

SECANT SHEAVES AND WEIL CLASSES ON ABELIAN VARIETIES

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ABSTRACT. Let K be a CM-field, i.e., a totally complex quadratic extension of a totally real field F . Let X be an abelian variety admitting an algebra embedding $F \rightarrow \text{End}_{\mathbb{Q}}(X)$, and let $\hat{X}$ be the dual abelian variety. We construct an embedding $\eta : K \rightarrow \text{End}_{\mathbb{Q}}(X \times \hat{X})$ associated to a choice of a polarization Θ in $\wedge_F^2 H^1(X, \mathbb{Q})$ and an element $q \in F$, such that $K = F(\sqrt{-q})$. We get the $[K : \mathbb{Q}]$ -dimensional subspace $HW(X \times \hat{X}, \eta)$ of Hodge Weil classes in $H^{\frac{d}{2}, \frac{d}{2}}(X \times \hat{X}, \mathbb{Q})$, where $d := 4 \dim(X)/[K : \mathbb{Q}]$.

The space $V_{\mathbb{C}} := H^1(X \times \hat{X}, \mathbb{C})$ admits a natural symmetric bilinear pairing and the even cohomology $S_{\mathbb{C}}^+ := H^{ev}(X, \mathbb{C})$ is the half-spin representation of $\text{Spin}(V_{\mathbb{C}})$. Hence, $\mathbb{P}(S_{\mathbb{C}}^+)$ contains a component of the Grassmannian of maximal isotropic subspaces of $V_{\mathbb{C}}$ known as the even spinorial variety. We associate to (Θ, q) a $2^{[F:\mathbb{Q}]}$ -dimensional subspace B of $S_{\mathbb{Q}}^+$ such that $\mathbb{P}(B)$ is secant to the spinorial variety. Associated to two coherent sheaves F_1 and F_2 on X with Chern characters in B we obtain the object $E := \Phi(F_1 \boxtimes F_2^\vee)$ in the derived category $D^b(X \times \hat{X})$, where $\Phi : D^b(X \times X) \rightarrow D^b(X \times \hat{X})$ is Orlov's equivalence. The flat deformations of the normalized Chern character $\kappa(E) := ch(E) \exp\left(-\frac{c_1(E)}{\text{rank}(E)}\right)$ of E remain of Hodge type under every deformation of $(X \times \hat{X}, \eta)$ as an abelian variety of Weil type (A', η') . The algebraicity of the Weil classes of every deformation (A', η') would thus follow if E is semiregular in the appropriate sense.

When $F = \mathbb{Q}$, so that K is an imaginary quadratic number field, the above construction was combined with the Semi-regularity theorem to prove the algebraicity of the Weil classes on abelian sixfolds of split Weil type. The algebraicity of the Weil classes on all abelian fourfold of Weil type follows. The Hodge conjecture for abelian varieties of dimension ≤ 5 is known to follow from the latter result.

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1. INTRODUCTION

1.1. Abelian varieties of Weil type. A *CM-field* K is a quadratic extension of a number field F , such that all embeddings of F in $\mathbb{C}$ are real and none of the embeddings of K in $\mathbb{C}$ are real. Set $e := [K : \mathbb{Q}]$. Let Σ be the set of all embeddings $\sigma : K \rightarrow \mathbb{C}$. The cardinality of Σ is e . Let ι be the involution in $\text{Gal}(K/F)$. Then $\sigma \circ \iota = \bar{\sigma}$, for all $\sigma \in \Sigma$, where $\bar{\sigma}$ is the complex conjugate embedding. Let A be a complex (projective) abelian variety and $\eta : K \rightarrow \text{End}_{\mathbb{Q}}(A) := \text{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ an algebra embedding. Note that η is equivalent to an embedding $\eta : K \rightarrow \text{End}_{Hdg}(H^1(A, \mathbb{Q}))$. Let $H_{\sigma}^1(A, \mathbb{C})$ be the subspace of $H^1(A, \mathbb{C})$ on which $\eta(K)$ acts via the character $\sigma \in \Sigma$ and set $H_{\sigma}^{1,0}(A) := H^{1,0}(A) \cap H_{\sigma}^1(A, \mathbb{C})$. Define $H_{\sigma}^{0,1}(A)$ analogously. Set $d := \dim_K H^1(A, \mathbb{Q})$. The pair (A, η) is said to be of *Weil type*, if $\dim H_{\sigma}^{1,0}(A) = \dim H_{\sigma}^{0,1}(A) = \frac{d}{2}$, for all $\sigma \in \Sigma$. In that case the subspace¹ $HW(A, \eta) := \wedge_K^d H^1(A, \mathbb{Q})$ of $\wedge^d H^1(A, \mathbb{Q})$ is an e -dimensional subspace of $H^{\frac{d}{2}, \frac{d}{2}}(A, \mathbb{Q})$, which is a 1-dimensional K -vector space [DM, Prop. 4.4]. Classes in $HW(A, \eta)$ are called *Weil classes*. Note that both e and d are necessarily even, and so $\dim_{\mathbb{C}}(A) = 2n$, where $n := de/4$ is an integer.

$$e := [K : \mathbb{Q}], \quad d := \dim_K H^1(A, \mathbb{Q}), \quad n := \dim_{\mathbb{C}}(A)/2 = de/4.$$

A *polarized abelian variety of Weil type* is a triple (A, η, h) , where² $h \in \wedge_F^2 H^1(A, \mathbb{Q}) \cap H^{1,1}(A, \mathbb{Q})$ is the class of a polarization satisfying $h(\eta_t(x), y) = h(x, \eta_{\iota(t)}(y))$, for all $t \in K$, $x, y \in H_1(A, \mathbb{Q})$. Such a polarization yields a non-degenerate K -valued hermitian form $H : H_1(A, \mathbb{Q}) \times H_1(A, \mathbb{Q}) \rightarrow K$ on the K vector space $H_1(X, \mathbb{Q})$ and (A, η, h) is said to be of *split Weil type*, if H has an isotropic subspace of half the dimension. The moduli space of polarized abelian varieties of Weil type is $ed^2/8$ -dimensional. Assume that $d \geq 4$. For a generic triple (A, η, h) in moduli, the rank of the Neron-Severi group $H^{1,1}(A, \mathbb{Z})$ is $e/2$, the Hodge ring of A is generated by $H^{1,1}(A, \mathbb{Q})$ and $HW(A, \eta)$, and $HW(A, \eta)$ intersects trivially the subalgebra generated by $H^{1,1}(A, \mathbb{Q})$. In particular, for a generic triple (A, η, h) in moduli,

$$(1.1) \quad H^{\frac{d}{2}, \frac{d}{2}}(A, \mathbb{Q}) = \text{Im}[\text{Sym}^{d/2}(H^{1,1}(A, \mathbb{Q}))] \oplus HW(A, \eta),$$

where the first summand is the image of $\text{Sym}^{d/2}(H^{1,1}(A, \mathbb{Q}))$ in $H^{\frac{d}{2}, \frac{d}{2}}(A, \mathbb{Q})$. The Hodge conjecture suggests the following.

Question 1.1. Does $HW(A, \eta)$ consist of algebraic classes?

¹We have the decomposition $K \otimes_{\mathbb{Q}} \mathbb{C} \cong \oplus_{\sigma \in \Sigma} \mathbb{C}$ yielding $H^1(A, \mathbb{C}) = H^1(A, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} = H^1(A, \mathbb{Q}) \otimes_K (K \otimes_{\mathbb{Q}} \mathbb{C}) \cong \oplus_{\sigma \in \Sigma} H_{\sigma}^1(A, \mathbb{C})$. The subspace $\oplus_{\sigma \in \Sigma} \wedge^d H_{\sigma}^1(A, \mathbb{C})$ of $H^d(A, \mathbb{C})$ is defined over $\mathbb{Q}$ and corresponds to the subspace $\wedge_K^d H^1(A, \mathbb{Q})$ of $\wedge^d H^1(A, \mathbb{Q})$.

²Let $\hat{\Sigma}$ be the set of embeddings of F into $\mathbb{R}$. We get the decomposition $H^1(A, \mathbb{R}) = \oplus_{\hat{\sigma} \in \hat{\Sigma}} H_{\hat{\sigma}}^1(A, \mathbb{R})$. The subspace $\oplus_{\hat{\sigma} \in \hat{\Sigma}} \wedge^2 H_{\hat{\sigma}}^1(A, \mathbb{R})$ is defined over $\mathbb{Q}$ and corresponds to the subspace $\wedge_F^2 H^1(A, \mathbb{Q})$.

The question of algebraicity of the Weil classes was considered by some as a first test case for the Hodge conjecture. The following historical note is part of [Mi1, footnote 14]: *Mumford and Tate tried to prove the Hodge conjecture for abelian varieties by showing that the $\mathbb{Q}$ -algebra of rational (p, p) classes is generated by those of type $(1, 1)$, for which the conjecture was known, but Mumford found a counterexample to this. When Tate told Weil of the example, he remarked that it is a special case of a slightly more generic example, namely a 4-dimensional family of examples, and then said “As you and Mumford seem to believe Hodge’s conjecture, it is up to you to exhibit algebraic cycles corresponding to these abnormal classes. I shall rather attempt to show there is no such cycle” (Letter from Tate to Serre, February 2, 1965.) ... See [Z] for a computer aided attempt to find counter examples to the algebraicity of the Weil classes via degeneration and tropical algebraic geometry.*

Weil’s 1977 paper [W] contains the construction mentioned in the quote above. He constructs the moduli spaces of abelian varieties of Weil type and their Weil classes for K an imaginary quadratic number field. The more general construction for CM-fields can be found in [DM].

Theorem 1.2. [M2, Theorem 1.5.1] *The Weil classes for abelian fourfolds of Weil type and abelian sixfolds of split Weil type with complex multiplication by a quadratic imaginary number field K are algebraic.*

A sketch of the proof of Theorem 1.2 is given in Section 11. The above result was proved earlier by Schoen in case $K = \mathbb{Q}(\sqrt{-3})$ [S2] and by Koike in case $K = \mathbb{Q}(\sqrt{-1})$ [Ko]. The algebraicity for abelian fourfolds of split Weil type was proven earlier for $K = \mathbb{Q}(\sqrt{-3})$ in [S1], for $K = \mathbb{Q}(\sqrt{-1})$ in [vG], and for all quadratic imaginary number fields in [M1]. An alternative proof for abelian fourfolds of split Weil type was obtained recently by Floccari and Fu [FF]. The proofs in [FF, M1] use the geometry of hyper-Kähler varieties.

As we shall see, the proof of Theorem 1.2 relies on developments which were not available at the time Weil made his remark. Among these developments are the works of Mukai, Polishchuk, and Orlov on equivalences of derived categories of abelian varieties and the works of Bloch, Buchweitz-Flenner, and Pridham on the Semi-regularity theorem [Blo, BF1, Mu, Or, Po, Pr]. Furthermore, the strategy of the proof is a generalization of the one in [M1], which in turn is inspired by O’Grady’s observation that the third intermediate Jacobians of projective hyper-Kähler varieties of Kummer type form complete 4-dimensional families of abelian fourfolds of split Weil type [O’G].

The Hodge conjecture is known for projective varieties of dimension ≤ 3 . The Hodge ring of abelian fourfolds is generated by divisor classes and Weil classes for complex multiplication by possibly more than one imaginary quadratic number field, by work of Moonen and Zarhin [MZ1, MZ2] combined with a result of Ramón Mari in the case of products of abelian surfaces [R]. The Hodge ring for simple abelian varieties of prime dimension is generated by divisor classes, by a result of Tankeev [Ta]. If X is a non-simple abelian variety of dimension 5, then the Hodge ring of X is generated by divisor classes and pull backs of Weil classes from quotient abelian fourfolds, by [MZ2, Theorem 0.2]. Combining these results with Theorem 1.2 we get:

Corollary 1.3. *The Hodge conjecture holds for abelian varieties of dimension ≤ 5 .*

We will present in this paper a general strategy for proving the algebraicity of the Weil classes on abelian varieties of split Weil type but implement it fully only for dimension ≤ 6 and K imaginary quadratic. Following is a class of abelian varieties for which the Hodge conjecture would follow, if the algebraicity of the Weil classes of split type would be proved. An abelian variety A is of *CM-type*, if it admits an algebra embedding $\eta : K \rightarrow \text{End}_{\mathbb{Q}}(A)$, such that $H^1(A, \mathbb{Q})$ is a 1-dimensional K -vector space. In this case $[K : \mathbb{Q}] = 2 \dim(A)$ and we have the decomposition $H^1(A, \mathbb{C}) = \bigoplus_{\sigma \in \Sigma} H_{\sigma}^1(A, \mathbb{C})$ by 1-dimensional subspaces. If $H_{\sigma}^{1,0}(A) \neq (0)$, then $H_{\bar{\sigma}}^{0,1}(A) \neq (0)$. Hence, $H^{1,0}(A) = \bigoplus_{\sigma \in T} H_{\sigma}^{1,0}(A)$, where $T \subset \Sigma$ consists of precisely one embedding of K out of each pair of complex conjugate embeddings. Such a pair (K, T) is called a *CM-type*. Every simple abelian variety of CM-type is of the form $\mathbb{C}^g/T(\mathfrak{a})$, where $2g = [K : \mathbb{Q}]$, $\mathfrak{a}$ is an ideal in the subring of K of algebraic integers, and $T(\mathfrak{a})$ is the rank $2g$ lattice in $\mathbb{C}^g$ consisting of $\{(\sigma_1(k), \dots, \sigma_g(k)) : k \in \mathfrak{a}\}$, where $T = \{\sigma_i : 1 \leq i \leq g\}$. Abelian varieties of CM-type are rigid. They correspond to special points in Shimura varieties and play a central role in the theory of Shimura varieties [Mi2, Ke]. André reduced the Hodge conjecture for abelian varieties of CM-type to the question of algebraicity of the Weil classes on abelian varieties of split Weil type.

Theorem 1.4. [A] *Let A be a complex abelian variety of CM-type. There exist abelian varieties A_i of split Weil type and homomorphisms $f_i : A \rightarrow A_i$, such that every Hodge class t on A can be written as a sum $t = \sum f_i^*(t_i)$ with t_i a Weil class on A_i .*

1.2. Organization of the paper. A general strategy for proving the algebraicity of the Weil classes in $HW(A, \eta)$, $\eta : K \rightarrow \text{End}_{\mathbb{Q}}(A)$, for imaginary quadratic number fields K , was developed in [M2] and implemented for abelian fourfolds of Weil type and abelian sixfolds of split Weil type. In sections 2 to 10 we present³ the natural generalization of this strategy for K a CM-field. In section 11 we survey the implementation of the strategy in the case of imaginary quadratic K carried out in [M2].

In Section 2 we recall the Semi-regularity theorem of Buchweitz and Flenner. In Section 3 we recall the results of Mukai, Polishchuk, and Orlov relating (1) the action on $H^*(X, \mathbb{Z})$ of the group of autoequivalences of an abelian variety X , and (2) the spin representation of the group $\text{Spin}(V)$ of the lattice $V = H^1(X, \mathbb{Z}) \oplus H^1(X, \mathbb{Z})^*$ endowed with its natural symmetric bilinear pairing. In Section 4 we formulate a general strategy for proving the algebraicity of Weil classes guided by the Semi-regularity theorem and Chevalley's theory of pure spinors.

Set $V_{\bullet} := V \otimes_{\mathbb{Z}} \bullet$, $\bullet = \mathbb{Z}$ or a field. In Section 5 we review how the theory of pure spinors relates classes in the spin representation $H^*(X, \mathbb{Q})$ of $\text{Spin}(V)$ to maximal isotropic subspaces of V_K . When X admits an algebra embedding $\hat{\eta} : F \rightarrow \text{End}_{\mathbb{Q}}(X)$ of the totally real subfield F of K , we associate to a maximal isotropic subspace W of V_K a collection of maximal isotropic subspaces W_T in $V_{\mathbb{C}}$, one for each CM-type T of K . We introduce the subspace B of $H^*(X, \mathbb{Q})$ spanned by the subset of the spinorial variety consisting of the pure spinors of $\{W_T\}$. By definition, $\mathbb{P}(B)$ is secant to the spinorial variety. In Section 6 we observe that if B is spanned by Hodge classes, then a natural

³Details will appear in [M3] due to page limitations for this contribution.

embedding $\eta : K \rightarrow V_{\mathbb{Q}}$ associated to W endows $X \times \hat{X}$ with the structure of an abelian variety of Weil type.

Let $\text{Spin}(V_{\mathbb{Q}})_B$ be the subgroup of $\text{Spin}(V_{\mathbb{Q}})$ fixing every class in B . In Section 7 we compute the subalgebra $\mathcal{A}$ of $\text{Spin}(V_{\mathbb{Q}})_B$ -invariant classes in $H^*(X \times \hat{X}, \mathbb{Q})$. We show that $\mathcal{A}$ is generated by its graded summand $\mathcal{A}^2 \subset H^{1,1}(X \times \hat{X}, \mathbb{Q})$ and by the subspace of Weil classes $HW(X \times \hat{X}, \eta)$. Let $K_- \subset K$ be the -1 -eigenspace of the involution $\iota \in \text{Gal}(K/F)$. We construct an isomorphism $\Xi : K_- \rightarrow \mathcal{A}^2$ and associate to $t \in K_-$ a K -valued hermitian form H_t on the K -vector space $(V_{\mathbb{Q}}, \eta)$. Let $m : \text{Spin}(V_{\mathbb{R}}) \rightarrow GL(H^*X, \mathbb{R})$ and $\rho : \text{Spin}(V_{\mathbb{R}}) \rightarrow SO_+(V_{\mathbb{R}})$ be the spin and vector representations. The complex structure I_X of X acts on the spin representation $H^*(X, \mathbb{R})$ via $m(\tilde{I})$, for an element $\tilde{I} \in \text{Spin}(V_{\mathbb{R}})_B$, and $\rho(\tilde{I}) \in SO_+(V_{\mathbb{R}})$ is the complex structure of $X \times \hat{X}$. The adjoint orbit in $\text{Spin}(V_{\mathbb{R}})_B$ of $\tilde{I}$ is shown to be the period domain for a complete family of polarized abelian varieties of Weil type deformation equivalent to $(X \times \hat{X}, \eta)$.

In Section 8 we associate to an embedding $\hat{\eta} : F \rightarrow \text{End}_{\mathbb{Q}}(X)$, a polarization $\Theta \in \wedge_F^2 H^1(X, \mathbb{Q})$, an element $q \in F$ with $K = F(\sqrt{-q})$, and an element $t \in K_-$, a complex multiplication $\eta : K \rightarrow \text{End}_{\mathbb{Q}}(X \times \hat{X})$ and a polarization Ξ_t , such that $(X \times \hat{X}, \eta, \Xi_t)$ is a polarized abelian variety of split Weil type.

In Section 9 we consider two coherent sheaves F_1, F_2 on X with $ch(F_i) \in B$. We refer to such sheaves as *secant sheaves*. We associate to F_1 and F_2 an object $\mathcal{G}$ in $D^b(X \times \hat{X})$, with a $\text{Spin}(V)_B$ -invariant characteristic class $\kappa(\mathcal{G}) := ch(\mathcal{G}) \exp(-c_1(\mathcal{G})/\text{rank}(\mathcal{G}))$, via a derived equivalence $\Phi : D^b(X \times X) \rightarrow D^b(X \times \hat{X})$ introduced by Orlov. The $\text{Spin}(V)_B$ -invariance of $\kappa(\mathcal{G})$ implies that the class $\kappa_{d/2}(\mathcal{G}) \in H^{\frac{d}{2}, \frac{d}{2}}(X \times \hat{X}, \mathbb{Q})$ is the sum $\gamma + \delta$, where γ is a Weil class and δ is a polynomial in classes in $\mathcal{A}^2$. In Section 10 we give a criterion in terms of $ch(F_i)$, $i = 1, 2$, for the Weil class γ associated to $\mathcal{G}$ not to vanish.

In Section 11 we specialize to the case where $K = \mathbb{Q}(\sqrt{-q})$ is an imaginary quadratic number field, q a positive integer, and X is the Jacobian of a genus 3 curve C . We show that the ideal sheaf of $q + 1$ translates of the Abel-Jacobi curve on X is a secant sheaf. This leads to a pair of secant sheaves F_1 and F_2 , for which the dual of $\mathcal{G}$ is a coherent sheaf $\mathcal{E}$ on $X \times \hat{X}$, which equivariantly satisfies the hypotheses of the Semi-regularity theorem. We then use the theorem to prove the algebraicity of the Weil classes on all polarized abelian sixfolds of split Weil type.

In Section 12 we express the hope that a speculative stronger version of the Semi-regularity theorem would lead to the proof of algebraicity of Weil classes on some abelian varieties of dimension ≥ 8 .

2. THE SEMI-REGULARITY THEOREM

Let K be a CM-field and F its totally real subfield. Let X be an abelian variety admitting an algebra embedding $\hat{\eta} : F \rightarrow \text{End}_{\mathbb{Q}}(X)$. Set $\hat{X} := \text{Pic}^0(X)$. The rough strategy is to (1) construct an embedding $\eta : K \rightarrow \text{End}_{\mathbb{Q}}(X \times \hat{X})$ making $X \times \hat{X}$ an abelian variety of Weil type. (2) Construct a coherent sheaf E on $X \times \hat{X}$ a characteristic class of which yields an algebraic Weil class in $HW(X \times \hat{X}, \eta)$. (3) Deform the sheaf E along with $(X \times \hat{X}, \eta)$ to all deformation equivalent abelian varieties of Weil type using the Semi-regularity theorem.

We state next the Semi-regularity theorem. Let Y be an N -dimensional compact Kähler manifold and let E be a coherent sheaf over Y . Denote by $at_E \in \text{Ext}^1(E, E \otimes \Omega_Y^1)$ the Atiyah class of E . Let σ_q , $0 \leq q \leq N-2$, be the composition

$$\text{Ext}^2(E, E) \xrightarrow{at_E^q/q!} \text{Ext}^{2+q}(E, E \otimes \Omega_Y^q) \xrightarrow{tr} H^{q,q+2}(Y).$$

The semi-regularity map is

$$\sigma_E := (\sigma_0, \dots, \sigma_{N-2}) : \text{Ext}^2(E, E) \rightarrow \bigoplus_{q=0}^{N-2} H^{q,q+2}(Y).$$

The sheaf E is *semi-regular*, if σ_E is injective. Note that if Y is a surface, then $\sigma_E = \sigma_0$. If, furthermore, Y is a $K3$ or abelian surface and E is a simple sheaf, i.e., $\text{End}(E) \cong \mathbb{C}$, then σ_0 is an isomorphism, by Serre's duality. Hence, every simple sheaf over a $K3$ or abelian surface is semi-regular.

Buchweitz and Flenner show that the following diagram is commutative

$$(2.1) \quad \begin{array}{ccc} H^1(Y, TY) & \xrightarrow{\downarrow at_E} & \text{Ext}^2(E, E) \\ & \searrow \downarrow ch(E) & \swarrow \sigma_E \\ & \bigoplus_{q=0}^{N-2} H^{q,q+2}(Y) & \end{array}$$

[BF1, Cor. 4.3]. A first order infinitesimal deformation $\xi \in H^1(TY)$ of Y belongs to the kernel of $\downarrow at_E$, if and only if it can be lifted to a first order deformation of the pair (Y, E) [To]. The kernel of $\downarrow ch(E)$ consists of $\xi \in H^1(TY)$, such that $ch(E)$ remains of Hodge type in the direction of ξ , by Griffiths' transversality. If E is semi-regular, the two conditions are equivalent, by the commutativity of the diagram. Buchweitz and Flenner extended this observation, generalizing Bloch's Semi-regularity theorem [Blo] as follows. Let $\pi : \mathcal{Y} \rightarrow B$ be a deformation of a compact Kähler manifold Y_0 over a smooth germ $(B, 0)$ and set $Y_b := \pi^{-1}(b)$ for $b \in B$. Let $\gamma = \sum_{p=0}^N \gamma_p$, where γ_p is a Hodge class in $H^{p,p}(Y_0, \mathbb{Q})$. We say that γ *remains of Hodge type over B* , if for all p the class γ_p extends to a horizontal section of the local system $R^{2p}\pi_*\mathbb{Q}$, which belongs to the direct summand $R^p\pi_*\Omega^p$ under the Hodge decomposition.

Theorem 2.1. [BF1, Th. 1.5] *Let E be a semi-regular coherent sheaf over Y_0 , such that $ch(E)$ remains of Hodge type over B . Then E extends to a coherent sheaf over $\pi^{-1}(U)$ for some open analytic neighborhood U of 0 in B .*

Assume instead that the base B is a smooth and connected analytic space. The theorem then implies that each class $ch_p(E)$ is algebraic over the whole of B , since the locus where it is algebraic is a countable union of Zariski closed analytic subsets [Vo, Sec. 4.2], and it contains the non-empty open subset U over which E deforms.

Assume that the rank r of E does not vanish. The condition that $ch(E)$ remains of Hodge type over B is equivalent to the conjunction of two independent conditions:

- (1) The class $c_1(E)$ remains of Hodge type over B .
- (2) The class $\kappa(E) := ch(E) \exp(-c_1(E)/r)$ remains of Hodge type over B .

Assume that (2) holds but (1) fails. If $c_1(E)/r$ is integral, then there exists a line bundle L over Y_0 with $c_1(L) = c_1(E)/r$ and $ch(E \otimes L^{-1}) = \kappa(E)$ remains of Hodge type over B . The tensor product of a semi-regular sheaf with a line-bundle is semi-regular and so

Theorem 2.1 applies to $E \otimes L^{-1}$ to conclude that $\kappa(E)$ remains algebraic in every fiber of π .

The above conclusion remains valid even if $c_1(E)/r$ is not integral. In that case we need to replace L above by a line bundle over a μ_r -gerb. Equivalently and more elementary, we replace L by a rank 1 locally free coherent sheaf twisted by a Čech 2-cocycle θ with coefficient in the local system μ_r of r -th roots of unity. See [Ca1] for basic facts about twisted sheaves. Represent the line bundle $\det(E)$ by a Čech 1-cocycle $\varphi_{ij} \in \mathcal{O}_{Y_0}^*(U_{ij})$ with respect to an open covering $\mathcal{U} := \{U_i\}_{i \in I}$ in the analytic topology, such that $U_{ij} := U_i \cap U_j$ is simply connected, for all $i, j \in I$. Let ψ_{ij} be an r -th root of φ_{ij} and set $\theta_{ijk} := \psi_{ij}\psi_{jk}\psi_{ki}^{-1}$. Then $\theta_{ijk}^r = 1$, since $\{\varphi_{ij}\}$ is a cocycle, and $\theta := \{\theta_{ijk}\}$ is a Čech 2-cocycle in $Z(\mathcal{U}, \mu_r)$. The 2-cocycle θ is the coboundary associated to the 1-cochain ψ_{ij} . The 1-cochain $\{\psi_{ij}\}$ glues $\{\mathcal{O}_{U_i}\}_{i \in I}$ to a rank 1 coherent sheaf L twisted by the cocycle θ . The tensor product $E \otimes L^{-1}$ is a coherent sheaf twisted by the cocycle θ^{-1} . The line-bundle $\det(E \otimes L^{-1})$ is twisted by the 2-cocycle θ^{-r} , which is trivial, and is represented by the gluing cocycle $\varphi_{ij}\psi_{ij}^{-r} = 1$, and so the determinant line bundle of $E \otimes L^{-1}$ is trivial. The definitions of the Atiyah class, Chern character, and the semi-regularity map, all extend to μ_r -twisted sheaves ([Li], [M2, Def. 7.3.5, 7.3.6]), the class $\kappa(E)$ is the Chern character $ch(E \otimes L^{-1})$ [M2, Lem. 7.3.7], and $E \otimes L^{-1}$ is semi-regular, since E is by assumption. The semi-regularity theorem was generalized by Pridham to the setting of derived stacks, which applies to our μ_r -gerb [Pr, Rem. 2.26] (see also [M2, Sec 7.4] for a proof in the case of families of abelian varieties). We thus conclude that if E is semi-regular and $\kappa(E)$ remains of Hodge type over B , then $\kappa(E)$ remains algebraic in every fiber of π .

3. ABELIAN VARIETIES AND SPIN GROUPS

We recall in this section the fundamental role spin groups play in the geometry of abelian varieties. Let X be an abelian n -fold and set $\hat{X} := \text{Pic}^0(X)$. Then $H^1(\hat{X}, \mathbb{Z})$ is naturally isomorphic to $H^1(X, \mathbb{Z})^*$ and so the abelian group $V := H^1(X, \mathbb{Z}) \oplus H^1(\hat{X}, \mathbb{Z})$ is endowed with the symmetric bilinear pairing $((w_1, \theta_1), (w_2, \theta_2))_V := \theta_1(w_2) + \theta_2(w_1)$, which is unimodular, even, of signature $(2n, 2n)$. Note that the two direct summands $H^1(X, \mathbb{Z})$ and $H^1(\hat{X}, \mathbb{Z})$ are each a maximal isotropic subgroup. Let $SO(V)$ be the special orthogonal group of V and let $SO(V_\bullet)$ be that of $V_\bullet := V \otimes_{\mathbb{Z}} \bullet$, where $\bullet$ is any field. Let $\text{Spin}(V)$ be the subgroup of $\text{Spin}(V_{\mathbb{Q}})$ leaving the lattice $V \subset V_{\mathbb{Q}}$ invariant. Let $\rho : \text{Spin}(V_\bullet) \rightarrow SO(V_\bullet)$ be the natural homomorphism and denote its image by $SO_+(V_\bullet)$, for $\bullet = \mathbb{Z}$ or a field.

The spin representation of $\text{Spin}(V_\bullet)$ is constructed as the exterior algebra of a maximal isotropic subspace of $V_\bullet$ [Ch]. We choose the subspace to be $H^1(X, \bullet)$, so that $S_\bullet := H^*(X, \bullet)$ is the spin representation and $S_\bullet^+ := H^{ev}(X, \bullet)$ and $S_\bullet^- := H^{odd}(X, \bullet)$ are the half-spin representations.

Set $S := H^*(X, \mathbb{Z})$. It comes with the natural bilinear pairing $(\alpha, \beta)_S := \int_X \tau(\alpha) \cup \beta$, where $\tau : H^i(X, \mathbb{Z}) \rightarrow H^i(X, \mathbb{Z})$ is multiplication by $(-1)^{i(i-1)/2}$. The pairing is symmetric, if n is even, and anti-symmetric, if n is odd. It extends naturally to a pairing on $S_{\mathbb{Q}}$. Given an object F in the bounded derived category of coherent sheaves $D^b(X)$ we have $\tau(ch(F)) = ch(F^\vee)$, where $F^\vee$ is the derived dual object $\mathcal{R}Hom(F, \mathcal{O}_X)$.

The spin representation is defined over $\mathbb{Z}$ as follows. The Clifford algebra $C(V)$ is the quotient of the tensor algebra $\bigoplus_{k=0}^{\infty} V^{\otimes k}$ by the two-sided ideal generated by $\{v_1 v_2 + v_2 v_1 - (v_1, v_2)_V : v_1, v_2 \in V\}$, where the integer $(v_1, v_2)_V$ is regarded in $V^{\otimes 0} := \mathbb{Z}$. The algebra $C(V)$ is $\mathbb{Z}/2\mathbb{Z}$ -graded, $C(V) = C(V)^{ev} \oplus C(V)^{odd}$. Let $m : V \rightarrow \text{End}(S)$ be the homomorphism mapping $(w, \theta) \in V$ to $w \wedge (\bullet) + \theta](\bullet)$, where $\theta] : S \rightarrow S$ is the contraction with $\theta \in H^1(\hat{X}, \mathbb{Z}) \cong H^1(X, \mathbb{Z})^*$. Then $m_{v_1} m_{v_2} + m_{v_2} m_{v_1} = (v_1, v_2) id_S$, and so m extends to a homomorphism

$$(3.1) \quad m : C(V) \rightarrow \text{End}(S),$$

which is in fact an isomorphism of algebras [GLO, Prop. 3.2.1(e)]. Note that V projects injectively into $C(V)$ and we denote its image by V as well. The Clifford group is the subgroup $G(V)$ of invertible elements of $C(V)$ conjugating V to itself

$$G(V) := \{g \in C(V)^{\times} : gVg^{-1} = V\}$$

and we denote by

$$(3.2) \quad \rho : G(V) \rightarrow O(V)$$

the natural homomorphism. Given an element $v \in V$ with $(v, v) = \pm 2$, then v is an element of $G(V)$ and ρ_v is minus the reflection in the co-rank 1 sublattice $v^{\perp} \subset V$ orthogonal to v . The *main anti-involution* $\tau : C(V) \rightarrow C(V)$ sends $v_1 \cdots v_r$ to $v_r \cdots v_1$. The main involution $\alpha : C(V) \rightarrow C(V)$ acts on $C(V)^{ev}$ as the identity and multiplies $C(V)^{odd}$ by -1 . The conjugation $x \mapsto x^*$ is the composition $\tau \circ \alpha$. Then

$$\text{Spin}(V) = \{g \in C(V)^{ev} : gg^* = 1 \text{ and } gVg^{-1} = V\}.$$

The spin representation

$$(3.3) \quad m : \text{Spin}(V) \rightarrow \text{Aut}(S, (\bullet, \bullet)_S)$$

is the restriction of (3.1) and is hence faithful. The vector representation $\rho : \text{Spin}(V) \rightarrow SO_+(V)$ is the restriction of (3.2) and its kernel is $\{\pm 1\}$.

Let $\text{Aut}(D^b(X))$ be the group of exact auto-equivalences of $D^b(X)$. Every equivalence of derived categories $\Phi : D^b(X) \rightarrow D^b(Y)$ between two smooth projective varieties X and Y is represented as the *Fourier-Mukai transform*

$$R\pi_{Y,*}(R\pi_X^*(\bullet) \otimes F) : D^b(X) \rightarrow D^b(Y),$$

where π_X and π_Y are the projections from $X \times Y$, by a theorem of Orlov. The object F in $D^b(X \times Y)$ is unique up to isomorphism and is called the *Fourier-Mukai kernel* of Φ . We denote by $\Phi^H : H^*(X, \mathbb{Q}) \rightarrow H^*(Y, \mathbb{Q})$ the correspondence homomorphism $[ch(F)]^*$. Denote by $\text{Spin}_{Hdg}(V) \subset \text{Spin}(V)$ the subgroup preserving the Hodge structure of $V = H^1(X \times \hat{X}, \mathbb{Z})$. Orlov proved that when X and Y are abelian varieties, then $\Phi^H : H^*(X, \mathbb{Z}) \rightarrow H^*(Y, \mathbb{Z})$ is an integral isomorphism [Or]. Furthermore, when $X = Y$, then the homomorphism $\Phi \mapsto \Phi^H$ factors through a surjective homomorphism $\text{Aut}(D^b(X)) \rightarrow \text{Spin}_{Hdg}(V)$ and the spin representation (3.3). We get the short exact sequence

$$(3.4) \quad 0 \rightarrow X \times \hat{X} \times 2\mathbb{Z} \rightarrow \text{Aut}(D^b(X)) \rightarrow \text{Spin}_{Hdg}(V) \rightarrow 0,$$

where the subgroup X corresponds to the auto-equivalences induced by translation automorphisms and the subgroup $\hat{X}$ corresponds to tensorization by line bundles in $\text{Pic}^0(X)$ (see [Or]). The factor $2\mathbb{Z}$ corresponds to even shifts $[2k]$, $k \in \mathbb{Z}$. The subgroup $X \times \hat{X}$ is the identity component of $\text{Aut}(D^b(X))$ and the latter acts on it by conjugation. The resulting homomorphism $\text{Aut}(D^b(X)) \rightarrow GL(H^1(X \times \hat{X}, \mathbb{Z}))$ factors through $\text{Spin}_{Hdg}(V)$ via the restriction of the vector representation ρ [Or, GLO].

4. A STRATEGY FOR PROVING THE ALGEBRAICITY OF WEIL CLASSES

Let K be a CM-field and let F be its totally real subfield. Assume that the abelian n -fold X admits an algebra embedding $\hat{\eta} : F \rightarrow \text{End}_{\mathbb{Q}}(X)$. For every rational representation $\psi : F \rightarrow \text{End}(\mathbb{Q}^{2n})$ (so that $n = de/4$), there exist non-empty $\frac{n}{2}(\frac{d}{2} + 1)$ -dimensional moduli spaces of polarized abelian n -folds X with an algebra embedding $\hat{\eta} : F \rightarrow \text{End}_{\mathbb{Q}}(X)$ equivalent to ψ [BL, Sec. 9.2]. See [E, Sh] and references therein for examples of Jacobians with real multiplication by F . We get an algebra embedding $F \rightarrow \text{End}_{\mathbb{Q}}(X \times \hat{X})$, which we denote by $\hat{\eta}$ as well. Let $\hat{\Sigma}$ be the set of all field embeddings $F \rightarrow \mathbb{R}$. The vector space structure of $V_{\mathbb{Q}} = H^1(X \times \hat{X}, \mathbb{Q})$ over F yields the decomposition $V_{\mathbb{R}} = \bigoplus_{\hat{\sigma} \in \hat{\Sigma}} V_{\hat{\sigma}, \mathbb{R}}$, where the pairing $(\bullet, \bullet)_V$ restricts to the natural pairing on $V_{\hat{\sigma}, \mathbb{R}} = H_{\hat{\sigma}}^1(X, \mathbb{R}) \oplus H_{\hat{\sigma}}^1(\hat{X}, \mathbb{R})$.

We would like to extend $\hat{\eta}$ to an embedding $\eta : K \rightarrow \text{End}_{\mathbb{Q}}(X \times \hat{X})$ endowing $X \times \hat{X}$ with the structure of an abelian variety of Weil type, establish the algebraicity of the Weil classes on $X \times \hat{X}$, and use the Semi-regularity theorem to deform them to all abelian varieties of Weil type in the same connected component in moduli. Let $\mathcal{A}^2 \subset H^{1,1}(X \times \hat{X}, \mathbb{Q})$ be the $e/2$ -dimensional subspace of classes that remain of Hodge type under all deformations of $(X \times \hat{X}, \eta)$ as an abelian variety of Weil type. An explicit description of $\mathcal{A}^2$ is given in Proposition 7.3 below. Following is our strategy.

- (1) Construct a complex multiplication $\eta : K \rightarrow \text{End}_{\mathbb{Q}}(X \times \hat{X})$, extending $\hat{\eta}$, and an η -compatible polarization h . Let $V_{\mathbb{C}} = \bigoplus_{\sigma \in \Sigma} V_{\sigma}$ be the decomposition associated to η . We require that for each $\sigma : K \rightarrow \mathbb{C}$, restricting to $\hat{\sigma} : F \rightarrow \mathbb{R}$, the summand V_{σ} is a maximal isotropic subspace of $V_{\hat{\sigma}, \mathbb{C}}$.
- (2) Construct a coherent sheaf E over $X \times \hat{X}$ of non-zero rank r satisfying:
 - (a) E is semi-regular.
 - (b) The class $\kappa(E) := ch(E) \exp(-c_1(E)/r)$ remains of Hodge type under all deformations of $(X \times \hat{X}, \eta, h)$ as a polarized abelian variety of Weil type.
 - (c) The class $\kappa_{d/2}(E)$ in $H^{\frac{d}{2}, \frac{d}{2}}(X \times \hat{X}, \mathbb{Q})$ does not belong to the image of $\text{Sym}^{d/2}(\mathcal{A}^2)$.

Conditions (2a) and (2b) and the Semi-regularity theorem imply that $\kappa(E)$ remains algebraic on every polarized abelian variety of Weil type (A, η', h') in the connected component of moduli containing $(X \times \hat{X}, \eta, h)$. It follows that $\kappa_{d/2}(E)$ belongs to the subspace $\text{Im}[\text{Sym}^{d/2}(\mathcal{A}^2)] \oplus HW(X \times \hat{X}, \eta)$, since these are the Hodge classes that remain of Hodge type on the generic abelian variety of Weil type, by Equation (1.1). It follows that $\gamma := \kappa_{d/2}(E) - \delta$ is a non-zero class in $HW(X \times \hat{X}, \eta)$, for some $\delta \in \text{Im}[\text{Sym}^{d/2}(\mathcal{A}^2)]$, by Condition (2c). The class γ remains algebraic on every (A, η', h')

deformation equivalent to $(X \times \hat{X}, \eta, h)$, since $\kappa_{d/2}(E)$ and δ do. Now K acts on $H^*(A, \mathbb{Q})$ via algebraic correspondences and $HW(A, \eta')$ is 1-dimensional over K . Hence every class in $HW(A, \eta')$ is algebraic.

Condition (2b) imposes a compatibility between the class $\kappa_{d/2}(E)$ and the complex multiplication η . The latter is equivalent to the data of the character subspaces $V_{\sigma, \mathbb{C}}$, $\sigma \in \Sigma$, of $V_{\mathbb{C}}$. The key to achieving this compatibility is Chevalley's theory of pure spinors, which is our next topic. We require the subspaces V_{σ} to be maximal isotropic in Condition (1) in order for the representation theory of spin groups to guide our construction.

5. PURE SPINORS

The Grassmannian $IGr(2n, V_{\bullet})$ of maximal isotropic subspaces of $V_{\bullet}$, $\bullet$ a field, has two connected components, $IGr^+(2n, V_{\bullet})$ and $IGr^-(2n, V_{\bullet})$. Given a maximal isotropic subspace $W \subset V_{\bullet}$, set

$$(5.1) \quad \ell_W := \{\lambda \in S_{\bullet} : m_v(\lambda) = 0, \forall v \in W\}.$$

Then ℓ_W is a one-dimensional subspace of $S_{\bullet}$, which is contained either in $S_{\bullet}^+$, if $W \in IGr^+(2n, V_{\bullet})$, or in $S_{\bullet}^-$, if $W \in IGr^-(2n, V_{\bullet})$ [Ch, III.1.4]. The morphism

$$\ell : IGr^{\pm}(2n, V_{\bullet}) \rightarrow \mathbb{P}(S_{\bullet}^{\pm})$$

is a $\text{Spin}(V_{\bullet})$ -equivariant embedding and its image is called the *even/odd spinorial variety*. A non-zero element $\lambda \in \ell_W$ is called an *even/odd pure spinor* and we will refer to ℓ_W as a pure spinor as well. It follows immediately from the definition (5.1) that

$$\ell_{H^1(\hat{X}, \bullet)} = H^0(X, \bullet) \quad \text{and} \quad \ell_{H^1(X, \bullet)} = H^{2n}(X, \bullet).$$

In particular, $1 \in H^0(X, \bullet) \subset S_{\bullet}^+$ is an even pure spinor. Given an element $g \in \text{Spin}(V_{\bullet})$, the class $m_g(1)$ is thus an even pure spinor.

Example 5.1. Let K be a CM-field with a totally real subfield F . Choose $q \in F$, such that $\hat{\sigma}(q) > 0$, for all $\hat{\sigma} \in \hat{\Sigma}$, so that $K = F(\sqrt{-q})$, where $\sqrt{-q}$ is a choice of a square root in K . Assume that the abelian n -fold X admits an embedding $\hat{\eta} : F \rightarrow \text{End}_{\mathbb{Q}}(X)$. Let $\Theta \in \wedge_F^2 H^1(X, \mathbb{Q})$ be a non-degenerate class. Set

$$(5.2) \quad \begin{aligned} \exp(\sqrt{-q}\Theta) &:= 1 + \sqrt{-q}\Theta + \dots + \frac{(\sqrt{-q})^k}{k!}\Theta^k \dots = \alpha + \sqrt{-q}\beta, \\ \alpha &:= 1 - \frac{q}{2}\Theta^2 + \dots + \frac{(-q)^j}{(2j)!}\Theta^{2j} + \dots \\ \beta &:= \Theta - \frac{q}{3!}\Theta^3 + \dots + \frac{(-q)^j}{(2j+1)!}\Theta^{2j+1} + \dots \end{aligned}$$

where here Θ^k denotes the element of $\wedge_F^{2k} H^1(X, \mathbb{Q})$. We have the natural isomorphism

$$\text{Hom}_F(H^1(X, \mathbb{Q}), F) \xrightarrow{\cong} \text{Hom}_{\mathbb{Q}}(H^1(X, \mathbb{Q}), \mathbb{Q}),$$

given by $f \mapsto \text{tr}_{F/\mathbb{Q}} \circ f$. Hence, the embedding $\hat{\eta}$ extends to an algebra embedding $\hat{\eta} : F \rightarrow \text{End}_F(V_{\mathbb{Q}})$ and we denote by $V_{\hat{\eta}}$ the vector space $V_{\mathbb{Q}}$ regarded as a vector space

over F . Denote by

$$(5.3) \quad (\bullet, \bullet)_{V_{\hat{\eta}}}$$

the natural F -valued pairing on $V_{\hat{\eta}}$, so that $(x, y)_V = \text{tr}((x, y)_{V_{\hat{\eta}}})$. We get the groups $\text{Spin}(V_{\hat{\eta}})$ and $\text{Spin}(V_{\hat{\eta}} \otimes_F K)$. Product with $\exp(\sqrt{-q}\Theta)$ in the exterior algebra $(\wedge_F^* V_{\hat{\eta}}) \otimes K \cong \wedge_K^*(V_{\hat{\eta}} \otimes_F K)$ is an element of $m(\text{Spin}(V_{\hat{\eta}} \otimes_F K))$. Evaluating it at the pure spinor 1 we see that the element $\exp(\sqrt{-q}\Theta)$ is an even pure spinor corresponding to a maximal isotropic subspace $W \subset V_{\hat{\eta}} \otimes_F K$. Let $g \in \text{Spin}(V_{\hat{\eta}} \otimes_F K)$ be the element satisfying $m_g = \exp(\sqrt{-q}\Theta) \cup (\bullet)$. One computes that

$$(5.4) \quad W = \rho_g(H^1(\hat{X}, \mathbb{Q}) \otimes_F K) = \{(-\sqrt{-q}(\theta] \Theta), \theta) : \theta \in \text{Hom}_F(H^1(X, \mathbb{Q}), F)_F \otimes K\}.$$

Note that $W \cap \iota(W) = (0)$, where ι acts on the second tensor factor of $V_{\hat{\eta}} \otimes_F K$ as the involution in $\text{Gal}(K/F)$, since Θ is assumed non-degenerate. $\square$

Assume given a maximal isotropic subspace $W \subset V_{\hat{\eta}} \otimes_F K$ such that $W \cap \iota(W) = (0)$. If $F = \mathbb{Q}$ and $K = \mathbb{Q}(\sqrt{-q})$, q a positive integer, then a choice of $\sqrt{-q} \in \mathbb{C}$ determines an embedding $\sigma : K \rightarrow \mathbb{C}$ and an embedding of W as a maximal isotropic subspace V_{σ} of $V_{\mathbb{C}}$. The complex conjugate embedding $\bar{\sigma}$ yields the maximal isotropic subspace $V_{\bar{\sigma}} = \bar{V}_{\sigma}$. For a general CM-field K we get a maximal isotropic subspace W_T of $V_{\mathbb{C}}$ associated to each choice of a CM-type $T : \hat{\Sigma} \rightarrow \Sigma$. We denote by T the two equivalent data:

- a subset of Σ consisting of a choice of one embedding for each pair of two complex conjugate embeddings, and
- a right inverse $T : \hat{\Sigma} \rightarrow \Sigma$ of the restriction map $\Sigma \rightarrow \hat{\Sigma}$.

We have the isomorphisms

$$(5.5) \quad V_{\hat{\eta}} \otimes_{\mathbb{Q}} \mathbb{C} \cong V_{\hat{\eta}} \otimes_F (F \otimes_{\mathbb{Q}} \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} = \bigoplus_{\hat{\sigma} \in \hat{\Sigma}} (V_{\hat{\eta}} \otimes_{F, \hat{\sigma}} \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}.$$

Denote by $W_{T, \hat{\sigma}}$ the subspace spanned over $\mathbb{C}$ by the image of W via the homomorphism

$$(5.6) \quad \text{id} \otimes \hat{\sigma} \otimes T(\hat{\sigma}) : V_{\hat{\eta}} \otimes_F F \otimes_F K \rightarrow V_{\hat{\eta}} \otimes_{F, \hat{\sigma}} \mathbb{R} \otimes_{\mathbb{R}} \mathbb{C}.$$

$W_{T, \hat{\sigma}}$ is a subspace of the direct summand corresponding to $\hat{\sigma}$ on the right hand side of (5.5). Set

$$W_T := \bigoplus_{\hat{\sigma} \in \hat{\Sigma}} W_{T, \hat{\sigma}}.$$

One checks that W_T is a maximal isotropic subspace of $V_{\mathbb{C}}$, for every CM-type T . Indeed, $W_{T, \hat{\sigma}}$ is a maximal isotropic subspace of $V_{\hat{\eta}} \otimes_{F, \hat{\sigma}} \mathbb{C}$, for each $\hat{\sigma} \in \hat{\Sigma}$.

The direct sum decomposition $H^1(X, \mathbb{C}) = \bigoplus_{\hat{\sigma} \in \hat{\Sigma}} H_{\hat{\sigma}}^1(X, \mathbb{C})$ yields the isomorphism

$$(5.7) \quad S_{\mathbb{C}} := \wedge^* H^1(X, \mathbb{C}) \cong \bigoplus_{\hat{\sigma} \in \hat{\Sigma}} \wedge^* H_{\hat{\sigma}}^1(X, \mathbb{C}),$$

where the tensor product is in the category of $\mathbb{Z}/2\mathbb{Z}$ graded algebras. Denote the even pure spinor $\ell_{W_{T, \hat{\sigma}}} \subset \wedge^{\text{ev}} H_{\hat{\sigma}}^1(X, \mathbb{C})$ by $\ell_{T, \hat{\sigma}}$. Then $\bigotimes_{\hat{\sigma} \in \hat{\Sigma}} \ell_{T, \hat{\sigma}} \subset S_{\mathbb{C}}^+$ is the pure spinor ℓ_{W_T} . Let $\mathcal{T}_K$ be the set of all CM-types for K . The linear subspace $B_{\mathbb{C}} \subset S_{\mathbb{C}}^+$ spanned by the lines $\{\ell_{W_T}\}_{T \in \mathcal{T}_K}$ is defined over $\mathbb{Q}$ and corresponds to a subspace

$$(5.8) \quad B \subset S_{\mathbb{Q}}^+$$

of dimension $2^{e/2}$ [M3, Lem. 7.1.3, Cor. 7.2.2].

Denote by $\bar{T}$ the CM-type complex conjugate to T given by $\bar{T}(\hat{\sigma}) = T(\hat{\sigma}) \circ \iota$. Then $W_{\bar{T}, \hat{\sigma}} = \bar{W}_{T, \hat{\sigma}}$. Let $P_{\hat{\sigma}} \subset \wedge^{\text{ev}} H_{\hat{\sigma}}^1(X, \mathbb{C})$ be the 2-dimensional subspace spanned by $\ell_{T, \hat{\sigma}}$ and $\ell_{\bar{T}, \hat{\sigma}}$. We have the equality $B_{\mathbb{C}} = \bigotimes_{\hat{\sigma} \in \hat{\Sigma}} P_{\hat{\sigma}}$ with respect to the factorization (5.7).

6. COMPLEX MULTIPLICATION

Assume given a maximal isotropic subspace $W \subset V_{\hat{\eta}} \otimes_F K$ such that $W \cap \iota(W) = (0)$. Then $V_{\mathbb{Q}} = \{w + \iota(w) : w \in W\}$. Define $\eta : K \rightarrow \text{End}_F(V_{\hat{\eta}} \otimes_F K)$ by letting η_t act on W by multiplication by t and on $\iota(W)$ by multiplication by $\iota(t)$, for all $t \in K$. Then for $w \in W$, $\eta_t(w + \iota(w)) = tw + \iota(tw)$ and so $\eta_t(V_{\mathbb{Q}}) = V_{\mathbb{Q}}$. We get the algebra embedding

$$(6.1) \quad \eta : K \rightarrow \text{End}(V_{\mathbb{Q}}).$$

Note that the subspace $W_{T, \hat{\sigma}}$ of $V_{\hat{\sigma}, \mathbb{R}}$ depends only on the value $T(\hat{\sigma})$, as the homomorphism (5.6) depends only on this value. Hence, given $t \in K$, η_t acts on $W_{T, \hat{\sigma}}$ via multiplication by $T(\hat{\sigma})(t)$. So the subspace V_{σ} of $V_{\mathbb{C}}$ on which η_t acts via $\sigma(t)$ is $W_{T, \hat{\sigma}}$, if and only if $T(\hat{\sigma}) = \sigma$ and

$$(6.2) \quad W_{T, \mathbb{C}} = \bigoplus_{\hat{\sigma} \in \hat{\Sigma}} V_{T(\hat{\sigma})}.$$

Let $\text{Spin}(V_{\mathbb{Q}})_B$ be the subgroup of $\text{Spin}(V_{\mathbb{Q}})$ fixing every point in the secant space $B \subset S_{\mathbb{Q}}^+$ associated to W in (5.8). Define $\text{Spin}(V_{\mathbb{R}})_B$ analogously. Note that ρ maps $\text{Spin}(V_{\mathbb{Q}})_B$ injectively into $SO_+(V_{\mathbb{Q}})$, since -1 does not belong to $\text{Spin}(V_{\mathbb{Q}})_B$.

Let $\text{Spin}(V_{\bullet})_{\eta}$ be the subgroup of $\text{Spin}(V_{\bullet})$ consisting of elements commuting with $\eta(K)$, for $\bullet = \mathbb{Q}$ or $\mathbb{R}$.

Lemma 6.1. *$\text{Spin}(V_{\bullet})_B$ is a subgroup of $\text{Spin}(V_{\bullet})_{\eta}$, for $\bullet = \mathbb{Q}$ or $\mathbb{R}$.*

Proof. Let g be an element of $\text{Spin}(V_{\bullet})_B$. Each ℓ_{W_T} is m_g -invariant, for all $T \in \mathcal{T}_K$. Thus, W_T is ρ_g -invariant, for all $T \in \mathcal{T}_K$. Let $\sigma \in \Sigma$ restrict to F as $\hat{\sigma}$. The subspace V_{σ} is the intersection of W_T and $W_{T'}$, if $T(\hat{\sigma}) = T'(\hat{\sigma}) = \sigma$ and $T(\hat{\sigma}') \neq T'(\hat{\sigma}')$, for all $\hat{\sigma}' \in \hat{\Sigma} \setminus \{\hat{\sigma}\}$. Hence, V_{σ} is ρ_g -invariant, for all $\sigma \in \Sigma$. Consequently, ρ_g commutes with $\eta(K)$. $\square$

Lemma 6.2. [M3, Lem. 7.4.1] *Assume that B is spanned by Hodge classes. Then $\eta(K)$ is contained in $\text{End}_{\text{Hdg}}(V_{\mathbb{Q}}) \cong \text{End}_{\mathbb{Q}}(X \times \hat{X})$. Furthermore, $\dim(V^{1,0})_{\sigma} = \dim(V^{0,1})_{\sigma}$, for all $\sigma \in \Sigma$. Consequently, $(X \times \hat{X}, \eta)$ is of Weil type.*

Sketch of proof that $\eta(K) \subset \text{End}_{\text{Hdg}}(V_{\mathbb{Q}})$. There exists an element $\tilde{I}$ of $\text{Spin}(V_{\mathbb{R}})$ satisfying $m_{\tilde{I}} = I_X$ and $\rho_{\tilde{I}} = I_{X \times \hat{X}}$. The assumption that B is spanned by Hodge classes implies that $\tilde{I}$ belongs to $\text{Spin}(V_{\mathbb{R}})_B$. Lemma 6.1 implies that $\rho_{\tilde{I}}$ commutes with $\eta(K)$. Hence, $\eta(K)$ is contained in $\text{End}_{\text{Hdg}}(V_{\mathbb{Q}})$. $\square$

7. THE $\text{Spin}(V_{\mathbb{Q}})_B$ -INVARIANT SUBALGEBRA OF $H^*(X \times \hat{X}, \mathbb{Q})$

Let K_- be the -1 eigenspace of $\iota : K \rightarrow K$.

Lemma 7.1. (1) [M3, Lem 5.1.2] *The endomorphism $\eta_{\iota(t)}$ is the adjoint of η_t with respect to $(\bullet, \bullet)_V$, for all $t \in K$.*

- (2) [M3, Cor. 5.2.1] *Given $t \in K_-$, the F -valued pairing $\Xi_t(x, y) := (\eta_t(x), y)_{V_{\hat{\eta}}}$, where $(x, y)_{V_{\hat{\eta}}}$ is given in (5.3), is anti-symmetric and $\text{Spin}(V_{\mathbb{Q}})_{\eta}$ -invariant. The resulting homomorphism⁴ $\Xi : K_- \rightarrow \wedge_F^2 V_{\mathbb{Q}}^* \cap (\wedge^2 V_{\mathbb{Q}}^*)^{\text{Spin}(V_{\mathbb{Q}})_{\eta}}$ is injective.*
- (3) [M3, Lem. 7.3.1] *Given $t \in K_-$, let $H_t : V_{\mathbb{Q}} \times V_{\mathbb{Q}} \rightarrow K$ be given by*

$$H_t(x, y) := (-t^2)(x, y)_{V_{\hat{\eta}}} + t\Xi_t(x, y).$$

Then H_t is a $\text{Spin}(V_{\mathbb{Q}})_B$ -invariant hermitian form.

Given $g \in SO_+(V_{\hat{\sigma}, \mathbb{R}})$ and $\sigma \in \Sigma$ restricting to $\hat{\sigma} \in \hat{\Sigma}$, g leaves the restriction of $\sigma \circ H_t$ invariant $\Leftrightarrow g$ leaves the restriction of $\sigma \circ \Xi_t$ invariant $\Leftrightarrow (\eta_t(x), y)_V = (\eta_t(g(x)), g(y))_V$, for all $x, y \in V_{\hat{\sigma}, \mathbb{R}}$, $\Leftrightarrow g$ commutes with the restriction of η_t to $V_{\hat{\sigma}, \mathbb{R}}$. Hence, if g leaves the restriction of $\sigma \circ H_t$ invariant, then g leaves V_{σ} and $V_{\bar{\sigma}}$ invariant. Let $SU(V_{\hat{\sigma}, \mathbb{R}})$ be the subgroup of $SO_+(V_{\hat{\sigma}, \mathbb{R}})$ of elements leaving the restriction of $\sigma \circ H_t$ invariant and restricting to each of $V_{\sigma, \mathbb{C}}$ and $V_{\bar{\sigma}, \mathbb{C}}$ with determinant 1.

Lemma 7.2. [M3, Lem 9.1.1] *The natural homomorphism $\text{Spin}(V_{\mathbb{R}})_B \rightarrow \prod_{\hat{\sigma} \in \hat{\Sigma}} SO_+(V_{\hat{\sigma}, \mathbb{R}})$ maps $\text{Spin}(V_{\mathbb{R}})_B$ isomorphically onto $\prod_{\hat{\sigma} \in \hat{\Sigma}} SU(V_{\hat{\sigma}, \mathbb{R}})$.*

Lemma 6.1 shows that V_{σ} is $\text{Spin}(V_{\mathbb{Q}})_B$ -invariant and Lemma 7.2 shows that ρ_g , $g \in \text{Spin}(V_{\mathbb{Q}})_B$, acts trivially on $\wedge^d V_{\sigma}$, for all $\sigma \in \Sigma$. Hence, $HW(X \times \hat{X}, \eta)$ consists of $\text{Spin}(V_{\mathbb{Q}})_B$ -invariant classes. Lemmas 7.1(2) and 6.1 show that $\Xi(K_-)$ is an $e/2$ -dimensional subspace of $\text{Spin}(V_{\mathbb{Q}})_B$ -invariant elements. Let $\mathcal{A} := (\wedge^* V_{\mathbb{Q}})^{\text{Spin}(V_{\mathbb{Q}})_B} = H^*(X \times \hat{X}, \mathbb{Q})^{\text{Spin}(V_{\mathbb{Q}})_B}$ be the $\text{Spin}(V_{\mathbb{Q}})_B$ -invariant subalgebra.

Proposition 7.3. (1) [M3, Prop. 8.0.1] *The subalgebra $\mathcal{A}$ is generated by $\Xi(K_-)$ and $HW(X \times \hat{X}, \eta)$.*

(2) *If B is spanned by Hodge classes, then so is $\mathcal{A}$.*

Proof of (2). Assume that B is spanned by Hodge classes. The subspace $HW(X \times \hat{X}, \eta)$ consists of Hodge classes and $I_{X \times \hat{X}}$ belongs to $\rho(\text{Spin}(V_{\mathbb{R}})_B)$ and commutes with $\eta(K)$, by Lemma 6.2. Now $I := I_{X \times \hat{X}}$ is an isometry with respect to $(\bullet, \bullet)_V$. Hence,

$$\Xi_t(I(x), I(y)) := (\eta_t(I(x)), I(y))_V = (I\eta_t(x), I(y))_V = (\eta_t(x), y)_V = \Xi_t(x, y)$$

and so Ξ_t is of type $(1, 1)$. Hence, $\mathcal{A}$ consists of Hodge classes, by part (1). $\square$

Lemma 7.4. *Assume that B is spanned by Hodge classes and $\Xi_t \in H^{1,1}(X \times \hat{X}, \mathbb{Q})$ is an ample class, for some $t \in K_-$. Then the adjoint orbit of $I_{X \times \hat{X}}$ in $\text{Spin}(V_{\mathbb{R}})_B$ consists of complex structures I on the differentiable manifold A underlying $X \times \hat{X}$ with respect to which*

- (1) [M3, Lem. 9.2.2, 9.2.3] *$((A, I), \eta, \Xi_t)$ is a polarized abelian variety of Weil type.*
- (2) *The subalgebra $\mathcal{A} := H^*(A, \mathbb{Q})^{\text{Spin}(V_{\mathbb{Q}})_B}$ of $\text{Spin}(V_{\mathbb{Q}})_B$ -invariant classes in $H^*(A, \mathbb{Q})$ consists of Hodge classes.*

Proof of (2). Note that the proof of Proposition 7.3(2) applies to any element I of $\text{Spin}(V_{\mathbb{R}})_B$ satisfying $I^2 = -id$. $\square$

⁴Here we again regard $\wedge_F^2 V_{\mathbb{Q}}^*$ as a subspace of $\wedge_{\mathbb{Q}}^2 V_{\mathbb{Q}}^*$. The subspace $\oplus_{\hat{\sigma} \in \hat{\Sigma}} \wedge^2 V_{\hat{\sigma}, \mathbb{R}}^*$ of $\wedge^2 V_{\mathbb{R}}^*$ is defined over $\mathbb{Q}$ and corresponds to the image of the injective homomorphism $\wedge_F^2 V_{\mathbb{Q}}^* \rightarrow \wedge_{\mathbb{Q}}^2 V_{\mathbb{Q}}^*$ given by $\theta(\bullet, \bullet) \mapsto \text{tr}_{F/\mathbb{Q}} \circ \theta(\bullet, \bullet)$.

An example of complex multiplication η on $X \times \hat{X}$ satisfying the hypothesis of the above Lemma is given in the next section. The adjoint orbit $\mathcal{O}$ of $I_{X \times \hat{X}}$ in $\text{Spin}(V_{\mathbb{R}})_B$ parametrizes a complete family of polarized abelian varieties of Weil type, by [M3, Rem. 9.2.4]. Let T be a CM-type and consider the map $\mathcal{O} \rightarrow \prod_{\hat{\sigma} \in \hat{\Sigma}} \text{Gr}(d/2, V_{T(\hat{\sigma})})$ sending a complex structure I to $\{V_{T(\hat{\sigma})}^{1,0}\}_{\hat{\sigma} \in \hat{\Sigma}}$. The above map is an embedding of $\mathcal{O}$ as an open subset, in the classical topology, of the product of grassmannians [M3, Sec. 9.2]. See also [DM] and [CS, Lemma 11.5.25].

The group $\text{Spin}(V_{\mathbb{Q}})_B$ is the special Mumford-Tate group⁵ of the generic abelian variety of Weil type deformation equivalent to $(X \times \hat{X}, \eta)$.

8. EXAMPLES OF POLARIZED $X \times \hat{X}$ OF SPLIT WEIL TYPE

Assume that the class $\Theta \in \wedge_F^2 H^1(X, \mathbb{Q})$ in Example 5.1 is such that $\text{tr}_{F/\mathbb{Q}} \circ \Theta(\bullet, \bullet)$ is an ample class in $H^{1,1}(X, \mathbb{Q})$. If X is simple and $\hat{\eta} : F \rightarrow \text{End}_{\mathbb{Q}}(X)$ is an isomorphism, then $H^{1,1}(X, \mathbb{Q}) \subset \wedge_F^2 H^1(X, \mathbb{Q})$ (see [M3, Lem. 11.1.1]) and so any polarization Θ would arise this way. Let $W \subset V_{\hat{\eta}} \otimes_F K$ be the maximal isotropic subspace associated to $(\Theta, \sqrt{-q})$ in (5.4). Let $\eta : K \rightarrow \text{End}_{Hdg} H^1(X \times \hat{X}, \mathbb{Q})$ be the embedding given in (6.1) and let $B \subset H^{ev}(X, \mathbb{Q})$ be the secant linear subspace given in (5.8), both associated to W . Let $t \in K_-$ be a non-zero element.

Lemma 8.1. [M3, Lemma 11.1.2]

- (1) $\eta(K) \subset \text{End}_{Hdg} H^1(X \times \hat{X}, \mathbb{Q})$,
- (2) H_t is of split-type, and
- (3) t can be chosen so that Ξ_t is an ample class and $(X \times \hat{X}, \eta, \Xi_t)$ is a polarized abelian variety of split Weil type.

Sketch of proof. (1) It suffices to prove that B is spanned by Hodge classes, by Lemma 6.2. We know that B is a rational subspace, so it suffices to prove that it is contained in $\oplus_{p=0}^{2n} H^{p,p}(X)$. The proof follows easily from the assumption that Θ is of type (1, 1).

(2) Let $\{y_1, \dots, y_d\}$ be an F -basis of $H^1(\hat{X}, \mathbb{Q})$, such that $\text{span}\{(y_1, \dots, y_{d/2})\}$ is a Θ -isotropic subspace. Let $g \in \text{Spin}(V_{\hat{\eta}} \otimes_F K)$ be the element in Example 5.1 satisfying $m_g(\bullet) = \exp(\sqrt{-q}\Theta) \cup (\bullet)$. Then $\{\rho_g(y_1), \dots, \rho_g(y_d)\}$ is a K -basis of W , by definition of W , and so $\{\rho_g(y_j) + (id_{V_{\hat{\eta}}} \otimes \iota)(\rho_g(y_j)) : 1 \leq j \leq d\}$ is a K -basis of $V_{\mathbb{Q}}$ with respect to η . We have seen in Equation (5.4) that $\rho_g(y_j) = (-\sqrt{-q}(y_j] \Theta), y_j)$. Hence,

$$\rho_g(y_j) + (id_{V_{\hat{\eta}}} \otimes \iota)(\rho_g(y_j)) = 2(0, y_j)$$

and $\{(0, y_1), \dots, (0, y_d)\}$ is a K -basis of $V_{\mathbb{Q}}$. The definition of η yields the equality $\eta_t(0, y) = (-\hat{\eta}_t \sqrt{-q}(y] \Theta), 0)$. We claim that the subspace $Z := \text{span}_K \{(0, y_j)\}_{j=1}^{d/2}$ is H_t isotropic. Indeed, it is $(\bullet, \bullet)_{V_{\hat{\eta}}}$ -isotropic, since $H^1(\hat{X}, \mathbb{Q})$ is $(\bullet, \bullet)_{V_{\hat{\eta}}}$ -isotropic, and it is Ξ_t isotropic, since

$$\begin{aligned} \Xi_t((0, y_j), (0, y_k)) &= (\eta_t(0, y_j), (0, y_k))_V = (\hat{\eta}_t \sqrt{-q}(-y_j] \Theta, 0), (0, y_k))_V = -t \sqrt{-q} \Theta(y_j, y_k) \\ &= 0, \text{ for } 1 \leq j, k \leq d/2. \end{aligned}$$

⁵The special Mumford-Tate group of $H^1((A, I), \mathbb{Q})$ is the smallest algebraic subgroup of $GL(H^1(A, \mathbb{R}))$, which is defined over $\mathbb{Q}$ and which contains the circle group $\{a + bI : a, b \in \mathbb{R}, a^2 + b^2 = 1\}$.

(3) We regard $tr_{F/\mathbb{Q}} \circ \Xi_t \in \wedge_{\mathbb{Q}}^2 V_{\mathbb{Q}}^*$ as an element of $\wedge_{\mathbb{R}}^2 V_{\mathbb{R}}^*$ by extending scalars. The symmetric bilinear form $(tr_{F/\mathbb{Q}} \circ \Xi_t)(\bullet, I_{X \times \hat{X}}(\bullet))$ restricts to a definite form on $V_{\hat{\sigma}, \mathbb{R}}$, for each $\hat{\sigma} \in \hat{\Sigma}$, by the argument of [M2, Lem. 3.1.1]. The choice of t affects the sign of this form. The image of the map $F \rightarrow \mathbb{R}^{\hat{\Sigma}}$, given by $f \mapsto (\hat{\sigma}(f))_{\hat{\sigma} \in \hat{\Sigma}}$, contains a full lattice. Hence, for every function $s : \hat{\Sigma} \rightarrow \{+, -\}$ there exists $f \in F$, such that the sign of $\hat{\sigma}(f)$ is $s(\hat{\sigma})$, for all $\hat{\sigma} \in \hat{\Sigma}$. It follows that $t \in K_-$ can be chosen, so that the bilinear pairing is positive definite on $V_{\hat{\sigma}, \mathbb{R}}$, for each $\hat{\sigma} \in \hat{\Sigma}$. The equality $\dim(V_{\sigma}^{1,0}) = \dim(V_{\sigma}^{0,1}) = \frac{d}{2}$, for all $\sigma \in \Sigma$, follows from Lemma 6.2. $\square$

Strategy revisited: Note that Lemma 8.1 achieves item (1) in the Strategy section 4. The compatibility condition (2b) in Strategy section 4, between $\kappa(E)$ and η , is equivalent to the condition that $\kappa(E)$ would be $\text{Spin}(V_{\mathbb{Q}})_B$ -invariant, by Proposition 7.3(1) and Lemma 7.4(2). If F_1 and F_2 are coherent sheaves on X with Chern characters $ch(F_i) \in B$, then $F_1 \boxtimes F_2 := \pi_1^* F_1 \otimes \pi_2^* F_2$ is a coherent sheaf on $X \times X$ with $ch(F_1 \boxtimes F_2)$ invariant with respect to the diagonal action of $\text{Spin}(V_{\mathbb{Q}})_B$. We would like to transform such a sheaf to an object E on $X \times \hat{X}$ with a $\text{Spin}(V_{\mathbb{Q}})_B$ -invariant $\kappa(E)$ via an equivalence $\Phi : D^b(X \times X) \rightarrow D^b(X \times \hat{X})$. This requires the induced isomorphism of cohomologies $\Phi^H : H^*(X \times X) \rightarrow H^*(X \times \hat{X})$ to be $\text{Spin}(V)$ -equivariant, up to cup product with $\exp(\alpha)$ for some $\alpha \in H^{1,1}(X \times \hat{X}, \mathbb{Q})$. In the next section we observe that Orlov introduced such an equivalence in [Or].

9. ORLOV'S DERIVED EQUIVALENCE $\Phi : D^b(X \times X) \rightarrow D^b(X \times \hat{X})$

Let $\mu : X \times X \rightarrow X \times X$ be the automorphism given by $\mu(x, y) = (x + y, y)$. Let $\mathcal{P}$ be the Poincaré line bundle over $\hat{X} \times X$, normalized so that it restricts trivially to $\hat{X} \times \{0\}$. Let $id \times \Phi_{\mathcal{P}} : D^b(X \times \hat{X}) \rightarrow D^b(X \times X)$ be the integral transform with Fourier-Mukai kernel $\pi_{13}^* \mathcal{O}_{\Delta_X} \otimes \pi_{24}^* \mathcal{P}$, where π_{ij} is the projection from $X \times \hat{X} \times X \times X$ to the product of the i -th and j -th factors. Orlov's derived equivalence $\Phi : D^b(X \times X) \rightarrow D^b(X \times \hat{X})$ is the inverse of $\mu_* \circ (id \times \Phi_{\mathcal{P}}) : D^b(X \times \hat{X}) \rightarrow D^b(X \times X)$. The Fourier-Mukai kernel $\mathcal{U} \in D^b(X \times \hat{X} \times X \times X)$ of $\mu_* \circ (id \times \Phi_{\mathcal{P}})$ restricts to $\{(x, L)\} \times X \times X$ as the Fourier-Mukai kernel $\mathcal{U}_{x,L} \in D^b(X \times X)$ of the auto-equivalence $L \otimes t_{x,*} : D^b(X) \rightarrow D^b(X)$ of push-forward via the translation automorphism $t_x(y) = x + y$ of X and tensorization by the line bundle L . So Φ^{-1} encodes the fact that $X \times \hat{X}$ is a subgroup of the group of auto-equivalences of $D^b(X)$.

Let $\Phi^H : H^*(X \times X, \mathbb{Z}) \rightarrow H^*(X \times \hat{X}, \mathbb{Z})$ be the correspondence induced by the Chern character of the Fourier-Mukai kernel of Φ . Let $\tau : H^i(X, \mathbb{Z}) \rightarrow H^i(X, \mathbb{Z})$ be multiplication by $(-1)^{i(i-1)/2}$. The Künneth theorem identifies $H^*(X \times X, \mathbb{Z})$ as the tensor square $S \otimes_{\mathbb{Z}} S$ of the spin representation and $H^*(X \times \hat{X}, \mathbb{Z})$ is the exterior algebra $\wedge^* V$ of the vector representation. Hence, both are endowed with the structure of an integral representation of $\text{Spin}(V)$. The two are not isomorphic, but they become isomorphic⁶ after tensoring with $\mathbb{Q}$ [Ch, Sec. III.3.3].

⁶The representations are reducible, so the set of isomorphisms is not a $\mathbb{Q}^\times$ -torsor.

Proposition 9.1. [M2, Prop. 6.1.2] *The composition*

$$\tilde{\phi} := \exp(-c_1(\mathcal{P})/2) \cup \Phi^H \circ (id \otimes \tau) : H^*(X \times X, \mathbb{Q}) \rightarrow H^*(X \times \hat{X}, \mathbb{Q})$$

is $\text{Spin}(V)$ -equivariant.

The integral isomorphism

$$\Phi^H \circ (id \otimes \tau) : S \otimes_{\mathbb{Z}} S \cong H^*(X \times X, \mathbb{Z}) \rightarrow H^*(X \times \hat{X}, \mathbb{Z}) \cong \wedge^* V$$

conjugates the diagonal action of m_g , $g \in \text{Spin}(V)$, on $S \otimes_{\mathbb{Z}} S$ to an automorphism of $\wedge^* V$ leaving invariant the decreasing filtration $F_k(\wedge^* V) := \bigoplus_{j \geq k} \wedge^j V$ and it was previously known that the induced action on the graded summand $\wedge^k V$ is $\wedge^k \rho_g$ [GLO, Prop. 4.3.7 and Cor. 4.3.8]. If g belongs to $\text{Spin}_{Hdg}(V)$, then the action of m_g on $H^*(X \times X)$ need not preserve the grading and is the diagonal cohomological action of an auto-equivalence of X , by the right exactness of Orlov's sequence (3.4), while the action of ρ_g is the cohomological action of an automorphism of $X \times \hat{X}$. The paper [GLO] refers to this correspondence of symmetries as an instance of homological Mirror-Symmetry.

Keep the assumptions and notation of Section 6. Let F_1 and F_2 be coherent sheaves on X with $ch(F_i)$ in the secant linear subspace $B \in H^{ev}(X, \mathbb{Q})$. We say that F_1 and F_2 are *secant sheaves*. Given a class γ in $H^*(X \times \hat{X}, \mathbb{Q})$ with graded summand γ_i in $H^{2i}(X \times \hat{X}, \mathbb{Q})$ and with non-zero $\gamma_0 \in \mathbb{Q}$, set $\kappa(\gamma) := \gamma \exp(-\gamma_1/\gamma_0)$. Given an object $E \in D^b(\bullet)$ of non-zero rank, we have $\kappa(E) := \kappa(ch(E))$.

Corollary 9.2. *Assume that the rank of the object $E := \Phi(F_1 \boxtimes F_2^\vee) \in D^b(X \times \hat{X})$ is non-zero. The class $\kappa(E)$ is $\text{Spin}(V_{\mathbb{Q}})_B$ -invariant.*

Proof. The class $\gamma := \tilde{\phi}(ch(F_1 \boxtimes F_2)) := \exp(-c_1(\mathcal{P})/2) \cup ch(E)$ is $\text{Spin}(V)_B$ -invariant, by the $\text{Spin}(V)$ -equivariance of $\tilde{\phi}$ in Proposition 9.1 and the assumed $\text{Spin}(V_{\mathbb{Q}})_B$ -invariance of $ch(F_1 \boxtimes F_2)$. It follows that γ_1 is $\text{Spin}(V_{\mathbb{Q}})_B$ -invariant and hence so is $\kappa(\gamma)$. Finally, $\kappa(\gamma) = \kappa(E)$. $\square$

10. THE HOMOMORPHISM $B \otimes B \rightarrow HW(X \times \hat{X}, \eta)$

Keep the notation of Corollary 9.2. We consider next Condition (2c) in the Strategy section 4. It requires the class $\kappa_{d/2}(E) \in \mathcal{A}^d \oplus HW(X \times \hat{X}, \eta)$ not to belong to the direct summand $\mathcal{A}^d$. We need to translate it to a condition on $ch(F_1)$ and $ch(F_2)$.

We define a natural grading $B \otimes_{\mathbb{Q}} B = \bigoplus_{k=0}^{e/2} BB_k$ on $B \otimes_{\mathbb{Q}} B$ as follows. Given two CM-types T, T' , set $T \cap T' := T(\hat{\Sigma}) \cap T'(\hat{\Sigma}) \subset \Sigma$ and let $|T \cap T'|$ be its cardinality. The subspace

$$BB_{k, \mathbb{C}} := \bigoplus_{(T_1, T_2) \in \mathcal{T}_K \times \mathcal{T}_K, |T_1 \cap T_2| = k} \ell_{W_{T_1}} \otimes \ell_{W_{T_2}}$$

is defined over $\mathbb{Q}$ and corresponds to a subspace $BB_k \subset B \otimes_{\mathbb{Q}} B$ of dimension $\binom{e/2}{k} 2^{e/2}$.

In particular, $\dim(BB_1) = e2^{(\frac{e}{2}-1)}$. The equality $W_T \cap W_{T'} = \bigoplus_{\hat{\sigma} \in T \cap T'} V_{T(\hat{\sigma})}$ holds, by Equation (6.2).

Lemma 10.1. [M3, Lemma 10.1.3] *If $|T \cap T'| = k$, then the line $\Phi^H(\ell_{W_T} \otimes \tau(\ell_{W_{T'}}))$ belongs to $F_{dk}(V_{\mathbb{C}}) := \bigoplus_{i \geq dk} (\wedge^i V_{\mathbb{C}})$ and it projects onto the line $\wedge^{dk}[W_T \cap W_{T'}]$ in $\wedge^{dk} V_{\mathbb{C}}$. In particular, the subspace $HW(X \times \hat{X}, \eta)$ is equal to the image of the composition*

$$(10.1) \quad BB_1 \xrightarrow{\Phi^H \circ (id \otimes \tau)} F_d(\wedge^* V_{\mathbb{Q}}) \rightarrow \wedge^d V_{\mathbb{Q}}.$$

Sketch of proof. The key is the analogous statement [Ch, III.3.3] relating the top exterior power $\wedge^{\dim(W+W')}(W + W')$ to the image of the tensor product $\ell_W \otimes \ell_{W'}$ of the two pure spinors via an isomorphism $\tilde{\varphi} : S \otimes_{\mathbb{Z}} S \rightarrow \wedge^* V$ Chevalley constructs in terms of the Clifford algebra $C(V)$. The equality $\Phi^H \circ (id \otimes \tau) = [\Phi_P^H \otimes (\Phi_P^H)^{-1}] \circ \tilde{\varphi}$ is proved in [M2, Lem. 6.1.1]. Let $\varsigma : X \times \hat{X} \rightarrow \hat{X} \times X$ be the transposition of the factors. The isomorphism $[\Phi_P^H \otimes (\Phi_P^H)^{-1}] \circ \varsigma_* : \wedge^* V \rightarrow \wedge^* V$ is an analogue of the Hodge star operator and it interchanges the two filtrations $F^\bullet(\wedge^* V)$ and $F_\bullet(\wedge^* V)$ [M2, Rem. 6.3.3]. $\square$

We have $\dim(HW(X \times \hat{X}, \eta)) = e$. Hence, $\dim(BB_1) = \dim(HW(X \times \hat{X}, \eta))$ and (10.1) is an isomorphism, if and only if $e = 2$, if and only if $F = \mathbb{Q}$. In that case $B_{\mathbb{C}} = \ell_W \oplus \ell_{\bar{W}}$ and Condition (2c) in section 4 requires that $ch(F_1) \otimes ch(F_2)$ does not belong to the 2-dimensional subspace $BB_0 := [\ell_W \otimes \ell_{\bar{W}}] \oplus [\ell_{\bar{W}} \otimes \ell_W]$ of the 4-dimensional $B \otimes_{\mathbb{Q}} B$. Classes in BB_0 are invariant under the larger group $\text{Spin}(V_{\mathbb{Q}})_{\eta}$, while classes in BB_1 are invariant only under the special Mumford-Tate group $\text{Spin}(V_{\mathbb{Q}})_B$ and so it suffices to check that $ch(F_1) \otimes ch(F_2)$ is not invariant under $\text{Spin}(V_{\mathbb{Q}})_{\eta}$ (see, for example, the proof of [M2, Lemma 8.3.1]).

More generally, write $ch(F_1) \otimes ch(F_2) = \sum_{k=0}^{e/2} \gamma_k$, with $\gamma_k \in BB_k$, and write $\gamma_1 = \sum_{(T, T') : |T \cap T'|=1} \gamma_{T, T'}$, with $\gamma_{T, T'} \in \ell_{W_T} \otimes \ell_{W_{T'}}$. Let Π be the composition (10.1).

Proposition 10.2. [M3, Prop. 10.2.1(3)] *Assume that $d > 2$, the rank of $\Phi(F_1 \boxtimes F_2^{\vee})$ is non-zero, and for some $\sigma \in \Sigma$*

$$(10.2) \quad \sum_{(T, T') : T \cap T' = \{\sigma\}} \Pi(\gamma_{T, T'}) \neq 0.$$

Then $\kappa_{d/2}(E)$ does not belong to the image of $\text{Sym}^{d/2} \mathcal{A}^2$ in $H^{d, d}(X \times \hat{X}, \mathbb{Q})$.

In the special case when K contains the subfield $\mathbb{Q}(\sqrt{-q})$, for some positive $q \in \mathbb{Q}$, then the sheaf F_1 can be chosen with $ch(F_1)$ in the plane P spanned by α and β in Equation (5.2). In that case the sum (10.2) consists of at most one non-zero summand (see [M3, Lem. 11.2.9]). Hence, for such F_1 the sum (10.2) does not vanish, for some $\sigma \in \Sigma$, if and only if the direct summand γ_1 of $ch(F_1) \otimes ch(F_2)$ in BB_1 does not vanish. An explicit example of coherent sheaves F_1 and F_2 on the Jacobian of a genus 4 curve satisfying the hypothesis of the above Proposition is given in [M3, Ex. 11.2.7 and Lem. 11.2.8].

11. A SEMI-REGULAR SHEAF OVER A SIXFOLD $(X \times \hat{X}, \eta)$ OF SPLIT WEIL TYPE

We finally deal with the semi-regularity condition 2a in the Strategy section 4, but we are currently able to handle it only for $\dim(X) \leq 3$, which forces $K = \mathbb{Q}(\sqrt{-q})$ to be a quadratic imaginary number field. We may choose q to be a positive integer. We may also choose $q \geq 4$, since $\mathbb{Q}(\sqrt{-q}) = \mathbb{Q}(\sqrt{-4q})$.

11.1. Two secant sheaves. Let C be a non hyperelliptic curve of genus 3. Set $X = \text{Pic}^0(C)$. Let $AJ : C \rightarrow \text{Pic}^1(C)$ be the Abel-Jacobi map. Let G_1 and G_2 be two cyclic subgroups of X of order $q+1$ satisfying $G_1 \cap G_2 = (0)$. Let $C_i \subset X$, $1 \leq i \leq q+1$, be $q+1$ disjoint translates of $AJ(C)$, which are transitively permuted by translations by elements of G_1 . Let $\Theta \subset X$ be a translate of the theta divisor in $\text{Pic}^2(C)$. Set $F_1 := \mathcal{I}_{\cup_{i=1}^{q+1} C_i}(\Theta)$, the ideal sheaf of the union tensored with $\mathcal{O}_X(\Theta)$. The class Poincaré dual to C_i is $\Theta^2/2 \in H^{2,2}(X, \mathbb{Z})$ and one checks that $ch(F_1) = (1 - \frac{q}{2}\Theta^2) + (\Theta - \frac{q}{3!}\Theta^3)$, which belongs to the secant $B = \text{span}_{\mathbb{Q}}\{\alpha, \beta\}$, $\alpha = 1 - \frac{q}{2}\Theta^2$, $\beta = \Theta - \frac{q}{3!}\Theta^3$, associated to the pure spinor $\exp(\sqrt{-q}\Theta)$ is Example 5.1 (see [M2, Lem 8.2.1] for the computation of $ch(F_1)$). We get the embedding $\eta : K \rightarrow \text{End}_{\mathbb{Q}}(X \times \hat{X})$ and a polarization Ξ_t , for some $t \in K_-$, such that $(X \times \hat{X}, \eta, \Xi_t)$ is a polarized abelian sixfold of split Weil type, by Lemma 8.1.

Set $C' := -AJ(C) \subset \text{Pic}^{-1}(C)$. Let $C'_i \subset X$, $1 \leq i \leq q+1$, be $q+1$ disjoint translates of C' , which are transitively permuted by G_2 . Set $F_2 = \mathcal{I}_{\cup_{i=1}^{q+1} C'_i}(\Theta)$. Then $ch(F_2) = ch(F_1)$ and so $ch(F_2)$ is in B as well. Note that $ch(F_i^\vee) = \alpha - \beta$ is in B as well, where $F_i^\vee$ is the derived dual object.

11.2. The transform of $F_2 \boxtimes F_1$ to $X \times \hat{X}$. If a translate $\tau_t(\Sigma_j)$ of Σ_j by a point $t \in X$ intersects C_i , then the two intersect along a length 2 subscheme and the canonical line bundle of $C_i \cup \tau_t(\Sigma_j)$ is the restriction of $L_{i,j,t}(2\Theta)$ for a unique line bundle $L_{i,j,t} \in \hat{X}$ (see [M2, Lem. 9.2.7] for an explicit computation of $L_{i,j,t}$). Define the morphism $f_{ij} : C_i \times \Sigma_j \rightarrow X \times \hat{X}$ by $f_{ij}(x, y) = (y - x, L_{i,j,x-y})$. Denote by $\tilde{\Theta}_{ij} \subset X \times \hat{X}$ the image of f_{ij} and set $\tilde{\Theta} := \cup_{1 \leq i,j \leq q+1} \tilde{\Theta}_{ij}$. Each $\tilde{\Theta}_{ij}$ projects isomorphically onto a translate of the divisor Θ in X , and Θ is isomorphic to the symmetric product $C^{(2)}$, by the assumption that C is not hyperelliptic. The genericity assumptions [M2, Assum. 9.1.1, 9.2.1] assure that the surfaces $\tilde{\Theta}_{ij}$ are pairwise disjoint, and so $\tilde{\Theta}$ is smooth. Let Φ be Orlov's equivalence.

Proposition 11.1. [M2, Lem. 9.1.4, 9.3.1 and Prop. 9.2.2] *For a suitable choice of the groups G_1 and G_2 and for a generic C , the cohomology sheaves $\mathcal{G}_i$ in degree i of the object $\mathcal{G} := \Phi(F_2 \boxtimes F_1)[-3]$ over $X \times \hat{X}$ satisfy:*

- (1) $\mathcal{G}_i$ vanishes for $i \notin \{1, 2\}$.
- (2) $\mathcal{G}_1$ is a reflexive sheaf of rank $8q$, which is locally free away from $\tilde{\Theta}$.
- (3) $\mathcal{G}_2$ is supported, set theoretically, over $\tilde{\Theta}$.
- (4) The object $\mathcal{G}^\vee[-1]$ is represented by a coherent sheaf $\mathcal{E}$, which is isomorphic to $\text{Hom}(\mathcal{G}_1, \mathcal{O}_{X \times \hat{X}})$ and $\mathcal{G}_2$ is isomorphic to $\mathcal{E}xt^1(\mathcal{E}, \mathcal{O}_{X \times \hat{X}})$.

The fiber of $\mathcal{G}_1$, at a point $(x, L) \in [X \times \hat{X}] \setminus \tilde{\Theta}$, is naturally isomorphic to

$$H^1(X, \mathcal{I}_{\cup_{i=1}^{q+1} C_i} \otimes \mathcal{I}_{\cup_{i=1}^{q+1} \tau_{-x}(C'_i)}(\Theta + \tau_{-x}(\Theta)) \otimes L^{-1}).$$

11.3. Equivariance of the object $\mathcal{G} \in D^b(X \times \hat{X})$. The object $\mathcal{G}[3] := \Phi(F_2 \boxtimes F_1)$ is the image of the $G_2 \times G_1$ -equivariant sheaf $\mathcal{I}_{\cup_{i=1}^{q+1} C'_i} \boxtimes \mathcal{I}_{\cup_{i=1}^{q+1} C_i}$ via the equivalence $\tilde{\Phi} := \Phi \circ (\mathcal{O}_{X \times \hat{X}}(\Theta \boxtimes \Theta) \otimes (\bullet))$. Denote by $G_2 \times G_1$ also its image in the identity component of $\text{Aut}(D^b(X \times X))$ and let $G := \tilde{\Phi} \circ (G_2 \times G_1) \circ \tilde{\Phi}^{-1}$ be its conjugate in the

identity component $(X \times \hat{X}) \times \text{Pic}^0(X \times \hat{X})$ of $\text{Aut}(D^b(X \times \hat{X}))$. The equivalence $\tilde{\Phi}$ transforms the natural linearization of $\mathcal{I}_{\cup_{i=1}^{q+1} C'_i} \boxtimes \mathcal{I}_{\cup_{i=1}^{q+1} C_i}$ to a linearization $\tilde{\lambda} := \{\tilde{\lambda}_g\}_{g \in G}$ of the object $\mathcal{G}$ with respect to the action of G on $D^b(X \times \hat{X})$, so that $(\mathcal{G}, \tilde{\lambda})$ is an object in the equivariant category $D_G^b(X \times \hat{X})$ in the sense of [BO]. Given $x \in X$, set $L_x := \mathcal{O}_X(\Theta - \tau_x(\Theta))$ and let $\mathcal{P}_x$ be the restriction of the Poincaré line bundle $\mathcal{P}$ to $\hat{X} \times \{x\}$. An explicit calculation yields

$$\tilde{\Phi} \circ (\tau_{x_1}, \tau_{x_2})_* \circ \tilde{\Phi}^{-1} = ((\pi_1^* L_{x_1} \otimes \pi_2^* \mathcal{P}_{-x_2}) \otimes) \circ (\tau_{x_1-x_2}, \tau_{L_{x_1+x_2}})_*$$

(see [M2, Eq. (9.3.1)]). We conclude that G projects injectively onto a subgroup $\bar{G}$ of translations in $X \times \hat{X}$, since $G_1 \cap G_2 = (0)$.

Let $D := \det(\mathcal{G})$ be the determinant line bundle of $\mathcal{G}$. Let a be a positive integer, such that $8qa \equiv -1 \pmod{q+1}$. Such an integer a exists, by our assumption that q is even.

Lemma 11.2. [M2, Lem. 9.3.5] *The object $\mathcal{G} \otimes D^a$ is $\bar{G}$ -equivariant. It admits $\bar{G}$ -linearization isomorphisms $\lambda_g : \mathcal{G} \otimes D^a \rightarrow \tau_g^*(\mathcal{G} \otimes D^a)$, $g \in \bar{G} \subset X \times \hat{X}$, satisfying the axioms of a linearization in [Pl, Sec. 1.1].*

Let $\pi : X \times \hat{X} \rightarrow A := (X \times \hat{X})/\bar{G}$ be the quotient morphism onto the quotient abelian variety. Let $\bar{\mathcal{G}}$ be the object in $D^b(A)$, which is mapped to $(\mathcal{G}, \lambda) \in D_G^b(X \times \hat{X})$ via the natural equivalence $\pi^* : D^b(A) \rightarrow D_G^b(X \times \hat{X})$. Then $\bar{\mathcal{G}}^\vee[-1]$ is represented by a coherent sheaf $\bar{\mathcal{E}}$, by Proposition 11.1. We have the isomorphisms

$$\begin{aligned} (11.1) \quad \text{Ext}^2(\bar{\mathcal{E}}, \bar{\mathcal{E}}) &\cong \text{Hom}(\mathcal{O}_A, \bar{\mathcal{E}}^\vee \otimes \bar{\mathcal{E}}[2]) \cong \text{Hom}(\mathcal{O}_A, \bar{\mathcal{G}} \otimes \bar{\mathcal{E}}[3]) \\ &\cong \text{Hom}(((\mathcal{O}_{X \times \hat{X}}, 1), (\mathcal{G} \otimes D^a, \lambda) \otimes (\mathcal{G} \otimes D^a, \lambda)^\vee[2]))^{\bar{G}} \\ &\cong \text{Ext}^2(\mathcal{G} \otimes D^a, \lambda), (\mathcal{G} \otimes D^a, \lambda))^{\bar{G}} \cong \text{Ext}^2((\mathcal{G}, \tilde{\lambda}), (\mathcal{G}, \tilde{\lambda}))^{\bar{G}} \\ &\cong \text{Ext}^2(\mathcal{I}_{\cup_{i=1}^{q+1} C'_i} \boxtimes \mathcal{I}_{\cup_{i=1}^{q+1} C_i}, \mathcal{I}_{\cup_{i=1}^{q+1} C'_i} \boxtimes \mathcal{I}_{\cup_{i=1}^{q+1} C_i})^{G_2 \times G_1} \end{aligned}$$

11.4. The sheaf $\mathcal{E}$ is equivariantly semi-regular. The second Hochschild cohomology $HH^2(Y)$ of a smooth projective variety Y consists of natural transformations from the identity endo-functor id of $D^b(Y)$ to $id[2]$. Given a coherent sheaf E over Y , let $ev_E : HH^2(Y) \rightarrow \text{Hom}(E, E[2])$ be the evaluation of a natural transformation on the object E . The Chern character $ch(E)$ corresponds to a class in the Hochschild homology $HH_*(Y)$, via the Hochschild-Kostant-Rosenberg (HKR) isomorphism [Ca2, Theorem 4.5], and $HH_*(Y)$ is a module over the ring $HH^*(Y)$. There is a sufficient criterion for E to be semi-regular [M2, Rem. 8.3.11], which we state for Y an abelian variety for simplicity:

Lemma 11.3. [M2, Lem. 8.3.10] *If the kernel of $ev_E : HH^2(Y) \rightarrow \text{Hom}(E, E[2])$ is equal to the annihilator of $ch(E)$ in $HH^2(Y)$ and ev_E is surjective, then E is semi-regular.*

Proof. We have the HKR isomorphism

$$HH^2(Y) \cong HT^2(Y) := H^2(Y, \mathcal{O}_Y) \oplus H^1(Y, TY) \oplus H^0(Y, \wedge^2 TY).$$

Diagram (2.1) has an analogous commutative diagram replacing $H^1(Y, TY)$ by the first order deformations $HH^2(Y)$ of the category of coherent sheaves on Y [BF2, Prop. 6.2.1 and Cor. 6.3.2]. The top horizontal homomorphism $\lrcorner at_E$ in diagram (2.1) is replaced $ev_E : HH^2(Y) \rightarrow \text{Hom}(E, E[2])$. The right arrow in the new version of diagram (2.1) is still the semi-regularity map σ_E and when Y is an abelian variety the left arrow is the composition of $HKR : HH^2(Y) \rightarrow HT^2(Y)$ with contraction with $ch(E)$.

$$\begin{array}{ccc} HH^2(Y) & \xrightarrow{ev_E} & \text{Ext}^2(E, E) \\ \downarrow HKR & & \downarrow \sigma_E \\ HT^2(E) & \xrightarrow{\lrcorner_{ch(E)}} & \oplus_{q=0}^{N-2} H^{q, q+2}(Y) \end{array}$$

If $\ker(\lrcorner_{ch(E)} \circ HKR)$ is equal to $\ker(ev_H)$ and ev_H is surjective, then the equality $\sigma_E \circ ev_E = \lrcorner_{ch(E)} \circ HKR$ implies that σ_E is injective. $\square$

Question 11.4. Is the surjectivity of ev_E needed in Lemma 11.3? Does the Semi-regularity Theorem 2.1 hold under the weaker assumption that σ_E restricts to the image of ev_E as an injective map (considering also twisted sheaves as we did in Section 2)?

The sheaf $\mathcal{E}$ of Proposition 11.1(4) cannot be semi-regular for all $q \geq 4$, since $\dim \text{Ext}^2(\mathcal{E}, \mathcal{E})$ grows quadratically with q , by [M2, Lem. 8.3.8], while the co-domain of $\sigma_{\mathcal{E}}$ is independent of q . Fortunately, $\sigma_{\mathcal{E}}$ restricts as an injective map to $\text{Ext}^2(\mathcal{E}, \mathcal{E})^{\bar{G}}$, which contains the image of $ev_{\mathcal{E}}$. Descending to the quotient abelian variety $A := (X \times \hat{X})/\bar{G}$ enables us to use the semi-regularity theorem and avoid Question 11.4.

Proposition 11.5. *The sheaf $\bar{\mathcal{E}}$ over A is semi-regular.*

Sketch of proof. We prove first that the sheaf $\mathcal{I}_{C_1}$ over X is semi-regular following [M2, Lem. 8.3.7]. The surjectivity of $ev_{\mathcal{I}_{C_1}}$ follows from the fact that $HH^1(X) \rightarrow \text{Ext}^1(\mathcal{I}_{C_1}, \mathcal{I}_{C_1})$ is an isomorphism, $HH^1(X)$ generates $HH^*(X)$, $\text{Ext}^1(\mathcal{I}_{C_1}, \mathcal{I}_{C_1})$ generates $\text{Ext}^*(\mathcal{I}_{C_1}, \mathcal{I}_{C_1})$, and so $ev_{\mathcal{I}_{C_1}}^* : HH^*(X) \rightarrow \text{Ext}^*(\mathcal{I}_{C_1}, \mathcal{I}_{C_1})$ is a surjective algebra homomorphism. Serre's duality yields $\text{Ext}^2(\mathcal{I}_{C_1}, \mathcal{I}_{C_1}) \cong \text{Ext}^1(\mathcal{I}_{C_1}, \mathcal{I}_{C_1})^*$. It follows that the rank of $ev_{\mathcal{I}_{C_1}} : HH^2(X) \rightarrow \text{Ext}^2(\mathcal{I}_{C_1}, \mathcal{I}_{C_1})$ is 6. One checks that the rank of $\lrcorner_{ch}(\mathcal{I}_{C_1})$ is 6 as well. The inclusion $\ker(ev_{\mathcal{I}_{C_1}}) \subset \ker(\lrcorner_{ch}(\mathcal{I}_{C_1}))$ holds, by [Hua, Th. B]. Hence, $\ker(ev_{\mathcal{I}_{C_1}}) = \ker(\lrcorner_{ch}(\mathcal{I}_{C_1}))$ and $\mathcal{I}_{C_1}$ is semi-regular, by Lemma 11.3. The curve C_1 maps isomorphically onto its image $\bar{C}$ in X/G_1 . The sheaf $\mathcal{I}_{\bar{C}}$ is semi-regular as well.

Note that the sheaf $\mathcal{I}_{\cup_{i=1}^{q+1} C_i}$ is naturally G_1 -equivariant and is the pullback of $\mathcal{I}_{\bar{C}}$. The group G_1 acts trivially on $HH^2(X)$, and so $ev_{\mathcal{I}_{\cup_{i=1}^{q+1} C_i}} : HH^2(X) \rightarrow \text{Ext}^2(\mathcal{I}_{\cup_{i=1}^{q+1} C_i}, \mathcal{I}_{\cup_{i=1}^{q+1} C_i})^{G_1}$ is surjective and its kernel is equal to the kernel of $\lrcorner_{ch}(\mathcal{I}_{\cup_{i=1}^{q+1} C_i})$ [M2, Cor. 8.3.12]. The analogous statement holds for $\mathcal{I}_{\cup_{i=1}^{q+1} C'_i}$. It follows, by the Künneth theorem, that $ev_{\mathcal{I}_{\cup_{i=1}^{q+1} C'_i} \boxtimes \mathcal{I}_{\cup_{i=1}^{q+1} C_i}} : HH^2(X \times X) \rightarrow \text{Ext}^2(\mathcal{I}_{\cup_{i=1}^{q+1} C'_i} \boxtimes \mathcal{I}_{\cup_{i=1}^{q+1} C_i}, \mathcal{I}_{\cup_{i=1}^{q+1} C'_i} \boxtimes \mathcal{I}_{\cup_{i=1}^{q+1} C_i})^{G_2 \times G_1}$ is surjective and its kernel is equal to that of $\lrcorner_{ch}(\mathcal{I}_{\cup_{i=1}^{q+1} C'_i} \boxtimes \mathcal{I}_{\cup_{i=1}^{q+1} C_i})$ [M2, Lem. 8.4.1(1) and Lem. 9.3.2]. Hence, $ev_{\mathcal{G}} : HH^2(X \times \hat{X}) \rightarrow \text{Ext}^2((\mathcal{G}, \tilde{\lambda}), (\mathcal{G}, \tilde{\lambda}))^G$ is surjective and its kernel is equal to that of $\lrcorner_{ch}(\mathcal{G})$. The analogous statement follows for the sheaf $\mathcal{E}$, by [M2, Lem. 8.4.1(2)], and for the sheaf $\bar{\mathcal{E}}$ by (11.1). Hence, $\bar{\mathcal{E}}$ is semi-regular, by Lemma 11.3. $\square$

11.5. Sketch of proof of Theorem 1.2. Step 1 (Weil classes on sixfolds of split Weil type): We have established the semi-regularity of $\bar{\mathcal{E}}$ in Proposition 11.5. The abelian variety A is isogenous to $X \times \hat{X}$ and is thus endowed with the embedding $\eta : K \rightarrow \text{End}_{Hdg}(H^1(X \times \hat{X})) \cong \text{End}_{Hdg}(H^1(A, \mathbb{Q}))$ and the polarization $\Xi_t \in H^{1,1}(X \times \hat{X}, \mathbb{Q}) \cong H^{1,1}(A, \mathbb{Q})$. This completes the implementation of the strategy for the proof of algebraicity of the Weil classes on all deformations of (A, η, Ξ_t) , and hence also for all deformations of $(X \times \hat{X}, \eta, \Xi_t)$, as explained in the strategy section 4. Two connected components of the moduli space of polarized abelian varieties of Weil type of dimension $2n$, the same imaginary quadratic number field, and the same discriminant, parametrize isogenous abelian varieties [vG, Th. 5.2(3)]. The discriminant takes values in $\mathbb{Q}^\times / Nm_{K/\mathbb{Q}}(K^\times)$ and is the coset of $(-1)^n$ if and only if the component parametrizes polarized abelian varieties of split Weil type [DM, Cor. 4.2]. We conclude that the Hodge Weil classes on all polarized abelian sixfolds of split Weil type are algebraic.

Step 2 (Weil classes on fourfolds): The discriminant invariant of polarized abelian varieties with complex multiplication by the same field is multiplicative under cartesian products. Every value in $\mathbb{Q}^\times / Nm_{K/\mathbb{Q}}(K^\times)$ is realized as the discriminant by some connected component of moduli in every even dimension [vG, Th. 5.2]. Hence, for every polarized abelian fourfold (A_1, η_1, h_1) of Weil type, of arbitrary discriminant, there exists a polarized abelian surface of Weil type (A_2, η_2, h_2) , such that the discriminant of their product polarized abelian sixfold of Weil type $(A_1 \times A_2, \eta, \pi_1^* h_1 + \pi_2^* h_2)$ is the coset of -1 . The sixfold is hence of split type and so its Weil classes are algebraic. It follows that the Weil classes of (A_1, η_1, h_1) are algebraic, by [S2, Prop. 10]. Hence, the Weil classes are algebraic on every abelian fourfold. $\square$

12. WHAT ABOUT WEIL CLASSES ON ABELIAN VARIETIES OF DIMENSION ≥ 8 ?

We expect that an affirmative answer to Question 11.4 would lead to a proof of the algebraicity of Weil classes on some higher dimensional abelian varieties, as well as for CM-fields K with $[K : \mathbb{Q}] > 2$, using the strategy outlined in Section 4. It would be interesting to find a systematic construction of examples of secant sheaves on abelian varieties with real multiplication by a totally real field F satisfying the weaker criterion proposed in Question 11.4. Examples of secant sheaves on the Jacobian J_C of a genus g curve C consisting of ideal sheaves of the union of translates in J_C of images in J_C^k of the symmetric products $C^{(k)}$, $0 \leq k \leq g - 2$, are considered in [M2, Ex. 8.2.3] and determined for $g = 4$. When $g = 4$ another example supported on a Θ divisor is given in [M2, Ex. 8.2.4]. It is yet to be checked if these secant sheaves satisfy the weaker criterion.

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