby proving the following theorem.

Theorem 4.1. *If the $S(3,6,42)$ design of Theorem 2.3 can be extended to a $S(4,7,43)$ design $\mathcal{E}$ by adding a point ∞, then the point stabilizer $\mathrm{Aut}(\mathcal{E})_\infty$ is trivial.*

Eliminating the trivial extension group would require solving an exact cover problem with 103 320 items and 288 504 options, out of which 2952 are to be chosen. We could not perform this computation, but we did manage to eliminate the trivial extension group for the following Steiner 5-design.

Theorem 4.2. *The Steiner system $S(5,7,28)$ constructed in* [12] *does not extend to a Steiner 6-design.*

Here, an exact cover problem with 343 980 options and 39 312 items, out of which 12 285 are to be chosen, was computationally proved to have no solutions. However, this result does not improve the '$t_{\max}\leq$' entry for $S(3,5,26)$ designs, because designs with these parameters are not known to be unique. Potentially, some other yet-undiscovered $S(3,5,26)$ design might admit an extension all the way to a Steiner 8-design.

The smallest parameters of a Steiner 3-design for which the extension problem is open are $S(3,4,20)$. We ran a lot of extension procedures using extension groups of orders $|G| \geq 10$, but did not find any $S(4,5,21)$ design. Our attempts to find other unknown extensions of the designs from Table 1 were also unsuccessful. Unfortunately, these calculations provide very little evidence that the sought-after Steiner 4-designs do not exist. We shall illustrate this with $S(3,4,22)$ designs, for which a few extensions are known.

$$\boxed{S(2,3,21)} \xrightarrow[:3 \qquad 22:]{1} \boxed{S(3,4,22)} \xrightarrow[:2 \qquad 23:]{11} \boxed{S(4,5,23)} \xrightarrow[:1 \qquad 24:]{253} \boxed{S(5,6,24)}$$

FIGURE 1. Derived designs of the three known $S(5,6,24)$ designs.

Denniston [12] established the existence of $S(5,6,24)$ designs by prescribing $\mathrm{PSL}(2,23)$ as a group of automorphisms. It is known that there are exactly three non-isomorphic designs under these assumptions. Figure 1 describes the derived designs of these three 5-designs. The order of the largest possible extension group G is written above the arrow, and below are the indices $[\mathrm{Aut}\,\mathcal{D} : G]$ and $[\mathrm{Aut}\,\mathcal{E} : G]$. Thus, there are three non-isomorphic $S(3,4,22)$ designs which we know to be extensible to $S(4,5,23)$ designs by an extension group of order 11. However, finding these extensions computationally amounts to exact cover problems requiring too much CPU time to be solved exhaustively. Each problem has 630 items and 1758 options, out of which 126 are to be chosen. Worse still, there are exactly 26 035 non-isomorphic $S(3,4,22)$ designs with an automorphism of order 11. If we go from smaller to larger t, we have no easy way to distinguish the three extensible designs in the list of 26 035 systems $S(3,4,22)$ with a group of order 11. In this case, it is much easier to construct the $S(4,5,23)$ and $S(5,6,24)$ designs directly, because larger prescribed groups of orders 253 and 6072 can be used.

To conclude, we believe that we have mostly exhausted the useful applications of extending Steiner designs with prescribed groups, as outlined in Sections 3 and 4. To make further progress, additional restrictions derived from new theoretical insights will be needed, or alternative computational techniques – such as the method of tactical decompositions [30, 38] – will have to be implemented and applied.

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