

Null integrability and photon phenomenology in the accelerated Schwarzschild black hole

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In this study we analyse null geodesics in an accelerated Schwarzschild black hole sustained by a cosmic string/strut (C-metric) in the sub-extremal regime. Using dimensionless variables and a Mino-type parameter we cast the equations into a complete first order form. For photons the conformal Hamilton-Jacobi equation separates and the radial and polar motion reduce to Bierman-Weierstrass form, yielding new closed analytic solutions. This provides a clean classification of photon trajectories, identifies a fixed photon cone that replaces equatorial symmetry when the acceleration is nonzero, and locates a single spherical photon surface shared by all latitudes. On the observational side, we prove that the shadow seen by any static observer is an exact circle: its screen radius depends on the acceleration and observer location but not on the conicity, so the local shadow cannot bound the string tension. We provide compact expressions for the photon ring orbital frequency and Lyapunov exponent and use them to obtain eikonal quasinormal estimates; we also show how a single shadow measurement at known position uniquely infers the dimensionless acceleration. As diagnostics of the background, we compute the (generally unequal) surface gravities of the black hole and acceleration horizons and derive an exact redshift law between static worldlines.

I. INTRODUCTION

Black holes undergoing uniform translatory acceleration are modelled by the static C-metric. This spacetime is a well known solution to Einstein's equation that describes Schwarzschild geometry attached to a string/strut that supplies the acceleration and introduces a conical deficit along the symmetry axis and has been studied in many disciplines, e.g., [1–12]. The spacetime originally belongs to the large class of solutions discovered by [13] and has a double horizon structure inside a static patch: the usual black hole horizon and an outer acceleration horizon. Beyond its exactness and conceptual clarity, this geometry provides a controlled setting in which to isolate how acceleration and conical defects alter geodesic motion and observables. In the static C-metric the null Hamilton-Jacobi equation separates, and null orbits admit analytic treatment in terms of elliptic functions. [14, 15] gave a detailed analysis of null and also timelike geodesics, including special orbits and integrability structures; later work extended the treatment to considering cosmological constant [15] and exhibited explicit solutions in terms of Jacobi elliptic functions. Also, [16, 17] discussed photon surfaces and provided the characteristic radius relation used widely in later work.

We recast the null problem so that both radial and polar equations reduce to the Biermann-Weierstrass form with explicit invariants following the method in [18]. This yields particularly compact and manifestly real expressions for the solutions and for the invariants, which

typically use Jacobi elliptics or work with the standard Weierstrass form without normalization. Further, we recover the photon cone angle and the photon surface radius in consistent dimensionless notation and show transparently how the broken equatorial symmetry is encoded by the single radius together with the cone latitude. For any static observer we prove that the shadow boundary is an exact circle. Also, we show that the Euler-Lagrange compatibility reduces to a radius dependent quadratic valid at any latitude. Prior detailed studies of circular orbits in the C-metric focused primarily on the equatorial case or on specific families; our approach gives a closed criterion at generic position (ξ, θ) . In addition, applying the Cardoso et.al. correspondence [19], we derive compact expressions for the orbital frequency and the Lyapunov exponent of the photon ring giving immediate quasi normal mode (QNM) eikonal estimates specialized to the accelerated Schwarzschild case.

Earlier analyses identified fixed-latitude photon motion (photon cones) and established the existence of photon surfaces in the C-metric (e.g. [15, 17]), often working with Jacobi elliptic functions or without a consistent dimensionless normalisation. Our contribution is a streamlined, fully dimensionless and parameter-consistent treatment that (a) puts the null Hamilton-Jacobi separation into Weierstrass form with explicit invariants for both sectors, (b) proves circularity of the shadow for any static observer and shows why the angular deficit drops out of the local boundary while still affecting global lensing, (c) supplies compact, closed expressions for the photon-ring frequency and instability that specialise the Cardoso et.al. correspondence to the accelerated Schwarzschild case [19, 20], and (d) provides an algebraic inversion from a single shadow radius to the

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acceleration parameter.

The paper is organized as follows: Section II reviews the metric, horizons, and constants of motion. Section III develops the dimensionless null geodesic equations and their Biermann-Weierstrass solutions, while Section IV discusses the photon cone and photon surface. Section V turns to observables: the circular shadow radius for static observers, photon ring dynamics and eikonal ringdown, the (non)role of the conical deficit in the local shadow, and a simple inversion for a . Section VI presents the surface gravities/temperatures of both horizons, and the redshift map between static worldlines. We conclude with a summary in Section VII. Throughout the paper we work in geometric units $G = c = 1$ and Lorentzian signature $(-, +, +, +)$. Greek indices run over (t, r, θ, ϕ) .

II. STATIC C-METRIC AND CONSTANTS OF MOTION

The static C-metric describes a Schwarzschild black hole uniformly accelerated along the z -axis, the acceleration being provided by a (generally unbalanced) cosmic string or strut on the symmetry axis. In Boyer-Lindquist type of coordinates (t, r, θ, ϕ) , its Lorentzian section is

$$ds^2 = \frac{1}{\Omega^2} \left[-\frac{\Delta}{r^2} dt^2 + \frac{r^2}{\Delta} dr^2 + \frac{r^2}{P} d\theta^2 + Pr^2 \sin^2 \theta d\phi^2 \right] \quad (1)$$

with

$$\Omega(r, \theta) = 1 + \alpha r \cos \theta, \quad (2a)$$

$$\Delta(r) = (r^2 - 2mr) (1 - \alpha^2 r^2), \quad (2b)$$

$$P(\theta) = 1 + 2\alpha m \cos \theta. \quad (2c)$$

Here, m is the black hole mass and α the (constant) proper acceleration. The functions $\Delta(r)$ and $P(\theta)$ are strictly positive in the "static patch"

$$0 \leq \alpha < \frac{1}{2m}, \quad 2m < r < \frac{1}{\alpha}, \quad 0 < \theta < \pi, \quad (3)$$

so that $g_{tt} < 0$, $g_{rr} > 0$, $g_{\theta\theta} > 0$, $g_{\phi\phi} > 0$. Moreover, $\Omega > 0$ throughout the patch since $\Omega_{\min} = 1 - \alpha r > 0$ (attained at $\cos \theta = -1$ and $r < 1/\alpha$).

Equation (1) is locally regular but may possess conical singularities on the symmetry axes ($\theta = 0, \pi$). To encode a possible imbalance one allows the azimuthal coordinate to cover the range $-\pi C < \phi < \pi C$, where the conicity parameter $C > 0$ is constant. After rescaling $\phi \mapsto C\phi$ the metric retains the form equation (1) but with $g_{\phi\phi} \rightarrow PC^2 r^2 \sin^2 \theta$.

Near $\theta = 0$ and $\theta = \pi$ the ratio of circumference to radius yields the (unsigned) conical deficits

$$\delta_N = 2\pi [1 - C(1 + 2\alpha m)], \quad \delta_S = 2\pi [1 - C(1 - 2\alpha m)].$$

Choosing $C_N = 1/(1 + 2\alpha m)$ regularizes the north axis and leaves a cosmic string of tension $\mu_S = \delta_S/(8\pi)$ on

the south axis; conversely $C_S = 1/(1 - 2\alpha m)$ regularizes the south axis. We keep C explicit because it enters the impact parameter of photons and the axial angular momentum of massive particles.

Regarding the horizons, the stationary Killing field $\xi_{(t)} = \partial_t$ becomes null at $g_{tt} = 0$, i.e. at the zeros of $\Delta(r)$,

$$\Delta(r) = (r^2 - 2mr) (1 - \alpha^2 r^2) = 0. \quad (4)$$

The roots are $r_H = 2m$, the Schwarzschild (black hole) horizon, and $r_A = 1/\alpha$, the acceleration horizon. The condition $\alpha < 1/(2m)$ guarantees $r_H < r_A$, so the region $2m < r < 1/\alpha$ is static.

A. Killing fields and conserved quantities

The C-metric (1) is stationary and axisymmetric, admitting the Killing vectors

$$\xi_{(t)} = \partial_t, \quad \xi_{(\phi)} = \partial_\phi. \quad (5)$$

For a geodesic $x^\mu(s)$ with $u^\mu = dx^\mu/ds$ the associated first integrals are

$$E := -\xi_{(t)} \cdot u = -g_{tt} \dot{t} = \frac{\Delta}{\Omega^2 r^2} \dot{t}, \quad (6a)$$

$$L := \xi_{(\phi)} \cdot u = g_{\phi\phi} \dot{\phi} = \frac{PC^2 r^2 \sin^2 \theta}{\Omega^2} \dot{\phi}, \quad (6b)$$

with $E > 0$ for future directed motion. The normalization $g_{\mu\nu} u^\mu u^\nu = \varepsilon$ distinguishes timelike ($\varepsilon = -1$) and null ($\varepsilon = 0$) trajectories.

III. NULL GEODESICS IN DIMENSIONLESS VARIABLES

Throughout this section we specialise to null geodesics ($g_{\mu\nu} u^\mu u^\nu = 0$) and convert all quantities to dimensionless form by scaling with the black hole mass m :

$$\xi := \frac{r}{m}, \quad \tau := \frac{t}{m}, \quad a := \alpha m, \quad \varepsilon := mE, \quad \ell := \frac{L}{m}, \quad \ell_C := \frac{\ell}{C}, \quad (7)$$

where $a = \alpha m \in (0, \frac{1}{2})$. The metric functions are

$$\Omega(\xi, \theta) = 1 + a\xi \cos \theta, \quad (8a)$$

$$\Delta(\xi) = \xi^2 \left(1 - \frac{2}{\xi} \right) (1 - a^2 \xi^2), \quad (8b)$$

$$P(\theta) = 1 + 2a \cos \theta, \quad (8c)$$

and the static patch is $2 < \xi < 1/a$, $0 < \theta < \pi$.

A. Mino-type parameter, Hamilton-Jacobi separation, and K -scaling

Because null motion is conformally invariant, we work with the conformal metric $\hat{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$. Let $\hat{\lambda}$ be an affine parameter for $\hat{g}_{\mu\nu}$.

