

A Combinatorial Technique for the Wedderburn Decomposition of Rational Group Algebras of Nested GVZ p -groups

Ram Karan Choudhary* and Sunil Kumar Prajapati

ABSTRACT. In this article, we present a combinatorial formula for the Wedderburn decomposition of rational group algebras of nested GVZ p -groups, where p is an odd prime. Using this formula, we derive an explicit combinatorial expression for the Wedderburn decomposition of rational group algebras of all two-generator p -groups of class 2. Additionally, we provide explicit combinatorial formulas for the Wedderburn decomposition of rational group algebras of certain families of nested GVZ p -groups with arbitrarily large nilpotency class. We also classify all nested GVZ p -groups of order at most p^5 and compute the Wedderburn decomposition of their rational group algebras. Finally, we determine a complete set of primitive central idempotents for the rational group algebras of nested GVZ p -groups.

1. Introduction

Throughout this paper, all groups are finite. For a group G , we write $\text{Irr}(G)$ for the set of irreducible complex characters of G . Let $N \trianglelefteq G$ and $\chi \in \text{Irr}(G)$. We say that χ is *fully ramified* over N if $\chi(g) = 0$ for all $g \in G \setminus N$. A group G is called of *central type* if there exists $\chi \in \text{Irr}(G)$ such that χ is fully ramified over $Z(G)$. This class of groups was first introduced by DeMeyer and Janusz [12], who showed that a group G is of central type if and only if each Sylow p -subgroup $\text{Syl}_p(G)$ of G is of central type and $Z(\text{Syl}_p(G)) = Z(G) \cap \text{Syl}_p(G)$. It is also known that groups of central type are solvable (see [18]). Central type groups have been studied extensively in the literature, including [12, 13, 15, 18]. Motivated by this definition, one says that a character $\chi \in \text{Irr}(G)$ is of *central type* if χ vanishes on $G \setminus Z(\chi)$, where

$$Z(\chi) = \{g \in G : |\chi(g)| = \chi(1)\}$$

is the center (or quasi-kernel) of χ . Equivalently, χ is of central type if and only if $\bar{\chi} \in \text{Irr}(G/\ker(\chi))$ is fully ramified over $Z(G/\ker(\chi))$, where $\bar{\chi}$ is the lift of χ to G , i.e., $G/\ker(\chi)$ is a group of central type with faithful character $\bar{\chi}$ (see [25]). Groups in which every irreducible complex character is of central type are called *GVZ-groups*. GVZ-groups were first investigated in [26] under the name *groups of Ono type*. For a group G , a conjugacy class $\mathcal{C}$ is said to be of *Ono type* if for every $\chi \in \text{Irr}(G)$ and every $g \in \mathcal{C}$, either $\chi(g) = 0$ or $|\chi(g)| = \chi(1)$. A group G is said to be of Ono type if every conjugacy class of G is of Ono type. Groups of Ono type were first introduced by Ono [31]. A group G is called *nested* if for all characters $\chi, \psi \in \text{Irr}(G)$, we have either $Z(\chi) \subseteq Z(\psi)$ or $Z(\psi) \subseteq Z(\chi)$. A GVZ-group G is called a *nested GVZ-group* if G is nested. Note that a nested GVZ-group G is a GVZ-group satisfying $Z(\psi) \subseteq Z(\chi)$ whenever $\chi(1) \leq \psi(1)$ for $\chi, \psi \in \text{Irr}(G)$ (see [25, Lemma 7.1]). Moreover, such a group G is strictly nested by degrees, i.e.,

$$Z(\psi) \subset Z(\chi) \iff \chi(1) < \psi(1), \quad \text{for } \chi, \psi \in \text{Irr}(G).$$

A group G has an irreducible character χ with $\chi(1) = |G/Z(\chi)|^{1/2}$ if and only if $\chi(g) = 0$ for all $g \in G \setminus Z(\chi)$ (see [19, Corollary 2.30]). Thus, the latter definition of a nested GVZ-group is equivalent to the one used by Nenciu in [27, 29], which was motivated by problems posed by Berkovich [5] (Research Problems 24 and

2020 *Mathematics Subject Classification.* primary 20C05; secondary 20C15, 20D15.

Key words and phrases. Rational group algebras, Wedderburn decomposition, GVZ-groups, nested groups.

* Corresponding author.

30 therein). Moreover, a nested GVZ-group is nilpotent and isomorphic to the direct product of a nested GVZ p -group and an abelian group, where p is a prime (see [27, Corollary 2.5]). For further background on such groups, see [6, 7, 8, 9, 25, 27, 29]. In this article, we investigate the Wedderburn decomposition of rational group algebras of nested GVZ p -groups, where p is an odd prime.

Let G be a finite group, and let $\mathbb{F}$ a field. According to the Wedderburn-Artin theorem, the group algebra $\mathbb{F}G$ is semisimple if and only if it decomposes into a direct sum of matrix algebras over division rings. More precisely, there is an isomorphism

$$\mathbb{F}G \cong \bigoplus_{i=1}^r M_{n_i}(D_i),$$

where each $M_{n_i}(D_i)$ is a full matrix ring of size n_i over a division ring D_i that is finite-dimensional over its center. These rings $M_{n_i}(D_i)$ are called the *simple components* of $\mathbb{F}G$. Furthermore, by the Brauer-Witt theorem (see [37]), each simple component is Brauer equivalent to a cyclotomic algebra. The Wedderburn decomposition of rational group algebras has garnered significant attention in recent years due to its importance in understanding various algebraic structures (see [17, 21, 35]). Recent studies have employed Shoda pair theory to analyze this decomposition, as documented in several works including [3, 4, 22, 23, 30]. Mathematical software implementations, most notably the WEDDERGA package in GAP [39], have been developed based on these methods. However, exact computations often remain challenging, especially for groups of large order. For finite abelian groups, Perlis and Walker [32] provided a well-known combinatorial formula for the Wedderburn decomposition of their rational group algebras.

THEOREM 1. [32, Theorem 1] *Let G be a finite abelian group of exponent m . Then the Wedderburn decomposition of $\mathbb{Q}G$ is given by*

$$\mathbb{Q}G \cong \bigoplus_{d|m} a_d \mathbb{Q}(\zeta_d),$$

where a_d is equal to the number of cyclic subgroups of G of order d .

A non-abelian group G is called a *VZ-group* if every $\chi \in \text{nl}(G)$ is fully ramified over $Z(G)$. It is easy to see that a VZ-group is a special case of a nested GVZ-group. In [10], we formulate the computation of the Wedderburn decomposition of rational group algebra of a VZ p -group G (where p is an odd prime), solely based on computing the number of cyclic subgroups of G/G' , $Z(G)$ and $Z(G)/G'$, which is similar to Theorem 1.

THEOREM 2. [10, Theorem 1] *Let G be a finite VZ p -group, where p is an odd prime. Let m and m' denote the exponents of $Z(G)$ and $Z(G)/G'$, respectively. Then the Wedderburn decomposition of $\mathbb{Q}G$ is given by*

$$\mathbb{Q}G \cong \mathbb{Q}(G/G') \bigoplus_{\substack{d|m \\ d \nmid m'}} a_d M_{|G/Z(G)|^{1/2}}(\mathbb{Q}(\zeta_d)) \bigoplus_{\substack{d|m \\ d \nmid m'}} (a_d - a'_d) M_{|G/Z(G)|^{1/2}}(\mathbb{Q}(\zeta_d)),$$

where a_d and a'_d are the number of cyclic subgroups of order d of $Z(G)$ and $Z(G)/G'$, respectively.

This work extends Theorem 2 by deriving a combinatorial formula for the Wedderburn decomposition of rational group algebras of nested GVZ p -groups, where p is an odd prime.

THEOREM 3. *Let G be a finite nested GVZ p -group, where p is an odd prime. Let $\text{cd}(G) = \{p^{\delta_i} : 0 \leq i \leq n, 0 = \delta_0 < \delta_1 < \delta_2 < \dots < \delta_n\}$, and for $0 \leq i \leq n$ define $Z_{\delta_i} := Z(\chi)$ for some $\chi \in \text{Irr}_{p^{\delta_i}}(G)$. For $1 \leq r \leq n$, let m_r and m'_r denote the exponents of $Z_{\delta_r}/[Z_{\delta_r}, G]$ and $Z_{\delta_r}/[Z_{\delta_{r-1}}, G]$, respectively. Then the Wedderburn decomposition of $\mathbb{Q}G$ is given by*

$$\mathbb{Q}G \cong \mathbb{Q}(G/G') \bigoplus_{r=1}^n \bigoplus_{\substack{d_r|m_r \\ d_r \nmid m'_r}} a_{d_r} M_{p^{\delta_r}}(\mathbb{Q}(\zeta_{d_r})) \bigoplus_{r=1}^n \bigoplus_{\substack{d_r|m_r \\ d_r \nmid m'_r}} (a_{d_r} - a'_{d_r}) M_{p^{\delta_r}}(\mathbb{Q}(\zeta_{d_r})),$$

where a_{d_r} and a'_{d_r} denote the number of cyclic subgroups of order d_r in $Z_{\delta_r}/[Z_{\delta_r}, G]$ and $Z_{\delta_r}/[Z_{\delta_{r-1}}, G]$, respectively.

Note that a VZ-group has nilpotency class 2 (see [14]). Furthermore, any nilpotent group of class 2 is a GVZ-group (see [19, Theorem 2.31]). Hence, any nested group of class 2 is necessarily a nested GVZ-group. In particular, all two-generator p -groups of class 2 are nested GVZ p -groups (see [28]). However, a two-generator p -group of class 2 need not be a VZ p -group. For example, the groups `SmallGroup(729, 24)`, `SmallGroup(729, 25)`, and `SmallGroup(729, 60)` in the GAP library are two-generator 3-groups of class 2 that are nested GVZ but not VZ (see Example 21). In this article, using Theorem 3, we derive explicit combinatorial formulas for the Wedderburn decomposition of rational group algebras of two-generator p -groups, where p is an odd prime.

Moreover, there exist nested GVZ p -groups of arbitrarily large nilpotency class. For every $n \geq 1$, Nenciu [29] constructed a family of nested GVZ p -groups of order p^{2n+1} , exponent p , and class $n+1$, where $p > n+1$ is prime. Similarly, for every $n \geq 1$, Lewis [25] constructed a family of nested GVZ p -groups of order p^{2n+1} , exponent p^{n+1} , and class $n+1$, where p is an odd prime. In this article, using Theorem 3, we also derive explicit combinatorial formulas for the Wedderburn decomposition of rational group algebras of these families of nested GVZ p -groups, where p is an odd prime.

Further, we prove Theorem 4, which shows that for a group, the property of being a GVZ-group (respectively, a nested GVZ-group) is preserved under the notion of *isoclinism*.

THEOREM 4. *Let G and H be two finite isoclinic groups. If G is a GVZ-group (respectively, a nested GVZ-group), then H is also a GVZ-group (respectively, a nested GVZ-group).*

Using Theorem 4, one can check that there are several isoclinic families of nested GVZ p -groups. In this article, we completely classify all nested GVZ p -groups of order at most p^5 (see Corollary 5), and derive the Wedderburn decomposition of rational group algebras of these groups.

COROLLARY 5. *Let G be a non-abelian group of order p^5 , where p is an odd prime. Then G is a nested GVZ-group if and only if $G \in \Phi_2 \cup \Phi_5 \cup \Phi_7 \cup \Phi_8$.*

Moreover, a non-abelian p -group G of order at most p^5 is a nested GVZ-group that is not a VZ-group if and only if $|G| = p^5$ and G belongs to either the isoclinic family Φ_7 or the isoclinic family Φ_8 .

Finally, we present a concise analysis of the primitive central idempotents and their associated simple components in the Wedderburn decomposition of rational group algebras of GVZ p -groups. In particular, we prove Theorem 6.

THEOREM 6. *Let G be a finite nested GVZ p -group, where p is an odd prime. Let $\chi \in \text{nl}(G)$ with $N = \ker(\chi)$. Then we have the following.*

- (1) $e_{\mathbb{Q}}(\chi) = \epsilon(Z(\chi), N)$.
- (2) $\mathbb{Q}G\epsilon(Z(\chi), N) \cong M_{|G/Z(\chi)|^{1/2}}(\mathbb{Q}(\zeta_{|Z(\chi)/N|}))$.

The structure of the article is as follows. In Section 2, we introduce notation, mostly following standard conventions, together with preliminary results that will be used throughout the paper. Section 3 focuses on rational group algebras of nested GVZ p -groups, where p is an odd prime, and contains the proof of Theorem 3. In Section 4, we discuss two-generator p -groups and derive explicit combinatorial formulas for the Wedderburn decomposition of their rational group algebras. Section 5 is devoted to the Wedderburn decomposition of rational group algebras corresponding to two families of nested GVZ p -groups of arbitrarily large nilpotency class. Section 6 contains the proofs of Theorem 4 and Corollary 5. In this section, we also classify all nested GVZ p -groups of order at most p^5 and compute the Wedderburn decomposition of their rational group algebras. Finally, Section 7 examines the structure of primitive central idempotents in rational group algebras of nested GVZ p -groups and provides the proof of Theorem 6.

2. Notation and Preliminaries

2.1. Notation. We adopt the following notation, consistent with standard conventions. Unless otherwise specified, p denotes an odd prime. For a finite group G , the notation introduced below will be used throughout.

$[g, h]$	$g^{-1}h^{-1}gh$ for $g, h \in G$
$[g, G]$	$\langle [g, x] : x \in G \rangle$ for $g \in G$
$[N, G]$	$\langle [n, g] : n \in N, g \in G \rangle$ for $N \trianglelefteq G$
G'	$[G, G]$, i.e., the commutator subgroup of G
$\text{cl}_G(g)$	the conjugacy class of $g \in G$
$\text{c}(G)$	the nilpotency class of G
$\text{Syl}_p(G)$	the Sylow p -subgroup of G
$ S $	the cardinality of a set S
$ z $	the absolute value of $z \in \mathbb{C}$
$\text{Irr}(G)$	the set of irreducible complex characters of G
$\text{lin}(G)$	$\{\chi \in \text{Irr}(G) : \chi(1) = 1\}$
$\text{nl}(G)$	$\{\chi \in \text{Irr}(G) : \chi(1) \neq 1\}$
$\text{Irr}_m(G)$	$\{\chi \in \text{Irr}(G) : \chi(1) = m\}$
$\text{cd}(G)$	$\{\chi(1) : \chi \in \text{Irr}(G)\}$
$\mathbb{F}(\chi)$	the field obtained by adjoining the values $\{\chi(g) : g \in G\}$ to the field $\mathbb{F}$ for some $\chi \in \text{Irr}(G)$
$m_{\mathbb{Q}}(\chi)$	the Schur index of $\chi \in \text{Irr}(G)$ over $\mathbb{Q}$
$\Omega(\chi)$	$m_{\mathbb{Q}}(\chi) \sum_{\sigma \in \text{Gal}(\mathbb{Q}(\chi)/\mathbb{Q})} \chi^\sigma$ for $\chi \in \text{Irr}(G)$
$\ker(\chi)$	$\{g \in G : \chi(g) = \chi(1)\}$ for $\chi \in \text{Irr}(G)$
$Z(\chi)$	$\{g \in G : \chi(g) = \chi(1)\}$ for $\chi \in \text{Irr}(G)$
$\text{Irr}(G N)$	$\{\chi \in \text{Irr}(G) : N \not\subseteq \ker(\chi)\}$, where $N \trianglelefteq G$
$\chi \downarrow_H$	the restriction of a character χ of G on H , where $H \leq G$
$\mathbb{F}G$	the group ring (algebra) of G with coefficients in $\mathbb{F}$
$M_n(D)$	a full matrix ring of order n over the skewfield D
$Z(B)$	the center of an algebraic structure B
ζ_m	an m -th primitive root of unity
C_n	the cyclic group of order n

2.2. Preliminaries. Here, we introduce some fundamental concepts and results that will be used repeatedly throughout this article. For a finite abelian p -group G , Yeh [38] obtained an explicit formula for the number of subgroups of a prescribed type, where p is any prime. More recently, an alternative expression was derived in [36], using a method based on certain matrices associated with the invariant factor decomposition of G . The following lemma provides the precise counting formula for cyclic subgroups.

LEMMA 7. [36, Theorem 4.3] *Let G be a finite abelian p -group such that $G \cong C_{p^{\alpha_1}} \times C_{p^{\alpha_2}} \times \cdots \times C_{p^{\alpha_k}}$, with $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_k$, where p is a prime. For each $1 \leq \alpha \leq \alpha_k$, denote by $\mathcal{G}_p^k(\alpha)$ the number of cyclic subgroups of order p^α in G . Then*

$$\mathcal{G}_p^k(\alpha) = \frac{p h_p^{k-1}(\alpha) - h_p^{k-1}(\alpha - 1)}{p - 1},$$

where $h_p^{k-1}(\alpha) = p^{(k-j)\alpha + \alpha_1 + \cdots + \alpha_{j-1}}$ whenever $\alpha_{j-1} \leq \alpha \leq \alpha_j$, for $j \in \{1, 2, \dots, k\}$ with the convention $\alpha_0 = 0$.

We next recall some essential results concerning characters and representations that will be required later.

LEMMA 8. [19, Corollary 10.14] *Let G be a p -group, where p is an odd prime. Then*

$$m_{\mathbb{Q}}(\chi) = 1 \quad \text{for all } \chi \in \text{Irr}(G).$$

Let G be a finite group. Define an equivalence relation on $\text{Irr}(G)$ by Galois conjugacy over $\mathbb{Q}$. Two irreducible characters $\chi, \psi \in \text{Irr}(G)$ are called *Galois conjugates* over $\mathbb{Q}$ if $\mathbb{Q}(\chi) = \mathbb{Q}(\psi)$ and there exists $\sigma \in \text{Gal}(\mathbb{Q}(\chi)/\mathbb{Q})$ such that $\chi^\sigma = \psi$.

LEMMA 9. [19, Lemma 9.17] *Let G be a finite group and $\chi \in \text{Irr}(G)$. Denote by $E(\chi)$ the Galois conjugacy class of χ over $\mathbb{Q}$. Then*

$$|E(\chi)| = [\mathbb{Q}(\chi) : \mathbb{Q}].$$

It follows that distinct Galois conjugacy classes correspond to distinct irreducible rational representations of G . Reiner [34, Theorem 3] further described the structure of the simple components in the Wedderburn decomposition of the rational group algebra $\mathbb{Q}G$ associated with these rational representations. We conclude this section by citing his result.

LEMMA 10. [34, Theorem 3] *Let $\mathbb{K}$ be a field of characteristic zero and let $\mathbb{K}^*$ denote its algebraic closure. Suppose T is an irreducible $\mathbb{K}$ -representation of G , extended linearly to a $\mathbb{K}G$ -representation of $\mathbb{K}G$. Set*

$$A = \{T(x) : x \in \mathbb{K}G\}.$$

Then A is a simple algebra over $\mathbb{K}$, and may be written as $A = M_n(D)$, where D is a division ring. Moreover,

$$Z(D) \cong \mathbb{K}(\chi_i) \quad \text{and} \quad [D : Z(D)] = (m_{\mathbb{K}}(\chi_i))^2 \quad (1 \leq i \leq k),$$

where U_i are irreducible $\mathbb{K}^$ -representations of G affording the characters χ_i ,*

$$T = m_{\mathbb{K}}(\chi_i) \bigoplus_{i=1}^k U_i,$$

with $k = [\mathbb{K}(\chi_i) : \mathbb{K}]$, and $m_{\mathbb{K}}(\chi_i)$ denoting the Schur index of χ_i over $\mathbb{K}$.

3. Rational group algebra

In this section, we establish the proof of Theorem 3. Let G be a nested GVZ-group. Then for every $\chi \in \text{Irr}(G)$, we have $\chi(1)^2 = |G/Z(\chi)|$ (see [19, Corollary 2.30]). Moreover, if $\chi, \psi \in \text{Irr}(G)$ satisfy $\chi(1) = \psi(1)$, then $Z(\chi) = Z(\psi)$. Let

$$\text{cd}(G) = \{p^{\delta_i} : 0 \leq i \leq n, 0 = \delta_0 < \delta_1 < \dots < \delta_n\},$$

and define $Z_{\delta_i} := Z(\chi)$ for some $\chi \in \text{Irr}_{p^{\delta_i}}(G)$, where $\text{Irr}_{p^{\delta_i}}(G) = \{\chi \in \text{Irr}(G) : \chi(1) = p^{\delta_i}\}$. Note that for each $\delta_r \in \{\delta_1, \dots, \delta_n\}$, the quotient group $Z_{\delta_r}/[Z_{\delta_r}, G]$ is abelian. With this notation, we recall the following result.

LEMMA 11. [27, Theorem 3.8] *Let G be a nested GVZ-group with $\text{cd}(G) = \{p^{\delta_i} : 0 \leq i \leq n, 0 = \delta_0 < \delta_1 < \dots < \delta_n\}$. Then for each $r \in \{1, \dots, n\}$, there is a bijection between the sets*

$$\text{Irr}(Z_{\delta_r}/[Z_{\delta_r}, G] \mid [Z_{\delta_{r-1}}, G]/[Z_{\delta_r}, G]) \quad \text{and} \quad \text{Irr}_{p^{\delta_r}}(G).$$

Moreover, for $\bar{\mu} \in \text{Irr}(Z_{\delta_r}/[Z_{\delta_r}, G] \mid [Z_{\delta_{r-1}}, G]/[Z_{\delta_r}, G])$, the associated character $\chi_{\mu} \in \text{Irr}_{p^{\delta_r}}(G)$ is given by

$$(1) \quad \chi_{\mu}(g) = \begin{cases} p^{\delta_r} \mu(g) & \text{if } g \in Z_{\delta_r}, \\ 0 & \text{otherwise,} \end{cases}$$

where μ denotes the lift of $\bar{\mu}$ to Z_{δ_r} .

Let G be a finite group, and let $\chi, \psi \in \text{Irr}(G)$ be Galois conjugate over $\mathbb{Q}$. Then necessarily $\ker(\chi) = \ker(\psi)$. The converse is true when $\chi, \psi \in \text{lin}(G)$. However, for general $\chi, \psi \in \text{nl}(G)$ with $\ker(\chi) = \ker(\psi)$, it need not follow that they are Galois conjugates. In the case of nested GVZ-groups, the converse does hold for nonlinear irreducible complex characters, as we show below.

LEMMA 12. *Let G be a nested GVZ-group with $\text{cd}(G) = \{p^{\delta_i} : 0 \leq i \leq n, 0 = \delta_0 < \delta_1 < \dots < \delta_n\}$. For $\chi, \psi \in \text{Irr}_{p^{\delta_r}}(G)$ with some $r \in \{1, \dots, n\}$, we have the following.*

$$\chi \text{ and } \psi \text{ are Galois conjugates over } \mathbb{Q} \iff \ker(\chi) = \ker(\psi).$$

PROOF. If χ and ψ are Galois conjugates over $\mathbb{Q}$, then $\ker(\chi) = \ker(\psi)$ is immediate. Conversely, suppose $\chi, \psi \in \text{Irr}_{p^{\delta_r}}(G)$ with $\ker(\chi) = \ker(\psi)$. By definition, $Z_{\delta_i} := Z(\phi)$ for some $\phi \in \text{Irr}_{p^{\delta_i}}(G)$. From (1), there exist $\mu, \nu \in \text{Irr}(Z_{\delta_r})$ such that

$$\chi \downarrow_{Z_{\delta_r}} = p^{\delta_r} \mu, \quad \psi \downarrow_{Z_{\delta_r}} = p^{\delta_r} \nu,$$

where μ and ν are the lifts of some $\bar{\mu} \in \text{Irr}(Z_{\delta_r}/[Z_{\delta_r}, G] \mid [Z_{\delta_{r-1}}, G]/[Z_{\delta_r}, G])$ to Z_{δ_r} and $\bar{\nu} \in \text{Irr}(Z_{\delta_r}/[Z_{\delta_r}, G] \mid [Z_{\delta_{r-1}}, G]/[Z_{\delta_r}, G])$ to Z_{δ_r} , respectively. Since $\ker(\chi) = \ker(\mu)$ and $\ker(\psi) = \ker(\nu)$, we deduce that μ and ν are Galois conjugates. Hence, $\mathbb{Q}(\mu) = \mathbb{Q}(\nu)$. Therefore, there exists $\sigma \in \text{Gal}(\mathbb{Q}(\mu)/\mathbb{Q})$ with $\mu^\sigma = \nu$. As $\mathbb{Q}(\chi) = \mathbb{Q}(\mu)$ and $\mathbb{Q}(\psi) = \mathbb{Q}(\nu)$, it follows that $\chi^\sigma = \psi$. Thus, χ and ψ are Galois conjugates over $\mathbb{Q}$. This completes the proof of Lemma 12. $\square$

Now, for $\chi, \psi \in \text{Irr}(G)$, we say that χ and ψ are *equivalent* if $\ker(\chi) = \ker(\psi)$.

LEMMA 13. [2, Lemma 1] *Let G be a finite abelian group of exponent m , and let $d \mid m$. If a_d denotes the number of cyclic subgroups of G of order d , then the number of inequivalent characters χ with $\mathbb{Q}(\chi) = \mathbb{Q}(\zeta_d)$ is a_d .*

Analogous to Lemma 13, and under the above hypotheses, we now determine the number of inequivalent irreducible characters of a nested GVZ-group corresponding to a given cyclotomic field.

LEMMA 14. *Let G be a nested GVZ-group with $\text{cd}(G) = \{p^{\delta_i} : 0 \leq i \leq n, 0 = \delta_0 < \delta_1 < \dots < \delta_n\}$. Define $Z_{\delta_i} := Z(\chi)$ for some $\chi \in \text{Irr}_{p^{\delta_i}}(G)$. For each $r \in \{1, \dots, n\}$, let d_r be a divisor of $\exp(Z_{\delta_r}/[Z_{\delta_r}, G])$. Let a_{d_r} and a'_{d_r} denote the number of cyclic subgroups of order d_r in $Z_{\delta_r}/[Z_{\delta_r}, G]$ and $Z_{\delta_r}/[Z_{\delta_{r-1}}, G]$, respectively. Finally, let m_{d_r} be the number of inequivalent characters $\chi \in \text{Irr}_{p^{\delta_r}}(G)$ with $\mathbb{Q}(\chi) = \mathbb{Q}(\zeta_{d_r})$. Then we have the following.*

- (1) *If $d_r \mid \exp(Z_{\delta_r}/[Z_{\delta_r}, G])$ but $d_r \nmid \exp(Z_{\delta_r}/[Z_{\delta_{r-1}}, G])$, then $m_{d_r} = a_{d_r}$.*
- (2) *If d_r divides both $\exp(Z_{\delta_r}/[Z_{\delta_r}, G])$ and $\exp(Z_{\delta_r}/[Z_{\delta_{r-1}}, G])$, then $m_{d_r} = a_{d_r} - a'_{d_r}$.*

PROOF. Let $\chi \in \text{Irr}_{p^{\delta_r}}(G)$. Then $\chi = \chi_\mu$ for some $\mu \in \text{Irr}(Z_{\delta_r})$, as in (1), where μ is the lift of $\bar{\mu} \in \text{Irr}(Z_{\delta_r}/[Z_{\delta_r}, G] \mid [Z_{\delta_{r-1}}, G]/[Z_{\delta_r}, G])$ to Z_{δ_r} . We observe that $\ker(\chi_\mu) = \ker(\mu)$ and $\mathbb{Q}(\chi_\mu) = \mathbb{Q}(\mu)$. By Lemma 11, there is a bijection between the sets $\text{Irr}(Z_{\delta_r}/[Z_{\delta_r}, G] \mid [Z_{\delta_{r-1}}, G]/[Z_{\delta_r}, G])$ and $\text{Irr}_{p^{\delta_r}}(G)$. Consequently, by applying Lemma 13, the results follow. This completes the proof of Lemma 14. $\square$

We are now ready to prove Theorem 3.

PROOF OF THEOREM 3. Let G be a finite nested GVZ p -group, where p is an odd prime, and let $\chi \in \text{Irr}(G)$. Suppose ρ is an irreducible $\mathbb{Q}$ -representation of G affording the character $\Omega(\chi)$. Denote by $A_{\mathbb{Q}}(\chi)$ the simple component in the Wedderburn decomposition of $\mathbb{Q}G$ corresponding to ρ , so that $A_{\mathbb{Q}}(\chi) \cong M_q(D)$ for some $q \in \mathbb{N}$ and a division algebra D . By Lemma 8, we have $m_{\mathbb{Q}}(\chi) = 1$. Moreover, Lemma 10 shows that $[D : Z(D)] = m_{\mathbb{Q}}(\chi)^2$ and $Z(D) = \mathbb{Q}(\chi)$. Hence, $D = Z(D) = \mathbb{Q}(\chi)$. Next, consider

$$\rho \cong \bigoplus_{i=1}^l \rho_i,$$

where $l = [\mathbb{Q}(\chi) : \mathbb{Q}]$ and each ρ_i is an irreducible complex representation of G affording the character χ^{σ_i} for some $\sigma_i \in \text{Gal}(\mathbb{Q}(\chi)/\mathbb{Q})$. Since $m_{\mathbb{Q}}(\chi) = 1$, it follows from [24, Theorem 3.3.1] that $q = \chi(1)$.

Note that $\text{lin}(G) \cong G/G'$. Therefore, the simple components in the Wedderburn decomposition of $\mathbb{Q}G$ corresponding to all inequivalent irreducible $\mathbb{Q}$ -representations of G affording the character $\Omega(\chi)$ with $\chi \in \text{lin}(G)$ are precisely isomorphic to $\mathbb{Q}(G/G')$.

Furthermore, we have

$$\text{cd}(G) = \{p^{\delta_i} : 0 \leq i \leq n, 0 = \delta_0 < \delta_1 < \dots < \delta_n\},$$

and for each i , let $Z_{\delta_i} := Z(\chi)$ for some $\chi \in \text{Irr}_{p^{\delta_i}}(G)$. Note that $Z_{\delta_i}/[Z_{\delta_i}, G]$ is abelian for every $i \in \{0, 1, \dots, n\}$.

Fix $r \in \{1, 2, \dots, n\}$, and let ρ be an irreducible $\mathbb{Q}$ -representation of G affording the character $\Omega(\chi_\mu)$, where $\chi_\mu \in \text{Irr}_{p^{\delta_r}}(G)$ (given in (1)). Here, $\chi_\mu(1) = p^{\delta_r}$ and $\mathbb{Q}(\chi_\mu) = \mathbb{Q}(\mu)$. Therefore, by the above discussion, we have

$$A_{\mathbb{Q}}(\chi_\mu) \cong M_{p^{\delta_r}}(\mathbb{Q}(\mu)).$$

Moreover, $\mathbb{Q}(\chi_\mu) = \mathbb{Q}(\mu) = \mathbb{Q}(\zeta_{d_r})$ for some d_r dividing $\exp(Z_{\delta_r}/[Z_{\delta_r}, G])$.

We now distinguish two cases.

Case 1. If $d_r \mid \exp(Z_{\delta_r}/[Z_{\delta_r}, G])$ but $d_r \nmid \exp(Z_{\delta_r}/[Z_{\delta_{r-1}}, G])$, then by Lemmas 12 and 14(1), the number of inequivalent irreducible $\mathbb{Q}$ -representations of G affording the character $\Omega(\chi_\mu)$ for some character $\chi_\mu \in \text{Irr}_{p^{\delta_r}}(G)$ with $\mathbb{Q}(\chi_\mu) = \mathbb{Q}(\zeta_{d_r})$ equals a_{d_r} , where a_{d_r} is the number of cyclic subgroups of order d_r in $Z_{\delta_r}/[Z_{\delta_r}, G]$.

Case 2. If $d_r \mid \exp(Z_{\delta_r}/[Z_{\delta_r}, G])$ and $d_r \mid \exp(Z_{\delta_r}/[Z_{\delta_{r-1}}, G])$, then by Lemmas 12 and 14(2), the number of inequivalent irreducible $\mathbb{Q}$ -representations of G affording the character $\Omega(\chi_\mu)$ for some character $\chi_\mu \in \text{Irr}_{p^{\delta_r}}(G)$ with $\mathbb{Q}(\chi_\mu) = \mathbb{Q}(\zeta_{d_r})$ equals $a_{d_r} - a'_{d_r}$, where a_{d_r} and a'_{d_r} denote the numbers of cyclic subgroups of order d_r in $Z_{\delta_r}/[Z_{\delta_r}, G]$ and $Z_{\delta_r}/[Z_{\delta_{r-1}}, G]$, respectively.

Now, let m_r and m'_r be the exponents of $Z_{\delta_r}/[Z_{\delta_r}, G]$ and $Z_{\delta_r}/[Z_{\delta_{r-1}}, G]$, respectively. By combining the two cases, the simple components of $\mathbb{Q}G$ corresponding to all inequivalent irreducible $\mathbb{Q}$ -representations of G affording the character $\Omega(\chi_\mu)$, where $\chi_\mu \in \text{Irr}_{p^{\delta_r}}(G)$, contribute

$$\bigoplus_{\substack{d_r \mid m_r \\ d_r \nmid m'_r}} a_{d_r} M_{p^{\delta_r}}(\mathbb{Q}(\zeta_{d_r})) \quad \bigoplus_{\substack{d_r \mid m_r \\ d_r \mid m'_r}} (a_{d_r} - a'_{d_r}) M_{p^{\delta_r}}(\mathbb{Q}(\zeta_{d_r}))$$

to the Wedderburn decomposition of $\mathbb{Q}G$.

Finally, by collecting the simple components corresponding to all inequivalent irreducible $\mathbb{Q}$ -representations of G affording the character $\Omega(\chi)$ with $\chi \in \text{Irr}_{p^{\delta_i}}(G)$ for all $i \in \{0, 1, \dots, n\}$, the result follows. This completes the proof of Theorem 3. $\square$

4. Two-generator p -groups of nilpotency class 2

The classification of two-generator p -groups of class 2 is given in [1]. In this section, we begin by introducing some notation and terminology that will allow us to restate the main result of [1]. Let p be a prime. For a given integer $n > 2$, define a set of 5-tuples

$$\tau_n = \{(\alpha, \beta, \gamma; \rho, \sigma) : \alpha \geq \beta \geq \gamma \geq 1, \alpha + \beta + \gamma = n, 0 \leq \rho \leq \gamma, 0 \leq \sigma \leq \gamma\}.$$

For each $(\alpha, \beta, \gamma; \rho, \sigma) \in \tau_n$, consider the group

$$(2) \quad G = G_{(\alpha, \beta, \gamma; \rho, \sigma)} = \langle a, b : [a, b]^{p^\gamma} = [a, b, a] = [a, b, b] = 1, a^{p^\alpha} = [a, b]^{p^\rho}, b^{p^\beta} = [a, b]^{p^\sigma} \rangle.$$

It is clear from the presentation that $G = G_{(\alpha, \beta, \gamma; \rho, \sigma)}$ is a two-generator p -group of class 2 with order p^n . Moreover, the derived subgroup is

$$G' = \langle [a, b] \rangle \cong C_{p^\gamma}.$$

and the center is

$$Z(G) = \langle a^{p^\gamma}, b^{p^\gamma}, [a, b] \rangle.$$

When $\rho \leq \sigma$, the subgroup $\langle a^{p^\gamma} \rangle \cong C_{p^{\alpha-\rho}}$ forms a cyclic direct factor $Z(G)$, and

$$Z(G)/\langle a^{p^\gamma} \rangle = \langle b^{p^\gamma} \langle a^{p^\gamma} \rangle, [a, b] \langle a^{p^\gamma} \rangle \rangle \cong C_{p^{\beta-\gamma}} \times C_{p^\rho}.$$

Similarly, if $\sigma < \rho$, then $\langle b^{p^\gamma} \rangle \cong C_{p^{\beta-\sigma}}$ is a cyclic direct factor of $Z(G)$, and

$$Z(G)/\langle b^{p^\gamma} \rangle = \langle a^{p^\gamma} \langle b^{p^\gamma} \rangle, [a, b] \langle b^{p^\gamma} \rangle \rangle \cong C_{p^{\alpha-\gamma}} \times C_{p^\sigma}.$$

Thus, we obtain

$$Z(G) \cong \begin{cases} C_{p^{\alpha-\rho}} \times C_{p^{\beta-\gamma}} \times C_{p^\rho} & \text{if } \rho \leq \sigma, \\ C_{p^{\alpha-\gamma}} \times C_{p^{\beta-\sigma}} \times C_{p^\sigma} & \text{if } \sigma < \rho. \end{cases}$$

Next, we introduce the following subsets of τ_n :

$$\begin{aligned}\tau_{n_1} &= \{(\alpha, \beta, \gamma; \rho, \gamma) \in \tau_n : \alpha > \beta \geq \gamma \geq \rho \geq 0\}, \\ \tau_{n_2} &= \{(\alpha, \beta, \gamma; \gamma, \sigma) \in \tau_n : \alpha > \beta \geq \gamma > \sigma \geq 0\}, \\ \tau_{n_3} &= \{(\alpha, \beta, \gamma; \rho, \sigma) \in \tau_n : \alpha > \beta \geq \gamma \text{ and } \min(\gamma, \sigma + \alpha - \beta) > \rho > \sigma \geq 0\}, \\ \tau_{n_4} &= \{(\alpha, \alpha, \gamma; \rho, \gamma) \in \tau_n : \alpha > \gamma \geq \rho \geq 0\}, \\ \tau_{n_5} &= \{(\gamma, \gamma, \gamma; \rho, \gamma) \in \tau_n : 0 \leq \rho \leq \gamma\}.\end{aligned}$$

For an odd prime p , a 5-tuple $(\alpha, \beta, \gamma; \rho, \sigma) \in \tau_n$ is called *p-good 5-tuple* if it belongs to $\tau_{n_1} \cup \tau_{n_2} \cup \tau_{n_3} \cup \tau_{n_4} \cup \tau_{n_5}$.

LEMMA 15. [1, Theorem 1.1] *Let G be a two-generator p -group of class 2 and order p^n , with p an odd prime. Then there exists a unique p -good 5-tuple $(\alpha, \beta, \gamma; \rho, \sigma) \in \tau_{n_1} \cup \tau_{n_2} \cup \tau_{n_3} \cup \tau_{n_4} \cup \tau_{n_5}$ such that $G \cong G_{(\alpha, \beta, \gamma; \rho, \sigma)}$.*

Nenciu [28] showed that every two-generator p -group of class 2 is a nested GVZ p -group. The next lemma describes some additional properties of such groups, which is a summary of [28, Theorems 3.1, 3.3 and 4.1].

LEMMA 16. *Let $G = G_{(\alpha, \beta, \gamma; \rho, \sigma)}$ as defined in (2) be a two-generator p -group of class 2 and order p^n , where p is an odd prime. Then we have the following.*

- (1) $\text{cd}(G) = \{1, p, \dots, p^\gamma\}$.
- (2) *If $\chi \in \text{Irr}(G)$ with $\chi(1) = p^\delta$ for some $\delta \in \{0, 1, \dots, \gamma\}$, then $Z(\chi) = \langle a^{p^\delta}, b^{p^\delta}, [a, b] \rangle$ and $[Z(\chi), G] = \langle [a, b]^{p^\delta} \rangle$.*

We are now ready to prove Theorem 17, which provides a combinatorial description of the Wedderburn decomposition of rational group algebras for the family of groups defined in (2).

THEOREM 17. *Let $G = G_{(\alpha, \beta, \gamma; \rho, \sigma)}$ (defined in (2)) be a two-generator p -group of class 2 and order p^n , where p is an odd prime. Then*

$$\mathbb{Q}G \cong \mathbb{Q} \bigoplus_{\lambda=1}^{\beta} (p^\lambda + p^{\lambda-1}) \mathbb{Q}(\zeta_{p^\lambda}) \bigoplus_{\lambda=\beta+1}^{\alpha} p^\beta \mathbb{Q}(\zeta_{p^\lambda}) \bigoplus_{\delta=1}^{\gamma} \bigoplus_{\lambda=1}^{m_\delta} (a_{p^\lambda} - a'_{p^\lambda}) M_{p^\delta}(\mathbb{Q}(\zeta_{p^\lambda})),$$

where a_{p^λ} and a'_{p^λ} are the number of cyclic subgroups of order p^λ of $\langle a^{p^\delta}, b^{p^\delta}, [a, b] \rangle / \langle [a, b]^{p^\delta} \rangle$ and $\langle a^{p^\delta}, b^{p^\delta}, [a, b] \rangle / \langle [a, b]^{p^{\delta-1}} \rangle$, respectively, and p^{m_δ} denotes the exponent of $\langle a^{p^\delta}, b^{p^\delta}, [a, b] \rangle / \langle [a, b]^{p^\delta} \rangle$.

PROOF. As $(\alpha, \beta, \gamma; \rho, \gamma) \in \tau_n$, we have $\alpha \geq \beta \geq \gamma \geq 1$. From (2), we have

$$G = G_{(\alpha, \beta, \gamma; \rho, \sigma)} = \langle a, b : [a, b]^{p^\gamma} = [a, b, a] = [a, b, b] = 1, a^{p^\alpha} = [a, b]^{p^\rho}, b^{p^\beta} = [a, b]^{p^\sigma} \rangle.$$

Note that $G' = \langle [a, b] \rangle \cong C_{p^\gamma}$ and the quotient $G/G' = \langle aG', bG' \rangle \cong C_{p^\alpha} \times C_{p^\beta}$. For each $1 \leq \lambda \leq \alpha$, let $\mathcal{G}/\mathcal{G}'_p(\lambda)$ denote the number of cyclic subgroups of order p^λ in G/G' . By Lemma 7, for λ with $1 \leq \lambda \leq \beta$, we have $k = 2$ and

$$\mathcal{G}/\mathcal{G}'_p(\lambda) = \frac{ph_p^1(\lambda) - h_p^1(\lambda - 1)}{p - 1},$$

where, $h_p^1(\lambda) = p^\lambda$ and $h_p^1(\lambda - 1) = p^{\lambda-1}$. Hence, we get

$$\mathcal{G}/\mathcal{G}'_p(\lambda) = \frac{p^{\lambda+1} - p^{\lambda-1}}{p - 1} = p^\lambda + p^{\lambda-1}.$$

Similarly, for λ with $\beta + 1 \leq \lambda \leq \alpha$, we have $k = 2$ and

$$\mathcal{G}/\mathcal{G}'_p(\lambda) = \frac{ph_p^1(\lambda) - h_p^1(\lambda - 1)}{p - 1} = \frac{p^{\beta+1} - p^\beta}{p - 1} = p^\beta$$

as $h_p^1(\lambda) = h_p^1(\lambda - 1) = p^\beta$ (see Lemma 7). Hence, from Theorem 1, we get

$$\mathbb{Q}(G/G') \cong \mathbb{Q} \bigoplus_{\lambda=1}^{\beta} (p^\lambda + p^{\lambda-1}) \mathbb{Q}(\zeta_{p^\lambda}) \bigoplus_{\lambda=\beta+1}^{\alpha} p^\beta \mathbb{Q}(\zeta_{p^\lambda}).$$

For each $\delta \in \{0, 1, \dots, \gamma\}$, let $Z_\delta = Z(\chi)$ for $\chi \in \text{Irr}(G)$ with $\chi(1) = p^\delta$. Then by Lemma 16, we get

$$Z_\delta = \langle a^{p^\delta}, b^{p^\delta}, [a, b] \rangle \quad \text{and} \quad [Z_\delta, G] = \langle [a, b]^{p^\delta} \rangle.$$

Moreover, $\text{cd}(G) = \{1, p, \dots, p^\gamma\}$. Suppose p^{m_δ} is the exponent of $Z_\delta/[Z_\delta, G]$ for each $\delta \in \{1, 2, \dots, \gamma\}$. We have

$$Z_\delta/[Z_{\delta-1}, G] = (Z_\delta/[Z_\delta, G]) / ([Z_{\delta-1}]/[Z_\delta, G]).$$

This implies that $\exp(Z_\delta/[Z_{\delta-1}, G])$ divides p^{m_δ} . Therefore, by applying Theorem 3, the result follows. This completes the proof of Theorem 17. $\square$

4.1. An explicit formula for $\mathbb{Q}G_{(\alpha, \beta, \gamma; \rho, \sigma)}$ with $(\alpha, \beta, \gamma; \rho, \sigma) \in \tau_{n_5}$. In this subsection, we derive an explicit combinatorial expression for the Wedderburn decomposition of rational group algebras corresponding to a particular subclass of two-generator p -groups of class 2. This serves as a concrete illustration of Theorem 17. Here, we restrict our attention to groups that are isomorphic to $G_{(\alpha, \beta, \gamma; \rho, \sigma)}$ with $(\alpha, \beta, \gamma; \rho, \sigma) \in \tau_{n_5}$. We begin with the following lemma.

LEMMA 18. *Let $G = G_{(\alpha, \beta, \gamma; \rho, \sigma)}$ as defined in (2) be a two generator p -group of class two and of order p^n for an odd prime p , such that $(\alpha, \beta, \gamma; \rho, \sigma) \in \tau_{n_5}$. Suppose $1 \leq \delta \leq \gamma$. Then we have the following.*

- (1) $\langle a^{p^\delta}, b^{p^\delta}, [a, b] \rangle / \langle [a, b]^{p^\delta} \rangle \cong \begin{cases} C_{p^{\gamma-\rho}} \times C_{p^{\gamma-\delta}} \times C_{p^\rho} & \text{if } \rho < \delta < \gamma - \rho, \\ C_{p^{\gamma-\delta}} \times C_{p^{\gamma-\delta}} \times C_{p^\delta} & \text{otherwise.} \end{cases}$
- (2) $\langle a^{p^\delta}, b^{p^\delta}, [a, b] \rangle / \langle [a, b]^{p^{\delta-1}} \rangle \cong \begin{cases} C_{p^{\gamma-\rho-1}} \times C_{p^{\gamma-\delta}} \times C_{p^\rho} & \text{if } \rho < \delta < \gamma - \rho, \\ C_{p^{\gamma-\delta}} \times C_{p^{\gamma-\delta}} \times C_{p^{\delta-1}} & \text{otherwise.} \end{cases}$

PROOF. Since $(\alpha, \beta, \gamma; \rho, \sigma) \in \tau_{n_5}$, we have $\alpha = \beta = \sigma = \gamma$ and $0 \leq \rho \leq \gamma$. Thus, from (2), we have

$$G = G_{(\gamma, \gamma, \gamma; \rho, \gamma)} = \langle a, b : [a, b]^{p^\gamma} = [a, b, a] = [a, b, b] = 1, a^{p^\gamma} = [a, b]^{p^\rho}, b^{p^\gamma} = 1 \rangle.$$

For $1 \leq \delta \leq \gamma$, we have

$$Z_\delta = \langle a^{p^\delta}, b^{p^\delta}, [a, b] \rangle; \quad [Z_\delta, G] = \langle [a, b]^{p^\delta} \rangle \quad \text{and} \quad [Z_{\delta-1}, G] = \langle [a, b]^{p^{\delta-1}} \rangle,$$

where $Z_\delta = Z(\chi)$ for some $\chi \in \text{Irr}_{p^\delta}(G)$ and $Z_{\delta-1} = Z(\psi)$ for some $\psi \in \text{Irr}_{p^{\delta-1}}(G)$ (see Lemma 16).

- (1) First note that $\langle b^{p^\delta} [Z_\delta, G] \rangle \cong C_{p^{\gamma-\delta}}$ is a cyclic factor of $Z_\delta/[Z_\delta, G]$. Further analysis proceeds in two cases.

Case 1 ($1 \leq \delta \leq \rho$). In this case, observe that $\langle a^{p^\delta} [Z_\delta, G] \rangle \cong C_{p^{\gamma-\delta}}$ and $\langle [a, b] [Z_\delta, G] \rangle \cong C_{p^\delta}$ are also cyclic factors of $Z_\delta/[Z_\delta, G]$, yielding

$$Z_\delta/[Z_\delta, G] \cong C_{p^{\gamma-\delta}} \times C_{p^{\gamma-\delta}} \times C_{p^\delta}.$$

Case 2 ($\rho < \delta \leq \gamma$). In this case,

$$(a^{p^\delta})^{p^{\gamma-\rho}} = (a^{p^\gamma})^{p^{\delta-\rho}} = ([a, b]^{p^\rho})^{p^{\delta-\rho}} = [a, b]^{p^\delta}.$$

Sub-case 2(a) ($\rho < \delta < \gamma - \rho$). In this sub-case, we have the following observations.

- $\text{Order}(a^{p^\delta} [Z_\delta, G]) = p^{\gamma-\rho} > \text{Order}([a, b] [Z_\delta, G]) = p^\delta$ in $Z_\delta/[Z_\delta, G]$.
- $\langle a^{p^\delta} [Z_\delta, G] \rangle \cong C_{p^{\gamma-\rho}}$ is also a cyclic factor of $Z_\delta/[Z_\delta, G]$.
- $(a^{p^\delta})^{p^{\gamma-\delta}} = a^{p^\gamma} = [a, b]^{p^\rho}$.

Therefore, in this sub-case,

$$Z_\delta/[Z_\delta, G] \cong C_{p^{\gamma-\rho}} \times C_{p^{\gamma-\delta}} \times C_{p^\rho}.$$

Sub-case 2(b) ($\gamma - \rho \leq \delta \leq \gamma$). In this sub-case, we have the following observations.

- $\text{Order}(a^{p^\delta} [Z_\delta, G]) = p^{\gamma-\rho} \leq \text{Order}([a, b] [Z_\delta, G]) = p^\delta$ in $Z_\delta/[Z_\delta, G]$.
- $\langle [a, b] [Z_\delta, G] \rangle \cong C_{p^\delta}$ is a cyclic factor of $Z_\delta/[Z_\delta, G]$.

$$\bullet (a^{p^\delta})^{p^{\gamma-\delta}} = a^{p^\gamma} = [a, b]^{p^\rho}.$$

Therefore, in this sub-case, we have

$$Z_\delta/[Z_\delta, G] \cong C_{p^{\gamma-\delta}} \times C_{p^{\gamma-\delta}} \times C_{p^\delta}.$$

(2) The proof proceeds in the same way by replacing $[Z_\delta, G]$ with $[Z_{\delta-1}, G]$. Note that $\langle b^{p^\delta}[Z_{\delta-1}, G] \rangle \cong C_{p^{\gamma-\delta}}$ is a cyclic factor of $Z_\delta/[Z_{\delta-1}, G]$. Further analysis again proceeds in two cases.

Case 1 ($1 \leq \delta \leq \rho$). In this case, observe that $\langle a^{p^\delta}[Z_{\delta-1}, G] \rangle \cong C_{p^{\gamma-\delta}}$ and $\langle [a, b][Z_{\delta-1}, G] \rangle \cong C_{p^{\delta-1}}$ are also cyclic factors of $Z_\delta/[Z_{\delta-1}, G]$. Therefore, we have

$$Z_\delta/[Z_{\delta-1}, G] \cong C_{p^{\gamma-\delta}} \times C_{p^{\gamma-\delta}} \times C_{p^{\delta-1}}.$$

Case 2 ($\rho < \delta \leq \gamma$). In this case, we have

$$(a^{p^\delta})^{p^{\gamma-\rho-1}} = (a^{p^\gamma})^{p^{\delta-\rho-1}} = ([a, b]^{p^\rho})^{p^{\delta-\rho-1}} = [a, b]^{p^{\delta-1}}.$$

Sub-case 2(a) ($\rho < \delta < \gamma - \rho$). In this sub-case, we have the following observations.

- $\text{Order}(a^{p^\delta}[Z_{\delta-1}, G]) = p^{\gamma-\rho-1} > \text{Order}([a, b][Z_{\delta-1}, G]) = p^{\delta-1}$ in $Z_\delta/[Z_{\delta-1}, G]$.
- $\langle a^{p^\delta}[Z_{\delta-1}, G] \rangle \cong C_{p^{\gamma-\rho}}$ is a cyclic factor of $Z_\delta/[Z_{\delta-1}, G]$.
- $(a^{p^\delta})^{p^{\gamma-\delta}} = a^{p^\gamma} = [a, b]^{p^\rho}$.

Hence, in this sub-case, we get

$$Z_\delta/[Z_{\delta-1}, G] \cong C_{p^{\gamma-\rho-1}} \times C_{p^{\gamma-\delta}} \times C_{p^\rho}.$$

Sub-case 2(b) ($\gamma - \rho \leq \delta \leq \gamma$). In this sub-case, we have the following observations.

- $\text{Order}(a^{p^\delta}[Z_{\delta-1}, G]) = p^{\gamma-\rho-1} \leq \text{Order}([a, b][Z_{\delta-1}, G]) = p^{\delta-1}$ in $Z_\delta/[Z_{\delta-1}, G]$.
- $\langle [a, b][Z_{\delta-1}, G] \rangle \cong C_{p^\delta}$ is a cyclic factor of $Z_\delta/[Z_{\delta-1}, G]$.
- $(a^{p^\delta})^{p^{\gamma-\delta}} = a^{p^\gamma} = [a, b]^{p^\rho}$.

Hence, in this sub-case, we get

$$Z_\delta/[Z_{\delta-1}, G] \cong C_{p^{\gamma-\delta}} \times C_{p^{\gamma-\delta}} \times C_{p^{\delta-1}}.$$

This completes the proof of Lemma 18. $\square$

We are now in a position to prove Theorem 19, which establishes an explicit combinatorial formula for the Wedderburn decomposition of rational group algebras associated with two-generator p -groups $G_{(\alpha, \beta, \gamma; \rho, \sigma)}$ satisfying $(\alpha, \beta, \gamma; \rho, \sigma) \in \tau_{n_5}$.

THEOREM 19. *Let $G = G_{(\alpha, \beta, \gamma; \rho, \sigma)}$ as defined in (2) be a two-generator p -group of class two and of order p^n for an odd prime p , such that $(\alpha, \beta, \gamma; \rho, \sigma) \in \tau_{n_5}$. Then we have the following.*

(1) **Case** ($\gamma \leq 2\rho + 1$). In this case, the Wedderburn decomposition of $\mathbb{Q}G$ is given by

$$\begin{aligned} \mathbb{Q}G \cong & \mathbb{Q} \bigoplus_{m=1}^{\gamma} (p^m + p^{m-1}) \mathbb{Q}(\zeta_{p^m}) \bigoplus_{\delta=1}^{\lfloor \frac{\gamma}{2} \rfloor} (p^{2\delta} + p^{2\delta-1}) M_{p^\delta}(\mathbb{Q}(\zeta_{p^\delta})) \\ & \bigoplus_{\delta=1}^{\lfloor \frac{\gamma}{2} \rfloor} \bigoplus_{m=\delta+1}^{\gamma-\delta} (p^{m+\delta} - p^{m+\delta-2}) M_{p^\delta}(\mathbb{Q}(\zeta_{p^m})) \bigoplus_{\delta=\lfloor \frac{\gamma}{2} \rfloor + 1}^{\gamma} p^{2(\gamma-\delta)} M_{p^\delta}(\mathbb{Q}(\zeta_{p^\delta})). \end{aligned}$$

(2) **Case** ($\gamma > 2\rho + 1$). In this case, the Wedderburn decomposition of $\mathbb{Q}G$ is given by

$$\begin{aligned} \mathbb{Q}G \cong & \mathbb{Q} \bigoplus_{m=1}^{\gamma} (p^m + p^{m-1}) \mathbb{Q}(\zeta_{p^m}) \bigoplus_{\delta=1}^{\rho} (p^{2\delta} + p^{2\delta-1}) M_{p^\delta}(\mathbb{Q}(\zeta_{p^\delta})) \\ & \bigoplus_{\delta=1}^{\rho} \bigoplus_{m=\delta+1}^{\gamma-\delta} (p^{m+\delta} - p^{m+\delta-2}) M_{p^\delta}(\mathbb{Q}(\zeta_{p^m})) \bigoplus_{\delta=\rho+1}^{\gamma-\rho-1} p^{\gamma+\rho-\delta} M_{p^\delta}(\mathbb{Q}(\zeta_{p^{\gamma-\rho}})) \\ & \bigoplus_{\delta=\gamma-\rho}^{\gamma} p^{2(\gamma-\delta)} M_{p^\delta}(\mathbb{Q}(\zeta_{p^\delta})). \end{aligned}$$

PROOF. Since $(\alpha, \beta, \gamma; \rho, \sigma) \in \tau_{n_5}$, we have $\alpha = \beta = \sigma = \gamma$ and $0 \leq \rho \leq \gamma$. Thus, from (2), we have

$$G = G_{(\gamma, \gamma, \gamma; \rho, \gamma)} = \langle a, b : [a, b]^{p^\gamma} = [a, b, a] = [a, b, b] = 1, a^{p^\gamma} = [a, b]^{p^\rho}, b^{p^\gamma} = 1 \rangle.$$

For $1 \leq \delta \leq \gamma$, we have

$$Z_\delta = \langle a^{p^\delta}, b^{p^\delta}, [a, b] \rangle; \quad [Z_\delta, G] = \langle [a, b]^{p^\delta} \rangle \quad \text{and} \quad [Z_{\delta-1}, G] = \langle [a, b]^{p^{\delta-1}} \rangle,$$

where $Z_\delta = Z(\chi)$ for some $\chi \in \text{Irr}_{p^\delta}(G)$ and $Z_{\delta-1} = Z(\psi)$ for some $\psi \in \text{Irr}_{p^{\delta-1}}(G)$ (see Lemma 16).

Furthermore, for $1 \leq \delta \leq \gamma$, let a_{p^m} and a'_{p^m} denote the number of cyclic subgroups of order p^m of $Z_\delta/[Z_\delta, G]$ and $Z_{\delta-1}/[Z_{\delta-1}, G]$, respectively. In view of Lemma 18, such groups are divided into two categories according to the values of γ and ρ , and the proof is therefore considered in the following two cases.

- (1) **Case** ($\gamma \leq 2\rho + 1$). In this case, observe that there does not exist any δ such that $\rho < \delta < \gamma - \rho$. Therefore, from Lemma 18), for each $\delta \in \{1, 2, \dots, \gamma\}$, we have

$$Z_\delta/[Z_\delta, G] \cong C_{p^{\gamma-\delta}} \times C_{p^{\gamma-\delta}} \times C_{p^\delta} \quad \text{and} \quad Z_{\delta-1}/[Z_{\delta-1}, G] \cong C_{p^{\gamma-\delta}} \times C_{p^{\gamma-\delta}} \times C_{p^{\delta-1}}.$$

We complete the rest of the proof in the following two sub-cases.

Sub-case ($\gamma - \delta \geq \delta$). Note that

$$\gamma - \delta \geq \delta \iff \delta \leq \left\lfloor \frac{\gamma}{2} \right\rfloor,$$

where $\lfloor \cdot \rfloor$ denotes the greatest integer function.

In this sub-case, we have $1 \leq \delta \leq \lfloor \frac{\gamma}{2} \rfloor$. By Lemma 7, one can check that $a_{p^m} = a'_{p^m}$ for $1 \leq m \leq \delta - 1$. Again from Lemma 7, we have $k = 3$ and

$$a_{p^\delta} = \frac{ph_p^2(\delta) - h_p^2(\delta - 1)}{p - 1},$$

where $h_p^2(\delta) = p^{2\delta}$ and $h_p^2(\delta - 1) = p^{2(\delta-1)}$. Hence,

$$a_{p^\delta} = \frac{p^{2\delta+1} - p^{2\delta-2}}{p - 1}.$$

Similarly,

$$a'_{p^\delta} = \frac{ph_p^2(\delta) - h_p^2(\delta - 1)}{p - 1} = \frac{p^{2\delta-1} - p^{2\delta-2}}{p - 1}.$$

as $h_p^2(\delta) = p^{2\delta-1}$ and $h_p^2(\delta - 1) = p^{2(\delta-1)}$ (see Lemma 7). Therefore,

$$a_{p^\delta} - a'_{p^\delta} = \frac{p^{2\delta+1} - p^{2\delta-1}}{p - 1} = p^{2\delta} + p^{2\delta-1}.$$

Now, for $\delta + 1 \leq m \leq \gamma - \delta$, Lemma 7 gives

$$a_{p^m} = \frac{p^{m+\delta+1} - p^{m+\delta-1}}{p - 1} \quad \text{and} \quad a'_{p^m} = \frac{p^{m+\delta} - p^{m+\delta-2}}{p - 1}.$$

Hence,

$$a_{p^m} - a'_{p^m} = p^{m+\delta} - p^{m+\delta-2}.$$

Sub-case ($\gamma - \delta < \delta$). Note that

$$\gamma - \delta < \delta \iff \delta > \left\lfloor \frac{\gamma}{2} \right\rfloor.$$

For each $\lfloor \frac{\gamma}{2} \rfloor + 1 \leq \delta \leq \gamma$, Lemma 7 shows that $a_{p^m} = a'_{p^m}$ for all $1 \leq m \leq \delta - 1$. Furthermore, $k = 3$ and

$$a_{p^\delta} = \frac{ph_p^2(\delta) - h_p^2(\delta - 1)}{p - 1} = \frac{p^{2(\gamma-\delta)+1} - p^{2(\gamma-\delta)}}{p - 1} = p^{2(\gamma-\delta)},$$

as $h_p^2(\delta) = h_p^2(\delta - 1) = p^{2(\gamma-\delta)}$ (see Lemma 7). Also, observe that $a'_{p^\delta} = 0$. Thus, the result follows from Theorem 17.

(2) **Case** ($\gamma > 2\rho + 1$). In this case, there exists some δ such that $\rho < \delta < \gamma - \rho$. We proceed the proof in the following three sub-cases.

Sub-case ($1 \leq \delta \leq \rho$). In this sub-case, from Lemma 18, we have

$$Z_\delta/[Z_\delta, G] \cong C_{p^{\gamma-\delta}} \times C_{p^{\gamma-\delta}} \times C_{p^\delta} \quad \text{and} \quad Z_\delta/[Z_{\delta-1}, G] \cong C_{p^{\gamma-\delta}} \times C_{p^{\gamma-\delta}} \times C_{p^{\delta-1}}.$$

Note that in this sub-case, $\gamma - \delta \geq \delta$. Hence, for each $1 \leq \delta \leq \rho$, by Lemma 7, we have $a_{p^m} = a'_{p^m}$ whenever $1 \leq m \leq \delta - 1$. Moreover, analogous to the previous case, we have

$$a_{p^\delta} = \frac{p^{2\delta+1} - p^{2\delta-2}}{p-1} \quad \text{and} \quad a'_{p^\delta} = \frac{p^{2\delta-1} - p^{2\delta-2}}{p-1}.$$

Therefore, we get

$$a_{p^\delta} - a'_{p^\delta} = \frac{p^{2\delta+1} - p^{2\delta-1}}{p-1} = p^{2\delta} + p^{2\delta-1}.$$

Further, for $\delta + 1 \leq m \leq \gamma - \delta$, analogous to the previous case, Lemma 7 yields

$$a_{p^m} = \frac{p^{m+\delta+1} - p^{m+\delta-1}}{p-1} \quad \text{and} \quad a'_{p^m} = \frac{p^{m+\delta} - p^{m+\delta-2}}{p-1}.$$

Hence,

$$a_{p^m} - a'_{p^m} = p^{m+\delta} - p^{m+\delta-2}.$$

Sub-case ($\rho + 1 \leq \delta \leq \gamma - \rho - 1$). In this sub-case, from Lemma 18, we have

$$Z_\delta/[Z_\delta, G] \cong C_{p^{\gamma-\rho}} \times C_{p^{\gamma-\delta}} \times C_{p^\rho} \quad \text{and} \quad Z_\delta/[Z_{\delta-1}, G] \cong C_{p^{\gamma-\rho-1}} \times C_{p^{\gamma-\delta}} \times C_{p^\rho}.$$

Since $\gamma - \rho > \rho + 1$ and $\delta \in \{\rho + 1, \dots, \gamma - \rho - 1\}$, one can see that $\rho + 1 \leq \gamma - \delta \leq \gamma - \rho - 1$. Hence, in this sub-case, $\rho < \gamma - \delta < \gamma - \rho$. By Lemma 7, we have $a_{p^m} = a'_{p^m}$ for $1 \leq m \leq \gamma - \rho - 1$. Again from Lemma 7, we have $k = 3$ and

$$a_{p^{\gamma-\rho}} = \frac{ph_p^2(\gamma - \rho) - h_p^2(\gamma - \rho - 1)}{p-1} = \frac{p^{\gamma-\delta+\rho+1} - p^{\gamma-\delta+\rho}}{p-1} = p^{\gamma+\rho-\delta},$$

as $h_p^2(\gamma - \rho) = h_p^2(\gamma - \rho - 1) = p^{\gamma-\delta+\rho}$. Further, note that $a'_{p^{\gamma-\rho}} = 0$.

Sub-case ($\gamma - \rho \leq \delta \leq \gamma$). In this sub-case, we have

$$Z_\delta/[Z_\delta, G] \cong C_{p^{\gamma-\delta}} \times C_{p^{\gamma-\delta}} \times C_{p^\delta} \quad \text{and} \quad Z_\delta/[Z_{\delta-1}, G] \cong C_{p^{\gamma-\delta}} \times C_{p^{\gamma-\delta}} \times C_{p^{\delta-1}}$$

(see Lemma 18). In this range of δ , we have $\gamma - \delta \leq \delta$. Now, by Lemma 7, we again obtain $a_{p^m} = a'_{p^m}$ for $1 \leq m \leq \delta - 1$. Furthermore, we have $k = 3$ and

$$a_{p^\delta} = \frac{ph_p^2(\delta) - h_p^2(\delta - 1)}{p-1} = \frac{p^{2(\gamma-\delta)+1} - p^{2(\gamma-\delta)}}{p-1} = p^{2(\gamma-\delta)},$$

as $h_p^2(\delta) = h_p^2(\delta - 1) = p^{2(\gamma-\delta)}$ (see Lemma 7). Again, note that $a'_{p^\delta} = 0$. Thus, the result follows from Theorem 17.

This completes the proof of Theorem 19. □

Next, we have Corollary 20.

COROLLARY 20. *Let $G_1 = G_{(\gamma, \gamma, \gamma; \rho_1, \gamma)}$ and $G_2 = G_{(\gamma, \gamma, \gamma; \rho_2, \gamma)}$ be p -groups of class two, each generated by two elements and of order p^n , for an odd prime p , as defined in (2). Suppose that $\frac{\gamma-1}{2} \leq \min\{\rho_1, \rho_2\}$. Then $\mathbb{Q}G_1 \cong \mathbb{Q}G_2$.*

PROOF. It follows directly from the first case of Theorem 19. □

We conclude this section with the following example. This example shows that two non-isomorphic, two-generator p -groups of class 2 can have isomorphic rational group algebras if they satisfy the hypothesis of Corollary 20. However, this need not hold if the hypothesis of Corollary 20 is relaxed.

EXAMPLE 21. Let

$$G_1 = \langle a, b : [a, b]^9 = [a, b, a] = [a, b, b] = 1, a^9 = 1, b^9 = 1 \rangle$$

and

$$G_2 = \langle a, b : [a, b]^9 = [a, b, a] = [a, b, b] = 1, a^9 = [a, b]^3, b^9 = 1 \rangle$$

be two groups. Note that both G_1 and G_2 are two-generator 3-groups of nilpotency class 2 and of order 729, with $G_1 = G_{(2,2,2;2,2)}$ and $G_2 = G_{(2,2,2;1,2)}$ (see (2)). Hence, by the first case of Theorem 19, we obtain

$$\mathbb{Q}G_1 \cong \mathbb{Q}G_2 \cong \mathbb{Q} \oplus 4\mathbb{Q}(\zeta_3) \oplus 12\mathbb{Q}(\zeta_9) \oplus 9M_3(\mathbb{Q}(\zeta_3)) \oplus M_9(\mathbb{Q}(\zeta_9)).$$

Next, let

$$G_3 = \langle a, b : [a, b]^9 = [a, b, a] = [a, b, b] = 1, a^9 = [a, b], b^9 = 1 \rangle$$

be a group. Note that G_3 is a two-generator 3-groups of nilpotency class 2 and of order 729, with $G_3 = G_{(2,2,2;0,2)}$ (see (2)). Hence, by the second case of Theorem 19, we obtain

$$\mathbb{Q}G_3 \cong \mathbb{Q} \oplus 4\mathbb{Q}(\zeta_3) \oplus 12\mathbb{Q}(\zeta_9) \oplus 3M_3(\mathbb{Q}(\zeta_9)) \oplus M_9(\mathbb{Q}(\zeta_9)).$$

Furthermore, G_1 , G_2 and G_3 correspond to SmallGroup(729, 24), SmallGroup(729, 25) and SmallGroup(729, 60), respectively, in the GAP SmallGroups library. The above Wedderburn decomposition can also be verified using the WEDDERGA package in GAP.

5. Nested GVZ p -groups of arbitrarily large nilpotency class

It is known that GVZ p -groups exist with arbitrarily large nilpotency class. In this section, we compute the Wedderburn decomposition of rational group algebras corresponding to two distinct families of nested GVZ p -groups with arbitrarily large nilpotency class. In Subsection 5.1, we determine the Wedderburn decomposition of rational group algebras of a family of nested GVZ p -groups of arbitrarily large nilpotency class and exponent p . In Subsection 5.2, we deal with the Wedderburn decomposition of rational group algebras for a family of nested GVZ p -groups of arbitrarily large nilpotency class and arbitrarily large exponent.

5.1. Nested GVZ p -groups of arbitrary large nilpotency class and exponent p . We begin with a family of nested GVZ p -groups introduced by Nenciu [29]. The members of this family are p -groups of exponent p . For every $n \geq 1$, we construct a group G_n , which is a nested GVZ-group of order p^{2n+1} , exponent p , and nilpotency class $n+1$, where $p > n+1$ is a prime.

Let G_1 be the extra-special p -group of order p^3 and exponent p . We may describe G_1 as

$$G_1 = (\langle b_0 \rangle \times \langle a_1 \rangle) \rtimes \langle b_1 \rangle,$$

with the relation $[b_1, a_1] = b_0$. Assume that G_{n-1} is defined. Now, we construct G_n by setting

$$G_n = (G_{n-1} \times \langle a_n \rangle) \rtimes \langle b_n \rangle,$$

where $a_n^p = b_n^p$, $[b_n, a_1] = b_{n-1}$, $[a_2, b_n] = b_{n-2}, \dots, [a_{n-1}, b_n] = b_1$, $[a_n, b_n] = b_0$, and all remaining commutators are trivial.

In terms of generators and relations, we obtain

$$(3) \quad G_n = \langle a_1, a_2, \dots, a_n, b_0, b_1, \dots, b_n \rangle,$$

where each generator has order p , $[a_i, a_j] = [b_i, b_j] = 1$ for all $0 \leq i < j \leq n$, and

$$[a_i, b_j] = \begin{cases} 1 & \text{if } i > j, \\ b_{j-1} & \text{if } 1 < i \leq j, \\ b_{j-1}^{p-1} & \text{if } 1 = i \leq j. \end{cases}$$

We now recall the following result from [29].

LEMMA 22. [29, Theorem 5.1] *For every $n \geq 1$, the group G_n defined in (3) is a nested GVZ-group with the following properties.*

- (1) $\text{cd}(G_n) = \{1, p, \dots, p^n\}$.
- (2) If $\chi \in \text{Irr}(G_n)$ with $\chi(1) = p^r$ for some $r \in \{0, 1, \dots, n\}$, then
 - (a) $Z(\chi) = \langle a_{r+1}, \dots, a_n, b_0, b_1, \dots, b_{n-r} \rangle$,
 - (b) $\langle b_0, b_1, \dots, b_{n-r-1} \rangle \subseteq \ker(\chi)$.

We are now ready to prove the following theorem, which gives a combinatorial formula for the Wedderburn decomposition of rational group algebras of the family of nested GVZ p -groups described in (3).

THEOREM 23. *Let $G = G_n$ be as defined in (3), a nested GVZ p -group of order p^{2n+1} , exponent p , and class $n+1$, where $p > n+1$ is an odd prime. Then the Wedderburn decomposition of $\mathbb{Q}G$ is given by*

$$\mathbb{Q}G \cong \mathbb{Q} \bigoplus (1 + p + p^2 + \dots + p^n) \mathbb{Q}(\zeta_p) \bigoplus_{r=1}^n p^{n-r} M_{p^r}(\mathbb{Q}(\zeta_p)).$$

PROOF. Let $G = G_n$ be defined as above. Note that $G' = \langle b_0, b_1, \dots, b_{n-1} \rangle$. Hence,

$$G/G' = \langle a_1 G', a_2 G', \dots, a_n G', b_n G' \rangle \cong \underbrace{C_p \times \dots \times C_p}_{n+1 \text{ factors}}.$$

By Theorem 1 and Lemma 7, it follows that

$$\mathbb{Q}(G/G') \cong \mathbb{Q} \bigoplus (1 + p + p^2 + \dots + p^n) \mathbb{Q}(\zeta_p).$$

Moreover, $\text{cd}(G) = \{1, p, \dots, p^n\}$. For $\chi \in \text{nl}(G)$ with $\chi(1) = p^r$ ($1 \leq r \leq n$), set $Z_r = Z(\chi)$. From Lemma 22, we obtain

$$Z_r = \langle a_{r+1}, \dots, a_n, b_0, b_1, \dots, b_{n-r} \rangle,$$

and

$$[Z_r, G] = \langle b_0, b_1, \dots, b_{n-r-1} \rangle.$$

Hence, we have the quotient group

$$Z_r/[Z_r, G] = \langle a_{r+1}[Z_r, G], \dots, a_n[Z_r, G], b_{n-r}[Z_r, G] \rangle \cong \underbrace{C_p \times \dots \times C_p}_{n-r+1 \text{ factors}}.$$

Also, since $[Z_{r-1}, G] = \langle b_0, b_1, \dots, b_{n-r} \rangle$, we obtain the quotient group

$$Z_r/[Z_{r-1}, G] \cong \underbrace{C_p \times \dots \times C_p}_{n-r \text{ factors}}.$$

Thus, by applying Theorem 3 together with Lemma 7, the claimed decomposition follows. This completes the proof of Theorem 23. $\square$

5.2. Nested GVZ p -groups of arbitrary large nilpotency class and arbitrary large exponent. In the previous subsection, we considered a family of nested GVZ p -groups of exponent p . We now turn to another important family, introduced by Lewis [25], which provides nested GVZ p -groups of arbitrarily large nilpotency class and simultaneously arbitrarily large exponent. Let p be an odd prime and n a positive integer. Denote by $C_{p^{n+1}}$ the cyclic group of order p^{n+1} . It is well known that the Sylow p -subgroup of $\text{Aut}(C_{p^{n+1}})$ is cyclic of order p^n . Define

$$G_n = C_{p^{n+1}} \rtimes \text{Syl}_p(\text{Aut}(C_{p^{n+1}})).$$

Then $|G_n| = p^{2n+1}$. Let $C_{p^{n+1}} = \langle x \rangle$ and $\text{Syl}_p(\text{Aut}(C_{p^{n+1}})) = \langle y \rangle$. With these generators, a presentation of G_n is

$$(4) \quad G_n = \langle x, y : x^{p^{n+1}} = y^{p^n} = 1, y^{-1}xy = x^{1+p} \rangle.$$

Here, $G'_n = \langle x^p \rangle \cong C_{p^n}$ and $Z(G_n) = \langle x^{p^n} \rangle \cong C_p$. Moreover, $\text{cd}(G) = \{1, p, p^2, \dots, p^n\}$. Next, we have Lemma 24, which summaries some properties of G_n .

LEMMA 24. [25, Example 3] *For every $n \geq 1$, the group $G = G_n$ defined in (4) is a nested GVZ p -group of order p^{2n+1} with the following properties:*

- (1) G has nilpotency class $n+1$ and exponent p^{n+1} .

$$(2) \text{ cd}(G) = \{1, p, \dots, p^n\}.$$

(3) If $\chi \in \text{Irr}(G)$ with $\chi(1) = p^r$ for some $r \in \{0, 1, \dots, n\}$, then

$$Z(\chi) = \langle x^{p^r}, y^{p^r} \rangle \quad \text{and} \quad [Z(\chi), G] = \langle x^{p^{r+1}} \rangle.$$

We now establish the Wedderburn decomposition of rational group algebras for this family.

THEOREM 25. *Let $G = G_n = \langle x, y : x^{p^{n+1}} = y^{p^n} = 1, y^{-1}xy = x^{1+p} \rangle$ be a nested GVZ p -group of order p^{2n+1} , exponent p^{n+1} , and nilpotency class $n+1$, where p is an odd prime. Then the Wedderburn decomposition of $\mathbb{Q}G$ is given by*

$$\mathbb{Q}G \cong \mathbb{Q} \bigoplus (p+1)\mathbb{Q}(\zeta_p) \bigoplus_{r=2}^n p\mathbb{Q}(\zeta_{p^r}) \bigoplus_{r=1}^{n-1} pM_{p^r}(\mathbb{Q}(\zeta_p)) \bigoplus_{r=1}^{n-2} \bigoplus_{m=2}^{n-r} (p-1)M_{p^r}(\mathbb{Q}(\zeta_{p^m})) \bigoplus M_{p^n}(\mathbb{Q}(\zeta_p)).$$

PROOF. Consider $G = G_n$ as defined in (4). Then $G' = \langle x^p \rangle \cong C_{p^n}$, and hence

$$G/G' = \langle xG', yG' \rangle \cong C_p \times C_{p^n}.$$

Therefore, by Theorem 1 and Lemma 7, we obtain

$$\mathbb{Q}(G/G') \cong \mathbb{Q} \bigoplus (p+1)\mathbb{Q}(\zeta_p) \bigoplus_{r=2}^n p\mathbb{Q}(\zeta_{p^r}).$$

Next, note that $\text{cd}(G) = \{1, p, \dots, p^n\}$. For $\chi \in \text{nl}(G)$ with $\chi(1) = p^r$ ($1 \leq r \leq n$), define $Z_r = Z(\chi)$. Then by Lemma 24, we have

$$Z_r = \langle x^{p^r}, y^{p^r} \rangle \quad \text{and} \quad [Z_{p^r}, G] = \langle x^{p^{r+1}} \rangle.$$

Thus, for $1 \leq r \leq n$, we have the quotient group

$$Z_r/[Z_r, G] = \langle x^{p^r}[Z_r, G], y^{p^r}[Z_r, G] \rangle \cong C_p \times C_{p^{n-r}}.$$

Similarly, we have $[Z_{r-1}, G] = \langle x^{p^r} \rangle$ (see Lemma 24). Thus, the quotient group

$$Z_r/[Z_{r-1}, G] = \langle y^{p^r}[Z_r, G] \rangle \cong C_{p^{n-r}}.$$

Let a_{p^s} and a'_{p^s} denote the number of cyclic subgroups of order p^s in $Z_r/[Z_r, G]$ and $Z_r/[Z_{r-1}, G]$, respectively. For $1 \leq r \leq n-1$ and $1 \leq s \leq n-r$, by Lemma 7, we have

$$a_{p^s} = \begin{cases} p+1 & \text{if } s=1, \\ p & \text{if } 2 \leq s \leq n-r \end{cases} \quad \text{and} \quad a'_{p^s} = 1.$$

Furthermore, for $r = n$, we have $a_p = 1$. Therefore, by applying Theorem 3, we obtain the claimed Wedderburn decomposition of $\mathbb{Q}G$. This completes the proof of Theorem 25. $\square$

REMARK 26. The family of groups described in (4) are split metacyclic p -groups. The decomposition formula for the Wedderburn decomposition of rational group algebras of this family of groups given in Theorem 25 can alternatively be deduced from [11, Theorem 1].

6. Isoclinic nested GVZ p -groups

In this section, we first establish that the property of being a GVZ-group (respectively, a nested GVZ-group) is preserved under *isoclinism*. Furthermore, we classify all nested GVZ p -groups of order at most p^5 . We then explicitly compute the Wedderburn decomposition of the rational group algebras of all nested GVZ p -groups of order $\leq p^5$, where p is an odd prime. These computations, carried out using our main theorem, provide one more concrete illustration of Theorem 3. We begin with the following definition.

DEFINITION 27. *Two finite groups G and H are said to be isoclinic if there exist two isomorphisms $\theta : G/Z(G) \rightarrow H/Z(H)$ and $\phi : G' \rightarrow H'$ such that for all $g_1, g_2 \in G$, if $h_1 \in \theta(g_1Z(G))$ and $h_2 \in \theta(g_2Z(G))$, then*

$$\phi([g_1, g_2]) = [h_1, h_2].$$

The pair (θ, ϕ) is referred to as an *isoclinism* from G onto H . The concept was originally introduced by Hall [16] in the context of classifying p -groups and may be viewed as a generalization of group isomorphism. It is a standard fact that two isoclinic nilpotent groups share the same nilpotency class.

We now proceed to the proof of Theorem 4.

PROOF OF THEOREM 4. Let G and H be two finite isoclinic groups. Suppose that G is a GVZ-group. Since G and H are isoclinic, there exist two isomorphisms $\theta : G/Z(G) \rightarrow H/Z(H)$ and $\phi : G' \rightarrow H'$ such that

$$\phi([g_1, g_2]) = [h_1, h_2]$$

whenever $\theta(g_1 Z(G)) = h_1 Z(H)$ and $\theta(g_2 Z(G)) = h_2 Z(H)$ for $g_1, g_2 \in G$ and $h_1, h_2 \in H$.

Let $h \in H$. Choose $g \in G$ such that $\theta(g Z(G)) = h Z(H)$. Since θ is surjective, we obtain

$$\phi(\{[g, x] : x \in G\}) = \{\phi([g, x]) : x \in G\} = \{[h, y] : y \in H\}.$$

Recall that G is a GVZ-group if and only if $\text{cl}_G(g) = g[g, G]$ for every $g \in G$ (see [9]). In particular,

$$[g, G] = \{[g, x] : x \in G\}.$$

Hence,

$$\phi([g, G]) = \{[h, y] : y \in H\}.$$

This shows that $\{[h, y] : y \in H\}$ is a subgroup of H' , i.e., $[h, H] = \{[h, y] : y \in H\}$. Consequently, $\text{cl}_H(h) = h[h, H]$, and therefore H is also a GVZ-group.

Next, let G is a nested GVZ-group. Let $h_1, h_2 \in H$, and choose $g_1, g_2 \in G$ such that $\theta(g_1 Z(G)) = h_1 Z(H)$ and $\theta(g_2 Z(G)) = h_2 Z(H)$. Since G is a nested GVZ-group, it follows that either $[g_1, G] \leq [g_2, G]$ or $[g_2, G] \leq [g_1, G]$ (see [6]). Without loss of generality, suppose $[g_1, G] \leq [g_2, G]$. Then we have

$$[g_1, G] \leq [g_2, G] \implies \phi([g_1, G]) \leq \phi([g_2, G]) \implies [h_1, H] \leq [h_2, H].$$

Thus, H is also a nested GVZ-group (see [6, Theorem 2]). This completes the proof of Theorem 4. $\square$

It is well known that a non-abelian p -group G of order at most p^4 is a nested GVZ-group if and only if the nilpotency class of G is 2. The groups of order p^5 for an odd prime p were classified by James [20] based on isoclinism, and we adopt the notations and presentations of these groups as given in [20]. This classification yields 10 isoclinism families, denoted by Φ_i for $1 \leq i \leq 10$. We now establish the proof of Corollary 5, which provides a classification of all nested GVZ p -groups of order p^5 . Before proceeding, we recall a fundamental result concerning GVZ-groups.

LEMMA 28. [9, Theorem B] *If G is a GVZ-group, then $c(G) \leq |\text{cd}(G)|$.*

PROOF OF COROLLARY 5. Let G be a non-abelian group of order p^5 , where p is an odd prime. Then $|\text{cd}(G)| = 2$ except when $G \in \Phi_7 \cup \Phi_8 \cup \Phi_{10}$ (see [20, Subsection 4.1]). Note that a nested GVZ p -group G with $|\text{cd}(G)| = 2$ is, in fact, a VZ p -group. It is well known that a non-abelian p -group G of order p^5 is a VZ-group if and only if $G \in \Phi_2 \cup \Phi_5$.

Next, suppose $G \in \Phi_{10}$. In this case, we have $\text{cd}(G) = \{1, p, p^2\}$ and the nilpotency class of G , denoted by $c(G)$, is 4. Thus, $|\text{cd}(G)| < c(G)$. Therefore, by Lemma 28, G is not a GVZ-group.

Finally, suppose $G \in \Phi_7 \cup \Phi_8$. Then we have $\text{cd}(G) = \{1, p, p^2\}$, $Z(G) \cong C_p$, and

$$G/Z(G) \cong \begin{cases} \Phi_2(1^4) & \text{if } G \in \Phi_7, \\ \Phi_2(22) & \text{if } G \in \Phi_8 \end{cases}$$

(see [20, Subsection 4.1]). Note that there is a bijection between the sets $\text{Irr}_p(G)$ and $\text{nl}(G/Z(G))$. Let $\chi \in \text{Irr}_p(G)$ be the lift of $\bar{\chi} \in \text{nl}(G/Z(G))$. Observe that $G/Z(G)$ is a VZ-group, and hence $\bar{\chi} \in \text{nl}(G/Z(G))$ is of central type. This implies that $\chi \in \text{Irr}_p(G)$ is of central type.

Furthermore, the pair $(G, Z(G))$ is a Camina pair (see [33, Lemmas 5.5 and 5.6]). Therefore, there is a bijection between the sets $\text{Irr}_{p^2}(G)$ and $\text{Irr}(Z(G)) \setminus \{1_{Z(G)}\}$, where $1_{Z(G)}$ denotes the trivial character of $Z(G)$. Moreover, for $1_{Z(G)} \neq \mu \in \text{Irr}(Z(G))$, the corresponding $\chi_\mu \in \text{Irr}_{p^2}(G)$ is given by

$$\chi_\mu(g) = \begin{cases} p^2 \mu(g) & \text{if } g \in Z(G), \\ 0 & \text{otherwise.} \end{cases}$$

Thus, $Z(\chi_\mu) = Z(G)$. This implies that

$$|G/Z(\chi_\mu)|^{\frac{1}{2}} = p^2 = \chi_\mu(1).$$

Therefore, $\chi_\mu \in \text{Irr}_{p^2}(G)$ is of central type. Moreover, notice that G is strictly nested by degrees. Hence, $G \in \Phi_7 \cup \Phi_8$ is a nested GVZ-group. This completes the proof of Corollary 5. $\square$

Recall that a nilpotent group of class 2 is a GVZ-group (see [19, Theorem 2.31]). The groups of order p^5 in the isoclinic family Φ_4 have nilpotency class 2 (see [20, Subsection 4.1]). Hence, these groups are GVZ-groups that are not nested GVZ-groups. We now prove Theorem 29, which gives the Wedderburn decomposition of rational group algebras for all nested GVZ p -groups of order at most p^5 .

THEOREM 29. *Let G be a non-abelian nested GVZ p -group of order $\leq p^5$, where p is an odd prime. Then $c(G) \in \{2, 3\}$. Moreover, we have the following.*

- (1) **Case** ($c(G) = 2$). *In this case, let p^m and p^n denote the exponents of G/G' and $Z(G)$, respectively. Then the Wedderburn decomposition of $\mathbb{Q}G$ is given by*

$$\mathbb{Q}G \cong \bigoplus_{\lambda=0}^m a_{p^\lambda} \mathbb{Q}(\zeta_{p^\lambda}) \bigoplus_{\lambda=1}^n (b_{p^\lambda} - c_{p^\lambda}) M_{|G/Z(G)|^{1/2}}(\mathbb{Q}(\zeta_{p^\lambda})),$$

where a_{p^λ} , b_{p^λ} and c_{p^λ} are the number of cyclic subgroups of order p^λ of G/G' , $Z(G)$ and $Z(G)/G'$, respectively.

- (2) **Case** ($c(G) = 3$). *In this case, let p^m denote the exponent of G/G' . Then the Wedderburn decomposition of $\mathbb{Q}G$ is given by*

$$\mathbb{Q}G \cong \bigoplus_{\lambda=0}^m a_{p^\lambda} \mathbb{Q}(\zeta_{p^\lambda}) \bigoplus p M_p(\mathbb{Q}(\zeta_p)) \bigoplus M_{p^2}(\mathbb{Q}(\zeta_p)),$$

where a_{p^λ} is the number of cyclic subgroups of order p^λ of G/G' .

PROOF. Let G be a non-abelian nested GVZ p -group of order at most p^5 , where p is an odd prime. Then $\text{cd}(G) \subseteq \{1, p, p^2\}$. Hence, by Lemma 28, we obtain that $c(G) \in \{2, 3\}$.

- (1) First, observe that every non-abelian nested GVZ p -group of order at most p^5 with nilpotency class 2 is a VZ-group. Therefore, by Theorems 1 and 2, the result follows in this case.
- (2) Now suppose that G is a nested GVZ-group of nilpotency class 3. Then necessarily $|G| = p^5$ and $G \in \Phi_7 \cup \Phi_8$ (see the proof of Corollary 5). For $i \in \{0, 1, 2\}$, let $Z_i = Z(\chi)$ for some $\chi \in \text{Irr}_{p^i}(G)$. In addition, let N be a normal subgroup of G such that

$$\{e\} < Z(G) < N < G$$

is part of the upper central series, where e denotes the identity element of G . Since $G/N = Z(G/N)$, the quotient G/N is abelian, which implies $G' \subseteq N$. Moreover, we have $N/Z(G) = Z(G/Z(G))$. By [20, Subsection 4.1],

$$G/Z(G) \cong \begin{cases} \Phi_2(1^4) & \text{if } G \in \Phi_7, \\ \Phi_2(22) & \text{if } G \in \Phi_8. \end{cases}$$

There is a natural bijection between $\text{Irr}_p(G)$ and $\text{nl}(G/Z(G))$. Since $G/Z(G)$ is a VZ-group, it follows that $Z_1 = N$. Next, $Z_2 = Z(G)$ as $\text{cd}(G) = \{1, p, p^2\}$. Finally, $Z_0 = G$.

From $Z(G/Z(G)) = N/Z(G)$, we deduce that $[N, G] = Z(G)$. Therefore,

$$[Z_0, G] = G', \quad [Z_1, G] = Z(G), \quad [Z_3, G] = \{e\}.$$

Next, note that

$$Z_1/[Z_1, G] = N/Z(G) = Z(G/Z(G)) \cong C_p \times C_p$$

(see [20, Subsections 4.1 and 4.5]). Furthermore, $|N| = p^3$ and $|G'| = p^2$. Hence, we get

$$Z_1/[Z_0, G] = N/G' \cong C_p.$$

Thus, by Theorems 1 and 3, the result follows.

This completes the proof of Theorem 29. $\square$

7. Primitive central idempotents

In this section, we begin by recalling some basic notions concerning primitive central idempotents in rational group algebras, and then proceed to the proof of Theorem 6.

Let G be a finite group. An element $e \in \mathbb{Q}G$ is called an *idempotent* if $e^2 = e$. A *primitive central idempotent* in $\mathbb{Q}G$ is an idempotent that lies in the center of $\mathbb{Q}G$ and cannot be decomposed into a sum of two nonzero orthogonal idempotents, i.e., there do not exist $e', e'' \in \mathbb{Q}G$ such that $e = e' + e''$ and $e'e'' = 0$.

It is a standard fact that the collection of primitive central idempotents of $\mathbb{Q}G$ determines the Wedderburn decomposition of $\mathbb{Q}G$ into simple components. More precisely, if e is a primitive central idempotent of $\mathbb{Q}G$, then $\mathbb{Q}Ge$ is a simple algebra. For $\chi \in \text{Irr}(G)$, the element

$$e(\chi) := \frac{\chi(1)}{|G|} \sum_{g \in G} \chi(g)g^{-1}$$

is a primitive central idempotent of $\mathbb{C}G$, and the set $\{e(\chi) : \chi \in \text{Irr}(G)\}$ forms a complete set of primitive central idempotents of $\mathbb{C}G$. Moreover, for $\chi \in \text{Irr}(G)$ one defines

$$e_{\mathbb{Q}}(\chi) := \sum_{\sigma \in \text{Gal}(\mathbb{Q}(\chi)/\mathbb{Q})} e(\chi^{\sigma}),$$

which gives a primitive central idempotent of $\mathbb{Q}G$.

For a subset $X \subseteq G$, we write

$$\hat{X} := \frac{1}{|X|} \sum_{x \in X} x \in \mathbb{Q}G.$$

If $N \trianglelefteq G$, define

$$\epsilon(G, N) := \begin{cases} \hat{G} & \text{if } N = G, \\ \prod_{D/N \in M(G/N)} (\hat{N} - \hat{D}) & \text{otherwise,} \end{cases}$$

where $M(G/N)$ denotes the set of minimal nontrivial normal subgroups D/N of G/N , with D a subgroup of G containing N .

In what follows, we determine a full set of primitive central idempotents of the rational group algebra of a nested GVZ p -group, where p is an odd prime. We begin with a general lemma.

LEMMA 30. [24, Lemma 3.3.2] *Let G be a finite group and $\chi \in \text{lin}(G)$ with kernel $N = \ker(\chi)$. Then*

- (1) $e_{\mathbb{Q}}(\chi) = \epsilon(G, N)$;
- (2) $\mathbb{Q}G\epsilon(G, N) \cong \mathbb{Q}(\zeta_{|G/N|})$.

We are now ready to establish Theorem 6.

PROOF OF THEOREM 6. Let G be a finite nested GVZ p -group, with p an odd prime. Suppose $\text{cd}(G) = \{p^{\delta_i} : 0 \leq i \leq n, 0 = \delta_0 < \delta_1 < \dots < \delta_n\}$, and define $Z_{\delta_i} := Z(\chi)$ for some $\chi \in \text{Irr}_{p^{\delta_i}}(G)$.

Take $\chi \in \text{nl}(G)$ with $\chi(1) = p^{\delta_r}$ for some $r \in \{1, 2, \dots, n\}$. By Lemma 11, there exists $\mu \in \text{lin}(Z_{\delta_r})$ such that

$$\chi = \chi_{\mu}(g) = \begin{cases} p^{\delta_r} \mu(g) & \text{if } g \in Z(\chi); \\ 0 & \text{otherwise,} \end{cases}$$

where μ is the lift of $\bar{\mu} \in \text{Irr}(Z_{\delta_r}/[Z_{\delta_r}, G] \mid [Z_{\delta_{r-1}}, G]/[Z_{\delta_r}, G])$ to Z_{δ_r} .

Let $N = \ker(\chi)$. Then $N = \ker(\chi_\mu) = \ker(\mu)$. Since $e(\chi) = e(\chi_\mu) = e(\mu)$, we compute

$$\begin{aligned}
 e_{\mathbb{Q}}(\chi) &= e_{\mathbb{Q}}(\chi_\mu) \\
 &= \sum_{\sigma \in \text{Gal}(\mathbb{Q}(\chi_\mu)/\mathbb{Q})} e(\chi_\mu^\sigma) \\
 &= \sum_{\sigma \in \text{Gal}(\mathbb{Q}(\mu)/\mathbb{Q})} e(\chi_{\mu^\sigma}) \\
 &= \sum_{\sigma \in \text{Gal}(\mathbb{Q}(\mu)/\mathbb{Q})} e(\mu^\sigma) \\
 &= e_{\mathbb{Q}}(\mu) \\
 &= \epsilon(Z(\chi), N),
 \end{aligned}$$

using Lemma 30.

It is known that G admits an irreducible character χ with $\chi(1) = |G/Z(\chi)|^{1/2}$ if and only if $\chi(g) = 0$ for all $g \in G \setminus Z(\chi)$ (see [19, Corollary 2.30]). Thus, $\chi(1) = \chi_\mu(1) = p^{\delta_r} = |G/Z(\chi)|^{1/2}$. By Theorem 3, the simple component of $\mathbb{Q}G$ corresponding to $e_{\mathbb{Q}}(\chi)$ is

$$\mathbb{Q}Ge_{\mathbb{Q}}(\chi) = \mathbb{Q}G\epsilon(Z(\chi), N) \cong M_{|G/Z(\chi)|^{1/2}}(\mathbb{Q}(\zeta_{|Z(\chi)/N|})).$$

This completes the proof of Theorem 6. $\square$

References

- [1] A. Ahmad, A. Magidin Viso and R. F. Morse, Two generator p -groups of nilpotency class 2 and their conjugacy classes, *Publ. Math. Debrecen* **81**(1-2) (2012), 145–166.
- [2] R. G. Ayoub and C. Ayoub, On the group rings of a finite abelian group, *Bull. Aust. Math. Soc.* **1** (1969) 245–261.
- [3] G. K. Bakshi and S. Maheshwary, The rational group algebra of a normally monomial group, *J. Pure Appl. Algebra* **218**(9) (2014), 1583–1593.
- [4] G. K. Bakshi, J. Garg and G. Olteanu, Rational group algebras of generalized strongly monomial groups: primitive idempotents and units, *Math. Comp.* **93**(350) (2024), 3027–3058.
- [5] Y. Berkovich, *Groups of prime power order Vol. 1* (Walter de Gruyter, Berlin, 2008).
- [6] S. T. Burkett and M. L. Lewis, A characterization of nested groups in terms of conjugacy classes, *C. R. Math. Acad. Sci. Paris* **358**(1) (2020), 109–112.
- [7] S. T. Burkett and M. L. Lewis, Groups where the centers of the irreducible characters form a chain II, *Monatsh. Math.* **192**(4) (2020), 783–812.
- [8] S. T. Burkett and M. L. Lewis, Partial GVZ-groups, *J. Algebra* **581** (2021), 50–62.
- [9] S. T. Burkett and M. L. Lewis, GVZ-groups, flat groups, and CM-groups, *C. R. Math. Acad. Sci. Paris* **359** (2021), 355–361.
- [10] R. K. Choudhary and S. K. Prajapati, Rational representations and rational group algebra of VZ p -groups, *J. Aust. Math. Soc.* **118**(1) (2025), 1–30.
- [11] R. K. Choudhary and S. K. Prajapati, A combinatorial formula for the Wedderburn decomposition of rational group algebras of split metacyclic p -groups, *J. Algebra Appl.* (2026) 2650068 (17 pages). doi.org/10.1142/S0219498826500684
- [12] F. DeMeyer and G. J. Janusz, Finite groups with an irreducible representation of large degree, *Math. Z.* **108** (1969), 145–153.
- [13] A. Espuelas, On certain groups of central type, *Proc. Amer. Math. Soc.* **97**(1) (1986), 16–18.
- [14] G. A. Fernández-Alcober and A. Moretó, Groups with two extreme character degrees and their normal subgroups, *Trans. Amer. Math. Soc.* **353**(6) (2001), 2171–2192.
- [15] S. M. Gagola Jr., Characters fully ramified over a normal subgroup, *Pacific J. Math.* **55** (1974), 107–126.
- [16] P. Hall, The classification of prime-power groups, *J. Reine Angew. Math.* **182** (1940), 130–141.
- [17] A. Herman, On the automorphism group of rational group algebras of metacyclic groups, *Comm. Algebra* **25**(7) (1997), 2085–2097.
- [18] R. B. Howlett and I. M. Isaacs, On groups of central type, *Math. Z.* **179**(4) (1982), 555–569.
- [19] I. M. Isaacs, *Character Theory of Finite Groups* (Dover Publications Inc., New York, 1994).
- [20] R. James, The groups of order p^6 (p an odd prime), *Math. Comp.* **34**(150) (1980), 613–637.
- [21] E. Jespers and Á. del Río, A structure theorem for the unit group of the integral group ring of some finite groups, *J. Reine Angew. Math.* **521** (2000), 99–117.
- [22] E. Jespers, G. Leal and A. Paques, Central idempotents in rational group algebras of finite nilpotent groups, *J. Algebra Appl.* **2**(1) (2003), 57–62.

- [23] E. Jespers, G. Olteanu and Á. del Río, Rational group algebras of finite groups: from idempotents to units of integral group rings, *Algebr. Represent. Theory* **15**(2) (2012), 359–377.
- [24] E. Jespers and Á. del Río, *Group Ring Groups, Volume 1: Orders and Generic Constructions of Units* (De Gruyter, