

Adiabatic and point-splitting regularization of spin- $\frac{1}{2}$ field in de Sitter space

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Abstract

We study the regularization of a spin- $\frac{1}{2}$ field in the vacuum state in de Sitter space. We find that the 2nd order adiabatic regularization is sufficient to remove all UV divergences for the spectral stress tensor, as well as for the power spectrum. The regularized vacuum stress tensors of the massive field is maximally symmetric with the energy density remaining negative, and behaves as a “negative” cosmological constant. In the massless limit it reduces smoothly to the zero stress tensor of the massless field, and there is no trace anomaly. We also perform the point-splitting regularization in coordinate space, and obtain the analytical, regularized correlation function and stress tensor, which agree with those from the adiabatic regularization. In contrast, the 4th order regularization is an oversubtraction, and changes the sign of the vacuum energy density. In the massless limit the 4th order regularized auto-correlation becomes singular and the regularized stress tensor does not reduce to the zero stress tensor of the massless field. These difficulties tell that the 4th order regularization is inadequate for the spin- $\frac{1}{2}$ massive field.

1 Introduction

Quantum fields in curved spacetime [1–5] have ultraviolet (UV) divergences in the stress tensor in the vacuum state. These vacuum UV divergences should not be simply dropped via the normal ordering of the field operators, because the finite part of the vacuum stress tensor can have gravitational effects in curved spacetime and may play a role of cosmological constant [6–8]. Several schemes of regularization have been proposed to remove the UV divergences, such as the adiabatic regularization in k -space [9–16], the point-splitting regularization in x -space [1, 17–20], and the dimensional regularization [1], etc.

In literature, the conventional 4th order regularization was adopted, by default, on the stress tensor of quantum fields, such as the scalar [9], the vector [21–23], the tensor fields [24, 25]. However, under the 4th order regularization the vacuum energy density would change its sign, and become unphysically negative, as in the cases of the scalar [20, 26, 27] and vector massive fields [28]. This is because the 4th order scheme would subtract off too much than necessary, not respecting the minimal subtraction rule [9]. Moreover, as an inconsistency, the massless limit of the 4th order regularized stress tensor of the massive fields does not reduce continuously to that of the massless fields [29, 30]. These are the difficulties of the conventional 4th order regularization.

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In fact, an adequate regularization depends on the coupling, the type of fields (the components), and the curved spacetimes. For the conformally coupling massive scalar field in de Sitter space, the 0th order regularization is sufficient to remove all divergences, and for the minimally coupling scalar field [26] the 2nd order regularization is sufficient. These have been worked out under both the adiabatic and point-splitting regularization [20]. For the tensor field (gravitational waves) in a flat Robertson-Walker spacetime, the stress tensor is actually equivalent to that of a pair of minimally coupling scalar fields [31], so that the 2nd order regularization is adequate [32]. For the Stueckelberg field (the massive vector fields with a gauge-fixing term), the transverse part is regularized at the 0th order, whereas the longitudinal, temporal, and gauge-fixing parts are regularized at the 2nd order [28]. It is interesting that the regularized vacuum stress tensors of these massive fields possess the maximal symmetry of the background spacetime and can be taken as a cosmological constant. Furthermore, the massless limit of these regularized stress tensors reduce smoothly to the zero regularized stress tensor of the massless fields [27, 29, 30], and there is no trace anomaly.

In this paper, we study regularization of the spin- $\frac{1}{2}$ field. In literature, the stress tensor of the spin- $\frac{1}{2}$ massive field was conventionally regularized at the 4th order [33–38]. Here the problems with the 4th order regularization are similar to those for the scalar and vector fields: more terms than necessary would be subtracted, the sign of the vacuum energy density would be changed, and the massless limit is inconsistent with that of massless field. As we shall show, the 2nd order regularization is sufficient to remove all UV divergences, the massless limit of the 2nd order regularized stress tensor reduces to the zero regularized stress tensor of the massless field, and there is no trace anomaly. We shall perform both the adiabatic and the point-splitting regularization, and show that the two schemes yield consistent results and are complementary [20, 26].

The paper is organized as follows. Sec. 2 presents the exact and adiabatic modes for spin-1/2 fields in de Sitter space. Sec. 3 gives the adiabatic regularization for the power spectrum. Sec. 4 presents the adiabatic regularization on the spectral stress tensor, and examine the difficulties of the 4th order regularization. Sec. 5 gives the point-splitting regularization and derives the analytic expressions for the regularized correlation function and stress tensor. Sect. 6 presents conclusions and discussion. Appendix A examines the WKB modes with the arbitrary functions up to the 4th order and the treatment differs from Refs. [33–35]. Appendix B shows that the arbitrary functions cancel out in the adiabatic power spectrum and spectral stress tensor. Appendix C performs the integrations for the analytical correlation function of the massive spin- $\frac{1}{2}$ fields in de Sitter space.

We use natural units $c = \hbar = 1$ throughout the paper.

2 The adiabatic solutions of spin-1/2 field

The Lagrangian density of a spin- $\frac{1}{2}$ field in curved spacetime is given by [3]

$$\mathcal{L} = \sqrt{-g}\bar{\psi}(i\bar{\gamma}^\mu\nabla_\mu - m)\psi, \quad (1)$$

where ψ is the spinor field and m is the mass. The spacetime dependent matrices $\bar{\gamma}^\mu(x)$ satisfy the anticommutation relation $\{\bar{\gamma}^\mu, \bar{\gamma}^\nu\} = 2g^{\mu\nu}$ and are defined by the tetrad fields $V_a{}^\mu$ as $\bar{\gamma}^\mu = V_a{}^\mu\gamma^a$, where γ^a are the 4×4 constant gamma matrices in Minkowski spacetime. The covariant derivative acting on the spinor field is defined by $\nabla_\mu \equiv \partial_\mu - \Gamma_\mu$, where the spin connection is given by $\Gamma_\nu = -\frac{1}{4}\gamma_a\gamma_b V^{a\lambda}V^b_{\lambda;\nu}$, with the semicolon denoting the covariant derivative acting on a tensor index. In this work, we adopt the standard Dirac-Pauli representation, where the gamma matrices take the form

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad (2)$$

and σ^i are the standard Pauli matrices. The metric of the fRW spacetime is

$$ds^2 = dt^2 - a(t)^2(dx^2 + dy^2 + dz^2), \quad (3)$$

where a is the scale factor and t is the cosmic time with $\dot{a} = da/dt$. In the fRW spacetime, the tetrad fields can be chosen as $V_\alpha^\mu = (1, a^{-1}, a^{-1}, a^{-1})$, which leads to the following spin connection components [4]

$$\Gamma^0 = 0, \quad \Gamma^i = \frac{1}{2} \frac{\dot{a}}{a^2} \gamma^0 \gamma^i, \quad (4)$$

and the spacetime dependent gamma matrices are

$$\bar{\gamma}^0 = \gamma^0, \quad \bar{\gamma}^i = a^{-1} \gamma^i. \quad (5)$$

From the Lagrangian density (1) follows the Dirac equation in curved spacetime

$$(i\bar{\gamma}^\mu \nabla_\mu - m)\psi = 0. \quad (6)$$

By multiplying $(i\bar{\gamma}^\nu \nabla_\nu + m)$ on (6) from the left and using the relation $[\bar{\gamma}^\mu, \bar{\gamma}^\nu][\nabla_\mu, \nabla_\nu]\psi = R\psi$ with R being the scalar curvature (see (5.271) in Ref. [3]), one has

$$(\nabla_\mu \nabla^\mu + m^2 + \frac{1}{4}R)\psi = 0. \quad (7)$$

This formally resembles the Klein-Gordon equation of the scalar field with a coupling constant $\xi = 1/4$ [26]. However, ψ is not simply a set of four arbitrary scalar functions, since ψ has to satisfy the spinor equation (6). Using (4) and (5), the Dirac equation (6) can be written as [5]

$$\left(i\gamma^0 \partial_0 + \frac{3i}{2} \frac{\dot{a}}{a} \gamma^0 + \frac{i}{a} \gamma^i \partial_i - m\right)\psi = 0. \quad (8)$$

The field operator can be expanded as [39, 40]

$$\psi(x, t) = \int \frac{d^3k}{(2\pi)^{3/2}} \sum_{\lambda=\pm\frac{1}{2}} (A_{\vec{k},\lambda} u_{\vec{k},\lambda}(t) e^{i\vec{k}\cdot\vec{x}} + B_{\vec{k},\lambda}^\dagger v_{\vec{k},\lambda}(t) e^{-i\vec{k}\cdot\vec{x}}), \quad (9)$$

where $A_{\vec{k},\lambda}$ and $B_{\vec{k},\lambda}^\dagger$ are the annihilation and creation operators respectively for electrons and positrons with helicity λ and momentum $\vec{k}$, and $u_{\vec{k},\lambda}$ and $v_{\vec{k},\lambda}$ are the mode spinors, the anti-commutation relations for these operators are

$$\{A_{\vec{k},\lambda}, A_{\vec{k}',\lambda'}^\dagger\} = \{B_{\vec{k},\lambda}, B_{\vec{k}',\lambda'}^\dagger\} = \delta_{\lambda\lambda'} \delta^{(3)}(\vec{k} - \vec{k}'). \quad (10)$$

Plugging (9) into (8) yields

$$i\gamma^0 \dot{u}_{\vec{k},\lambda} + \frac{3i}{2} \frac{\dot{a}}{a} \gamma^0 u_{\vec{k},\lambda} - \frac{1}{a} \gamma^i k_i u_{\vec{k},\lambda} - m u_{\vec{k},\lambda} = 0, \quad (11)$$

the spinor $v_{\vec{k},\lambda}$ can be obtained by charge conjugation, $v_{\vec{k},\lambda} = -i\gamma^2 u_{\vec{k},\lambda}^*$ [40]. The spinor $u_{\vec{k},\lambda}$ can be expressed in terms of the two-component spinors $\xi_{\lambda,\vec{k}}$ as the following [33–36]

$$u_{\vec{k},\lambda}(t) = \frac{1}{a^{\frac{3}{2}}} \begin{pmatrix} h_k^I(t) \xi_{\lambda,\vec{k}} \\ h_k^{II}(t) \frac{\sigma^i k_i}{k} \xi_{\lambda,\vec{k}} \end{pmatrix}, \quad (12)$$

where

$$\xi_{\frac{1}{2},\vec{k}} = \begin{pmatrix} \cos(\frac{\theta_k}{2}) e^{-i\phi_k} \\ \sin(\frac{\theta_k}{2}) \end{pmatrix}, \quad \xi_{-\frac{1}{2},\vec{k}} = \begin{pmatrix} -\sin(\frac{\theta_k}{2}) e^{-i\phi_k} \\ \cos(\frac{\theta_k}{2}) \end{pmatrix}, \quad (13)$$

with θ_k and ϕ_k being the polar and azimuthal angles of $\vec{k}$ in momentum space. The spinors $\xi_{\lambda,\vec{k}}$ satisfy the eigenvalue equation $\frac{\sigma^i k_i}{2k} \xi_{\lambda,\vec{k}} = \lambda \xi_{\lambda,\vec{k}}$ with the normalization $\xi_{\lambda,\vec{k}}^\dagger \xi_{\lambda',\vec{k}} = \delta_{\lambda\lambda'}$. Using the equal-time anticommutation relation $\{\psi_a(x, t), \pi_b(x', t)\} = i\delta(x - x')\delta_{ab}$, where the canonical momentum is defined by $\pi \equiv \frac{\partial \mathcal{L}}{\partial(\partial_0 \psi)} = \sqrt{-g} i \psi^\dagger$, and with (a, b) denoting spinor indices, together with (10), one finds the normalization condition

$$|h_k^I|^2 + |h_k^{II}|^2 = 1. \quad (14)$$

Plugging (12) into (11) yields two coupled first order differential equations

$$h_k^I(t) = i \frac{a}{k} (\partial_0 - im) h_k^{II}(t), \quad (15)$$

$$h_k^{II}(t) = i \frac{a}{k} (\partial_0 + im) h_k^I(t). \quad (16)$$

The functions h_k^I and h_k^{II} contain the variable t , and also depend on the parameter m . Eqs.(15) and (16) imply the following relation

$$h_k^I(t; -m) = h_k^{II}(t; m). \quad (17)$$

In de Sitter space, the scale factor is given by

$$a(t) = e^{Ht}, \quad (18)$$

where H is the Hubble parameter. Eqs. (15) and (16) can be rewritten as two decoupled second order differential equations

$$z^2 \frac{\partial^2 h_k^I(z)}{\partial z^2} + z \frac{\partial h_k^I(z)}{\partial z} + (z^2 - (-i\mu + \frac{1}{2})^2) h_k^I(z) = 0, \quad (19)$$

$$z^2 \frac{\partial^2 h_k^{II}(z)}{\partial z^2} + z \frac{\partial h_k^{II}(z)}{\partial z} + (z^2 - (-i\mu - \frac{1}{2})^2) h_k^{II}(z) = 0, \quad (20)$$

where the dimensionless variable $z \equiv k/(aH)$ and the dimensionless parameter $\mu \equiv m/H$. Eqs. (19) and (20) can be alternatively derived by substituting (9) and (12) into (7).

The positive frequency solutions of (19) and (20) are [34, 44]

$$h_k^I(z) = i \frac{\sqrt{\pi z}}{2} e^{\frac{\pi\mu}{2}} H_{-i\mu + \frac{1}{2}}^{(1)}(z), \quad (21)$$

$$h_k^{II}(z) = \frac{\sqrt{\pi z}}{2} e^{\frac{\pi\mu}{2}} H_{-i\mu - \frac{1}{2}}^{(1)}(z), \quad (22)$$

where $H_\nu^{(1)}(z)$ is the Hankel function of the first kind. The modes h_k^I and h_k^{II} satisfy the relation (17) and the normalization condition (14). In the massless limit $\mu \rightarrow 0$, the exact modes (21) and (22) reduce to

$$\lim_{\mu \rightarrow 0} h_k^I(z) = \frac{1}{\sqrt{2}} e^{iz}, \quad (23)$$

$$\lim_{\mu \rightarrow 0} h_k^{II}(z) = \frac{1}{\sqrt{2}} e^{iz}. \quad (24)$$

The solutions (21) — (24) will be used to compute the unregularized vacuum correlation function and stress tensor.

To analyze the vacuum UV divergences, we next examine the high frequency behavior of h_k^I and h_k^{II} . The WKB modes will be used in adiabatic regularization [9] because they approximate

the exact modes at high k adequately and respect the conservation to each adiabatic order. Assuming the n th order WKB approximations for h_k^I and h_k^{II} have the following form [33–35]

$$g_k^{I(n)}(t) = \sqrt{\frac{\omega + m}{2\omega}} e^{-i \int^t \Omega(t') dt'} F(t), \quad (25)$$

$$g_k^{II(n)}(t) = \sqrt{\frac{\omega - m}{2\omega}} e^{-i \int^t \Omega(t') dt'} G(t), \quad (26)$$

where $\omega = \sqrt{k^2/a^2 + m^2}$, and the functions

$$\Omega(t) = \sum_{n=0} (\omega + \omega^{(1)} + \dots + \omega^{(n)}), \quad (27)$$

$$F(t) = \sum_{n=0} (F^{(0)} + F^{(1)} + \dots + F^{(n)}), \quad (28)$$

$$G(t) = \sum_{n=0} (G^{(0)} + G^{(1)} + \dots + G^{(n)}). \quad (29)$$

At the 0th order, $\Omega^{(0)} = \omega$, $G^{(0)} = F^{(0)} = 1$. At each order the WKB modes $g_k^{I(n)}$ and $g_k^{II(n)}$ satisfy the equations similar to (15) and (16), and satisfy the normalization condition

$$|g^{I(n)}|^2 + |g^{II(n)}|^2 = 1, \quad (30)$$

and the relation

$$g^{I(n)}(t; -m) = g^{II(n)}(t; m), \quad (31)$$

in analogy to (14) and (17). Plugging (27) (28) (29) into (31) leads to the following relations

$$F^{(n)}(t; -m) = G^{(n)}(t; m), \quad (32)$$

$$\omega^{(n)}(t; -m) = \omega^{(n)}(t; m). \quad (33)$$

As shown in Appendix A, in determining $\Omega^{(n)}$, $F^{(n)}$ and $G^{(n)}$, some arbitrary functions appear. In Appendix B, we show that all these arbitrary functions cancel in the power spectrum and spectral stress tensor, so they can be set to zero without affecting the physical results.

In the massless limit, (25) and (26) become

$$g_k^{I(0)} = g_k^{I(2)} = g_k^{I(4)} = \frac{1}{\sqrt{2}} e^{-i \int^t \frac{k}{eHt'} dt'} = \frac{1}{\sqrt{2}} e^{iz}, \quad (34)$$

$$g_k^{II(0)} = g_k^{II(2)} = g_k^{II(4)} = \frac{1}{\sqrt{2}} e^{-i \int^t \frac{k}{eHt'} dt'} = \frac{1}{\sqrt{2}} e^{iz}, \quad (35)$$

ie, the WKB modes of all orders are equal to the exact (23) and (24). This is an important property of the massless WKB modes.