With the additive ansatz $S = -\varepsilon\tau + \ell_C\phi + S_r(\xi) + S_\theta(\theta)$, the conformal Hamilton-Jacobi equation $\hat{g}^{\mu\nu}\partial_\mu S\partial_\nu S = 0$ separates,

$$(S'_r)^2 - \xi^4 \varepsilon^2 = -\Delta(\xi)K, \quad (9a)$$

$$(S'_\theta)^2 + \frac{\ell_C^2}{\sin^2 \theta} = P(\theta)K, \quad (9b)$$

where primes denote ordinary derivatives and $K \geq 0$ is a separation constant (the analogue of Carter's constant in Kerr; nonnegativity follows from equation (9b) since both terms are nonnegative in the static patch). Introducing the impact parameter $b := L/E$ and $Q := K/E^2$, the first order null equations become

$$\left(\frac{d\xi}{d\tilde{\lambda}}\right)^2 = \varepsilon^2 \left[1 - \frac{\Delta(\xi)}{\xi^4} Q\right], \quad (10a)$$

$$\left(\frac{d\theta}{d\tilde{\lambda}}\right)^2 = \frac{1}{\xi^4} \left[P(\theta)K - \frac{\ell_C^2}{\sin^2 \theta}\right], \quad (10b)$$

$$\frac{d\phi}{d\tilde{\lambda}} = \frac{\ell_C}{P(\theta)C\xi^2 \sin^2 \theta} \quad (10c)$$

$$\frac{d\tau}{d\tilde{\lambda}} = \frac{\xi^2 \varepsilon}{\Delta(\xi)}. \quad (10d)$$

Radial turning points satisfy $\xi^4 = \Delta(\xi)Q$; angular turning points satisfy

$$Q = \frac{b^2}{C^2 P(\theta) \sin^2 \theta}. \quad (11)$$

To complete decouple the above equations, we define a Mino-type parameter λ by

$$d\lambda = \frac{1}{\xi^2} d\tilde{\lambda}. \quad (12)$$

For later convenience, additionally we absorb $K > 0$ by a constant reparametrisation and scaled constants:

$$\lambda_K := \sqrt{K}\lambda, \quad \hat{\varepsilon} := \frac{\varepsilon}{\sqrt{K}}, \quad \hat{\ell} := \frac{\ell_C}{\sqrt{K}}. \quad (13)$$

The degenerate case $K = 0$ forces (from equation (9b)) $\ell_C = 0$ and $S'_\theta = 0$, i.e. $\theta = \text{const}$ and $\dot{\phi} = 0$: these are radial null rays at arbitrary fixed latitude. Such rays should be treated in the unscaled Mino system by setting $K = 0 = \ell_C$ before the K -rescaling (13). The first order equations become

$$\left(\frac{d\xi}{d\lambda_K}\right)^2 = \xi^4 \hat{\varepsilon}^2 - \Delta(\xi), \quad (14a)$$

$$\left(\frac{d\theta}{d\lambda_K}\right)^2 = P(\theta) - \frac{\hat{\ell}^2}{\sin^2 \theta}, \quad (14b)$$

$$\frac{d\phi}{d\lambda_K} = \frac{\hat{\ell}}{C P(\theta) \sin^2 \theta}, \quad \frac{d\tau}{d\lambda_K} = \frac{\xi^4 \hat{\varepsilon}}{\Delta(\xi)}. \quad (14c)$$

Radial turning points are the positive roots of $\xi^4 \hat{\varepsilon}^2 - \Delta(\xi) = 0$, and polar motion is allowed where

$$P(\theta) - \frac{\hat{\ell}^2}{\sin^2 \theta} \geq 0 \iff \sin^2 \theta \geq \frac{\hat{\ell}^2}{P(\theta)}. \quad (15)$$

Unless $a = 0$, the reflection symmetry $\theta \mapsto \pi - \theta$ is broken.

B. Radial quartic and Weierstrass reduction

From equation (14a) we define

$$R(\xi) := \xi^4 \hat{\varepsilon}^2 - \Delta(\xi) \quad (16)$$

with $\Delta(\xi)$ given in equation (8c). Expanding $R(\xi)$ yields the explicit quartic

$$R(\xi) = (\hat{\varepsilon}^2 + a^2) \xi^4 - 2a^2 \xi^3 - \xi^2 + 2\xi. \quad (17)$$

Zeros of $R(\xi)$ determine radial turning points. Following [18, 21], see also the appendix, we solve the radial equation using the Biermann-Weierstrass approach. For any initial value ξ_0 the solution then reads

$$\xi(\lambda_K) = \xi_0 + \Pi, \quad (18)$$

where

$$\Pi = \frac{R'_0(\wp(\lambda_K) - \frac{1}{24}R''_0) - 2\varepsilon_r \sqrt{R_0} \wp'(\lambda_K) + \frac{1}{12}R_0 R'''_0}{4(\wp(\lambda_K) - \frac{1}{24}R''_0)^2 - \frac{1}{24}R_0 R''''_0}. \quad (19)$$

Here the index 0 indicates evaluation at the initial value ξ_0 , e.g. $R'_0 = R'(\xi_0)$, and the primes indicate a derivative of R with respect to ξ . The symbol $\varepsilon_r = \pm 1$ encodes the initial propagation direction: $\varepsilon_r = +1$ ($\varepsilon_r = -1$) for initially increasing (decreasing) radius. The Weierstrass function $\wp(\lambda_K) = \wp(\lambda_K; g_2, g_3)$ depends on the invariants g_2, g_3 , that are independent of the chosen initial value. The general expression for a generic quartic can be found in the appendix. For the radial case at hand they are

$$g_2^{(r)} = \frac{1}{12} + a^2, \quad (20a)$$

$$g_3^{(r)} = \frac{1}{216} - \frac{a^2}{6} - \frac{\hat{\varepsilon}^2}{4}. \quad (20b)$$

Because of the K -scaling, these invariants are K independent. In the Schwarzschild case $a = 0$ this reduces to the known expressions [18].

The manifestly real general solution above has the advantage that we do not need to solve the quartic to find any turning points; knowledge of the K -scaled energy is sufficient. However, for the particular case that ξ_0 is a turning point, $R(\xi_0) = R_0 = 0$, the general solution (19) simplifies significantly to

$$\Pi = \frac{R'_0}{4\wp(\lambda_K) - R''_0/6}. \quad (21)$$

Since $R(0) = 0$, in the case that $\xi_0 = 0$ is a physically feasible initial value the solution can be rewritten in the compact form

$$\xi(\lambda_K) = \frac{1}{2\wp(\lambda_K) + \frac{1}{6}}. \quad (22)$$

In addition, spherical photon orbits occur when $R(\xi)$ has a positive double root, i.e. $R(\xi_{\text{ph}}) = R'(\xi_{\text{ph}}) = 0$, which yields the known radius $\xi_{\text{ph}}(a) = 6/(1 + \sqrt{1 + 12a^2})$ used later. See Figure 2 as an example of such an orbit.

C. Polar equation and Weierstrass reduction

Let $v := \cos \theta$, so that $P(\theta) = 1 + 2av$ and $\sin^2 \theta = 1 - v^2$. From equation (14b) and using $dv/d\lambda_K = -\sin \theta d\theta/d\lambda_K$ we obtain

$$\left(\frac{dv}{d\lambda_K}\right)^2 = (1 + 2av)(1 - v^2) - \hat{\ell}^2 =: \Theta_v(v). \quad (23)$$

Turning latitudes satisfy

$$(1 + 2av_*)(1 - v_*^2) = \hat{\ell}^2,$$

i.e. $P(\theta_*) \sin^2 \theta_* = \hat{\ell}^2$. Since $0 < a < \frac{1}{2}$ one has $P(\theta) > 0$ on $(0, \pi)$, so polar motion is allowed precisely where the right hand side of equation (23) is non-negative. See Figure 4 as an example.

The Biermann-Weierstrass solution of equation (23) is analogous to the radial case. We find

$$v(\lambda_K) = v_0 + \chi, \quad (24)$$

where

$$\chi = \frac{\Theta'_0(\wp(\lambda_K) - \frac{1}{24}\Theta'_0) - 2\varepsilon_v \sqrt{\Theta_0}\wp'(\lambda_K) + \frac{1}{12}\Theta_0\Theta'''_0}{4(\wp(\lambda_K) - \frac{1}{24}\Theta'_0)^2 - \frac{1}{24}\Theta_0\Theta'''_0}.$$

Here v_0 is the initial values of $v = \cos \theta$, primes are derivatives with respect to v , and an index 0 indicates evaluation at the initial value, e.g. $\Theta'_0 = \Theta'_v(v_0)$. As in the radial case, $\varepsilon_v = +1$ ($\varepsilon_v = -1$) for initially increasing (decreasing) $\cos \theta$. The invariants of the Weierstrass $\wp$ function are

$$g_2^{(\theta)} = \frac{1}{12} + a^2, \quad (25a)$$

$$g_3^{(\theta)} = \frac{1}{216} + a^2 \left(\frac{\hat{\ell}^2}{4} - \frac{1}{6} \right). \quad (25b)$$

Notice $g_2^{(\theta)}$ coincides with the radial $g_2^{(r)}$, cf. equation (20b). As before, the general equation simplifies significantly, if we choose a turning point as the initial value. In this case,

$$\chi = \frac{\Theta'_0}{4\wp(\lambda_K) - \Theta'_0/6}. \quad (26)$$

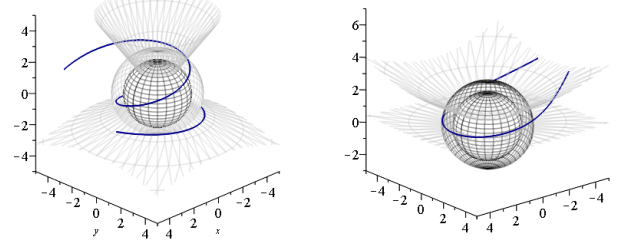


FIG. 1. Null trajectories in the static C-metric for $a = 0.375$ and $C = 1$. Left: Region I (equator-crossing) example with $\hat{e} = E/\sqrt{K} = 0.07$ and $\hat{\ell} = L/\sqrt{K} = 0.75$ ($\hat{\ell}^2 = 0.5625$). Right: Region II (hemisphere-confined) example with $\hat{e} = E/\sqrt{K} = 0.05$ and $\hat{\ell} = L/\sqrt{K} = 1.02$ ($\hat{\ell}^2 = 1.0404$), and $\hat{e}_{\text{crit}}^2 = \Delta(\xi_{\text{ph}})/\xi_{\text{ph}}^4 \approx 6.37 \times 10^{-3}$. In both panels, $\hat{e}_{\text{crit}}^2 \leq \hat{e}^2$, hence the radial motion has a single turning point located near $\xi_{\text{ph}} \approx 2.273$. The black sphere marks the black hole horizon at $\xi = 2$. A faint plane or ring at $\xi_A = 1/a \approx 2.667$ for acceleration horizon. Translucent cones show the polar turning latitudes obtained from $P(\theta) \sin^2 \theta = \hat{\ell}^2$. The left trajectory crosses the equator; the right trajectory remains entirely in the northern hemisphere. Coordinates: $(x, y, z) = (\xi \sin \theta \cos \varphi, \xi \sin \theta \sin \varphi, \xi \cos \theta)$.

D. Azimuth and coordinate time as Weierstrass quadratures

The remaining first integrals in equation (14c) involve rational functions of ξ or θ . After substituting the Weierstrass solutions for the polar and radial sectors they reduce to integrals of the type $\int dz (\wp(z) - c)^{-1}$. We use the standard antiderivative:

$$\begin{aligned} \mathcal{F}_c^{(\cdot)}(z) &:= \int^z \frac{dw}{\wp(w; g_2^{(\cdot)}, g_3^{(\cdot)}) - c} \\ &= \frac{1}{\wp(z_c^{(\cdot)}; g_2^{(\cdot)}, g_3^{(\cdot)})} \left[\log \frac{\sigma(z - z_c^{(\cdot)})}{\sigma(z + z_c^{(\cdot)})} + 2z \zeta(z_c^{(\cdot)}) \right], \end{aligned} \quad (27)$$

where $z_c^{(\cdot)}$ is any preimage satisfying $\wp(z_c^{(\cdot)}; g_2^{(\cdot)}, g_3^{(\cdot)}) = c$ (the result is independent of that choice up to periods).

(i) Azimuth $\phi(\lambda_K)$. A partial fraction decomposition of the azimuthal equation reads

$$\frac{d\phi}{d\lambda_K} = \frac{\hat{\ell}}{C} \sum_{i=1}^3 \frac{\beta_i}{v - c_i}, \quad (28)$$

where

$$c_1 = 1, \quad c_2 = -1, \quad c_3 = -\frac{1}{2a}, \quad (29)$$

and

$$\beta_1 = \frac{1}{2 - 4a}, \quad \beta_2 = \frac{-1}{2 + 4a}, \quad \beta_3 = \frac{2a}{(4a^2 - 1)}. \quad (30)$$

Following [21], we can rewrite the decomposition in terms of the Weierstrass functions by noting that $p_c = \frac{1}{v - c}$ as

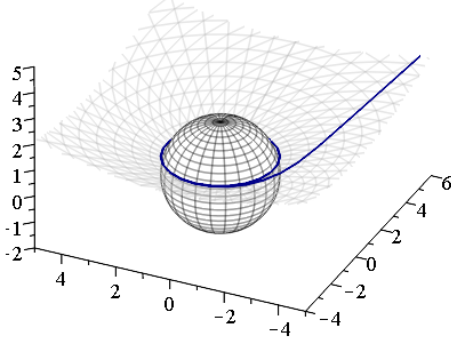


FIG. 2. Orbit which comes from infinity and spirals on an off-equatorial circular orbit. The radius of the circular orbit was chosen as $\xi = 2.1m$. It lies on a cone with $\cos \theta \approx 0.3$.

a function of λ_K satisfies the differential equation

$$\begin{aligned} \frac{dp_c}{d\lambda_K} &= -p_c^2 \sqrt{\Theta_v(v)} \\ &= -\sqrt{\Theta_v(c)p_c^3 + \Theta'_v(c)p_c^2 + \Theta''_v(c)p_c + \Theta'''_v(c)}. \end{aligned} \quad (31)$$

Therefore, $p(\lambda_K)$ can be written in terms of $\wp$ in the same way as for the radial and polar case. This gives

$$\frac{1}{v - c_i} = \frac{1}{v_0 - c_i} + \frac{\frac{1}{4}P'_i}{\wp(\lambda_K) - \frac{1}{24}P'_i}, \quad (32)$$

assuming that v_0 is a turning point, and $P'_i = -\Theta'_0/(v_0 - c_i)^2$, $P''_i = \Theta''_0 - 6\Theta'_0/(v_0 - c_i)$. The solution for ϕ can then be written as

$$\phi(\lambda_K) = \phi_0 + \frac{\hat{\ell}\lambda_K}{CP(\theta_0)\sin^2\theta_0} - \sum_{i=1}^3 \frac{\Theta'_0\beta_i\mathcal{F}_{d_i}(\lambda_K)}{4(v_0 - c_i)^2}, \quad (33)$$

where

$$d_i = \frac{P''_i}{24} = \frac{\Theta''_0}{24} - \frac{\Theta'_0}{4(v_0 - c_i)}, \quad (34)$$

and each primitive uses the polar invariants and preimages, i.e. $\wp(z_{d_i}; g_2^{(\theta)}, g_3^{(\theta)}) = d_i$.

(ii) Coordinate time $\tau(\lambda_K)$. We find for the radial sector

$$\frac{d\tau}{d\lambda_K} = \frac{\hat{\ell}}{a^2} \left[-1 + \sum_{i=1}^3 \frac{\gamma_i}{r - h_i} \right], \quad (35)$$

with coefficients

$$\gamma_1 = \frac{1}{2a(2a-1)}, \quad \gamma_2 = \frac{1}{2a(2a+1)}, \quad \gamma_3 = \frac{8a^2}{1-4a^2}, \quad (36)$$

and h_i given by the horizons

$$h_1 = \xi_A = \frac{1}{a}, \quad h_2 = -\xi_A, \quad h_3 = \xi_H = 2. \quad (37)$$

Assuming that ξ_0 is a turning point, we then obtain the solution for the dimensionless coordinate time $\tau = t/m$ as

$$\tau(\lambda_K) = \tau_0 + \frac{\xi_0 \hat{\ell}}{\Delta(\xi_0)} \lambda_K - \frac{\hat{\ell}}{a^2} \sum_{i=1}^3 \frac{R'_0 \gamma_i \mathcal{F}_{H_i}(\lambda_K)}{4(r_0 - h_i)^2}, \quad (38)$$

where

$$H_i = \frac{R''_0}{24} - \frac{R'_0}{4(r_0 - h_i)}, \quad (39)$$

and each primitive uses the radial invariants and preimages, i.e. $\wp(z_{H_i}; g_2^{(r)}, g_3^{(r)}) = H_i$.

Equations (21), (26), (33), and (38) provide a complete analytic solution for null geodesics in the static C-metric throughout the subextremal domain $a \in (0, \frac{1}{2})$. In Figure 1, we plotted two examples for a very large value of a , to clearly show the impact of the acceleration parameter. The two horizons are very close together, and it is clearly visible how the flyby orbits deviate from plane motion. Also, Figure 2 shows an example of an orbit that comes from infinity and spirals on an off-equatorial circular orbit.

In summary, qualitative structure and turning points are as follows:

- i) Radial motion. With $R(\xi) := \xi^4 \hat{\ell}^2 - \Delta(\xi)$ [equation (14a)], R is a quartic with vanishing constant term ($R(0) = 0$). Real, positive roots classify fly-by, plunge, and spherical photon orbits. Spherical (constant- ξ) orbits occur at double roots: $R(\xi_*) = 0 = R'(\xi_*)$.
- ii) Polar motion. From equation (14b) and since $P(\theta) = 1 + 2a \cos \theta > 0$ for $0 < a < \frac{1}{2}$, motion is allowed precisely where $P(\theta) \sin^2 \theta \geq \hat{\ell}^2$. Turning latitudes satisfy $P(\theta_*) \sin^2 \theta_* = \hat{\ell}^2$. Because $P(\theta)$ is not symmetric under $\theta \mapsto \pi - \theta$ for $a \neq 0$, the allowed latitude band is generally not symmetric about the equator.
- iii) Genus. The radial and polar motions obey independent Weierstrass equations, generally with different invariants. Therefore, each sector is elliptic (genus 1), and the full null motion is quasi periodic on the 2-torus $\mathbb{T}^2$. The observables $\phi(\lambda_K)$ and $\tau(\lambda_K)$ are sums of elliptic (third kind) integrals of the two arguments. A single hyperelliptic (genus 2) curve does not arise unless one eliminates λ_K to express, say, $\theta(r)$, which artificially mixes the two square roots. When θ is constant (cone orbits at $\theta = \theta_\gamma(a)$) the polar phase is fixed and the quadratures reduce to a single Weierstrass argument; $\phi(\lambda_K)$ is then linear in λ_K .

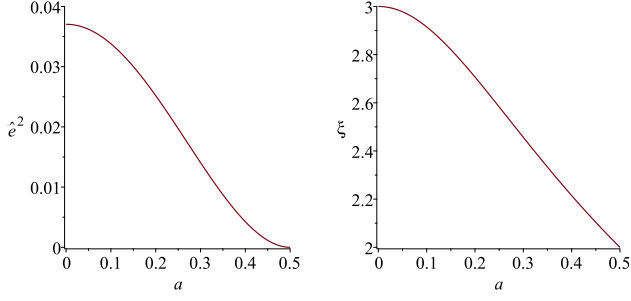


FIG. 3. Radial motion and photon surface. Left: the critical ratio $\hat{\epsilon}^2/K = \Delta(\xi_{ph})/\xi_{ph}^4$ as a function of a . Below the curve $R(\xi) = \hat{\epsilon}^2\xi^4 - \Delta K$ has radial turning points; on the curve the orbit is spherical (double root); above it there are no turning points. Right: photon surface radius $\xi_{ph}(a) = 6/[1 + \sqrt{1 + 12a^2}]$.

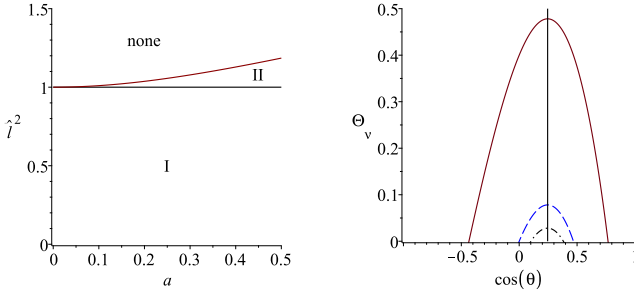


FIG. 4. Polar motion of null geodesics. Left: classification in the plane $(a, \hat{\ell})$. The black line is $\hat{\ell} = 1$ (turning point at the equator). The red line is the fixed cone bound $\hat{\ell} = P(\theta_\gamma) \sin^2 \theta_\gamma$ with equation (45), $\cos \theta_\gamma = 2a/[1 + \sqrt{1 + 12a^2}]$; above it no null geodesic motion is allowed. Right: $\Theta_v(v)$ vs $v = \cos \theta$ for $a = 0.3$ and $\hat{\ell} \in \{0.6, 1.0, 1.05\}$. Region I ($\hat{\ell} < 1$) oscillates about the equator; region II ($1 < \hat{\ell} < P(\theta_\gamma) \sin^2 \theta_\gamma$) is confined to one hemisphere. The vertical black line indicates the value of the photon cone $\cos \theta_\gamma$, that is an extremum of Θ_v independent of the value of $\hat{\ell}$.

IV. PHOTON SURFACES AND ORBITAL CONES

A. Photon cones and breakdown of equatorial symmetry

We continue with the K -scaled first integrals introduced above. In particular, from equation (14b) we write

$$\left(\frac{d\theta}{d\lambda_K}\right)^2 = \Theta(\theta) := P(\theta) - \frac{\hat{\ell}^2}{\sin^2 \theta}. \quad (40)$$

with $P(\theta) = 1 + 2a \cos \theta$, $0 < a < \frac{1}{2}$. A null geodesic remains at a fixed latitude $\theta = \theta_\gamma$ iff (i) $\Theta(\theta_\gamma) = 0$ and (ii) the θ sector of Euler-Lagrange equation is satisfied at $\dot{\theta} = 0$. Condition (i) fixes the (scaled) axial momentum to

$$\hat{\ell}^2 = P(\theta_\gamma) \sin^2 \theta_\gamma. \quad (41)$$

Therefore, the sign of $\hat{\ell}$ sets the azimuthal orientation: $\hat{\ell} > 0$ ($\hat{\ell} < 0$) corresponds to increasing (decreasing) ϕ .

For (ii) we use the θ equation in the conformal metric $\hat{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$. At $\dot{\theta} = 0$ (but with r arbitrary) the only θ -dependence entering the Lagrangian is through $\hat{g}_{\phi\phi} = PC^2 r^2 \sin^2 \theta$, hence the equation reduces to

$$\partial_\theta [P(\theta) \sin^2 \theta] \Big|_{\theta=\theta_\gamma} = 0. \quad (42)$$

Equivalently, one may differentiate $\Theta(\theta) = P(\theta) - \hat{\ell}^2/\sin^2 \theta$ and, using $\Theta(\theta_\gamma) = 0 \Leftrightarrow \hat{\ell}^2 = P(\theta_\gamma) \sin^2 \theta_\gamma$, impose $\Theta'(\theta_\gamma) = 0$; this yields exactly the same condition as equation (42). Writing $v = \cos \theta$ gives (for $0 < a < \frac{1}{2}$)

$$\frac{d}{dv} [(1 + 2av)(1 - v^2)] = 0 \implies 3av_\gamma^2 + v_\gamma - a = 0. \quad (43)$$

The physically admissible root (the only one with $|v_\gamma| \leq 1$) is

$$\cos \theta_\gamma = v_\gamma = \frac{-1 + \sqrt{1 + 12a^2}}{6a} = \frac{2a}{1 + \sqrt{1 + 12a^2}}, \quad (44)$$

where $a = \alpha m$ is dimensionless. Its small- a expansion is

$$\cos \theta_\gamma = a - 3a^3 + \mathcal{O}(a^5) \quad (a \ll 1), \quad (45)$$

so $\theta_\gamma \rightarrow \frac{\pi}{2}$ as $a \rightarrow 0$. Along the cone, equation (41) fixes the magnitude of $\hat{\ell}$, and hence $d\phi/d\lambda_K = \hat{\ell}/[P(\theta_\gamma) \sin^2 \theta_\gamma]$ is constant (its sign is the azimuthal orientation fixed by cone condition). Thus, except in the Schwarzschild limit, equatorial symmetry is broken: non-axial constant latitude null geodesics lie on the photon cone $\theta = \theta_\gamma(a)$. The symmetry axes ($\theta = 0, \pi$) remain excluded by the conical defect on one half-axis (see section II). To be more precise, along the cone the azimuthal constant is fixed in magnitude by equation (41), thus

$$\hat{\ell}^2 = \frac{2(2 + \sqrt{1 + 12a^2})^2}{9(1 + \sqrt{1 + 12a^2})} = 1 + a^2 - 2a^4 + \mathcal{O}(a^6), \quad (46)$$

while the sign of $\hat{\ell}$ (equivalently, of L) sets the sense of revolution via $d\phi/d\lambda_K = \hat{\ell}/[P(\theta_\gamma) \sin^2 \theta_\gamma] = \text{const}$.

B. Photon surface (constant radius) and critical radius

We repeat the radial first integral here for convenience

$$\left(\frac{d\xi}{d\lambda_K}\right)^2 = R(\xi) := \xi^4 \hat{\epsilon}^2 - \Delta(\xi), \quad (47)$$

cf. equations (14a) and (8c). A ξ -constant null orbit requires a double root of R :

$$R(\xi_{ph}) = 0, \quad R'(\xi_{ph}) = 0. \quad (48)$$

Eliminating $\hat{\epsilon}^2$ between the two conditions gives an equation independent of the constants of motion,

$$\frac{4\Delta(\xi)}{\xi} = \Delta'(\xi), \implies \xi_{ph}(a) = \frac{6}{1 + \sqrt{1 + 12a^2}}. \quad (49)$$

Restoring dimensions ($r = m\xi$, $a = \alpha m$), gives the (aspherical) photon surface radius inside the static patch

$$r_{\text{ph}}(\alpha) = \frac{6m}{1 + \sqrt{1 + 12\alpha^2 m^2}},$$

which expands as

$$r_{\text{ph}} = 3m \left[1 - 3(\alpha m)^2 + 18(\alpha m)^4 + \mathcal{O}((\alpha m)^6) \right]. \quad (50)$$

Energy parameter on the photon surface. From $R(\xi_{\text{ph}}) = 0$ we have

$$\hat{e}^2 = \frac{\Delta(\xi_{\text{ph}})}{\xi_{\text{ph}}^4}. \quad (51)$$

Using the photon surface condition (49) one can show that on $\xi = \xi_{\text{ph}}$ the factor $(1 - a^2 \xi^2)$ equals $\xi - 2$, hence

$$\Delta(\xi_{\text{ph}}) = \xi_{\text{ph}}(\xi_{\text{ph}} - 2)^2. \quad (52)$$

Substituting (52) into (51) gives the compact expression

$$\hat{e}^2 = \frac{(\xi_{\text{ph}} - 2)^2}{\xi_{\text{ph}}^3}. \quad (53)$$

Expanding for $a \ll 1$ (so $\xi_{\text{ph}} = 3 - 9a^2 + 54a^4 + \dots$) yields

$$\hat{e}^2 = \frac{1}{27} - \frac{(\alpha m)^2}{3} + (\alpha m)^4 + \mathcal{O}((\alpha m)^6). \quad (54)$$

as we see in Figure 3 (left). Note that, in unscaled constants this corresponds to $K = \varepsilon^2 \xi_{\text{ph}}^4 / \Delta(\xi_{\text{ph}})$ and hence $\hat{e}^2 = \varepsilon^2 / K$, in agreement with the spherical orbit relations used elsewhere in the paper.

C. Impact parameters

Let $b := L/E$ be the physical impact parameter and $\beta := \ell/\varepsilon = b/m$ its dimensionless counterpart.

(i) Circular ring on the photon cone ($\xi = \xi_{\text{ph}}$, $\theta = \theta_\gamma$). At fixed latitude the polar first integral enforces

$$K = \frac{\ell_C^2}{P(\theta) \sin^2 \theta}, \quad (55)$$

while the radial double root condition gives

$$K = \frac{\varepsilon^2 \xi^4}{\Delta(\xi)}. \quad (56)$$

Combining these at $(\xi, \theta) = (\xi_{\text{ph}}, \theta_\gamma)$ and restoring dimensions yields

$$b_{\text{ring}}^2(a, C) = C^2 P(\theta_\gamma) \sin^2 \theta_\gamma \frac{r_{\text{ph}}^4}{\Delta(r_{\text{ph}})} \quad (57)$$

where $P(\theta_\gamma) = 1 + 2a \cos \theta_\gamma$ and $\cos \theta_\gamma$ is given by equation (45). For $a \ll 1$, one has (using $\cos \theta_\gamma = a - 3a^3 + \mathcal{O}(a^5)$)

$$P(\theta_\gamma) \sin^2 \theta_\gamma = 1 + a^2 - 2a^4 + \mathcal{O}(a^6), \quad (58)$$

and using equation (50) for $a \ll 1$ one finds

$$b_{\text{ring}} = 3\sqrt{3}mC \left[1 + 5a^2 + \mathcal{O}(a^4) \right]. \quad (59)$$

Or equivalently, eliminating θ_γ and ξ_{ph} and using equations (46) and (54) one finds a closed form,

$$\frac{b_{\text{ring}}}{mC} = \frac{2\sqrt{3}(2 + \sqrt{1 + 12a^2})}{(1 + \sqrt{1 + 12a^2})(2 - \sqrt{1 + 12a^2})}, \quad (60)$$

which expands also as $b_{\text{ring}} = 3\sqrt{3}mC[1 + 5a^2 + \mathcal{O}(a^4)]$.

The C -dependence enters only through the azimuthal constant (via $g_{\phi\phi} \propto C^2$); for a fixed cone latitude the true capture/escape threshold is b_{ring} , not a radius-only proxy.

It is sometimes convenient to factor out the inclination/conicity and quote the radius-only scale impact parameter b_{rad} , which equals the true threshold in spherical symmetry but serves only as a baseline in the anisotropic C-metric:

(ii) Radius-only proxy. The radius-only impact parameter scale is given by

$$b_{\text{rad}}(r_{\text{ph}}) := \frac{r_{\text{ph}}^2}{\sqrt{\Delta(r_{\text{ph}})}}, \quad (61)$$

which is obtained by ignoring the latitude/conicity factor in the angular barrier. In spherically symmetric spacetimes (e.g., Schwarzschild, where $P \equiv 1$ and $C \equiv 1$) this coincides with the true critical impact parameter for photon capture, because the spherical photon orbit fixes a single radius r_{ph} and the effective potential depends only on r . In the static C-metric the spherical photon surface still sits at a single radius $r_{\text{ph}}(\alpha)$ [equation (49)], but the angular sector is anisotropic through $P(\theta) = 1 + 2a \cos \theta$ and the conical parameter C . As a result, the true threshold depends on the latitude of motion and reads (our equation (57))

$$\begin{aligned} b_{\text{ring}}^2(a, C; \theta_\gamma) &= C^2 P(\theta_\gamma) \sin^2 \theta_\gamma \frac{r_{\text{ph}}^4}{\Delta(r_{\text{ph}})} \\ &= \underbrace{C^2 P(\theta_\gamma) \sin^2 \theta_\gamma}_{\text{inclination/conicity}} \underbrace{b_{\text{rad}}^2}_{\text{radius-only scale}}. \end{aligned} \quad (62)$$

showing explicitly how orientation (θ_γ) and conicity (C) enter only through b_{ring} , whereas b_{rad} captures the radius-only part common to all latitudes¹. Equivalently, in

¹ Since $b_{\text{ring}} = b_{\text{rad}} \sqrt{P(\theta_\gamma) \sin^2 \theta_\gamma C}$, their small- a series are consistent: $\sqrt{P(\theta_\gamma) \sin^2 \theta_\gamma} = 1 + \frac{1}{2}a^2 + \mathcal{O}(a^4)$, hence the net $+5a^2$ coefficient in b_{ring} follows from $\frac{9}{2} + \frac{1}{2} = 5$.

dimensionless form with $\beta := b/m$, one has

$$\begin{aligned}\beta_{\text{rad}}^2 &= \frac{b_{\text{rad}}^2}{m^2} = \frac{\xi_{\text{ph}}^4}{\Delta(\xi_{\text{ph}})} =: Q_{\text{ph}}(a), \\ \beta_{\text{ring}}^2 &= C^2 P(\theta_\gamma) \sin^2 \theta_\gamma Q_{\text{ph}}(a),\end{aligned}\quad (63)$$

therefore, $Q_{\text{ph}}(a)$ is the orientation-independent baseline set by the spherical photon surface, while the factor $C^2 P(\theta_\gamma) \sin^2 \theta_\gamma$ encodes inclination and conical deficit. Thus:

- In Schwarzschild ($a = 0$, $C = 1$) one recovers $b_{\text{ring}} = b_{\text{rad}} = 3\sqrt{3}m$.
- In the static C-metric, b_{rad} remains a useful reference scale that isolates the radial physics of the photon surface, but it does not by itself decide capture/escape; thus the latitude-aware threshold is b_{ring} .

For small a , by using equations (49)-(50) one finds

$$b_{\text{rad}} = 3\sqrt{3}m \left[1 + \frac{9}{2}a^2 + \mathcal{O}(a^4) \right]. \quad (64)$$

Note that: (i) Constant latitude null geodesics at the cone angle equation (45) are elementary in Hong-Teo coordinates (fixed $x = \cos \theta$). (ii) The constant radius condition (49) matches optical metric arguments for photon surfaces in accelerated black holes; see, e.g., [15, 17] for complementary perspectives.²

V. ASTROPHYSICAL IMPLICATIONS

We specialise to a static observer at (ξ_o, θ_o) inside the static patch $2 < \xi_o < 1/a$ (with $a = \alpha m \in (0, \frac{1}{2})$). An orthonormal tetrad aligned with the coordinate axes is

$$e_{(t)} = \frac{\Omega \xi_o}{\sqrt{\Delta_o}} \partial_t, \quad (65a)$$

$$e_{(r)} = \frac{\sqrt{\Delta_o}}{\Omega \xi_o} \partial_r, \quad (65b)$$

$$e_{(\theta)} = \frac{\sqrt{P_o}}{\Omega \xi_o} \partial_\theta, \quad (65c)$$

$$e_{(\phi)} = \frac{\Omega}{C \xi_o \sin \theta_o \sqrt{P_o}} \partial_\phi, \quad (65d)$$

with $\Omega = 1 + a\xi \cos \theta$, $P = 1 + 2a \cos \theta$, and $\Delta = \xi^2(1 - \frac{2}{\xi})(1 - a^2\xi^2)$. A subscript o means “evaluate at the observer”. One can check that $g(e_{(t)}, e_{(t)}) = -1$ and $g(e_{(i)}, e_{(j)}) = \delta_{ij}$.

² In Hong-Teo coordinates (x, y) one has $x = \cos \theta$ and, in the Ricci-flat case, $r = -(\alpha y)^{-1}$. Fixed x null orbits arise for specific ratios of the conserved quantities; see [15].

For a photon with conserved (E, L) and separation constant K (section III), we define the screen coordinates (α, β) in the observer’s sky by

$$\iota := -\frac{k_{(\phi)}}{k_{(t)}}, \quad \vartheta := +\frac{k_{(\theta)}}{k_{(t)}}, \quad k_{(a)} := g_{\mu\nu} k^\mu e_{(a)}^\nu. \quad (66)$$

Using equation (65d) together with the first integrals in (14) gives

$$\iota = \frac{\sqrt{\Delta_o}}{C \xi_o^2 \sin \theta_o \sqrt{P_o}} \frac{L}{E}, \quad (67a)$$

$$\beta = \frac{\sqrt{\Delta_o}}{\xi_o^2} \sqrt{Q - \frac{1}{C^2 P_o \sin^2 \theta_o} \left(\frac{L}{E} \right)^2}, \quad (67b)$$

with $Q := K/E^2$ as in the section III. (Our sign choice makes α increase toward $+e_{(\phi)}$ and β toward $+e_{(\theta)}$.) Adding relations (67a) and (67b) yields the useful identity

$$\iota^2 + \vartheta^2 = \frac{\Delta_o}{\xi_o^2} Q. \quad (68)$$

A. Shadow boundary from the photon surface

From equation (49), the (spherical) photon surface sits at the constant radius

$$\xi_{\text{ph}}(a) = \frac{6}{1 + \sqrt{1 + 12a^2}}. \quad (69)$$

For any spherical null orbit one has (section III)

$$Q = \frac{\xi_{\text{ph}}^4}{\Delta(\xi_{\text{ph}})} =: Q_{\text{ph}}(a), \quad (70)$$

which is independent of L/E . Every generator of the shadow boundary arises from such an orbit, hence satisfies (70). Eliminating L/E between (67a)-(67b) using (70) collapses the locus to

$$\iota^2 + \vartheta^2 = \frac{\Delta(\xi_o)}{\xi_o^4} \frac{\xi_{\text{ph}}(a)^4}{\Delta(\xi_{\text{ph}}(a))}. \quad (71)$$

For any static observer in the static C-metric, the black hole shadow boundary is a circle in the local sky. Its (screen) radius is

$$R_{\text{sh}}(\xi_o, \theta_o; a) = \frac{\sqrt{\Delta(\xi_o)}}{\xi_o^2} \frac{\xi_{\text{ph}}(a)^2}{\sqrt{\Delta(\xi_{\text{ph}}(a))}}, \quad (72)$$

with $\xi_{\text{ph}}(a)$ in equation (69). Neither the centre nor the radius depends on the conicity C . Indeed, as $a \rightarrow 0$ one recovers the Schwarzschild result

$$R_{\text{sh}} = 3\sqrt{3} \sqrt{1 - \frac{2}{\xi_o}} / \xi_o, \quad (73)$$

corresponding to the usual $3\sqrt{3}m$ scaled by the redshift to the observer at ξ_o .

Using equation (72) and the photon surface identity $\Delta(\xi_{\text{ph}}) = \xi_{\text{ph}}(\xi_{\text{ph}} - 2)^2$ (which follows immediately from the double root condition $4\Delta/\xi = \Delta'$ at $\xi = \xi_{\text{ph}}$), the shadow radius can be factorised as

$$R_{\text{sh}}(\xi_o, \theta_o; a) = \frac{\sqrt{(1 - 2/\xi_o)(1 - a^2\xi_o^2)}}{\xi_o} \frac{\xi_{\text{ph}}(a)^{3/2}}{\xi_{\text{ph}}(a) - 2}, \quad (74)$$

where ξ_{ph} is given by equation (69) (or equation (49)). This makes the competing effects explicit: a local redshift factor $\sqrt{1 - a^2\xi_o^2}$ that decreases with a (and vanishes as the acceleration horizon $\xi_A = 1/a$ approaches ξ_o), and a photon-surface factor $\xi_{\text{ph}}^{3/2}/(\xi_{\text{ph}} - 2)$ that increases as $\xi_{\text{ph}} \downarrow 2$ when $a \uparrow 1/2$.

A direct differentiation gives the clean, fully reduced identity

$$\frac{d}{da} \ln R_{\text{sh}} = -\frac{a\xi_o^2}{1 - a^2\xi_o^2} + \frac{a\xi_{\text{ph}}(a)^2}{\xi_{\text{ph}}(a) - 2}. \quad (75)$$

Equivalently, using $1 + 2a^2\xi_{\text{ph}} = (6 - \xi_{\text{ph}})/\xi_{\text{ph}}$, which follows from the photon-surface relation $a^2 = (3 - \xi_{\text{ph}})/\xi_{\text{ph}}^2$, one may write

$$\frac{d}{da} \ln R_{\text{sh}} = -\frac{a\xi_o^2}{1 - a^2\xi_o^2} + \frac{a\xi_{\text{ph}}(a)[6 - \xi_{\text{ph}}(a)]}{[\xi_{\text{ph}}(a) - 2][1 + 2a^2\xi_{\text{ph}}(a)]}.$$

In this form one sees that the photon-surface contribution is positive for $2 < \xi_{\text{ph}} < 3$, while the local term is always negative on $0 < a < 1/\xi_o$. Their competition yields the following:

- Small- a behaviour. Expanding (72) and (69) one finds

$$\frac{R_{\text{sh}}(a)}{R_{\text{sh}}(0)} = 1 + \frac{1}{2}(9 - \xi_o^2)a^2 + \mathcal{O}(a^4). \quad (76)$$

Hence R_{sh} initially increases with a if $2 < \xi_o < 3$, is flat at $\xi_o = 3$, and decreases if $\xi_o > 3$.

- Global trend on the allowed range $0 < a < 1/\xi_o$. Because the prefactor $\sqrt{1 - a^2\xi_o^2}$ in equation (74) vanishes as $a \rightarrow (1/\xi_o)^-$, one always has $R_{\text{sh}}(a) \rightarrow 0$ as the observer approaches the acceleration horizon. Consequently, for $\xi_o > 3$ the shadow radius decreases monotonically with a . For $2 < \xi_o < 3$, $R_{\text{sh}}(a)$ increases at small a (by equation (76)), reaches a single maximum at some $a_*(\xi_o) \in (0, 1/\xi_o)$, and then decreases to zero as $a \rightarrow (1/\xi_o)^-$.

In all cases the centre of the shadow remains unshifted and the boundary remains exactly circular; only its angular scale varies with a and the observer position.

In summary, acceleration ($a > 0$) changes the size but not the shape (still a circle) and does not shift the centre. Although the C-metric breaks equatorial symmetry in

the photon dynamics (cone angle, polar turning points), the entire family of spherical photon orbits sits at a single radius $\xi_{\text{ph}}(a)$. Inserting the corresponding $Q_{\text{ph}}(a)$ into equation (68) yields the circular boundary (71).

Importantly, all spherical photon orbits share a single radius in the static C-metric. More precisely, in the K -scaled Mino system the radial equation reads

$$\left(\frac{d\xi}{d\lambda_K}\right)^2 = R(\xi) := \xi^4 \hat{e}^2 - \Delta(\xi), \quad \hat{e} := \varepsilon/\sqrt{K}, \quad (77)$$

cf. equation (14a). A constant radius "spherical" null orbit requires a double root of R :

$$R(\xi_{\text{ph}}) = 0, \quad R'(\xi_{\text{ph}}) = 0. \quad (78)$$

From $R(\xi_{\text{ph}}) = 0$ one finds $\hat{e}^2 = \Delta(\xi_{\text{ph}})/\xi_{\text{ph}}^4$. Inserting this into $R'(\xi) = 4\xi^3\hat{e}^2 - \Delta'(\xi)$ gives

$$\frac{4\Delta(\xi_{\text{ph}})}{\xi_{\text{ph}}} = \Delta'(\xi_{\text{ph}}), \quad (79)$$

which is exactly the photon surface condition (49). Thus $\xi_{\text{ph}} = \xi_{\text{ph}}(a)$ is uniquely fixed by the metric function Δ and depends only on the acceleration parameter a .

Structurally, this uniqueness follows because in the conformal metric $\hat{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$ the radial potential equation (77) depends on a single combination of constants of motion ($\hat{e}^2$) and on ξ alone; the two double root conditions then determine $(\xi_{\text{ph}}, \hat{e}^2)$ with no residual dependence on the polar constant. This contrasts with Kerr, where the radial potential depends on two independent ratios (e.g. L_z/E and Q/E^2), so the double-root system solves for those ratios as functions of the radius, leaving a one-parameter family of spherical photon radii. Here, by staticity and the resulting radial decoupling, all spherical photon orbits lie on the single photon surface $\xi = \xi_{\text{ph}}(a)$.

It is worth mentioning here that the stationary, axisymmetric barotropic thick accretion discs in this set-up exist only for relatively small α and vanish as α increases [22]. The disc affects the brightness map (which photons are emitted or absorbed), not the circular capture boundary itself. For typical observers with $\xi_o > 3$ R_{sh} decreases monotonically as a increases, while the band of stable circular geodesics (and hence the geometric support for a thick torus) also shrinks and disappears at a_{crit} . Thus the two findings are consistent: larger acceleration simultaneously reduces the observable shadow scale and suppresses stationary thick discs. However, to quantitatively relate disc morphology to observed images one must fix an emission model (optically thin flow vs. optically thick torus), but the geometric inputs are already provided here. A minimal, self-consistent pipeline is: (i) choose a stationary torus family (e.g. constant ℓ_C); (ii) locate the centre and cusp and the Hessian of effective potential [22]; (iii) ray-trace photons using the closed null solution (section III) and the observer tetrad (65d). The resulting images will always contain the circular shadow boundary (71); the disc merely modulates the brightness across it.

B. Photon ring dynamics and eikonal ringdown

As we had in equation (69) unstable spherical photon orbits sit at the constant radius, and for constant latitude motion, on the photon cone

$$\cos \theta_\gamma = \frac{-1 + \sqrt{1 + 12a^2}}{6a}, \quad (80)$$

as derived in sections IV B and IV A. We use $\Delta(\xi) = \xi^2(1 - \frac{2}{\xi})(1 - a^2\xi^2)$ and $P(\theta) = 1 + 2a\cos\theta$ throughout.

In the K -scaled Mino system (Section III),

$$\frac{d\phi}{d\lambda_K} = \frac{\hat{\ell}}{CP(\theta)\sin^2\theta}, \quad \frac{d\tau}{d\lambda_K} = \frac{\xi^4\hat{e}}{\Delta(\xi)}.$$

On a spherical cone orbit $(\xi, \theta) = (\xi_{\text{ph}}, \theta_\gamma)$ the constants obey

$$K = \frac{\varepsilon^2 \xi_{\text{ph}}^4}{\Delta(\xi_{\text{ph}})} = \frac{\ell_C^2}{P(\theta_\gamma)\sin^2\theta_\gamma} \\ \implies \hat{e}^2 = \frac{\Delta(\xi_{\text{ph}})}{\xi_{\text{ph}}^4}, \quad \hat{\ell}^2 = P(\theta_\gamma)\sin^2\theta_\gamma. \quad (81)$$

Hence the dimensionless coordinate time orbital frequency $\Omega_{\text{orb}} := d\phi/d\tau$ is

$$\Omega_{\text{orb}} = \frac{\sqrt{\Delta(\xi_{\text{ph}})}}{\xi_{\text{ph}}^2 C \sqrt{P(\theta_\gamma)\sin\theta_\gamma}}, \quad (82)$$

(the reparametrisation $\lambda \leftrightarrow \lambda_K$ cancels in $d\phi/d\tau$). In the Schwarzschild limit $a \rightarrow 0$, $\Omega_{\text{orb}} \rightarrow 1/(3C\sqrt{3})$.

1. Lyapunov exponent.

Let $R(\xi) = \varepsilon^2 \xi^4 - \Delta(\xi)K$ be the radial polynomial (section III). For a small radial perturbation about the spherical orbit, $(d\xi/d\lambda)^2 = R(\xi)$ about the spherical orbit gives, in Mino time, $\delta\xi'' = \frac{1}{2}R''(\xi_{\text{ph}})\delta\xi$. Converting to coordinate time τ using $(d\tau/d\lambda)|_{\xi_{\text{ph}}} = \xi_{\text{ph}}^4\varepsilon/\Delta(\xi_{\text{ph}})$ yields the (dimensionless) Lyapunov exponent

$$\Lambda = \sqrt{\frac{R''(\xi_{\text{ph}})}{2(d\tau/d\lambda)^2}} \\ = \frac{\Delta(\xi_{\text{ph}})}{\xi_{\text{ph}}^4} \sqrt{\frac{1}{2} \left[12\xi_{\text{ph}}^2 - \frac{\xi_{\text{ph}}^4}{\Delta(\xi_{\text{ph}})} \Delta''(\xi_{\text{ph}}) \right]}. \quad (83)$$

with $\Delta''(\xi) = -12a^2\xi^2 + 12a^2\xi + 2$. For $a = 0$ one obtains $\Lambda = 1/(3\sqrt{3})$, as expected.

Small- a expansions. Using $\xi_{\text{ph}} = 3[1 - 3a^2 + \mathcal{O}(a^4)]$, $\cos\theta_\gamma = a - 3a^3 + \mathcal{O}(a^5)$, $\sin\theta_\gamma = 1 - \frac{1}{2}a^2 + \mathcal{O}(a^4)$, and

$P(\theta_\gamma) = 1 + 2a^2 + \mathcal{O}(a^4)$, one finds

$$\Omega_{\text{orb}} = \frac{1}{3C\sqrt{3}} \left[1 - 5a^2 + \mathcal{O}(a^4) \right], \quad (84a)$$

$$\Lambda = \frac{1}{3\sqrt{3}} \left[1 - \frac{3}{2}a^2 + \mathcal{O}(a^4) \right]. \quad (84b)$$

Thus, acceleration reduces both the pattern speed and the instability rate at quadratic order in a ; only the azimuthal frequency carries the explicit $1/C$ inherited from $g_{\phi\phi} \propto C^2$.

2. Eikonal quasinormal estimate and conventions.

Linear perturbations (scalar, electromagnetic, gravitational) on a stationary black hole background satisfy wave equations of the schematic form

$$\partial_\tau^2 \Psi - \partial_{r_*}^2 \Psi + \mathcal{V}_m(x; \omega) \Psi = 0, \quad (85)$$

where $\tau = t/m$ is our dimensionless time, r_* is a tortoise type coordinate, and m is the azimuthal number (in spherical cases one would also have l , but in the static C-metric axisymmetry singles out m). In the eikonal (geometric optics) limit (see e.g., [19, 20]) $m \gg 1$ the wavelength is short compared to curvature scales, and a WKB ansatz

$$\Psi(\tau, x) \sim A(x) \exp[i m S(\tau, x)] \quad (86)$$

reduce, at leading order, the Hamilton-Jacobi equation $g^{\mu\nu} \partial_\mu S \partial_\nu S = 0$, i.e. wavefronts propagate along null geodesics. A wave packet initially localised near the unstable photon ring stays trapped for many cycles; its pattern speed is the geodesic angular frequency while the decay rate is controlled by the local instability (Lyapunov exponent) of neighbouring null rays.

The standard matching across the potential peak at $\xi = \xi_{\text{ph}}(a)$ (with purely ingoing/outgoing boundary conditions at the two Killing horizons in the static patch) yields the eikonal spectrum (see, e.g., [19, 20])

$$\omega_{mn} \approx m\Omega_{\text{orb}} - i\left(n + \frac{1}{2}\right)\Lambda, \quad m \gg 1, \quad n = 0, 1, 2, \dots, \quad (87)$$

with Ω_{orb} and Λ as above equations (82)-(83), measured with respect to $\tau = t/m$; the physical frequency is $\omega_{mn}^{\text{phys}} = \omega_{mn}/m$. Note that Λ carries no explicit C factor, whereas Ω_{orb} contains an overall $1/C$ coming from $g_{\phi\phi} \propto C^2$.

For the static C-metric the unstable spherical photon orbits sit at the single radius $\xi = \xi_{\text{ph}}(a)$ and at the cone latitude $\theta = \theta_\gamma(a)$ (section IV A). In the radial WKB problem the effective potential $\mathcal{V}_m$ develops a sharp maximum at $\xi = \xi_{\text{ph}}(a)$.³ A 1D WKB expansion about this

³ Formally, one introduces a tortoise coordinate adapted to the spherical orbit, $dr_*/d\xi > 0$, e.g. the standard choice along $\theta = \theta_\gamma$, so that the potential peak sits at a finite r_* . The precise definition of r_* is irrelevant for the leading eikonal result.

maximum, matched to purely ingoing (outgoing) boundary conditions at the black hole (acceleration) horizon in the static patch, yields a quantisation condition whose leading real and imaginary parts are set by classical photon ring data (see, e.g., [19, 20]).

Because the static C-metric breaks equatorial symmetry through $\theta_\gamma(a)$ and $P(\theta_\gamma)$, the eikonal spectrum depends on the acceleration a and on the cone latitude via (82)-(83).

At fixed coordinate harmonic label m (with $\phi \in [0, 2\pi)$), since the separated perturbations are $e^{im\phi}$ with integer m , inserting (82) into (87) gives

$$\Re \omega_{mn} \approx \frac{m}{C} \frac{\sqrt{\Delta(\xi_{\text{ph}})}}{\xi_{\text{ph}}^2 \sqrt{P(\theta_\gamma)} \sin \theta_\gamma}, \quad (88)$$

i.e. the leading real part scales as $1/C$ when m is held fixed.

There is an equivalent (and often more physical) convention by using the physical azimuth $\Phi := C\phi$, whose period is $2\pi C$, and expand the perturbations as $e^{im\Phi}$ with integer $\tilde{m}$. Then

$$\frac{d\Phi}{d\tau} = C\Omega_{\text{orb}}, \quad m\Omega_{\text{orb}} = \tilde{m} \frac{d\Phi}{d\tau}, \quad (89)$$

so the explicit C cancels in the product, and one can write the eikonal estimate as

$$\omega_{\tilde{m}n} \approx \tilde{m} \frac{d\Phi}{d\tau} - i\left(n + \frac{1}{2}\right)\Lambda, \quad (90)$$

with a C -independent leading coefficient. This is simply labeling harmonics by the number of waves around the physical circumference (period $2\pi C$) rather than around the coordinate angle $\phi \in [0, 2\pi)$. Both conventions are mathematically consistent; one must state which is being used. Therefore, at the eikonal order,

- The imaginary part is set by Λ and is independent of C .
- The real part is $m\Omega_{\text{orb}}$. With the standard coordinate label m (period 2π in ϕ) it scales as $1/C$; with the physical azimuthal label $\tilde{m}$ (period $2\pi C$ in $\Phi = C\phi$) the leading coefficient is independent of C .

Thus, (i) Equation (87) captures the high m ringdown controlled by the unstable photon ring seen by static observers in the wedge $2 < \xi < 1/a$. The boundary conditions at the black hole and acceleration horizons fix the quasinormal mode character, but they do not alter the leading eikonal relation above. (ii) The estimate is accurate for $m \gg 1$ and low overtones $n = \mathcal{O}(1)$; subleading corrections (e.g. beyond-WKB and spin-dependent terms) do not affect the leading identification $\Re \omega = m\Omega_{\text{orb}}$ and $\Im \omega = -(n + \frac{1}{2})\Lambda$.

C. Cosmic string tension: what can (and cannot) be constrained

The conical parameter C enters only through $g_{\phi\phi} = \Omega^{-2}PC^2\xi^2\sin^2\theta$. Locally, the shadow boundary is obtained from the screen variables (67a)-(67b). Combining them gives

$$\iota^2 + \vartheta^2 = \frac{\Delta_o}{\xi_o^4} Q, \quad (91)$$

see equation (68). On the photon surface $Q = Q_{\text{ph}}$ is fixed by equation (70), so the shadow locus satisfies equation (71). Crucially, the C -dependence cancels between α and β , and therefore:

For any static observer, the shadow boundary is a circle with radius (72) that is independent of C . Thus, from the shape and size of the local shadow alone one cannot infer the string tension. In fact, the conical deficit changes the spacetime at large radii to a wedge like "conical" geometry with azimuth reduced by $2\pi(1-C)$. This produces global lensing signals (double images separated by the deficit, azimuthal discontinuities across the string, distinct caustic structure) that do not show up in the local shadow boundary. Realistic strategies therefore are:

- infer a from the shadow radius via equation (72) at a known observer position (ξ_o, θ_o) ;
- constrain C from global lensing (multiple images, azimuthal jumps), not from the circle outline of the shadow.

A joint analysis of the bright photon ring (which depends on transport and emissivity) together with global lensing offers a practical path to bounds on C .

D. Inferring the acceleration from a single shadow

From equations (71)-(72), a measured screen radius R_{sh} at known (ξ_o, θ_o) fixes

$$Q_{\text{ph}} := \frac{\xi_{\text{ph}}^4}{\Delta(\xi_{\text{ph}})} = \frac{\xi_o^4}{\Delta(\xi_o)} R_{\text{sh}}^2. \quad (92)$$

On the photon surface, the C-metric identity $\Delta(\xi_{\text{ph}}) = \xi_{\text{ph}}(\xi_{\text{ph}} - 2)^2$ holds exactly (it follows from the double root condition defining ξ_{ph} and eliminates a). Hence

$$Q_{\text{ph}} = \frac{\xi_{\text{ph}}^3}{(\xi_{\text{ph}} - 2)^2}, \quad (93)$$

a cubic equation for ξ_{ph} . Therefore, given Q_{ph} from equation (92), one can solve equation (93) and select the physical root in the static range $2 < \xi_{\text{ph}} < 3$.

Finally, the acceleration parameter follows algebraically from the photon surface relation (equivalent to the double root condition)

$$a^2 = \frac{3 - \xi_{\text{ph}}}{\xi_{\text{ph}}^2}, \quad 0 < a < \frac{1}{2} \iff 2 < \xi_{\text{ph}} < 3. \quad (94)$$

Therefore, (i) If (ξ_o, θ_o) is not independently known, the single measurement R_{sh} leaves a degeneracy between a and ξ_o ; an additional observable (e.g. flux centroid shift, time delay, or independent distance/mass estimate) breaks the degeneracy. (ii) Error propagation is straightforward: since $Q_{\text{ph}} \propto R_{\text{sh}}^2$, $\delta Q_{\text{ph}}/Q_{\text{ph}} = 2\delta R_{\text{sh}}/R_{\text{sh}}$; then δa follows from $\partial a/\partial \xi_{\text{ph}}$ using equation (94) together with the derivative of the cubic equation (93).

VI. CLASSICAL INVARIANTS AND DIAGNOSTICS

Beyond individual geodesics it is useful to collect simple, coordinate invariant quantities that characterise the static C-metric itself and that can be used as quick checks in analytic and numerical work. In this section, we present briefly two such diagnostics, working throughout in the dimensionless variables $(\xi, a = \alpha m)$ and within the static patch $2 < \xi < 1/a$ (with $\Omega > 0$): (i) the surface gravities and hence Hawking temperatures of the two Killing horizons (black hole and acceleration), which depend only on the horizon locations and are independent of the conical parameter; and (ii) an exact redshift law between any two static worldlines, isolating the combined gravitational and accelerative contributions via $\Omega(\xi, \theta) = 1 + a\xi \cos \theta$. We adopt the same normalisation of the stationary Killing vector $\xi_{(t)} = \partial_t$ used throughout the paper, so all expressions below are directly comparable to the geodesic results and to the $a \rightarrow 0$ (Schwarzschild) limit.

A. Surface gravities and (non) equilibrium of horizons

In the static patch the metric components relevant here are

$$g_{tt} = -\frac{\Delta}{\Omega^2 r^2}, \quad g_{rr} = \frac{r^2}{\Omega^2 \Delta}, \quad (95)$$

$$\Delta = (r^2 - 2mr)(1 - \alpha^2 r^2), \quad \Omega = 1 + \alpha r \cos \theta.$$

The two Killing horizons of $\xi_{(t)}$ are at $r_{\text{H}} = 2m$ (black hole) and $r_{\text{A}} = 1/\alpha$ (acceleration). For any static metric of the form above, the surface gravity associated with $\xi_{(t)}$ at a Killing horizon $r = r_{\text{h}}$ is

$$\kappa = \frac{1}{2} \partial_r(-g_{tt}) \sqrt{\frac{g^{rr}}{-g_{tt}}} \Big|_{r=r_{\text{h}}} = \frac{1}{2r_{\text{h}}^2} \Delta'(r_{\text{h}}). \quad (96)$$

Thus, κ is independent of θ and of the canonical parameter C . Hence, differentiating Δ gives $\Delta'(r) = (2r - 2m)(1 - \alpha^2 r^2) - 2\alpha^2 r(r^2 - 2mr)$, evaluating at the two horizons gives

$$\Delta'(r_{\text{H}}) = 2m(1 - 4\alpha^2 m^2), \quad (97a)$$

$$\Delta'(r_{\text{A}}) = -2(r_{\text{A}} - 2m). \quad (97b)$$

Hence, leads to

$$\kappa_{\text{H}} = \frac{1 - 4a^2}{4m}, \quad \kappa_{\text{A}} = \frac{r_{\text{A}} - 2m}{r_{\text{A}}^2} = \frac{a - 2a^2}{m}. \quad (98)$$

For the acceleration horizon we report the positive magnitude: with the outward normal one has $\Delta'(r_{\text{A}}) < 0$, so $|\kappa_{\text{A}}|$ is as in equation (98). The associated Hawking temperatures are $T_{\text{H}} = |\kappa_{\text{H}}|/(2\pi)$ and $T_{\text{A}} = |\kappa_{\text{A}}|/(2\pi)$.

In non-asymptotically flat spacetimes the overall scale of the Killing vector matters. Throughout we use the canonical choice $\xi_{(t)} = \partial_t$ already adopted in the geodesic analysis. Any common rescaling of $\xi_{(t)}$ rescales both κ and T by the same constant and does not affect ratios or the inequivalence of the two temperatures.

Near-horizon Rindler form. Fix a latitude θ with $\Omega_{\text{h}} := \Omega(r_{\text{h}}, \theta) > 0$ (away from the south pole of the acceleration horizon), and expand the lapse function $\Delta(r) = \Delta'(r_{\text{h}})(r - r_{\text{h}}) + \mathcal{O}((r - r_{\text{h}})^2)$. To leading order the (t, r) -sector of the metric is

$$ds_{(t,r)}^2 \simeq -\frac{\Delta'(r_{\text{h}})(r - r_{\text{h}})}{\Omega_{\text{h}}^2 r_{\text{h}}^2} dt^2 + \frac{r_{\text{h}}^2}{\Omega_{\text{h}}^2 \Delta'(r_{\text{h}})(r - r_{\text{h}})} dr^2. \quad (99)$$

Defining the proper radial distance ρ by integrating

$$d\rho^2 = g_{rr} dr^2 = \frac{r_{\text{h}}^2}{\Omega_{\text{h}}^2 \Delta'(r_{\text{h}})} \frac{dr^2}{r - r_{\text{h}}}, \quad (100)$$

so that

$$\rho = \frac{2r_{\text{h}}}{\Omega_{\text{h}} \sqrt{|\Delta'(r_{\text{h}})|}} \sqrt{|r - r_{\text{h}}|} [1 + \mathcal{O}(r - r_{\text{h}})], \quad (101)$$

Therefore, up to leading order we have

$$r - r_{\text{h}} = \frac{\Omega_{\text{h}}^2 |\Delta'(r_{\text{h}})|}{4r_{\text{h}}^2} \rho^2. \quad (102)$$

Substituting this back into g_{tt} gives

$$g_{tt} \simeq -\frac{\Delta'(r_{\text{h}})}{\Omega_{\text{h}}^2 r_{\text{h}}^2} (r - r_{\text{h}}) = -\frac{\Delta'(r_{\text{h}}) |\Delta'(r_{\text{h}})|}{4r_{\text{h}}^4} \rho^2 = -\kappa^2 \rho^2, \quad (103)$$

where $\kappa := \frac{|\Delta'(r_{\text{h}})|}{2r_{\text{h}}^2}$. Hence the local (t, ρ) -geometry is Rindler,

$$ds^2 \simeq -\kappa^2 \rho^2 dt^2 + d\rho^2 + \underbrace{\frac{r_{\text{h}}^2}{\Omega_{\text{h}}^2 P(r_{\text{h}}, \theta)} d\theta^2 + \frac{P(r_{\text{h}}, \theta) r_{\text{h}}^2 \sin^2 \theta}{\Omega_{\text{h}}^2} \frac{d\phi^2}{C^2}}_{\text{finite angular block}}, \quad (104)$$

with non-singular angular coefficients taking the standard non negative magnitude for κ . This reproduces equation (96)⁴.

Additionally, for $0 < a < \frac{1}{2}$, we have

$$\kappa_{\text{H}} - \kappa_{\text{A}} = \frac{(1 - 2a)^2}{4m} \geq 0, \quad (105)$$

⁴ At the south pole of the acceleration horizon, where $\Omega_{\text{h}} = 0$, the above change of variables is not admissible, and a separate corner analysis is required.

with equality only at $a = \frac{1}{2}$ where the two horizons merge and both temperatures vanish. Thus the static C-metric has two distinct temperatures, $T_H > T_A$ for $0 < a < \frac{1}{2}$, and cannot be in global thermal equilibrium with respect to $\xi_{(t)}$ (Tolman law). In the Schwarzschild limit $a \rightarrow 0$ one recovers $\kappa_H \rightarrow (4m)^{-1}$ and $\kappa_A \rightarrow 0$ (the acceleration horizon recedes). For small a ,

$$\kappa_H = \frac{1}{4m}(1 - 4a^2) + \mathcal{O}(a^4), \quad \kappa_A = \frac{a}{m}(1 - 2a) + \mathcal{O}(a^3).$$

This worth mention that g_{tt} and g_{rr} do not contain C , so horizon surface gravities and temperatures are unaffected by the conical deficit.

B. Redshift between static worldlines

Let a photon with four-momentum k^μ be emitted by a static source at (ξ_e, θ_e) and received by a static observer at (ξ_o, θ_o) within the static patch. Static worldlines are proportional to ∂_t and are normalised by $u^t = 1/\sqrt{-g_{tt}}$. The measured frequencies are $\omega = -k \cdot u$, since $k_t = -E$ is conserved, so the exact redshift factor is

$$\frac{\omega_o}{\omega_e} = \sqrt{\frac{-g_{tt}(\xi_e, \theta_e)}{-g_{tt}(\xi_o, \theta_o)}} = \sqrt{\frac{\Delta(\xi_e)}{\Delta(\xi_o)}} \frac{\Omega(\xi_o, \theta_o)}{\Omega(\xi_e, \theta_e)} \frac{\xi_o}{\xi_e}. \quad (106)$$

The factor $\sqrt{\Delta(\xi_e)/\Delta(\xi_o)}$ is the familiar Schwarzschild type gravitational redshift in the conformal frame, while $[\Omega(\xi, \theta)\xi]^{-1}$ encodes the latitude-dependent accelerative redshift arising from the overall conformal factor Ω^{-2} and the r scaling in g_{tt} . In the Schwarzschild limit $a \rightarrow 0$, $\Omega \rightarrow 1$ and $\Delta \rightarrow \xi^2(1 - 2/\xi)$, recovering the usual static redshift.

Small- a expansion and anisotropy. At fixed latitudes one finds the first order modulation by acceleration,

$$\frac{\omega_o}{\omega_e} = \sqrt{\frac{1 - \frac{2}{\xi_e}}{1 - \frac{2}{\xi_o}}} \left[1 + a(\xi_o \cos \theta_o - \xi_e \cos \theta_e) + \mathcal{O}(a^2) \right], \quad (107)$$

i.e. the Schwarzschild redshift multiplied by a linear north-south factor from $\Omega(\xi, \theta) = 1 + a\xi \cos \theta$: a blueshift towards the north ($\cos \theta > 0$) and a redshift towards the south ($\cos \theta < 0$).

Equation (106) is exact and algebraic;. Thus, it is convenient for validating ray-tracing codes, for analytic flux transfer problems with static emitters/observers, and for quick checks of the interplay between gravitational and accelerative effects as a function of (ξ, θ) .

VII. SUMMARY AND CONCLUSIONS

We studied test particle dynamics in the (sub-extremal) static C-metric, adopting dimensionless variables and a Mino type parameter to put null motion into

first order form. We also introduced a K -scaling for null motion, so that the reduced system depends only on two dimensionless ratios and all sector equations share a uniform normalisation across the paper. The principal results are as follows.

Because null motion is conformally invariant, the Hamilton-Jacobi (HJ) equation separates. We reduced the radial and polar equations to the Biermann-Weierstrass form; the general solution follows in terms of $\wp, \zeta, \sigma$. This yields (i) a classification of photon trajectories via the quartic radial polynomial and the polar potential; (ii) the latitude of the photon cone, which replaces equatorial symmetry for $a \neq 0$; and (iii) the spherical photon surface at a single radius $\xi_{ph}(a)$. For generic orbits the azimuth and time quadratures involve rational functions of two independent Weierstrass arguments (from the r and θ sectors), that admit closed expressions in σ -logarithms. Cone (or spherical) orbits collapse to a single sector, leading to linear azimuth (or time).

Furthermore, for any static observer in the static patch the black hole shadow boundary is an exact circle. Its screen radius depends on a and the observer's position but is independent of the conicity C , so the local silhouette does not constrain the string tension. The photon ring orbital frequency and Lyapunov exponent were obtained in compact closed form, providing eikonal quasi-normal estimates (real and imaginary parts). A useful inversion procedure shows how a single shadow radius measurement at known (ξ_o, θ_o) determines the dimensionless acceleration. We gave small a expansions showing that acceleration reduces both the orbital frequency and the instability rate at quadratic order, and we explained the geodesic QNM correspondence so that readers can interpret the eikonal estimate without recourse to the full perturbation problem.

We also collected invariant quantities useful for interpretation: the surface gravities (and hence temperatures) of the black hole and acceleration horizons are generically unequal, precluding global thermal equilibrium and an exact redshift law between static emitters/observers anywhere in the static patch. Both diagnostics are independent of the conical parameter C and provide simple checks for analytical formulas and numerical ray tracing.

To sum up, acceleration modifies photon dynamics in a transparent way, shifting the photon surface and replacing equatorial symmetry by a fixed photon cone. The shadow remains a circle but with an a -dependent scale set jointly by the photon surface and the observer's location. The conical deficit affects global lensing but not the local shadow boundary. Taken together, these results offer a unified, closed form system for dynamics and observables in the static C-metric and suggest concrete parameter inference strategies: extract the acceleration from the shadow radius (for a known observer), and constrain the string tension from global lensing rather than from the local shadow boundary.

Several extensions are natural. (1) Add a charge and track how the photon cone, shadow radius, and sta-

bility band deform. (2) Include slow rotation (accelerated Kerr) to study the interplay of frame dragging and translatory acceleration in the eikonal spectrum and in shadow asymmetry for nonstatic observers. (3) Couple the geodesic machinery to radiative transfer or plasma refraction to predict intensity maps beyond the geometric boundary. (4) Develop a global lensing analysis to convert global image distortions into quantitative bounds on the string tension. (5) Calibrate the eikonal formulas against full perturbation theory for test fields in the static C-metric wedge, clarifying subleading $1/m$ corrections and boundary condition effects at the acceleration horizon. These directions can build directly on the analytic structures developed here.

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Appendix A: Null geodesics: Weierstrass toolkit (details)

For completeness, we collect the explicit variable changes and σ, ζ integrals underlying sections III–IV A, using the dimensionless notation (7) $a = \alpha m$, $\varepsilon = mE$, $\ell = L/m$, $\ell_C = \ell/C$, and $\xi = r/m$.

1. Weierstrass reduction

The first order null equations in Mino gauge are

$$\left(\frac{d\xi}{d\lambda}\right)^2 = R(\xi) = \varepsilon^2 \xi^4 - \Delta(\xi) K, \quad (\text{A1})$$

$$\Delta(\xi) = \xi^2 \left(1 - \frac{2}{\xi}\right) (1 - a^2 \xi^2), \quad (\text{A2})$$

$$\left(\frac{d\theta}{d\lambda}\right)^2 = \Theta(\theta) = P(\theta) K - \frac{\ell_C^2}{\sin^2 \theta}, \quad (\text{A3})$$

$$P(\theta) = 1 + 2a \cos \theta. \quad (\text{A4})$$

a. Radial sector. The classical approach to solve the equation is as follows: Set $u = 1/\xi$ and then $u = y + \frac{1}{6}$, followed by $Y = \frac{K}{2}y$. One obtains the standard Weierstrass form

$$\left(\frac{dY}{d\lambda}\right)^2 = 4Y^3 - g_2^{(r)} Y - g_3^{(r)}, \quad (\text{A5})$$

$$g_2^{(r)} = K^2 \left(a^2 + \frac{1}{12}\right), \quad (\text{A6})$$

$$g_3^{(r)} = \frac{K^3}{216} - \frac{K^3 a^2}{6} - \frac{K^2 \varepsilon^2}{4}. \quad (\text{A7})$$

Hence $Y(\lambda) = \wp(\lambda - \lambda_{r0}; g_2^{(r)}, g_3^{(r)})$ and

$$\xi(\lambda) = \frac{1}{\frac{2}{K} Y(\lambda) + \frac{1}{6}} = \frac{1}{\frac{2}{K} \wp(\lambda - \lambda_{r0}) + \frac{1}{6}}. \quad (\text{A8})$$

The Biermann-Weierstrass approach is based on Taylor's theorem,

$$R(r) = \sum_{n=0}^4 \frac{1}{n!} \frac{d^n R}{dr^n}(r_0) (r - r_0)^n. \quad (\text{A9})$$

If r_0 is a zero of R , then it is straightforward to find that a substitution $r = r_0 + \frac{R'_0}{4y - R'_0/6}$, with $R'_0 = R'(r_0)$ and $R''_0 = R''(r_0)$ leads to the Weierstrass cubic with invariants

$$g_2 = \frac{1}{48} (R''_0 - 2R'_0 R'''_0), \quad (\text{A10})$$

$$g_3 = \frac{1}{12^3} (48R'_0 R''_0 R'''_0 - R''_0{}^3 - 29 \cdot 12^2 R_0'^2 R_0'''''). \quad (\text{A11})$$

These invariants are actually independent from the choice of r_0 and can be expressed directly in terms of

the coefficients of R as

$$g_2 = b_0 b_4 - \frac{1}{4} b_1 b_3 + \frac{1}{12} b_2^2, \quad (\text{A12})$$

$$g_3 = \frac{1}{6} b_0 b_2 b_4 - \frac{1}{16} b_0 b_3^2 - \frac{1}{216} b_2^3 - \frac{1}{16} b_1^2 b_4 + \frac{1}{48} b_1 b_2 b_3 \quad (\text{A13})$$

where $R(r) = \sum_{n=0}^4 b_n r^n$. The result is the same as above, where $r_0 = 0$ was used.

Note that the solution (A8) is of course equivalent to the solution given in the main body of the paper. This is due to the fact that for orbital motion that does not reach $r = 0$ the argument of the Weierstrass function in (A8) is complex. It is given by $\lambda - \lambda_{r0} = x + \omega_2$, where x is some real number and ω_2 is the complex half-period of $\wp$. Using the addition theorems for $\wp$ the equivalence to (21) can be explicitly demonstrated.

The above approach can be further generalized to any initial value r_0 . For details of the derivation, we refer to [18] (Appendix A) and references therein.

b. Polar sector. With $v = \cos \theta$, define Y_θ by $v = -(2Y_\theta + \frac{1}{6})/a$ and $\gamma = \sqrt{K}\lambda$. Then

$$\left(\frac{dY_\theta}{d\gamma}\right)^2 = 4Y_\theta^3 - g_2^{(\theta)} Y_\theta - g_3^{(\theta)}, \quad (\text{A14})$$

$$g_2^{(\theta)} = \frac{1}{12} + a^2, \quad (\text{A15})$$

$$g_3^{(\theta)} = \frac{1}{216} + a^2 \left(\frac{\ell_C^2}{4K} - \frac{1}{6}\right), \quad (\text{A16})$$

hence $Y_\theta(\gamma) = \wp(\gamma - \gamma_0; g_2^{(\theta)}, g_3^{(\theta)})$ and

$$\theta(\lambda) = \arccos \left[-\frac{1}{a} \left(2Y_\theta(\sqrt{K}\lambda) + \frac{1}{6} \right) \right]. \quad (\text{A17})$$

2. ϕ and t : σ, ζ -integrals

For the azimuth and coordinate time,

$$\frac{d\phi}{d\lambda} = \frac{\ell_C}{P(\theta) \sin^2 \theta}, \quad \frac{d\tau}{d\lambda} = \frac{\xi^4 \varepsilon}{\Delta(\xi)}, \quad (\tau = t/m), \quad (\text{A18})$$

the substitution rules above produce integrals of the form $\int dz (\wp(z) - c)^{-1}$. It is convenient to introduce the poles

$$c_1 = \frac{a}{2} - \frac{1}{12}, \quad c_2 = -\frac{a}{2} - \frac{1}{12}, \quad c_3 = \frac{1}{6}, \quad (\text{A19})$$

and, for each c , a pair of preimages z_1, z_2 with $\wp(z_s) = c$. Define

$$f(z; z_1, z_2) = \sum_{s=1}^2 \frac{\log \sigma(z - z_s) - \log \sigma(z_0 - z_s) - \zeta(z_s)(z - z_0)}{\wp'(z_s)}, \quad (\text{A20})$$

where σ, ζ are the standard Weierstrass functions and z_0 is the initial argument (from the radial or polar sector as

indicated below). The coefficients

$$\beta_i = \left\{ \frac{1}{4(2a-1)}, \frac{1}{4(2a+1)}, -\frac{a}{4a^2-1} \right\}, \quad (\text{A21})$$

$$\gamma_i = \left\{ -\frac{1}{4a(2a-1)}, -\frac{1}{4a(2a+1)}, \frac{1}{4a^2-1} \right\} \quad (\text{A22})$$

yield the compact expressions

$$\phi(\lambda) = \phi_0 + \frac{a\ell}{C^2} \sum_{i=1}^3 \beta_i f(\sqrt{K}\lambda - \gamma_0; z_{i1}, z_{i2}), \quad (\text{A23})$$

$$\tau(\lambda) = \tau_0 + \varepsilon \sum_{i=1}^3 \gamma_i f(\lambda - \lambda_{r0}; z_{i1}, z_{i2}). \quad (\text{A24})$$

For cone orbits ($\theta = \theta_\gamma$ fixed) the arguments coincide and both ϕ and τ are functions of a single Weierstrass phase; for generic C-metric motion the radial and polar arguments differ (hyperelliptic structure), but the σ, ζ representation remains valid.

Limits. As $a \rightarrow 0$ the two sets of invariants merge and the above formulas reduce to the standard Schwarzschild expressions. Setting $K = \ell^2$ reproduces equatorial motion in this limit (for $a \neq 0$ equatorial photon orbits do not exist, consistent with section IV A).