3 Adiabatic regularization of power spectrum

Now we study the power spectrum of the spin- $\frac{1}{2}$ field ψ , and examine its UV divergences. Given ψ and the vacuum state $|0\rangle$ defined by

$$A_{\vec{k},\lambda}|0\rangle = B_{\vec{k},\lambda}|0\rangle = 0, \quad (36)$$

one considers the vacuum expectation value as the following

$$\langle 0 | \bar{\psi}(x) \psi(x) | 0 \rangle. \quad (37)$$

This is a scalar, and referred to as the auto-correlation function of the field ψ . As shall be seen later, the vacuum stress tensor is related to the auto-correlation function in de Sitter space.

Using (9) into (37), we get

$$\langle 0|\bar{\psi}(x)\psi(x)|0\rangle = \int \Delta_k^2 \frac{dk}{k}, \quad (38)$$

where the vacuum power spectrum is

$$\Delta_k^2 \equiv -\frac{k^3}{a^3\pi^2}(|h_k^I|^2 - |h_k^{II}|^2), \quad (39)$$

with an overall minus sign. In de Sitter space, using the exact modes (21) and (22), the power spectrum is

$$\Delta_k^2 = -\frac{H^3}{\pi^2} z^3 \left(\left| \frac{\sqrt{\pi z}}{2} e^{\frac{\pi\mu}{2}} H_{-i\mu+\frac{1}{2}}^{(1)}(z) \right|^2 - \left| \frac{\sqrt{\pi z}}{2} e^{\frac{\pi\mu}{2}} H_{-i\mu-\frac{1}{2}}^{(1)}(z) \right|^2 \right). \quad (40)$$

Fig.1 shows that Δ_k^2 is negative and proportional to $-k^2$ at high k . (For illustration the

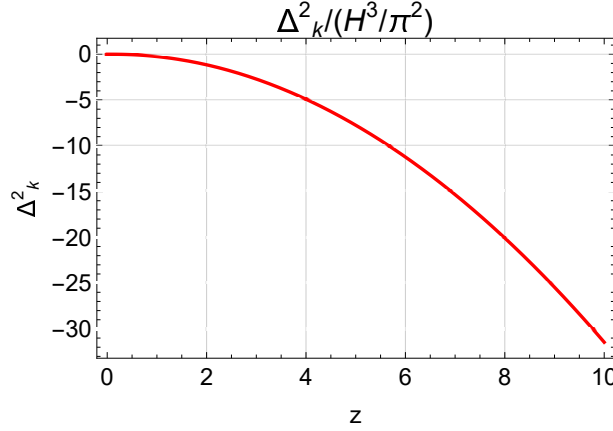


Figure 1: Unregularized power spectrum Δ_k^2 . The parameter $\mu^2 = \frac{m^2}{H^2} = 0.1$ is taken.

parameter $\mu^2 = 0.1$ will be taken in all the figures except for Fig. 6.) This Δ_k^2 will yield a UV divergent auto-correlation of (38) at its upper limit of integration, and in this sense the power spectrum is said to be UV divergent. A negative vacuum power spectrum is due to the anticommutation relations (10) for the spin- $\frac{1}{2}$ field operators. This is in contrast to the scalar and vector fields that have a positive vacuum power spectrum.

The high- z expansion of (40) is

$$\begin{aligned} \lim_{z \rightarrow \infty} \Delta_k^2 \simeq & -\frac{H^3}{\pi^2} z^3 \left(\frac{\mu}{z} - \frac{\mu(1+\mu^2)}{2z^3} + \frac{3\mu(4+5\mu^2+\mu^4)}{8z^5} - \frac{5\mu(1+\mu^2)(4+\mu^2)(9+\mu^2)}{16z^7} \right. \\ & + \frac{35\mu(1+\mu^2)(4+\mu^2)(9+\mu^2)(16+\mu^2)}{128z^9} - \frac{63\mu(1+\mu^2)(4+\mu^2)(9+\mu^2)(16+\mu^2)(25+\mu^2)}{256z^{11}} \\ & \left. + \frac{231\mu(1+\mu^2)(4+\mu^2)(9+\mu^2)(16+\mu^2)(25+\mu^2)(36+\mu^2)}{1024z^{13}} \right), \end{aligned} \quad (41)$$

where the first two leading terms

$$-\frac{H^3}{\pi^2} z^3 \left(\frac{\mu}{z} - \frac{\mu(1+\mu^2)}{2z^3} \right)$$

are respectively quadratic and logarithmic divergent. The low- z expansion of (40) is

$$\lim_{z \rightarrow 0} \Delta_k^2 \simeq -\frac{H^3}{\pi^2} \tanh(\pi\mu) z^3 \leq 0, \quad (42)$$

which is infrared convergent and negative.

Now we are to remove the UV divergences in the power spectrum of the spin- $\frac{1}{2}$ field by the adiabatic regularization. For that purpose, firstly we use the WKB approximate modes (25) and (26) to construct the adiabatic power spectra $\Delta_{k,ad}^{2(n)}$ of n th order. The adiabatic power spectra for $n = 0, 2, 4$ are listed in (B.1), (B.3), and (B.5). (See Appendix B for details.) Then, subtracting $\Delta_{k,ad}^{2(n)}$ from the unregularized spectrum Δ_k^2 , we get the regularized power spectra as the following

$$\Delta_{k\text{ reg}}^{2(n)} \equiv \Delta_k^2 - \Delta_{k\text{ ad}}^{2(n)}, \quad n = 0, 2, 4, \dots \quad (43)$$

In de Sitter space, the adiabatic subtraction terms are explicitly

$$\Delta_{k\text{ ad}}^{2(0)} = -\frac{H^3}{\pi^2} z^3 \left(\frac{\mu}{\bar{\omega}} \right), \quad (44)$$

$$\Delta_{k\text{ ad}}^{2(2)} = -\frac{H^3}{\pi^2} z^3 \left(\frac{\mu}{\bar{\omega}} - \frac{\mu}{2\bar{\omega}^3} + \frac{9\mu^3}{8\bar{\omega}^5} - \frac{5\mu^5}{8\bar{\omega}^7} \right), \quad (45)$$

$$\Delta_{k\text{ ad}}^{2(4)} = -\frac{H^3}{\pi^2} z^3 \left(\frac{\mu}{\bar{\omega}} - \frac{\mu}{2\bar{\omega}^3} + \frac{3\mu(4+3\mu^2)}{8\bar{\omega}^5} - \frac{5\mu^3(37+2\mu^2)}{16\bar{\omega}^7} + \frac{3535\mu^5}{128\bar{\omega}^9} - \frac{1701\mu^7}{64\bar{\omega}^{11}} + \frac{1155\mu^9}{128\bar{\omega}^{13}} \right), \quad (46)$$

where $\bar{\omega} \equiv \omega/H = (z^2 + \mu^2)^{1/2}$. The high- z limit of (44), (45), and (46) are

$$\lim_{z \rightarrow \infty} \Delta_{k\text{ ad}}^{2(0)} \simeq -\frac{H^3}{\pi^2} z^3 \left(\frac{\mu}{z} - \frac{\mu^3}{2z^3} + \frac{3\mu^5}{8z^5} - \frac{5\mu^7}{16z^7} + \frac{35\mu^9}{128z^9} - \frac{63\mu^{11}}{256z^{11}} + \frac{231\mu^{13}}{1024z^{13}} \right), \quad (47)$$

$$\begin{aligned} \lim_{z \rightarrow \infty} \Delta_{k\text{ ad}}^{2(2)} \simeq & -\frac{H^3}{\pi^2} z^3 \left(\frac{\mu}{z} - \frac{\mu(1+\mu^2)}{2z^3} + \frac{3\mu^3(5+\mu^2)}{8z^5} - \frac{5\mu^5(14+\mu^2)}{16z^7} \right. \\ & \left. + \frac{35\mu^7(30+\mu^2)}{128z^9} - \frac{63\mu^9(55+\mu^2)}{256z^{11}} + \frac{231\mu^{11}(91+\mu^2)}{1024z^{13}} \right), \end{aligned} \quad (48)$$

$$\begin{aligned} \lim_{z \rightarrow \infty} \Delta_{k\text{ ad}}^{2(4)} \simeq & -\frac{H^3}{\pi^2} z^3 \left(\frac{\mu}{z} - \frac{\mu(1+\mu^2)}{2z^3} + \frac{3\mu(4+5\mu^2+\mu^4)}{8z^5} \right. \\ & - \frac{5\mu^3(7+\mu^2)^2}{16z^7} + \frac{35\mu^5(273+30\mu^2+\mu^4)}{128z^9} \\ & \left. - \frac{63\mu^7(1023+55\mu^2+\mu^4)}{256z^{11}} + \frac{231\mu^9(3003+91\mu^2+\mu^4)}{1024z^{13}} \right). \end{aligned} \quad (49)$$

Clearly, the first two leading terms of the 0th order (47) do not cancel all the divergences in the unregularized Δ_k^2 of (41), and there still remains a logarithmic divergence $\frac{H^3}{\pi^2} \frac{\mu}{2}$. The first two leading terms of the 2nd order (48) successfully cancel all the divergences in Δ_k^2 of (41), yielding a UV convergent regularized power spectrum $\Delta_{k\text{ reg}}^{2(2)}$. According to the minimal subtraction rule [9], the 2nd order regularization is sufficient. The 2nd order regularized power spectrum at high z is

$$\begin{aligned} \lim_{z \rightarrow \infty} \Delta_{k\text{ reg}}^{2(2)} \simeq & -\frac{H^3}{\pi^2} z^3 \left(\frac{3\mu}{2z^5} - \frac{5\mu(36+49\mu^2)}{16z^7} + \frac{35\mu(576+820\mu^2+273\mu^4)}{128z^9} \right. \\ & - \frac{63\mu(14400+11\mu^2(1916+695\mu^2+93\mu^4))}{256z^{11}} \\ & \left. + \frac{231\mu(518400+13\mu^2(59472+22792\mu^2+3421\mu^4+231\mu^6))}{1024z^{13}} \right), \end{aligned} \quad (50)$$

where the leading term,

$$-\frac{H^3}{\pi^2} z^3 \left(\frac{3\mu}{2z^5} \right) \propto -\frac{1}{z^2}, \quad (51)$$

is UV convergent and remains negative at high z . So the 2nd order adiabatic regularization not only removes all UV divergences, but also preserves the negative sign of the vacuum power spectrum at high z . In literature the 2nd order adiabatic regularization was first applied upon the power spectrum of a minimally coupling massive scalar field Ref. [41].

The 4th order subtraction term (49) would subtract more than necessary and result in an improper regularized power spectrum. Let us examine the 4th order regularized power spectrum at high z

$$\lim_{z \rightarrow \infty} \Delta_{k \text{ reg}}^{2(4)} \simeq -\frac{H^3}{\pi^2} z^3 \left(-\frac{45\mu}{4z^7} + \frac{35\mu(144 + 205\mu^2)}{32z^9} - \frac{63\mu(14400 + 21076\mu^2 + 7645\mu^4)}{256z^{11}} + \frac{231\mu(518400 + 773136\mu^2 + 296296\mu^4 + 44473\mu^6)}{1024z^{13}} \right), \quad (52)$$

where the leading term,

$$-\frac{H^3}{\pi^2} z^3 \left(-\frac{45\mu}{4z^7} \right) \propto \frac{1}{z^4} > 0, \quad (53)$$

is over-convergent and positive. This is because the 4th order regularization subtracts too much, so that the convergent term $\propto -1/z^2$ of Δ_k^2 has been subtracted.

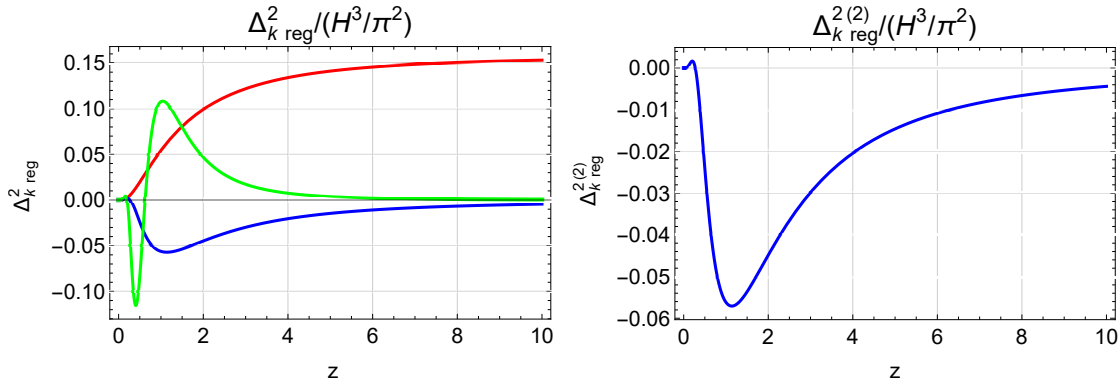


Figure 2: (a) Regularized power spectrum: the 0th order $\Delta_{k \text{ reg}}^{2(0)}$ (red), the 2nd order $\Delta_{k \text{ reg}}^{2(2)}$ (blue), the 4th order $\Delta_{k \text{ reg}}^{2(4)}$ (green). (b) The enlarged 2nd order $\Delta_{k \text{ reg}}^{2(2)}$.

We plot the three regularized power spectra, $\Delta_{k \text{ reg}}^{2(0)}$, $\Delta_{k \text{ reg}}^{2(2)}$, and $\Delta_{k \text{ reg}}^{2(4)}$ in Fig. 2 (a) with the parameter $\mu^2 = 0.1$. The 0th order regularized $\Delta_{k \text{ reg}}^{2(0)}$ (red) is UV logarithmically divergent and positive. The 4th order regularized $\Delta_{k \text{ reg}}^{2(4)}$ (green) is over-convergent and positive at high z , showing an irregular infrared behavior. The 2nd order regularized $\Delta_{k \text{ reg}}^{2(2)}$ (blue) is UV convergent and negative. Fig. 2 (b) shows an enlarged view of $\Delta_{k \text{ reg}}^{2(2)}$, which at convergent and becomes positive at very small z ,

$$\lim_{z \rightarrow 0} \Delta_{k \text{ reg}}^{2(2)} \simeq \frac{H^3}{\pi^2} z^3 (1 - \tanh(\pi\mu)). \quad (54)$$

This infrared distortion is caused by the inaccuracy of WKB modes at small k under the adiabatic regularization. This issue has been addressed in the schemes of the inside-horizon regularization [42] and the energy-dependent regularization [43].

The massless limit ($\mu \rightarrow 0$) of the unregularized power spectrum (40) vanishes

$$\lim_{\mu \rightarrow 0} \Delta_k^2 = -\frac{H^3}{\pi^2} z^3 \times 0 = 0. \quad (55)$$

The massless limits of $\Delta_{kad}^{2(2)}$ in (45) is vanishing

$$\lim_{\mu \rightarrow 0} \Delta_{kad}^{2(2)} = 0, \quad (56)$$

so that the massless limit of the 2nd order regularized spectrum is also vanishing,

$$\lim_{\mu \rightarrow 0} \Delta_{kreg}^{2(2)} = 0. \quad (57)$$

From (45) and (46) it is seen that the massless limits (56) and (57) are valid for all the orders ($n = 0, 2, 4, \dots$). If one starts with the massless field, one also obtains (56) and (57). Thus, under the adiabatic regularization, the regularized power spectrum in the massless limit is zero, and equals to that of the massless spin- $\frac{1}{2}$ field.

The correlation function is the Fourier transformation of the power spectrum, and will be presented in Sect. 5 and Appendix C for the point-splitting scheme.

4 Adiabatic regularization of stress tensor

In this section, we calculate the vacuum stress tensor of the spin- $\frac{1}{2}$ field ψ and remove the UV divergences by the adiabatic regularization. Like for the power spectrum, the 2nd order regularization will suffice to remove all UV divergences of the vacuum stress tensor, and the associated, regularized spectral energy density will keep the same sign as the unregularized one. On the other hand, the conventional 4th order regularization [35] would change the sign of the energy density and would lead to the trace anomaly, because it does not respect the minimal subtraction rule and subtracts off more terms than necessary.

The stress tensor of the field ψ in curved spacetimes is defined by [2, 33]

$$T_{\mu\nu} = \frac{1}{2}i[\bar{\psi}\gamma_{(\mu}\nabla_{\nu)}\psi - (\nabla_{(\mu}\bar{\psi})\gamma_{\nu)}\psi]. \quad (58)$$

The trace of (58) is

$$T^\mu_\mu = g^{\mu\nu}T_{\mu\nu} = m\bar{\psi}\psi, \quad (59)$$

where the Dirac equation (6) has been used for the second equality. Taking the vacuum expectation value of (58) gives the vacuum energy density and pressure as the following

$$\rho = \langle 0|T^0_0|0\rangle = \int \frac{dk}{k}\rho_k, \quad (60)$$

$$p = -\frac{1}{3}\langle 0|T^i_i|0\rangle = \int \frac{dk}{k}p_k, \quad (61)$$

where

$$\rho_k = \frac{k^3}{2\pi^2a^3}i(h_k^{II}\dot{h}_k^{II*} + h_k^I\dot{h}_k^{I*} - \dot{h}_k^{II}h_k^{II*} - \dot{h}_k^Ih_k^{I*}), \quad (62)$$

$$p_k = \frac{k^4}{2\pi^2a^4}(-\frac{2}{3})(h_k^{II}\dot{h}_k^{I*} + h_k^I\dot{h}_k^{II*}), \quad (63)$$

are the vacuum spectral energy density and spectral pressure, respectively. By use of the equations (15) and (16), the spectral energy density (62) can be rewritten as

$$\rho_k = -\frac{k^4}{\pi^2a^4}(h_k^{II}\dot{h}_k^{I*} + h_k^I\dot{h}_k^{II*}) - m\frac{k^3}{\pi^2a^3}(|h_k^I|^2 - |h_k^{II}|^2). \quad (64)$$

From (64) it is seen that

$$\rho_k - 3p_k = m\Delta_k^2, \quad (65)$$

where p_k is given by (63) and Δ_k^2 is given by (39). Eq.(65) also follows from the vacuum expectation value of eq.(59). Using the modes (21) and (22) into (62) and (63) gives

$$\begin{aligned} \rho_k = & -i\frac{H^4}{\pi^2}z^4\frac{\sqrt{\pi z}}{4}e^{\pi\mu}\left(H_{-i\mu-\frac{1}{2}}^{(1)}\frac{\partial}{\partial z}\left(\frac{\sqrt{\pi z}}{2}H_{i\mu-\frac{1}{2}}^{(2)}\right) - H_{i\mu-\frac{1}{2}}^{(2)}\frac{\partial}{\partial z}\left(\frac{\sqrt{\pi z}}{2}H_{-i\mu-\frac{1}{2}}^{(1)}\right)\right. \\ & \left.+ H_{-i\mu+\frac{1}{2}}^{(1)}\frac{\partial}{\partial z}\left(\frac{\sqrt{\pi z}}{2}H_{i\mu+\frac{1}{2}}^{(2)}\right) - H_{i\mu+\frac{1}{2}}^{(2)}\frac{\partial}{\partial z}\left(\frac{\sqrt{\pi z}}{2}H_{-i\mu+\frac{1}{2}}^{(1)}\right)\right), \end{aligned} \quad (66)$$

$$p_k = \frac{H^4}{2\pi^2}z^4\frac{2}{3}i\frac{\pi z}{4}e^{\pi\mu}\left(H_{-i\mu-\frac{1}{2}}^{(1)}H_{i\mu+\frac{1}{2}}^{(2)} - H_{i\mu-\frac{1}{2}}^{(2)}H_{-i\mu+\frac{1}{2}}^{(1)}\right). \quad (67)$$

Fig. 3 shows that both ρ_k and p_k are negative, and UV divergent at high z . A negative vacuum energy density is an intrinsic feature of the spin- $\frac{1}{2}$ field, and originates from the anticommutation relations, unlike the scalar and vector fields that have a positive vacuum energy density.

At high z , the spectral energy density and pressure are

$$\begin{aligned} \lim_{z \rightarrow \infty} \rho_k \simeq & -\frac{H^4}{\pi^2}z^4\left(1 + \frac{\mu^2}{2z^2} - \frac{\mu^2 + \mu^4}{8z^4} + \frac{\mu^2(4 + 5\mu^2 + \mu^4)}{16z^6}\right. \\ & \left. - \frac{5\mu^2(1 + \mu^2)(4 + \mu^2)(9 + \mu^2)}{128z^8}\right), \end{aligned} \quad (68)$$

$$\begin{aligned} \lim_{z \rightarrow \infty} p_k \simeq & \frac{H^4}{\pi^2}\left(-\frac{1}{3}\right)z^4\left(1 - \frac{\mu^2}{2z^2} + \frac{3(\mu^2 + \mu^4)}{8z^4} - \frac{5\mu^2(4 + 5\mu^2 + \mu^4)}{16z^6}\right. \\ & \left.+ \frac{35\mu^2(1 + \mu^2)(4 + \mu^2)(9 + \mu^2)}{128z^8}\right), \end{aligned} \quad (69)$$

where the first three terms are, respectively, quartic, quadratic, and logarithmically divergent.

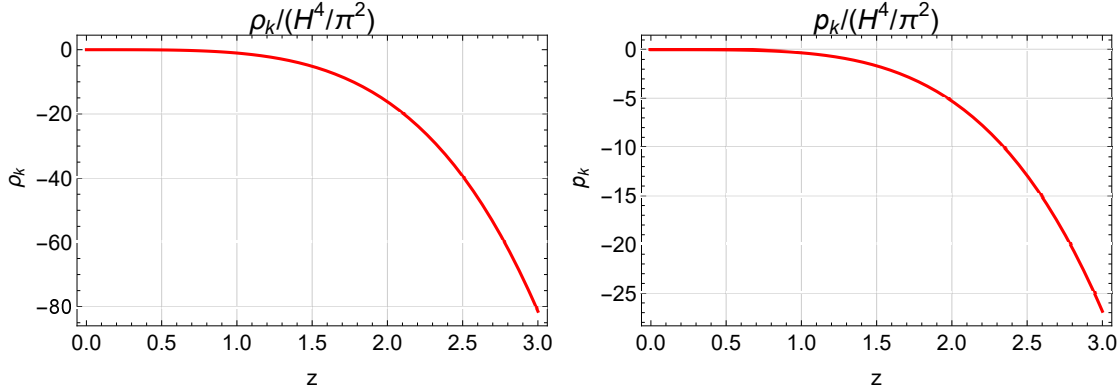


Figure 3: (a) Unregularized spectral energy density ρ_k . (b) Unregularized spectral pressure p_k .

At small z , the spectral energy density and pressure are infrared convergent,

$$\lim_{z \rightarrow 0} \rho_k \simeq -\frac{H^4}{2\pi^2}z^4\left(\frac{2\mu \tanh(\pi\mu)}{z}\right) < 0, \quad (70)$$

$$\lim_{z \rightarrow 0} p_k \simeq -\frac{H^4}{\pi^2}z^4 \text{Re}\left(\frac{2^{-1+2i\mu}z^{-2i\mu}\Gamma(\frac{1}{2} + i\mu)}{3\Gamma(\frac{1}{2} - i\mu)\cosh(\pi\mu)}\right), \quad (71)$$

with ρ_k of (70) being negative, and the sign of p_k of (71) depending on the magnitude of μ .

Now we shall remove the UV divergences in the unregularized ρ_k and p_k . In analogy to the power spectrum, the regularized spectral energy density and pressure are defined as the difference

$$\rho_{k \text{ reg}}^{(n)} \equiv \rho_k - \rho_{k \text{ ad}}^{(n)}, \quad n = 0, 2, 4, \dots \quad (72)$$

$$p_{k \text{ reg}}^{(n)} \equiv p_k - p_{k \text{ ad}}^{(n)}, \quad n = 0, 2, 4, \dots \quad (73)$$

where $\rho_{k \text{ ad}}^{(n)}$ and $p_{k \text{ ad}}^{(n)}$ are the adiabatic spectral energy density and pressure, listed in Appendix B. (See (B.6), (B.8), (B.10), (B.11), (B.12), (B.13).) In de Sitter space, they are given by the following

$$\rho_{k \text{ ad}}^{(0)} = -\frac{H^4}{\pi^2} z^4 \frac{1}{z} \bar{\omega}, \quad (74)$$

$$p_{k \text{ ad}}^{(0)} = -\frac{1}{3} \frac{H^4}{\pi^2} z^4 \frac{z}{\bar{\omega}}, \quad (75)$$

$$\rho_{k \text{ ad}}^{(2)} = -\frac{H^4}{\pi^2} z^4 \frac{1}{z} \left(\bar{\omega} - \frac{\mu^2}{8\bar{\omega}^3} + \frac{\mu^4}{8\bar{\omega}^5} \right), \quad (76)$$

$$p_{k \text{ ad}}^{(2)} = -\frac{1}{3} \frac{H^4}{\pi^2} z^4 \frac{z}{\bar{\omega}} \left(1 + \frac{3\mu^2}{8\bar{\omega}^4} - \frac{5\mu^4}{8\bar{\omega}^6} \right), \quad (77)$$

$$\rho_{k \text{ ad}}^{(4)} = -\frac{H^4}{\pi^2} z^4 \frac{1}{z} \left(\bar{\omega} + \frac{\mu^4}{8\bar{\omega}^5} - \frac{\mu^2}{8\bar{\omega}^3} + \frac{\mu^2}{4\bar{\omega}^5} - \frac{165\mu^4}{128\bar{\omega}^7} + \frac{119\mu^6}{64\bar{\omega}^9} - \frac{105\mu^8}{128\bar{\omega}^{11}} \right), \quad (78)$$

$$p_{k \text{ ad}}^{(4)} = -\frac{1}{3} \frac{H^4}{\pi^2} z^4 \frac{z}{\bar{\omega}} \left(1 + \frac{3\mu^2}{8\bar{\omega}^4} - \frac{5\mu^2(2+\mu^2)}{8\bar{\omega}^6} + \frac{1155\mu^4}{128\bar{\omega}^8} - \frac{1071\mu^6}{64\bar{\omega}^{10}} + \frac{1155\mu^8}{128\bar{\omega}^{12}} \right). \quad (79)$$

The 0th order (74) and (75) contain a single divergent term, the 2nd order (76) and (77) and the 4th order (78) and (79) contain two divergent terms. To compare with the unregularized (68) and (69) at high z , we expand the adiabatic (74) — (79) at high z as follows

$$\lim_{z \rightarrow \infty} \rho_{k \text{ ad}}^{(0)} \simeq -\frac{H^4}{\pi^2} z^4 \left(1 + \frac{\mu^2}{2z^2} - \frac{\mu^4}{8z^4} + \frac{\mu^6}{16z^6} - \frac{5\mu^8}{128z^8} \right), \quad (80)$$

$$\lim_{z \rightarrow \infty} p_{k \text{ ad}}^{(0)} \simeq -\frac{1}{3} \frac{H^4}{\pi^2} z^4 \left(1 - \frac{\mu^2}{2z^2} + \frac{3\mu^4}{8z^4} - \frac{5\mu^6}{16z^6} + \frac{35\mu^8}{128z^8} \right), \quad (81)$$

$$\lim_{z \rightarrow \infty} \rho_{k \text{ ad}}^{(2)} \simeq -\frac{H^4}{\pi^2} z^4 \left(1 + \frac{\mu^2}{2z^2} - \frac{\mu^2(1+\mu^2)}{8z^4} + \frac{\mu^4(5+\mu^2)}{16z^6} - \frac{5\mu^6(14+\mu^2)}{128z^8} \right), \quad (82)$$

$$\lim_{z \rightarrow \infty} p_{k \text{ ad}}^{(2)} \simeq -\frac{1}{3} \frac{H^4}{\pi^2} z^4 \left(1 - \frac{\mu^2}{2z^2} + \frac{3(\mu^2+\mu^4)}{8z^4} - \frac{5\mu^4(5+\mu^2)}{16z^6} + \frac{35\mu^6(14+\mu^2)}{128z^8} \right), \quad (83)$$

$$\lim_{z \rightarrow \infty} \rho_{k \text{ ad}}^{(4)} \simeq -\frac{H^4}{\pi^2} z^4 \left(1 + \frac{\mu^2}{2z^2} - \frac{\mu^2(1+\mu^2)}{8z^4} + \frac{\mu^2(4+5\mu^2+\mu^4)}{16z^6} - \frac{5\mu^4(7+\mu^2)^2}{128z^8} \right), \quad (84)$$

$$\lim_{z \rightarrow \infty} p_{k \text{ ad}}^{(4)} \simeq -\frac{1}{3} \frac{H^4}{\pi^2} z^4 \left(1 - \frac{\mu^2}{2z^2} + \frac{3(\mu^2+\mu^4)}{8z^4} - \frac{5\mu^2(4+5\mu^2+\mu^4)}{16z^6} + \frac{35\mu^4(7+\mu^2)^2}{128z^8} \right). \quad (85)$$

We calculate the regularized spectral stress tensor for each order in the following. The 0th order regularized spectral stress tensor at high z is given by

$$\lim_{z \rightarrow \infty} \rho_{k \text{ reg}}^{(0)} \simeq -\frac{H^4}{\pi^2} z^4 \left(-\frac{\mu^2}{8z^4} + \frac{\mu^2(4+5\mu^2)}{16z^6} - \frac{5\mu^2(36+49\mu^2+14\mu^4)}{128z^8} \right), \quad (86)$$

$$\lim_{z \rightarrow \infty} p_{k \text{ reg}}^{(0)} \simeq -\frac{1}{3} \frac{H^4}{\pi^2} z^4 \left(\frac{3\mu^2}{8z^4} - \frac{5\mu^2(4+5\mu^2)}{16z^6} + \frac{35\mu^2(36+49\mu^2+14\mu^4)}{128z^8} \right), \quad (87)$$

still having the logarithmic divergence. So we are not interested in it.

The 2nd order regularized spectral stress tensor at high z is given by

$$\lim_{z \rightarrow \infty} \rho_{k \text{ reg}}^{(2)} \simeq -\frac{H^4}{\pi^2} z^4 \left(\frac{\mu^2}{4z^6} - \frac{5\mu^2(36+49\mu^2)}{128z^8} \right), \quad (88)$$

$$\lim_{z \rightarrow \infty} p_{k \text{ reg}}^{(2)} \simeq -\frac{1}{3} \frac{H^4}{\pi^2} z^4 \left(-\frac{5\mu^2}{4z^6} + \frac{35\mu^2(36+49\mu^2)}{128z^8} \right), \quad (89)$$

being UV convergent. Furthermore, $\rho_{k \text{ reg}}^{(2)}$ remains negative at high z . Thus, the 2nd order regularization is sufficient to remove all UV divergences in the spectral stress tensor, and preserves the negative sign, like the case for the power spectrum.

The 4th order regularized spectral stress tensor at high z is given by

$$\lim_{z \rightarrow \infty} \rho_{k \text{ reg}}^{(4)} \simeq \frac{H^4}{\pi^2} z^4 \left(\frac{45\mu^2}{32z^8} \right), \quad (90)$$

$$\lim_{z \rightarrow \infty} p_{k \text{ reg}}^{(4)} \simeq -\frac{1}{3} \frac{H^4}{\pi^2} z^4 \left(\frac{315\mu^2}{32z^8} \right), \quad (91)$$

and $\rho_{k \text{ reg}}^{(4)}$ becomes positive. This is because the 4th order regularization subtracts more than necessary, and the convergent z^{-2} terms have been subtracted.

Fig.4 (a) plots the spectral energy density, $\rho_{k \text{ reg}}^{(0)}$, $\rho_{k \text{ reg}}^{(2)}$, $\rho_{k \text{ reg}}^{(4)}$, and Fig.5 (a) plots the spectral pressure, $p_{k \text{ reg}}^{(0)}$, $p_{k \text{ reg}}^{(2)}$, $p_{k \text{ reg}}^{(4)}$. The enlarged $\rho_{k \text{ reg}}^{(2)}$ and $p_{k \text{ reg}}^{(2)}$ are plotted in Fig.4 (b) and Fig.5 (b). It is seen that the 2nd order $\rho_{k \text{ reg}}^{(2)}$ is negative except at very small z , and $p_{k \text{ reg}}^{(2)}$ is positive except at small z . Like the power spectrum, the infrared behavior of $\rho_{k \text{ reg}}^{(2)}$ and $p_{k \text{ reg}}^{(2)}$ is due to the inaccuracy of WKB modes at small k .

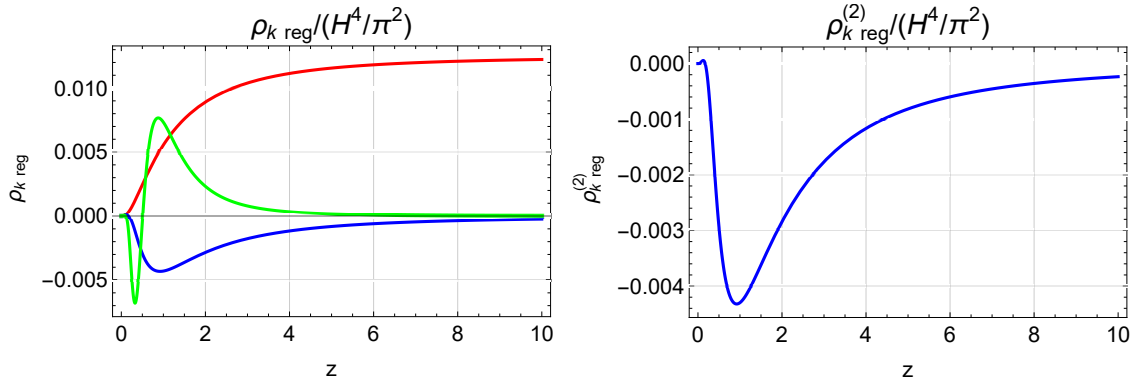


Figure 4: (a) Regularized spectral energy density: the 0th order $\rho_{k \text{ reg}}^{(0)}$ (red), the 2nd order $\rho_{k \text{ reg}}^{(2)}$ (blue), the 4th order $\rho_{k \text{ reg}}^{(4)}$ (green). (b) The enlarged 2nd order $\rho_{k \text{ reg}}^{(2)}$.

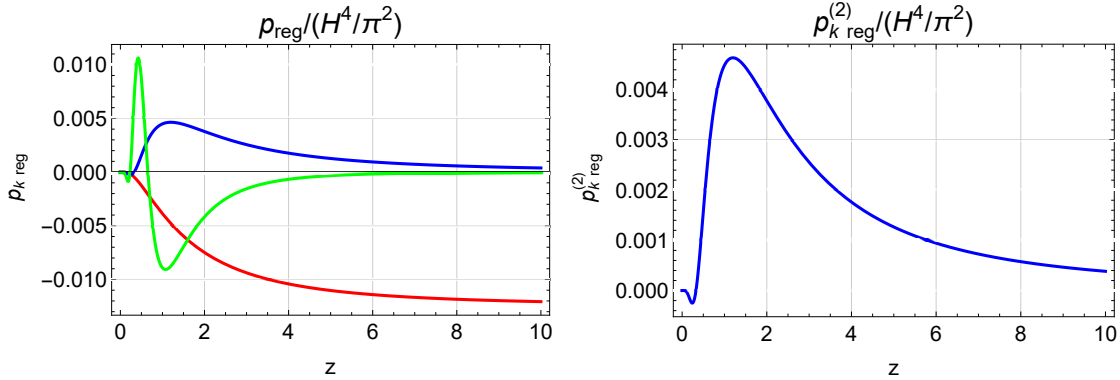


Figure 5: (a) Regularized spectral pressure: the 0th order $p_{k \text{ reg}}^{(0)}$ (red), the 2nd order $p_{k \text{ reg}}^{(2)}$ (blue), the 4th order $p_{k \text{ reg}}^{(4)}$ (green). (b) The enlarged 2nd order $p_{k \text{ reg}}^{(2)}$.

Obviously, the regularized vacuum spectral energy density and pressure are not minus to each other,

$$\rho_{k \text{ reg}}^{(2)} \neq -p_{k \text{ reg}}^{(2)}. \quad (92)$$

Nevertheless, integrating the spectra over k ,

$$\rho_{\text{reg}}^{(2)} \equiv \int_0^\infty \rho_{k \text{ reg}}^{(2)} \frac{dk}{k}, \quad (93)$$

$$p_{reg}^{(2)} \equiv \int_0^\infty p_{k\ reg}^{(2)} \frac{dk}{k}, \quad (94)$$

we find that the regularized vacuum energy density and pressure are opposite to each other

$$\rho_{reg}^{(2)} = -p_{reg}^{(2)}. \quad (95)$$

For example, for $\mu^2 = 0.1$, the numerical integration gives

$$\rho_{reg}^{(2)} = -0.00837 \frac{H^4}{\pi^2} = -p_{reg}^{(2)}, \quad (96)$$

and the numerical $\rho_{reg}^{(2)}$ for other values of μ are plotted (in the blue dots) in Fig. 6. It is remarkable that the outcome vacuum energy density and pressure as in (95) are equal in magnitude but with opposite signs. (For a simplified model, Ref [7] showed that the finite, regularized energy density and pressure given by a generic scheme of regularization also satisfies the relation (95).) Thus, the regularized vacuum stress tensor is proportional to the metric of background spacetime

$$\langle 0 | T_{\mu\nu} | 0 \rangle_{reg}^{(2)} = \frac{1}{4} g_{\mu\nu} \langle 0 | T^\beta{}_\beta | 0 \rangle_{reg}^{(2)}, \quad (97)$$

and possesses the maximal symmetry in de Sitter space [8]. The vacuum stress tensor (97) has a form of a “negative” cosmological constant, due to $\rho_{reg}^{(2)} < 0$. This property of the spin- $\frac{1}{2}$ field is distinguished from the scalar and vector fields which have a positive vacuum energy density. In short, by the adiabatic regularization, we have proven that the 2nd order regularized vacuum stress tensor is finite, and maximally symmetric, and that the sign of the 2nd order regularized energy density remains negative, the same as the unregularized energy density.

By numerical integration, we also find that the 4th order $\rho_{reg}^{(4)} = -p_{reg}^{(4)}$, like the 2nd order one, and nevertheless that $\rho_{reg}^{(4)}$ can be either positive or negative, depending on the parameter μ . We plot the numerical $\rho_{reg}^{(4)}$ (in the red dots) in Fig. 6.

The massless limit of the unregularized spectral stress tensor, (66) and (67), is

$$\lim_{\mu \rightarrow 0} \rho_k = \lim_{\mu \rightarrow 0} 3p_k = -\frac{H^4}{\pi^2} z^4. \quad (98)$$

The massless limit of the adiabatic spectral stress tensor, (76) (77), is given by

$$\rho_{k\ ad}^{(2)} = 3p_{k\ ad}^{(2)} = -\frac{H^4}{\pi^2} z^4, \quad (99)$$

equal to the unregularized (98), so that the massless limit of the regularized spectral stress tensor is vanishing,

$$\rho_{k\ reg}^{(2)} = p_{k\ reg}^{(2)} = 0. \quad (100)$$

It should be mentioned that (98) — (100) are actually also valid for all the orders ($n = 0, 2, 4, \dots$). The massless limit of the regularized spectral stress tensor is equal to the regularized spectral stress tensor of the massless spin- $\frac{1}{2}$ field, both are zero. Thus, integrating (100) over k , we get the zero trace

$$\langle 0 | T^\mu{}_\mu | 0 \rangle_{reg}^{(2)} = \rho_{reg}^{(2)} - 3p_{reg}^{(2)} = 0, \quad (101)$$

and there is no trace anomaly for the massless spin-1/2 field.

However, if the k -integration is taken on the 4th order regularized spectral stress tensor preceding the massless limit, the resultant 4th order regularized stress tensor will be nonzero

and the trace anomaly will appear. Let us show this. Since the 2nd order regularization does not give rise to the trace anomaly, we consider only the trace difference between the 4th order and 2nd order subtraction terms,

$$\lim_{m \rightarrow 0} \int ((\rho_{k \text{ ad}}^{(4)} - \rho_{k \text{ ad}}^{(2)}) - 3(p_{k \text{ ad}}^{(4)} - p_{k \text{ ad}}^{(2)})) \frac{dk}{k} = \lim_{m \rightarrow 0} m \int (\Delta_{k \text{ ad}}^{(4)} - \Delta_{k \text{ ad}}^{(2)}) \frac{dk}{k}, \quad (102)$$

where the relation (65) has been used. From the adiabatic terms (B.3) and (B.5), the difference reads

$$\begin{aligned} \Delta_{k \text{ ad}}^{(4)} - \Delta_{k \text{ ad}}^{(2)} = & -\frac{k^3}{a^3 \pi^2} \left(\left(\frac{\dot{a}^4}{16a^4} + \frac{11\dot{a}^2\ddot{a}}{16a^3} + \frac{7\dot{a}\ddot{a}}{16a^2} + \frac{\ddot{a}^2}{4a^2} + \frac{\ddot{a}}{16a} \right) \frac{m}{\omega^5} \right. \\ & - \left(\frac{43\dot{a}^4}{16a^4} + \frac{211\dot{a}^2\ddot{a}}{32a^3} + \frac{29\ddot{a}^2}{32a^2} + \frac{21\dot{a}\ddot{a}}{16a^2} + \frac{\ddot{a}}{16a} \right) \frac{m^3}{\omega^7} \\ & + \left(\frac{1659\dot{a}^4}{128a^4} + \frac{105\dot{a}^2\ddot{a}}{8a^3} + \frac{21\ddot{a}^2}{32a^2} + \frac{7\dot{a}\ddot{a}}{8a^2} \right) \frac{m^5}{\omega^9} \\ & \left. - \left(\frac{1239m^7\dot{a}^4}{64a^4\omega^{11}} + \frac{231m^7\dot{a}^2\ddot{a}}{32a^3\omega^{11}} \right) \frac{m^7}{\omega^{11}} + \frac{1155m^9\dot{a}^4}{128a^4\omega^{13}} \right). \end{aligned} \quad (103)$$

Performing k -integration and using the formula

$$\int_0^\infty \frac{x^2}{(1+x^2/b^2)^{\frac{n}{2}}} dx = \frac{\sqrt{\pi}}{4} \frac{\Gamma(\frac{n}{2} - \frac{3}{2})}{\Gamma(\frac{n}{2})} b^3,$$

one gets

$$\int (\Delta_{k \text{ ad}}^{(4)} - \Delta_{k \text{ ad}}^{(2)}) \frac{dk}{k} = \frac{1}{240\pi^2 m} \left(4 \frac{\dot{a}^2 \ddot{a}}{a^2} - 3 \frac{\ddot{a}^2}{a^2} - 9 \frac{\dot{a} \ddot{a}}{a^2} - 3 \frac{\ddot{a}}{a} \right) \quad (104)$$

$$= -\frac{11H^4}{240\pi^2 m}, \quad (105)$$

which is singular at zero mass. Multiplying the above by m and taking the massless limit, one gets

$$\lim_{m \rightarrow 0} m \int (\Delta_{k \text{ ad}}^{(4)} - \Delta_{k \text{ ad}}^{(2)}) \frac{dk}{k} = \frac{1}{240\pi^2} \left(4 \frac{\dot{a}^2 \ddot{a}}{a^2} - 3 \frac{\ddot{a}^2}{a^2} - 9 \frac{\dot{a} \ddot{a}}{a^2} - 3 \frac{\ddot{a}}{a} \right) = -\frac{11H^4}{240\pi^2}. \quad (106)$$

The outcome (106) corresponds to the trace anomaly in Ref. [35]. Thus, the trace anomaly is an artifact of the improper 4th order regularization with the k -integration preceding the massless limit. This is the case for the scalar fields [16], the vector fields [21, 22], as well as the spin- $\frac{1}{2}$ field [33–38].

In sum, for the massive spin- $\frac{1}{2}$ field, the 2nd order adiabatic regularization is sufficient to remove all the UV divergences in both the power spectrum and the stress tensor. The massless limit of the 2nd order regularized spectral stress tensor is zero, and equal to that of the massless field. The 4th order regularization subtracts more than necessary and changes the sign of the spectral energy density, as it does not respect the minimal subtraction rule. The difficulties of the 4th order regularization will also be analyzed by the point-splitting method in the next section.

5 Point-splitting regularization in coordinate space

The point-splitting regularization as a method works in coordinate space [1, 17, 18, 20, 26], and can give the analytical, regularized correlation function and stress tensor, whereas the adiabatic regularization in k -space can give the regularized power spectrum and spectral stress tensor. The

two methods are complementary. We shall derive the analytic regularized correlation function and stress tensor, and examine the difficulty of the 4th order regularization in the massless limit.

The unregularized vacuum correlation function is defined by

$$\begin{aligned} \langle 0 | \bar{\psi}(x) \psi(x') | 0 \rangle &= \frac{1}{a(t)^{\frac{3}{2}} a(t')^{\frac{3}{2}} |\vec{x} - \vec{x}'|} \\ &\times \int_0^\infty \frac{k^3}{\pi^2} (h_k^{II}(t) h_k^{II*}(t') - h_k^I(t) h_k^{I*}(t')) \frac{\sin k |\vec{x} - \vec{x}'|}{k} \frac{dk}{k}, \end{aligned} \quad (107)$$

and its coincidence limit ($x' \rightarrow x$) is the UV divergent auto-correlation (37). To remove the UV divergences, one constructs the adiabatic correlation function

$$\langle 0 | \bar{\psi}(x) \psi(x') | 0 \rangle_{ad}^{(n)}, \quad n = 0, 2, 4, \dots \quad (108)$$

which is formed by using the adiabatic modes $g_k^{I(n)}$ and $g_k^{II(n)}$ to replace the exact modes h_k^I and h_k^{II} in (107). Then one subtracts the adiabatic correlation from the unregularized correlation, and takes the coincidence limit,

$$\langle 0 | \bar{\psi}(x) \psi(x) | 0 \rangle_{reg}^{(2)} \equiv \lim_{x' \rightarrow x} \langle 0 | \bar{\psi}(x) \psi(x') | 0 \rangle - \lim_{x' \rightarrow x} \langle 0 | \bar{\psi}(x) \psi(x') | 0 \rangle_{ad}^{(2)}, \quad (109)$$

where the 2nd order regularization is adopted. (109) defines the regularized auto-correlation in the point-splitting scheme, and is analogous to the adiabatic regularization (43) of the power spectrum in k -space. From the maximal symmetry (97) and the relation $T^\beta_\beta = m \bar{\psi} \psi$, the regularized vacuum stress tensor can be expressed in terms of the regularized auto-correlation

$$\langle 0 | T_{\mu\nu} | 0 \rangle_{reg}^{(2)} = \frac{1}{4} g_{\mu\nu} m \langle 0 | \bar{\psi}(x) \psi(x) | 0 \rangle_{reg}^{(2)}. \quad (110)$$

We first consider the simple case of the massless field. By the massless modes (23) and (24), one has

$$h_k^{II}(t) h_k^{II*}(t') = h_k^I(t) h_k^{I*}(t'), \quad (111)$$

so the correlation function of the massless spin- $\frac{1}{2}$ field is zero

$$\langle 0 | \bar{\psi}(x) \psi(x') | 0 \rangle = 0. \quad (112)$$

Since the massless WKB modes (34) and (35) are equal to the exact modes, the adiabatic correlation function is also zero

$$\langle 0 | \bar{\psi}(x) \psi(x') | 0 \rangle_{ad}^{(n)} = 0, \quad n = 0, 2, 4, \dots, \quad (113)$$

which holds for all adiabatic orders. Thus, the regularized correlation function of the massless field is zero

$$\lim_{x' \rightarrow x} \langle 0 | \bar{\psi}(x) \psi(x') | 0 \rangle_{reg}^{(n)} = 0, \quad n = 0, 2, 4, \dots \quad (114)$$

The result (114) is consistent with the vanishing regularized power spectrum (57) that has been derived from the adiabatic regularization. The regularized vacuum stress tensor of the massless field is also zero

$$\langle 0 | T_{\mu\nu} | 0 \rangle_{reg}^{(n)} = 0, \quad n = 0, 2, 4, \dots \quad (115)$$

This result from the point-splitting scheme agrees with the results (100) and (101) from the adiabatic scheme.

Next consider the massive case. Inserting the exact modes (21) and (22) into (107) and performing the integration, we get the unregularized correlation function (see Appendix C for the details)

$$\begin{aligned} \langle 0|\bar{\psi}(x)\psi(x')|0\rangle &= \frac{H^3}{\pi^2}\Gamma(\nu + \frac{3}{2})\Gamma(\frac{3}{2} - \nu)\left(-\frac{i}{4}\nu {}_2F_1(\frac{3}{2} - \nu, \frac{3}{2} + \nu; 2; 1 + \frac{\sigma_2}{2})\right. \\ &\quad + \frac{i}{16}(\frac{9}{4} - \nu^2) {}_2F_1(\frac{5}{2} - \nu, \frac{5}{2} + \nu; 3; 1 + \frac{\sigma_2}{2}) \\ &\quad - \frac{i}{32}(\nu + \frac{5}{2})(\nu + \frac{3}{2}) {}_2F_1(\frac{3}{2} - \nu, \frac{7}{2} + \nu; 3; 1 + \frac{\sigma_2}{2}) \\ &\quad \left. - \frac{i}{32}(\frac{5}{2} - \nu)(\frac{3}{2} - \nu) {}_2F_1(\frac{7}{2} - \nu, \nu + \frac{3}{2}; 3; 1 + \frac{\sigma_2}{2})\right), \end{aligned} \quad (116)$$

where ${}_2F_1(a, b; c; d)$ is the hypergeometric function, $\nu \equiv -\frac{1}{2} - i\mu$, and

$$\sigma_2 \equiv \frac{1}{2} \frac{(a(t) - a(t'))^2}{a(t)a(t')} - \frac{1}{2} a(t)a(t')H^2|\vec{x} - \vec{x}'|^2 \quad (117)$$

is one-half of the squared geodesic interval in de Sitter space.

The 2nd order adiabatic correlation function of the massive field can be derived as the following. Use the adiabatic modes $(g_k^{I(2)}, g_k^{II(2)})$ to replace (h_k^I, h_k^{II}) in (107), where the integrand

$$g_k^{II(2)}(t)g_k^{II(2)*}(t) - g_k^{II(2)}(t)g_k^{II(2)*}(t) \quad (118)$$

at the equal time ($t = t'$) is the 2nd order adiabatic power spectrum $\Delta_{kad}^{2(2)}$ of (B.3). Carrying out the integration, we obtain the 2nd order adiabatic correlation of the massive field

$$\begin{aligned} \langle 0|\bar{\psi}(x)\psi(x')|0\rangle_{ad}^{(2)} &= \frac{-H^3}{\pi^2} \frac{1}{\sqrt{-2\sigma_2}} \left(\mu^2 K_1(\mu\sqrt{-2\sigma_2}) - \frac{1}{2} \mu\sqrt{-2\sigma_2} K_0(\mu\sqrt{-2\sigma_2}) \right. \\ &\quad \left. + \frac{3}{8} (\mu\sqrt{-2\sigma_2})^2 K_1(\mu\sqrt{-2\sigma_2}) - \frac{1}{24} (\mu\sqrt{-2\sigma_2})^3 K_2(\mu\sqrt{-2\sigma_2}) \right). \end{aligned} \quad (119)$$

In deriving (119), the following formula has been used [45]

$$\int_0^\infty dz \frac{z \sin(zy)}{(z^2 + \mu^2)^{n+\frac{1}{2}}} = \frac{-\sqrt{\pi}}{2^n \mu^n \Gamma(n + 1/2)} \frac{d}{dy} (y^n K_n(y\mu)), \quad (120)$$

with $K_n(x)$ being the modified Bessel function and satisfying the relations $\frac{d}{dx} K_0(x) = -K_1(x)$, $\frac{1}{x} \frac{d}{dx} (x K_1(x)) = -K_0(x)$, $\frac{1}{x} \frac{d}{dx} (x^2 K_2(x)) = -x K_1(x)$, $\frac{1}{x} \frac{d}{dx} (x^3 K_3(x)) = -x^2 K_2(x)$ [44].

We are more interested in the coincidence limit ($\sigma_2 \rightarrow 0$). The unregularized correlation function (116) becomes

$$\lim_{\sigma_2 \rightarrow 0} \langle 0|\bar{\psi}(x)\psi(x')|0\rangle \simeq \frac{H^3}{\pi^2} \left(\frac{1}{2} \frac{\mu}{\sigma_2} - \frac{1}{4} \mu(1 + \mu^2)(-1 + 2\gamma + \log(-\frac{\sigma_2}{2})) + \psi(2 + i\mu) + \psi(2 - i\mu) \right), \quad (121)$$

where ψ on the rhs is the digamma function defined by $\psi(y) \equiv \frac{d}{dy} \ln \Gamma(y)$, and a formula $\psi(1 - i\mu) = \psi(2 - i\mu) - \frac{1}{1 - i\mu}$ has been used. The 2nd order adiabatic correlation (119) becomes

$$\lim_{\sigma_2 \rightarrow 0} \langle 0|\bar{\psi}(x)\psi(x')|0\rangle_{ad}^{(2)} \simeq \frac{H^3}{\pi^2} \left(\frac{1}{2} \frac{\mu}{\sigma_2} - \frac{13\mu}{24} - \frac{1}{4} \mu(1 + \mu^2)(-1 + 2\gamma + \log \mu^2 + \log(-\frac{\sigma_2}{2})) \right). \quad (122)$$

The difference between (121) and (122) gives the 2nd order regularized auto-correlation function

$$\begin{aligned}\langle 0|\bar{\psi}(x)\psi(x)|0\rangle_{reg}^{(2)} &= \int \Delta_{k\ reg}^{2(2)} \frac{dk}{k} \\ &= \frac{H^3}{\pi^2} \left(\frac{13}{24}\mu - \frac{1}{4}\mu(1+\mu^2)(\psi(2+i\mu) + \psi(2-i\mu) - \log \mu^2) \right),\end{aligned}\quad (123)$$

where the UV divergences, $\frac{1}{\sigma_2}$ and $\log(-\frac{\sigma_2}{2})$, have been subtracted off. Multiplying (123) by $\frac{1}{4}mg_{\mu\nu}$ yields the 2nd order regularized stress tensor

$$\langle 0|T_{\mu\nu}|0\rangle_{reg}^{(2)} = \frac{1}{4}g_{\mu\nu} \frac{H^4}{\pi^2} \left(\frac{13}{24}\mu^2 - \frac{1}{4}\mu^2(1+\mu^2)(\psi(2+i\mu) + \psi(2-i\mu) - \log \mu^2) \right), \quad (124)$$

and the corresponding regularized energy density and pressure are

$$\rho_{reg}^{(2)} = -p_{reg}^{(2)} = \frac{1}{4} \frac{H^4}{\pi^2} \left(\frac{13}{24}\mu^2 - \frac{1}{4}\mu^2(1+\mu^2)(\psi(2+i\mu) + \psi(2-i\mu) - \log \mu^2) \right). \quad (125)$$

(119) (123) (124) are our main result of the point-splitting scheme. Given the expression, we plot the analytical $\rho_{reg}^{(2)}$ (in the blue line) vs the scaled mass μ in Fig. 6. $\rho_{reg}^{(2)}$ is negative, like the unregularized ρ_k . For comparison, the numerical $\rho_{reg}^{(2)}$ (in the blue dots) from the adiabatic regularization is also plotted in Fig. 6. The results from the two schemes of regularization match consistently.

As a consistency check, the massless limit of (116), (119), (123), (124) are vanishing,

$$\lim_{\mu \rightarrow 0} \langle 0|\bar{\psi}(x)\psi(x')|0\rangle = 0, \quad (126)$$

$$\lim_{\mu \rightarrow 0} \langle 0|\bar{\psi}(x)\psi(x')|0\rangle_{ad}^{(2)} = 0, \quad (127)$$

$$\lim_{\mu \rightarrow 0} \langle 0|\bar{\psi}(x)\psi(x)|0\rangle_{reg}^{(2)} = 0, \quad (128)$$

$$\lim_{\mu \rightarrow 0} \langle 0|T_{\mu\nu}|0\rangle_{reg}^{(2)} = 0, \quad (129)$$

agreeing with (112), (113), (114), (115) of the massless field. In particular, (129) shows that under the 2nd order regularization the trace anomaly never appears. In computing (128) (129), we have used the following formula for the di-gamma functions

$$\lim_{\mu \rightarrow 0} (\psi(2+i\mu) + \psi(2-i\mu)) \simeq (2-2\gamma) - \mu^2\psi^{(2)}(2), \quad (130)$$

where the Euler number $\gamma \simeq 0.577$ and $\psi^{(2)}(2) \equiv d^2\psi(z)/dz^2|_{z=2} \simeq 0.404$. From the above it is seen that, for the 2nd order regularization, the ordering of the massless limit and the k -integration can be exchanged, yielding the same outcome,

$$\lim_{m \rightarrow 0} \int \Delta_{k\ reg}^{2(2)} \frac{dk}{k} = \int \lim_{m \rightarrow 0} \Delta_{k\ reg}^{2(2)} \frac{dk}{k} = 0, \quad (131)$$

$$\lim_{m \rightarrow 0} \int \rho_{k\ reg}^{(2)} \frac{dk}{k} = \int \lim_{m \rightarrow 0} \rho_{k\ reg}^{(2)} \frac{dk}{k} = 0, \quad (132)$$

$$\lim_{m \rightarrow 0} \int p_{k\ reg}^{(2)} \frac{dk}{k} = \int \lim_{m \rightarrow 0} p_{k\ reg}^{(2)} \frac{dk}{k} = 0. \quad (133)$$

The 4th order regularization is improper for the massive spin- $\frac{1}{2}$ field, as is known in Sect 4. Still we will reveal its difficulty via the point-splitting scheme. Similarly to the 2nd order case, using the 4th order adiabatic modes ($g_k^{I(4)}$, $g_k^{II(4)}$) to replace (h_k^I , h_k^{II}) in (107), and carrying out the integration, we get the 4th order adiabatic correlation function

$$\langle 0|\bar{\psi}(x)\psi(x')|0\rangle_{ad}^{(4)} = \frac{-H^3}{\pi^2} \frac{1}{\sqrt{-2\sigma_2}} \left[- \left(\frac{17}{960}\mu^3(\sqrt{-2\sigma_2})^5 + \frac{\mu^3}{24}(\sqrt{-2\sigma_2})^3 + \frac{109\mu}{480}(\sqrt{-2\sigma_2})^3 \right. \right.$$

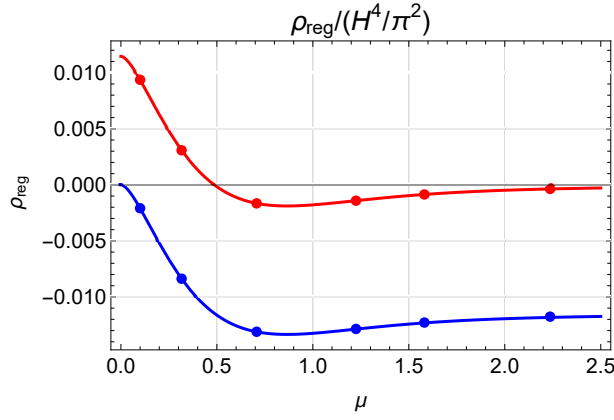


Figure 6: Blue line: the analytical 2nd order $\rho_{reg}^{(2)}$ of (125) from the point-splitting. Blue dots: the numerical $\rho_{reg}^{(2)}$ of (93) from the adiabatic regularization. Red line: the analytical 4th order $\rho_{reg}^{(4)}$ of (140) from the point-splitting. Red dots: the numerical $\rho_{reg}^{(4)}$ from the adiabatic regularization.

$$\begin{aligned}
& + \frac{\mu}{2}(\sqrt{-2\sigma_2}) K_0(\mu\sqrt{-2\sigma_2}) + \left(\frac{\mu^4}{1152}(\sqrt{-2\sigma_2})^6 + \frac{193\mu^2}{1920}(\sqrt{-2\sigma_2})^4 \right. \\
& \left. + \frac{7\mu^2}{24}(\sqrt{-2\sigma_2})^2 + \frac{11}{240}(\sqrt{-2\sigma_2})^2 + \mu^2 \right) K_1(\mu\sqrt{-2\sigma_2}) \Big], \quad (134)
\end{aligned}$$

and its coincidence limit

$$\lim_{\sigma_2 \rightarrow 0} \langle 0 | \bar{\psi}(x) \psi(x') | 0 \rangle_{ad}^{(4)} \simeq \frac{H^3}{\pi^2} \left(\frac{1}{2} \frac{\mu}{\sigma_2} - \frac{11}{240\mu} - \frac{13\mu}{24} - \frac{1}{4} \mu(1 + \mu^2)(-1 + 2\gamma + \log \mu^2 + \log(-\frac{\sigma_2}{2})) \right). \quad (135)$$

The difference between (121) and (135) gives the 4th order regularized auto-correlation

$$\begin{aligned}
\langle 0 | \bar{\psi}(x) \psi(x) | 0 \rangle_{reg}^{(4)} &= \int \Delta_k^{2(4)} \frac{dk}{k} \\
&= \frac{H^3}{\pi^2} \left(\frac{11}{240\mu} + \frac{13\mu}{24} - \frac{1}{4} \mu(1 + \mu^2)(\psi(2 + i\mu) + \psi(2 - i\mu) - \log \mu^2) \right) \quad (136)
\end{aligned}$$

$$= \frac{H^3}{\pi^2} \frac{11}{240\mu} + \langle 0 | \bar{\psi}(x) \psi(x) | 0 \rangle_{reg}^{(2)}, \quad (137)$$

which corresponds to the result (105). Multiplying the above by $\frac{1}{4}mg_{\mu\nu}$ yields the 4th order regularized stress tensor

$$\langle 0 | T_{\mu\nu} | 0 \rangle_{reg}^{(4)} = \frac{1}{4} g_{\mu\nu} \frac{H^4}{\pi^2} \left(\frac{11}{240} + \frac{13\mu^2}{24} - \frac{1}{4} \mu^2(1 + \mu^2)(\psi(2 + i\mu) + \psi(2 - i\mu) - \log \mu^2) \right) \quad (138)$$

$$= g_{\mu\nu} \frac{H^4}{\pi^2} \frac{11}{960} + \langle 0 | T_{\mu\nu} | 0 \rangle_{reg}^{(2)}. \quad (139)$$

Ref. [34] derived (138) by use of a regulator of integration, without giving the full expressions (116) (134). The 4th order regularized energy density and pressure are

$$\rho_{reg}^{(4)} = -p_{reg}^{(4)} = \frac{H^4}{\pi^2} \frac{11}{960} + \rho_{reg}^{(2)}. \quad (140)$$

Fig. 6 shows that $\rho_{reg}^{(4)}$ is higher than $\rho_{reg}^{(2)}$ by $\frac{H^4}{\pi^2} \frac{11}{960}$, and becomes positive at small μ . This is due to the over-subtraction under the 4th order regularization.

Let us examine the difficulties associated with the massless limit of the 4th order regularization. Firstly, the massless limit of the 4th order regularized auto-correlation (137) is singular

$$\lim_{\mu=0} \langle 0 | \bar{\psi}(x) \psi(x) | 0 \rangle_{reg}^{(4)} = \frac{H^3}{\pi^2} \frac{11}{240\mu} = \infty, \quad (141)$$

in contradiction to the zero correlation function (114) of the massless field. Next, the massless limit of the 4th order regularized stress tensor (139) is

$$\lim_{\mu=0} \langle 0 | T_{\mu\nu} | 0 \rangle_{reg}^{(4)} = g_{\mu\nu} \frac{H^4}{\pi^2} \frac{11}{960}, \quad (142)$$

in contradiction to the zero stress tensor (115) of the massless field, too. So, for the 4th order regularization, the ordering of the massless limit and the k -integration may not be exchanged

$$\lim_{m \rightarrow 0} \int \Delta_{k \text{ reg}}^{(4)} \frac{dk}{k} \neq \int \lim_{m \rightarrow 0} \Delta_{k \text{ reg}}^{(4)} \frac{dk}{k} = 0, \quad (143)$$

$$\lim_{m \rightarrow 0} \int \rho_{k \text{ reg}}^{(4)} \frac{dk}{k} \neq \int \lim_{m \rightarrow 0} \rho_{k \text{ reg}}^{(4)} \frac{dk}{k} = 0, \quad (144)$$

$$\lim_{m \rightarrow 0} \int p_{k \text{ reg}}^{(4)} \frac{dk}{k} \neq \int \lim_{m \rightarrow 0} p_{k \text{ reg}}^{(4)} \frac{dk}{k} = 0, \quad (145)$$

unlike the 2nd order case. The trace anomaly will appear only in the 4th order regularization with the k -integration preceding the massless limit [33–35], but will disappear when the massless limit is taken first. These inconsistencies tell that the 4th order regularization is inadequate for the massive spin- $\frac{1}{2}$ field.

6 Conclusion and Discussion

We have studied the regularization of the spin- $\frac{1}{2}$ field in de Sitter space under both the adiabatic and point-splitting schemes. This is part of our serial study on the regularization of quantum fields in curved spacetimes.

The 2nd order regularization is sufficient to remove all divergences for the massive field, whereas the 0th order regularization is insufficient. We have derived the regularized vacuum power spectrum and spectral stress tensor under the adiabatic scheme, as well as the analytical, regularized vacuum correlation and stress tensor under the point-splitting scheme. The outcomes from the two schemes agree with each other consistently. The regularized vacuum stress tensor is maximally symmetric, and the associated energy density remains negative, as the unregularized vacuum energy density. Moreover, the 2nd order regularized stress tensor in the massless limit smoothly reduces to the vanishing regularized stress tensor of the massless field, and there is no trace anomaly. The 2nd order regularization is adequate to the spin- $\frac{1}{2}$ massive field, just like the minimally coupling massive scalar field [20, 26, 27], the longitudinal, temporal, and gauge-fixing parts of the massive vector field [28–30], and the gravitational waves [32].

The conventional 4th order regularization does not respect the minimal subtraction rule, subtracts more terms than necessary, and thus changes the signs of the vacuum spectral energy density. In the massless limit the 4th order regularized auto-correlation function is singular, and the 4th order regularized stress tensor does not reduce to the vanishing regularized stress tensor of the massless field. The so-called trace anomaly will appear only in the 4th order regularization with the k -integration preceding the massless limit. If the massless limit is taken before k -integration (or starting with a massless field), the regularized stress tensor will be zero for each adiabatic order, so that the trace anomaly will not appear. These inconsistencies tell that the 4th order regularization is inadequate for the spin- $\frac{1}{2}$ massive field. The trace anomaly is an artifact of the 4th order regularization.

Due to the anticommutation relations, the spin- $\frac{1}{2}$ massive field possesses a negative vacuum energy density $\rho_{reg} < 0$ which behaves as a “negative” cosmological constant, unlike the massive scalar and vector fields that have a positive $\rho_{reg} > 0$ [20, 26, 28]. In this regard, the cosmological constant that occurs in the observational cosmology is presumably contributed by a sum of the regularized vacuum stress tensors of various quantum fields, among which the boson fields are dominant over the fermion fields. This will provide a pertinent mechanism of quantum origin of the cosmological constant, as advocated by Refs. [7, 8].

We also examined the WKB modes with the arbitrary functions up to the 4th order, and found that these arbitrary functions are actually canceled out in the adiabatic power spectrum and spectral stress tensor.

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A WKB modes

We shall derive the WKB modes g_k^I of (25) and g_k^{II} of (26) up to the 4th adiabatic orders. Our treatments on the arbitrary functions of Ω , F , and G are different from those in Refs. [33–35].

Replacing the exact functions h_k^I and h_k^{II} with the WKB functions g_k^I and g_k^{II} in (14), (15) and (16) yields

$$g_k^I(t) = i\frac{a}{k}(\partial_0 - im)g_k^{II}(t), \quad (\text{A.1})$$

$$g_k^{II}(t) = i\frac{a}{k}(\partial_0 + im)g_k^I(t), \quad (\text{A.2})$$

$$|g_k^I(t)|^2 + |g_k^{II}(t)|^2 = 1, \quad (\text{A.3})$$

Plugging (25) and (26) into (A.1), (A.2), and (A.3), one gets the equations of Ω , F and G as the following

$$\Omega G + i\dot{G} + i\frac{G}{2}\frac{d\omega}{dt}\left(\frac{1}{\omega - m} - \frac{1}{\omega}\right) + mG = (\omega + m)F, \quad (\text{A.4})$$

$$\Omega F + i\dot{F} + i\frac{F}{2}\frac{d\omega}{dt}\left(\frac{1}{\omega + m} - \frac{1}{\omega}\right) - mF = (\omega - m)G, \quad (\text{A.5})$$

$$(\omega + m)FF^* + (\omega - m)GG^* = 2\omega, \quad (\text{A.6})$$

agreeing with (15) in [35]. Decompose F and G into the real and imaginary parts as the following

$$F \equiv RF + iIF = \sum_n (RF^{(n)} + iIF^{(n)}), \quad (\text{A.7})$$

$$G \equiv RG + iIG = \sum_n (RG^{(n)} + iIG^{(n)}). \quad (\text{A.8})$$

Then the relations (32) and (33) lead to the following

$$RF^{(n)}(t; -m) = RG^{(n)}(t; m), \quad (\text{A.9})$$

$$IF^{(n)}(t; -m) = IG^{(n)}(t; m), \quad (\text{A.10})$$

$$\omega^{(n)}(t; -m) = \omega^{(n)}(t; m). \quad (\text{A.11})$$

Plugging (A.7) and (A.8) into (A.4), (A.5), and (A.6) yields

$$\Omega RG - i\dot{G} - \frac{IG}{2}\frac{d\omega}{dt}\left(\frac{1}{\omega - m} - \frac{1}{\omega}\right) + mRG = (\omega + m)RF, \quad (\text{A.12})$$

$$\Omega RF - i\dot{F} - \frac{IF}{2}\frac{d\omega}{dt}\left(\frac{1}{\omega + m} - \frac{1}{\omega}\right) - mRF = (\omega - m)RG, \quad (\text{A.13})$$

$$\Omega IG + i\dot{G} + \frac{RG}{2}\frac{d\omega}{dt}\left(\frac{1}{\omega - m} - \frac{1}{\omega}\right) + mIG = (\omega + m)IF, \quad (\text{A.14})$$

$$\Omega IF + i\dot{F} + \frac{RF}{2}\frac{d\omega}{dt}\left(\frac{1}{\omega + m} - \frac{1}{\omega}\right) - mIF = (\omega - m)IG, \quad (\text{A.15})$$

$$(\omega + m)(RF^2 + IF^2) + (\omega - m)(RG^2 + IG^2) = 2\omega. \quad (\text{A.16})$$

In the following we shall solve the set of equations (A.12) $\sim$ (A.16) order by order. Substituting (29), (A.7), and (A.8) into (A.12) $\sim$ (A.16), we get the following, for the respective order,

0th order:

$$(\omega + m) = (\omega + m), \quad (\text{A.17})$$

$$(\omega - m) = (\omega - m), \quad (\text{A.18})$$

$$0 = 0, \quad (\text{A.19})$$

$$0 = 0, \quad (\text{A.20})$$

$$(\omega + m) + (\omega - m) = 2\omega. \quad (\text{A.21})$$

which are the identities.

1st order:

$$\omega^{(1)} + \omega RG^{(1)} + mRG^{(1)} = (\omega + m)RF^{(1)}, \quad (\text{A.22})$$

$$\omega^{(1)} + \omega RF^{(1)} - mRF^{(1)} = (\omega - m)RG^{(1)}, \quad (\text{A.23})$$

$$\omega IG^{(1)} - \frac{1}{2} \frac{\dot{a}}{a} \frac{m(m + \omega)}{\omega^2} + mIG^{(1)} = (\omega + m)IF^{(1)}, \quad (\text{A.24})$$

$$\omega IF^{(1)} - \frac{1}{2} \frac{\dot{a}}{a} \frac{m(m - \omega)}{\omega^2} - mIF^{(1)} = (\omega - m)IG^{(1)}, \quad (\text{A.25})$$

$$(RF^{(1)} + RG^{(1)}) + \frac{m}{\omega}(RF^{(1)} - RG^{(1)}) = 0, \quad (\text{A.26})$$

where $IG^{(0)} = RG^{(0)} = 1$ (see (29)) and $d\omega/dt = \frac{\dot{a}}{a} \frac{1}{\omega}(m^2 - \omega^2)$ have been used. (A.24) and (A.25) give

$$IG^{(1)} - IF^{(1)} = \frac{1}{2} \frac{\dot{a}}{a} \frac{m}{\omega^2}. \quad (\text{A.27})$$

Simplifying (A.22) and (A.23) yields

$$RG^{(1)} - RF^{(1)} = -\frac{\omega^{(1)}}{m + \omega}, \quad (\text{A.28})$$

$$RG^{(1)} - RF^{(1)} = -\frac{\omega^{(1)}}{m - \omega}, \quad (\text{A.29})$$

which imply

$$\omega^{(1)} = 0, \quad (\text{A.30})$$

$$RG^{(1)} - RF^{(1)} = 0. \quad (\text{A.31})$$

Combining (A.26) with (A.31) leads to

$$RG^{(1)} = RF^{(1)} = 0. \quad (\text{A.32})$$

so $F^{(1)}$ and $G^{(1)}$ are imaginary and can be written as [33]

$$F^{(1)} = i(-A \frac{\dot{a}}{a} \frac{m}{\omega^2} + K \frac{\dot{a}}{a}), \quad (\text{A.33})$$

$$G^{(1)} = i(B \frac{\dot{a}}{a} \frac{m}{\omega^2} + K \frac{\dot{a}}{a}), \quad (\text{A.34})$$

where (A, B, K) are some real functions depending on m and ω with the appropriate dimensions. (A.27) leads to a constraint $A + B = \frac{1}{2}$. In fact, $A = B = \frac{1}{4}$, as we shall see later from the 2nd adiabatic order. Thus, (A.33) and (A.34) become

$$F^{(1)} = -\frac{1}{4} i \frac{\dot{a}}{a} \frac{m}{\omega^2} + i K \frac{\dot{a}}{a}, \quad G^{(1)} = \frac{1}{4} i \frac{\dot{a}}{a} \frac{m}{\omega^2} + i K \frac{\dot{a}}{a}. \quad (\text{A.35})$$

By the relation (A.10), one has $K(t; m) = K(t; -m)$, so K is even in m and can be nonzero in general. Our calculation differs from Ref. [34] which assumed $IF^{(1)}(m) = -IG^{(1)}(m)$.

2nd order

$$\omega^{(2)} - \dot{IG}^{(1)} - \frac{IG^{(1)}}{2} \frac{d\omega}{dt} \left(\frac{1}{\omega - m} - \frac{1}{\omega} \right) = (\omega + m)(RF^{(2)} - RG^{(2)}), \quad (\text{A.36})$$

$$\omega^{(2)} - \dot{IF}^{(1)} - \frac{IF^{(1)}}{2} \frac{d\omega}{dt} \left(\frac{1}{\omega + m} - \frac{1}{\omega} \right) = (\omega - m)(RG^{(2)} - RF^{(2)}), \quad (\text{A.37})$$

$$\omega IG^{(2)} + mIG^{(2)} = (\omega + m)IF^{(2)}, \quad (\text{A.38})$$

$$\omega IF^{(2)} - mIF^{(2)} = (\omega - m)IG^{(2)}, \quad (\text{A.39})$$

$$(\omega + m)(2RF^{(2)} + IF^{(1)2}) + (\omega - m)(2RG^{(2)} + IG^{(1)2}) = 0, \quad (\text{A.40})$$

where $\omega^{(1)} = RG^{(1)} = RF^{(1)} = IG^{(0)} = IF^{(0)} = 0$ have been used. (A.38) and (A.39) yield the equation

$$IG^{(2)} = IF^{(2)}. \quad (\text{A.41})$$

Ref. [33] set $K = IF^{(2)} = IG^{(2)} = 0$ based on an assumption $IF^{(n)}(t; m) = -IG^{(n)}(t; m)$. But, as we see, only the relation $IF^{(2)}(t; m) = IG^{(2)}(t; -m)$ will follow from (32), and $IG^{(2)}$ and $IF^{(2)}$ can be nonzero in general. Eqs. (A.36)~(A.40) reduce to the following inhomogeneous linear equations,

$$\omega^{(2)} - (\omega + m)(RF^{(2)} - RG^{(2)}) = \dot{IG}^{(1)} + \frac{IG^{(1)}}{2} \frac{\dot{a}}{a} \frac{1}{\omega} (m^2 - \omega^2) \left(\frac{1}{\omega - m} - \frac{1}{\omega} \right), \quad (\text{A.42})$$

$$\omega^{(2)} - (\omega - m)(RG^{(2)} - RF^{(2)}) = \dot{IF}^{(1)} + \frac{IF^{(1)}}{2} \frac{\dot{a}}{a} \frac{1}{\omega} (m^2 - \omega^2) \left(\frac{1}{\omega + m} - \frac{1}{\omega} \right), \quad (\text{A.43})$$

$$2(\omega + m)RF^{(2)} + 2(\omega - m)RG^{(2)} = -(\omega + m)IF^{(1)2} - (\omega - m)IG^{(1)2}, \quad (\text{A.44})$$

and the solutions are

$$\omega^{(2)} = K \left(\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} \right) + \dot{K} \frac{\dot{a}}{a} + (2A - \frac{1}{2}) \left(\frac{m^3 \dot{a}^2}{a^2 \omega^4} - \frac{m \dot{a}^2}{2a^2 \omega^2} - \frac{m \ddot{a}}{2a \omega^2} \right) + \left(\frac{5m^4 \dot{a}^2}{8a^2 \omega^5} - \frac{3m^2 \dot{a}^2}{8a^2 \omega^3} - \frac{m^2 \ddot{a}}{4a \omega^3} \right), \quad (\text{A.45})$$

$$RF^{(2)} = -K \left(K - 2A \frac{m}{\omega^2} \right) \frac{\dot{a}^2}{2a^2} - A^2 \frac{m^2 \dot{a}^2}{2a^2 \omega^4} + \frac{m^2 R}{48 \omega^4} - \frac{5m^4 \dot{a}^2}{16a^2 \omega^6} - \frac{mR}{48 \omega^3} + \frac{5m^3 \dot{a}^2}{16a^2 \omega^5}, \quad (\text{A.46})$$

$$RG^{(2)} = -K \left(K - (2A - 1) \frac{m}{\omega^2} \right) \frac{\dot{a}^2}{2a^2} - \left(A^2 - A + \frac{1}{4} \right) \frac{m^2 \dot{a}^2}{2a^2 \omega^4} + \frac{m^2 R}{48 \omega^4} - \frac{5m^4 \dot{a}^2}{16a^2 \omega^6} + \frac{mR}{48 \omega^3} - \frac{5m^3 \dot{a}^2}{16a^2 \omega^5}, \quad (\text{A.47})$$

where $B = \frac{1}{2} - A$ has been used, and $R = 6(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2})$ is the Ricci scalar. Imposing the constraint (A.11) on (A.45) and the constraint (A.9) on (A.46) and (A.47), we find

$$A = \frac{1}{4}. \quad (\text{A.48})$$

Then, (A.45), (A.46), and (A.47) become

$$\omega^{(2)} = K \left(\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} \right) + \omega \frac{dK}{d\omega} \left(\frac{m^2}{\omega^2} - 1 \right) \frac{\dot{a}^2}{a^2} + \frac{5m^4 \dot{a}^2}{8a^2 \omega^5} - \frac{3m^2 \dot{a}^2}{8a^2 \omega^3} - \frac{m^2 \ddot{a}}{4a \omega^3}, \quad (\text{A.49})$$

$$RF^{(2)} = -K \left(K - \frac{1}{2} \frac{m}{\omega^2} \right) \frac{\dot{a}^2}{2a^2} + \frac{m^2 R}{48 \omega^4} - \frac{5m^4 \dot{a}^2}{16a^2 \omega^6} - \frac{m^2 \dot{a}^2}{32a^2 \omega^4} - \frac{mR}{48 \omega^3} + \frac{5m^3 \dot{a}^2}{16a^2 \omega^5}, \quad (\text{A.50})$$

$$RG^{(2)} = -K \left(K + \frac{1}{2} \frac{m}{\omega^2} \right) \frac{\dot{a}^2}{2a^2} + \frac{m^2 R}{48\omega^4} - \frac{5m^4 \dot{a}^2}{16a^2 \omega^6} - \frac{m^2 \dot{a}^2}{32a^2 \omega^4} + \frac{mR}{48\omega^3} - \frac{5m^3 \dot{a}^2}{16a^2 \omega^5}, \quad (\text{A.51})$$

where $\dot{K} = \frac{dK}{d\omega} \frac{d\omega}{dt}$ has been used.

3rd order

$$(\omega + m)(RG^{(3)} - RF^{(3)}) + \omega^{(3)} = \dot{IG}^{(2)} + \frac{IG^{(2)}}{2} \frac{d\omega}{dt} \left(\frac{1}{\omega - m} - \frac{1}{\omega} \right), \quad (\text{A.52})$$

$$(\omega - m)(RF^{(3)} - RG^{(3)}) + \omega^{(3)} = \dot{IF}^{(2)} + \frac{IF^{(2)}}{2} \frac{d\omega}{dt} \left(\frac{1}{\omega + m} - \frac{1}{\omega} \right), \quad (\text{A.53})$$

$$(\omega + m)(IG^{(3)} - IF^{(3)}) = -\omega^{(2)} IG^{(1)} - \dot{RG}^{(2)} - \frac{RG^{(2)}}{2} \frac{d\omega}{dt} \left(\frac{1}{\omega - m} - \frac{1}{\omega} \right), \quad (\text{A.54})$$

$$(\omega - m)(IF^{(3)} - IG^{(3)}) = -\omega^{(2)} IF^{(1)} - \dot{RF}^{(2)} - \frac{RF^{(2)}}{2} \frac{d\omega}{dt} \left(\frac{1}{\omega + m} - \frac{1}{\omega} \right), \quad (\text{A.55})$$

$$(\omega + m)(RF^{(3)} + IF^{(1)} IF^{(2)}) + (\omega - m)(RG^{(3)} + IG^{(1)} IG^{(2)}) = 0, \quad (\text{A.56})$$

where the relations $\omega^{(1)} = RG^{(1)} = RF^{(1)} = IG^{(0)} = IF^{(0)} = 0$ have been used. Solving (A.54) and (A.55) yields

$$IF^{(3)} - IG^{(3)} = \frac{65m^5 \dot{a}^3}{32a^3 \omega^8} - \frac{97m^3 \dot{a}^3}{64a^3 \omega^6} + \frac{m \dot{a}^3}{8a^3 \omega^4} - \frac{19m^3 \ddot{a}}{16a^2 \omega^6} + \frac{m \dot{a} \ddot{a}}{2a^2 \omega^4} + \frac{m \ddot{a}}{8a \omega^4} + K \left(\frac{5m^3 \dot{a}^3}{8a^3 \omega^5} - \frac{m \dot{a}^3}{4a^3 \omega^3} + \frac{K m \dot{a}^3}{4a^3 \omega^2} - \frac{m \dot{a} \ddot{a}}{4a^2 \omega^3} \right). \quad (\text{A.57})$$

By $IF^{(3)}(t; -m) = IG^{(3)}(t; m)$, we can write $IF^{(3)}$ and $IG^{(3)}$ as

$$\begin{aligned} IF^{(3)} &= \frac{65m^5 \dot{a}^3}{64a^3 \omega^8} - \frac{97m^3 \dot{a}^3}{128a^3 \omega^6} + \frac{m \dot{a}^3}{16a^3 \omega^4} - \frac{19m^3 \ddot{a}}{32a^2 \omega^6} + \frac{m \dot{a} \ddot{a}}{4a^2 \omega^4} + \frac{m \ddot{a}}{16a \omega^4} \\ &\quad + K \left(\frac{5m^3 \dot{a}^3}{16a^3 \omega^5} - \frac{m \dot{a}^3}{8a^3 \omega^3} + \frac{K m \dot{a}^3}{8a^3 \omega^2} - \frac{m \dot{a} \ddot{a}}{8a^2 \omega^3} \right) + L \frac{\ddot{a}}{a} + M \frac{\ddot{a} \ddot{a}}{a^2} + N \frac{\dot{a}^3}{a^3}, \\ IG^{(3)} &= -\frac{65m^5 \dot{a}^3}{64a^3 \omega^8} + \frac{97m^3 \dot{a}^3}{128a^3 \omega^6} - \frac{m \dot{a}^3}{16a^3 \omega^4} + \frac{19m^3 \ddot{a}}{32a^2 \omega^6} - \frac{m \dot{a} \ddot{a}}{4a^2 \omega^4} - \frac{m \ddot{a}}{16a \omega^4} \\ &\quad - K \left(\frac{5m^3 \dot{a}^3}{16a^3 \omega^5} - \frac{m \dot{a}^3}{8a^3 \omega^3} + \frac{K m \dot{a}^3}{8a^3 \omega^2} - \frac{m \dot{a} \ddot{a}}{8a^2 \omega^3} \right) + L \frac{\ddot{a}}{a} + M \frac{\ddot{a} \ddot{a}}{a^2} + N \frac{\dot{a}^3}{a^3}, \end{aligned} \quad (\text{A.58})$$

where L , M , and N are arbitrary functions even in m ,

$$L(t; -m) = L(t; m), \quad (\text{A.59})$$

$$M(t; -m) = M(t; m), \quad (\text{A.60})$$

$$N(t; -m) = N(t; m), \quad (\text{A.61})$$

and can be nonzero in general. These arbitrary functions were set to zero in Refs. [34, 35].

From (A.52), (A.53), and (A.56), we have the following

$$\omega^{(3)} = \dot{IF}^{(2)}, \quad (\text{A.62})$$

$$RF^{(3)} = -IF^{(2)} \left(K - \frac{m}{4\omega^2} \right) \frac{\dot{a}}{a} = -IF^{(2)} IF^{(1)}, \quad (\text{A.63})$$

$$RG^{(3)} = -IG^{(2)} \left(K + \frac{m}{4\omega^2} \right) \frac{\dot{a}}{a} = -IG^{(2)} IG^{(1)}, \quad (\text{A.64})$$

where (A.35) has been used. In general, $RF^{(3)}$ and $RG^{(3)}$ can be nonzero.

4th order

$$(\omega RG^{(4)} + \omega^{(2)} RG^{(2)} + \omega^{(4)}) - \dot{IG}^{(3)} - \frac{IG^{(3)}}{2} \frac{d\omega}{dt} \left(\frac{1}{\omega - m} - \frac{1}{\omega} \right) + mRG^{(4)} = (\omega + m)RF^{(4)}, \quad (\text{A.65})$$

$$(\omega RF^{(4)} + \omega^{(2)} RF^{(2)} + \omega^{(4)}) - \dot{IF}^{(3)} - \frac{IF^{(3)}}{2} \frac{d\omega}{dt} \left(\frac{1}{\omega + m} - \frac{1}{\omega} \right) - mRF^{(4)} = (\omega - m)RG^{(4)}, \quad (\text{A.66})$$

$$(\omega^{(2)} IG^{(2)} + \omega^{(3)} IG^{(1)}) + \dot{RG}^{(3)} + \frac{RG^{(3)}}{2} \frac{d\omega}{dt} \left(\frac{1}{\omega - m} - \frac{1}{\omega} \right) = (\omega + m)(IF^{(4)} - IG^{(4)}), \quad (\text{A.67})$$

$$(\omega^{(2)} IF^{(2)} + \omega^{(3)} IF^{(1)}) + \dot{RF}^{(3)} + \frac{RF^{(3)}}{2} \frac{d\omega}{dt} \left(\frac{1}{\omega + m} - \frac{1}{\omega} \right) = (\omega - m)(IG^{(4)} - IF^{(4)}), \quad (\text{A.68})$$

$$(\omega + m)(2RF^{(4)} + 2IF^{(1)}IF^{(3)} + RF^{(2)}RF^{(2)} + IF^{(2)}IF^{(2)}), \\ + (\omega - m)(2RG^{(4)} + 2IG^{(1)}IG^{(3)} + RG^{(2)}RG^{(2)} + IG^{(2)}IG^{(2)}) = 0, \quad (\text{A.69})$$

where we have used $IF^{(0)} = IG^{(0)} = 0$, $RF^{(0)} = RG^{(0)} = 1$, $\omega^{(1)} = RF^{(1)} = RG^{(1)} = 0$. (A.67) and (A.68) lead to the following equation

$$IF^{(4)} - IG^{(4)} = IF^{(2)} \left(\frac{5m^3 \dot{a}^2}{8a^2 \omega^5} - \frac{m \dot{a}^2}{4a^2 \omega^3} + K \frac{m \dot{a}^2}{2a^2 \omega^2} - \frac{m \ddot{a}}{4a \omega^3} \right), \quad (\text{A.70})$$

where $IF^{(4)}$ and $IG^{(4)}$ remain undetermined. Solving the equations (A.65), (A.66), and (A.69), one gets

$$\omega^{(4)} = -\frac{1105m^8 \dot{a}^4}{128a^4 \omega^{11}} + \frac{337m^6 \dot{a}^4}{32a^4 \omega^9} - \frac{377m^4 \dot{a}^4}{128a^4 \omega^7} + \frac{3m^2 \dot{a}^4}{32a^4 \omega^5} + \frac{221m^6 \dot{a}^2 \ddot{a}}{32a^3 \omega^9} - \frac{389m^4 \dot{a}^2 \ddot{a}}{64a^3 \omega^7} + \frac{13m^2 \dot{a}^2 \ddot{a}}{16a^3 \omega^5} - \frac{19m^4 \ddot{a}^2}{32a^2 \omega^7} \\ + \frac{m^2 \ddot{a}^2}{4a^2 \omega^5} - \frac{7m^4 \dot{a} a^{(3)}}{8a^2 \omega^7} + \frac{15m^2 \dot{a} \ddot{a}}{32a^2 \omega^5} + \frac{m^2 \ddot{a} \ddot{a}}{16a \omega^5} \\ + K \frac{23m^4 \dot{a}^2 \ddot{a}}{16a^3 \omega^6} - K \frac{17m^2 \dot{a}^2 \ddot{a}}{32a^3 \omega^4} - K \frac{3m^2 \dot{a}^4}{32a^4 \omega^4} + K^3 \frac{\dot{a}^2 \ddot{a}}{2a^3} - M \frac{2\dot{a}^2 \ddot{a}}{a^3} + N \frac{3\dot{a}^2 \ddot{a}}{a^3} - K \frac{m^2 \ddot{a}^2}{8a^2 \omega^4} - L \frac{\dot{a} \ddot{a}}{a^2} + M \frac{\dot{a} \ddot{a}}{a^2} \\ - K \frac{m^2 \dot{a} \ddot{a}}{8a^2 \omega^4} + M \frac{\ddot{a}^2}{a^2} + L \frac{\ddot{a} \ddot{a}}{a} - K^3 \frac{\dot{a}^4}{2a^4} + K \frac{21m^4 \dot{a}^4}{16a^4 \omega^6} - N \frac{3\dot{a}^4}{a^4} - K \frac{15m^6 \dot{a}^4}{8a^4 \omega^8} + K \dot{K} \frac{\dot{a}^3}{2a^3} + \dot{M} \frac{\ddot{a} \ddot{a}}{a^2} + \dot{N} \frac{\dot{a}^3}{a^3} \\ + \dot{L} \frac{\ddot{a}}{a} + \dot{K} \frac{5m^4 \dot{a}^3}{16a^3 \omega^6} - \dot{K} \frac{3m^2 \dot{a}^3}{32a^3 \omega^4} - \dot{K} \frac{m^2 \ddot{a} \ddot{a}}{8a^2 \omega^4}, \quad (\text{A.71})$$

$$RF^{(4)} = +\frac{2285m^8 \dot{a}^4}{512a^4 \omega^{12}} - \frac{565m^7 \dot{a}^4}{128a^4 \omega^{11}} - \frac{1263m^6 \dot{a}^4}{256a^4 \omega^{10}} + \frac{2611m^5 \dot{a}^4}{512a^4 \omega^9} + \frac{2371m^4 \dot{a}^4}{2048a^4 \omega^8} - \frac{333m^3 \dot{a}^4}{256a^4 \omega^7} - \frac{3m^2 \dot{a}^4}{128a^4 \omega^6} + \frac{m \dot{a}^4}{32a^4 \omega^5} \\ - \frac{457m^6 \dot{a}^2 \ddot{a}}{128a^3 \omega^{10}} + \frac{113m^5 \dot{a}^2 \ddot{a}}{32a^3 \omega^9} + \frac{725m^4 \dot{a}^2 \ddot{a}}{256a^3 \omega^8} - \frac{749m^3 \dot{a}^2 \ddot{a}}{256a^3 \omega^7} - \frac{19m^2 \dot{a}^2 \ddot{a}}{64a^3 \omega^6} + \frac{11m \dot{a}^2 \ddot{a}}{32a^3 \omega^5} + \frac{41m^4 \ddot{a}^2}{128a^2 \omega^8} \\ - \frac{5m^3 \ddot{a}^2}{16a^2 \omega^7} - \frac{17m^2 \ddot{a}^2}{128a^2 \omega^6} + \frac{m \ddot{a}^2}{8a^2 \omega^5} + \frac{7m^4 \dot{a} \ddot{a}}{16a^2 \omega^8} - \frac{7m^3 \dot{a} \ddot{a}}{16a^2 \omega^7} - \frac{13m^2 \dot{a} \ddot{a}}{64a^2 \omega^6} + \frac{7m \dot{a} \ddot{a}}{32a^2 \omega^5} - \frac{m^2 \ddot{a} \ddot{a}}{32a \omega^6} + \frac{m \ddot{a} \ddot{a}}{32a \omega^5} \\ - \frac{1}{2} IF^{(2)2} - K^4 \frac{\dot{a}^4}{8a^4} - KN \frac{\dot{a}^4}{a^4} - K \frac{15m^5 \dot{a}^4}{16a^4 \omega^8} + K \frac{47m^3 \dot{a}^4}{64a^4 \omega^6} - K^2 \frac{5m^4 \dot{a}^4}{32a^4 \omega^6} - K^2 \frac{5m^3 \dot{a}^4}{32a^4 \omega^5} \\ - K \frac{m \dot{a}^4}{16a^4 \omega^4} + K^2 \frac{3m^2 \dot{a}^4}{64a^4 \omega^4} + K^2 \frac{m \dot{a}^4}{16a^4 \omega^3} + N \frac{m \dot{a}^4}{4a^4 \omega^2} - KM \frac{\dot{a}^2 \ddot{a}}{a^3} + K \frac{9m^3 \dot{a}^2 \ddot{a}}{16a^3 \omega^6} - K \frac{m \dot{a}^2 \ddot{a}}{4a^3 \omega^4}$$

$$+ K^2 \frac{m^2 \dot{a}^2 \ddot{a}}{16a^3 \omega^4} + K^2 \frac{m \dot{a}^2 \ddot{a}}{16a^3 \omega^3} + M \frac{m \dot{a}^2 \ddot{a}}{4a^3 \omega^2} - KL \frac{\dot{a} \ddot{a}}{a^2} - K \frac{m \dot{a} \ddot{a}}{16a^2 \omega^4} + L \frac{m \dot{a} \ddot{a}}{4a^2 \omega^2}, \quad (\text{A.72})$$

$$\begin{aligned} RG^{(4)} = & + \frac{2285m^8 \dot{a}^4}{512a^4 \omega^{12}} + \frac{565m^7 \dot{a}^4}{128a^4 \omega^{11}} - \frac{1263m^6 \dot{a}^4}{256a^4 \omega^{10}} - \frac{2611m^5 \dot{a}^4}{512a^4 \omega^9} + \frac{2371m^4 \dot{a}^4}{2048a^4 \omega^8} + \frac{333m^3 \dot{a}^4}{256a^4 \omega^7} \\ & - \frac{3m^2 \dot{a}^4}{128a^4 \omega^6} - \frac{m \dot{a}^4}{32a^4 \omega^5} - \frac{457m^6 \dot{a}^2 \ddot{a}}{128a^3 \omega^{10}} - \frac{113m^5 \dot{a}^2 \ddot{a}}{32a^3 \omega^9} + \frac{725m^4 \dot{a}^2 \ddot{a}}{256a^3 \omega^8} + \frac{749m^3 \dot{a}^2 \ddot{a}}{256a^3 \omega^7} \\ & - \frac{19m^2 \dot{a}^2 \ddot{a}}{64a^3 \omega^6} - \frac{11m \dot{a}^2 \ddot{a}}{32a^3 \omega^5} + \frac{41m^4 \ddot{a}^2}{128a^2 \omega^8} + \frac{5m^3 \ddot{a}^2}{16a^2 \omega^7} - \frac{17m^2 \ddot{a}^2}{128a^2 \omega^6} - \frac{m \ddot{a}^2}{8a^2 \omega^5} \\ & + \frac{7m^4 \dot{a} \ddot{a}}{16a^2 \omega^8} + \frac{7m^3 \dot{a} \ddot{a}}{16a^2 \omega^7} - \frac{13m^2 \dot{a} \ddot{a}}{64a^2 \omega^6} - \frac{7m \dot{a} \ddot{a}}{32a^2 \omega^5} - \frac{m^2 \ddot{a} \ddot{a}}{32a \omega^6} - \frac{m \ddot{a} \ddot{a}}{32a \omega^5} \\ & - \frac{1}{2} IF^{(2)2} - K^4 \frac{\dot{a}^4}{8a^4} - KN \frac{\dot{a}^4}{a^4} + K \frac{15m^5 \dot{a}^4}{16a^4 \omega^8} - K \frac{47m^3 \dot{a}^4}{64a^4 \omega^6} - K^2 \frac{5m^4 \dot{a}^4}{32a^4 \omega^6} + K^2 \frac{5m^3 \dot{a}^4}{32a^4 \omega^5} \\ & + K \frac{m \dot{a}^4}{16a^4 \omega^4} + K^2 \frac{3m^2 \dot{a}^4}{64a^4 \omega^4} - K^2 \frac{m \dot{a}^4}{16a^4 \omega^3} - N \frac{m \dot{a}^4}{4a^4 \omega^2} - KM \frac{\dot{a}^2 \ddot{a}}{a^3} - K \frac{9m^3 \dot{a}^2 \ddot{a}}{16a^3 \omega^6} + K \frac{m \dot{a}^2 \ddot{a}}{4a^3 \omega^4} \\ & + K^2 \frac{m^2 \dot{a}^2 \ddot{a}}{16a^3 \omega^4} - K^2 \frac{m \dot{a}^2 \ddot{a}}{16a^3 \omega^3} - M \frac{m \dot{a}^2 \ddot{a}}{4a^3 \omega^2} - KL \frac{\dot{a} \ddot{a}}{a^2} + K \frac{m \dot{a} \ddot{a}}{16a^2 \omega^4} - L \frac{m \dot{a} \ddot{a}}{4a^2 \omega^2}, \quad (\text{A.73}) \end{aligned}$$

which contain arbitrary functions $K, L, M, N, IF^{(2)}$.

We have shown that, some arbitrary functions appear in the WKB modes at each order, and can not be completely determined by the conditions (A.9), (A.10), and (A.11). Nevertheless, as we shall show in Appendix B, these arbitrary functions will cancel out in the power spectrum and the spectral stress tensor.

B Adiabatic spectra

Using the WKB modes given in Appendix A, we shall calculate the adiabatic power spectrum and spectral stress tensor, and show that the arbitrary functions cancel out in the results.

Adiabatic power spectrum

The formula of power spectrum is (39). The adiabatic power spectrum is given by using the WKB modes g_k^I and g_k^{II} of (25) and (26) to replace h_k^I and h_k^{II} . Keeping terms up to each order, we get the following, respectively,

$$\Delta_{k \text{ ad}}^{2(0)} = -\frac{k^3}{a^3 \pi^2} \left(\frac{\omega + m}{2\omega} - \frac{\omega - m}{2\omega} \right) = -\frac{k^3}{a^3 \pi^2} \frac{m}{\omega}, \quad (\text{B.1})$$

$$\Delta_{k \text{ ad}}^{2(1)} = \Delta_{k \text{ ad}}^{2(0)}, \quad (\text{B.2})$$

$$\begin{aligned} \Delta_{k \text{ ad}}^{2(2)} = & -\frac{H^3}{\pi^2} z^3 \left(\frac{\omega + m}{2\omega} (IF^{(1)2} + 2RF^{(2)}) - \frac{\omega - m}{2\omega} (IG^{(1)2} + 2RG^{(2)}) \right) + \Delta_{k \text{ ad}}^{2(0)} \\ = & -\frac{k^3}{a^3 \pi^2} \left(\frac{m}{\omega} - \frac{5m^5 \dot{a}^2}{8a^2 \omega^7} + \frac{7m^3 \dot{a}^2}{8a^2 \omega^5} - \frac{m \dot{a}^2}{4a^2 \omega^3} + \frac{m^3 \ddot{a}}{4a \omega^5} - \frac{m \ddot{a}}{4a \omega^3} \right), \quad (\text{B.3}) \end{aligned}$$

$$\Delta_{k \text{ ad}}^{2(3)} = \Delta_{k \text{ ad}}^{2(2)}, \quad (\text{B.4})$$

being independent of the arbitrary functions $K, IF^{(2)}, IG^{(2)}$,

$$\begin{aligned} \Delta_{k \text{ ad}}^{2(4)} = & -\frac{k^3}{a^3 \pi^2} \left(\frac{\omega + m}{2\omega} 2(RF^{(4)} + IF^{(1)} IF^{(3)} + \frac{1}{2}(IF^{(2)2} + RF^{(2)2})) \right. \\ & \left. - \frac{\omega - m}{2\omega} 2(RG^{(4)} + IG^{(1)} IG^{(3)} + \frac{1}{2}(IG^{(2)2} + RG^{(2)2})) \right) + \Delta_{k \text{ ad}}^{2(2)} \\ = & -\frac{k^3}{a^3 \pi^2} \left(\frac{m}{\omega} - \frac{5m^5 \dot{a}^2}{8a^2 \omega^7} + \frac{7m^3 \dot{a}^2}{8a^2 \omega^5} - \frac{m \dot{a}^2}{4a^2 \omega^3} + \frac{m^3 \ddot{a}}{4a \omega^5} - \frac{m \ddot{a}}{4a \omega^3} \right) \end{aligned}$$

$$\begin{aligned}
& + \frac{1155m^9\dot{a}^4}{128a^4\omega^{13}} - \frac{1239m^7\dot{a}^4}{64a^4\omega^{11}} + \frac{1659m^5\dot{a}^4}{128a^4\omega^9} - \frac{43m^3\dot{a}^4}{16a^4\omega^7} + \frac{m\dot{a}^4}{16a^4\omega^5} - \frac{231m^7\dot{a}^2\ddot{a}}{32a^3\omega^{11}} \\
& + \frac{105m^5\dot{a}^2\ddot{a}}{8a^3\omega^9} - \frac{211m^3\dot{a}^2\ddot{a}}{32a^3\omega^7} + \frac{11m\dot{a}^2\ddot{a}}{16a^3\omega^5} + \frac{21m^5\ddot{a}^2}{32a^2\omega^9} - \frac{29m^3\ddot{a}^2}{32a^2\omega^7} + \frac{m\ddot{a}^2}{4a^2\omega^5} \\
& + \frac{7m^5\dot{a}\ddot{a}}{8a^2\omega^9} - \frac{21m^3\dot{a}\ddot{a}}{16a^2\omega^7} + \frac{7m\dot{a}\ddot{a}}{16a^2\omega^5} - \frac{m^3\ddot{a}}{16a\omega^7} + \frac{m\ddot{a}}{16a\omega^5} \Big), \tag{B.5}
\end{aligned}$$

being independent of the arbitrary functions $K, L, M, N, IG^{(2)}, IF^{(2)}, IG^{(4)}, IF^{(4)}$.

Adiabatic spectral stress tensor

We now compute the adiabatic spectral stress tensor up to the 4th order, and show that it is independent of the arbitrary functions appearing in the WKB modes.

The formula of the spectral pressure is (63). The adiabatic spectral pressure is given by using the WKB modes g_k^I and g_k^{II} to replace h_k^I and h_k^{II} . To each order, we get the following, respectively

$$p_{k\ ad}^{(0)} = \frac{k^4}{2\pi^2 a^4} \left(-\frac{2}{3}\right) \frac{\sqrt{\omega^2 - m^2}}{\omega}, \tag{B.6}$$

$$p_{k\ ad}^{(1)} = p_{k\ ad}^{(0)}, \tag{B.7}$$

$$\begin{aligned}
p_{k\ ad}^{(2)} &= \frac{k^4}{2\pi^2 a^4} \left(-\frac{2}{3}\right) \frac{\sqrt{\omega^2 - m^2}}{\omega} (RF^{(2)} + RG^{(2)} + IG^{(1)}IF^{(1)} + RG^{(1)}RF^{(1)}) + p_k^{(0)} \\
&= \frac{k^4}{2\pi^2 a^4} \left(-\frac{2}{3}\right) \frac{\sqrt{\omega^2 - m^2}}{\omega} \left(1 - \frac{5m^4\dot{a}^2}{8a^2\omega^6} + \frac{m^2\dot{a}^2}{8a^2\omega^4} + \frac{m^2\ddot{a}}{4a\omega^4}\right), \tag{B.8}
\end{aligned}$$

$$p_{k\ ad}^{(3)} = p_{k\ ad}^{(2)}, \tag{B.9}$$

being independent of the arbitrary functions $K, IF^{(2)}, IG^{(2)}$,

$$\begin{aligned}
p_{k\ ad}^{(4)} &= \frac{k^4}{2\pi^2 a^4} \left(-\frac{2}{3}\right) \frac{\sqrt{\omega^2 - m^2}}{\omega} (RF^{(4)} + RG^{(4)} + IF^{(2)2} + IF^{(3)}IG^{(1)} + IF^{(1)}IG^{(3)} + RF^{(2)}RG^{(2)}) + p_k^{(3)} \\
&= \frac{k^4}{2\pi^2 a^4} \left(-\frac{2}{3}\right) \frac{\sqrt{\omega^2 - m^2}}{\omega} \left(1 - \frac{5m^4\dot{a}^2}{8a^2\omega^6} + \frac{m^2\dot{a}^2}{8a^2\omega^4} + \frac{m^2\ddot{a}}{4a\omega^4} \frac{1155m^8\dot{a}^4}{128a^4\omega^{12}} - \frac{609m^6\dot{a}^4}{64a^4\omega^{10}} \right. \\
&\quad + \frac{259m^4\dot{a}^4}{128a^4\omega^8} - \frac{m^2\dot{a}^4}{32a^4\omega^6} - \frac{231m^6\dot{a}^2\ddot{a}}{32a^3\omega^{10}} + \frac{175m^4\dot{a}^2\ddot{a}}{32a^3\omega^8} - \frac{m^2\dot{a}^2\ddot{a}}{2a^3\omega^6} + \frac{21m^4\ddot{a}^2}{32a^2\omega^8} \\
&\quad \left. - \frac{9m^2\ddot{a}^2}{32a^2\omega^6} + \frac{7m^4\dot{a}\ddot{a}}{8a^2\omega^8} - \frac{3m^2\dot{a}\ddot{a}}{8a^2\omega^6} - \frac{m^2\ddot{a}}{16a\omega^6}\right), \tag{B.10}
\end{aligned}$$

being independent of the arbitrary functions $K, L, M, N, IF^{(2)}, IG^{(2)}, IF^{(4)}, IG^{(4)}$.

The adiabatic spectral energy density can be written as $\rho_{k\ ad}^{(n)} = m\Delta_{k\ ad}^{2(n)} + 3p_{k\ ad}^{(n)}$ according to the relation (65). So we get

$$\rho_{k\ ad}^{(0)} = -\frac{k^3}{\pi^2 a^3} \omega, \tag{B.11}$$

$$\rho_{k\ ad}^{(2)} = -\frac{k^3}{\pi^2 a^3} \left(\omega + \left(\frac{m^4}{8\omega^5} - \frac{m^2}{8\omega^3}\right) \frac{\dot{a}^2}{a^2}\right), \tag{B.12}$$

$$\begin{aligned}
\rho_{k\ ad}^{(4)} &= -\frac{k^3}{\pi^2 a^3} \left(\omega + \left(\frac{m^4}{8\omega^5} - \frac{m^2}{8\omega^3}\right) \frac{\dot{a}^2}{a^2} - \left(\frac{105m^8}{128\omega^{11}} - \frac{91m^6}{64\omega^9} + \frac{81m^4}{128\omega^7} - \frac{m^2}{32\omega^5}\right) \frac{\dot{a}^4}{a^4} \right. \\
&\quad \left. + \left(\frac{7m^6}{16\omega^9} - \frac{5m^4}{8a^3\omega^7} + \frac{3m^2}{16\omega^5}\right) \frac{\dot{a}^2\ddot{a}}{a^3} + \left(\frac{m^4}{32\omega^7} - \frac{m^2}{32\omega^5}\right) \frac{\ddot{a}^2}{a^2} - \left(\frac{m^4}{16\omega^7} - \frac{m^2}{16\omega^5}\right) \frac{\dot{a}\ddot{a}}{a^2}\right), \tag{B.13}
\end{aligned}$$

which are independent of the arbitrary functions, too.

Thus, to the each order, the arbitrary functions cancel out in the adiabatic power spectrum and in the adiabatic spectral stress tensor. Therefore, in practice, these functions can be set to zero, $K = L = M = N = IF^{(2)} = IG^{(2)} = IF^{(4)} = IG^{(4)} = 0$, as in Refs. [33–35].

We have also verified that the adiabatic spectral stress tensor is conserved, to each order,

$$\dot{\rho}_{k\text{ ad}}^{(n)} + 3\frac{\dot{a}}{a}(\rho_{k\text{ ad}}^{(n)} + p_{k\text{ ad}}^{(n)}) = 0, \quad n = 0, 2, 4. \quad (\text{B.14})$$

So the regularized spectral stress tensor is also conserved to each order.

C The correlation function in de Sitter space

In this appendix, we derive the analytic expression (116) of the unregularized correlation function in de Sitter space. The integration involved is similar to that for the scalar field [19, 26, 46]. We first consider the equal-time case $t = t'$ for convenience, and extend the result to the general case of $t \neq t'$ by using the maximal symmetry of de Sitter space. Plugging the modes $h_k^I(z)$ of (21) and $h_k^{II}(z)$ of (22) into the correlation function (107) yields

$$\begin{aligned} \langle 0 | \bar{\psi}(x) \psi(x') | 0 \rangle &= \frac{-i}{4\pi H a^4 |\vec{x} - \vec{x}'|} \int_0^\infty k^2 \left(H_{-i\mu-\frac{1}{2}}^{(1)}(z) H_{-i\mu+\frac{1}{2}}^{(2)}(z) + H_{-i\mu+\frac{1}{2}}^{(1)}(z) H_{-i\mu-\frac{1}{2}}^{(2)}(z) \right) \\ &\times \sin k |\vec{x} - \vec{x}'| dk. \end{aligned} \quad (\text{C.1})$$

By the recurrence relations of the Hankel functions, (C.1) can be written as

$$\begin{aligned} \langle 0 | \bar{\psi}(x) \psi(x') | 0 \rangle &= i \frac{H^3}{4\pi\sigma} \int_0^\infty \frac{d}{dz} \left(H_\nu^{(1)}(z) H_\nu^{(2)}(z) \right) z^2 \sin(\sigma z) dz \\ &+ \frac{H^3}{4\pi\sigma} (i - 2\mu) \int_0^\infty H_\nu^{(1)}(z) H_\nu^{(2)}(z) z \sin(\sigma z) dz, \end{aligned} \quad (\text{C.2})$$

where $\sigma \equiv aH |\vec{x} - \vec{x}'|$, and $\nu \equiv -\frac{1}{2} - i\mu$. We now calculate the first integral in (C.2).

$$Int_1 \equiv \int_0^\infty \frac{d}{dz} \left(H_\nu^{(1)}(z) H_\nu^{(2)}(z) \right) z^2 \sin(\sigma z) dz. \quad (\text{C.3})$$

By use of the following formulae (see Ref. [44] and (6.671.5) in Ref. [45])

$$J_\nu(z)^2 + Y_\nu(z)^2 = \frac{8}{\pi^2} \int_0^\infty \cosh(2\nu t) K_0(2z \sinh t) dt, \quad (\text{C.4})$$

$$\frac{dK_0(x)}{dx} = -K_1(x), \quad (\text{C.5})$$

$$\int_0^\infty K_\nu(az) \sin(bz) dz = \frac{1}{4} \pi a^{-\nu} \csc\left(\frac{\nu\pi}{2}\right) \frac{1}{\sqrt{a^2 + b^2}} \left([(b^2 + a^2)^{\frac{1}{2}} + b]^\nu - [(b^2 + a^2)^{\frac{1}{2}} - b]^\nu \right), \quad (\text{C.6})$$

the integration (C.3) can be written as

$$Int_1 = -\frac{8}{\pi^2} \left(-\frac{\partial^2}{\partial \sigma^2} \right) \int_0^\infty 2 \sinh t \cosh(2\nu t) \left(\frac{\pi}{4} (2 \sinh t)^{-1} \frac{2\sigma}{\sqrt{4 \sinh^2 t + \sigma^2}} \right) dt. \quad (\text{C.7})$$

Further simplification gives

$$Int_1 = -\frac{12\sigma}{\pi} \int_0^\infty \frac{4 \sinh^2 t \cosh(2\nu t)}{(4 \sinh^2 t + \sigma^2)^{\frac{5}{2}}} dt$$

$$= 3 \times 2^{-\frac{1}{2}} \frac{\sigma}{\pi} \int_0^\infty \frac{(\cosh(\nu T) - 2^{-1} \cosh((1+\nu)T)) - 2^{-1} \cosh((\nu-1)T)}{((\frac{\sigma^2}{2} - 1) + \cosh T)^{\frac{5}{2}}} dT, \quad (\text{C.8})$$

where the integration variable $T \equiv 2t$. Using relations (14.3.15), (15.1.1) and (15.1.2) in Ref. [44] in (C.8) leads to

$$\begin{aligned} Int_1 = & \frac{\sigma}{4\pi} \Gamma(\nu + \frac{3}{2}) \Gamma(\frac{3}{2} - \nu) \left((\frac{9}{4} - \nu^2) {}_2F_1(\frac{5}{2} - \nu, \frac{5}{2} + \nu; 3; 1 - \frac{\sigma^2}{4}) \right. \\ & - \frac{1}{2}(\nu + \frac{5}{2})(\nu + \frac{3}{2}) {}_2F_1(\frac{3}{2} - \nu, \frac{7}{2} + \nu; 3; 1 - \frac{\sigma^2}{4}) \\ & \left. - \frac{1}{2}(\frac{5}{2} - \nu)(\frac{3}{2} - \nu) {}_2F_1(\frac{7}{2} - \nu, \nu + \frac{3}{2}; 3; 1 - \frac{\sigma^2}{4}) \right), \end{aligned} \quad (\text{C.9})$$

where $\Gamma(x+1) = x\Gamma(x)$ has been used and ${}_2F_1(a, b; c; d)$ is the hypergeometric function.

Now calculate the second integration in (C.2)

$$Int_2 \equiv \int_0^\infty H_\nu^{(1)}(z) H_\nu^{(2)}(z) z \sin(\sigma z) dz. \quad (\text{C.10})$$

Using (C.4) and the following formula [45]

$$\int_0^\infty K_0(\beta z) \cos(\alpha z) dz = \frac{\pi}{2\sqrt{\alpha^2 + \beta^2}}, \quad (\text{C.11})$$

the integration (C.10) can be expressed as

$$Int_2 = \frac{8}{\pi^2} \left(-\frac{\partial}{\partial \sigma}\right) \int_0^\infty \cosh(2\nu t) \frac{\pi}{2\sqrt{4\sinh^2 t + \sigma^2}} dt, \quad (\text{C.12})$$

which is written as

$$Int_2 = \frac{2^{-\frac{1}{2}} \sigma}{\pi} \int_0^\infty \frac{\cosh(\nu T)}{((\frac{\sigma^2}{2} - 1) + \cosh T)^{\frac{3}{2}}} dT, \quad (\text{C.13})$$

where the integration variable t has changed from t to $T/2$. Using the relations (14.12.4), (14.3.15), (15.1.1), and (15.1.2) in Ref. [44] in (C.13) yields

$$Int_2 = \frac{\sigma}{2\pi} \Gamma(\nu + \frac{3}{2}) \Gamma(\frac{3}{2} - \nu) \Gamma(2)^{-1} {}_2F_1(\frac{3}{2} - \nu, \nu + \frac{3}{2}; 2; 1 - \frac{\sigma^2}{4}). \quad (\text{C.14})$$

Plugging (C.9) and (C.14) into (C.2) yields the equal-time correlation function

$$\begin{aligned} & \langle 0 | \bar{\psi}(\vec{x}, t) \psi(\vec{x}', t) | 0 \rangle \\ &= \frac{H^3}{\pi^2} \Gamma(\nu + \frac{3}{2}) \Gamma(\frac{3}{2} - \nu) \left(-\frac{i}{4} \nu {}_2F_1(\frac{3}{2} - \nu, \frac{3}{2} + \nu; 2; 1 - \frac{\sigma^2}{4}) \right. \\ & \quad + \frac{i}{16} (\frac{9}{4} - \nu^2) {}_2F_1(\frac{5}{2} - \nu, \frac{5}{2} + \nu; 3; 1 - \frac{\sigma^2}{4}) \\ & \quad - \frac{i}{32} (\nu + \frac{5}{2})(\nu + \frac{3}{2}) {}_2F_1(\frac{3}{2} - \nu, \frac{7}{2} + \nu; 3; 1 - \frac{\sigma^2}{4}) \\ & \quad \left. - \frac{i}{32} (\frac{5}{2} - \nu)(\frac{3}{2} - \nu) {}_2F_1(\frac{7}{2} - \nu, \nu + \frac{3}{2}; 3; 1 - \frac{\sigma^2}{4}) \right). \end{aligned} \quad (\text{C.15})$$

By the maximal symmetry of de Sitter space, the correlation function depends in general on the one-half of the squared geodesic interval σ_2 of (117), so we can replace

$$-\frac{1}{2}\sigma^2 \rightarrow \sigma_2, \quad (\text{C.16})$$

in (C.15) to give the non-equal time correlation function (116).

In the massless limit $\mu = 0$, $\nu = -\frac{1}{2}$, the correlation function (C.15) reduces to zero.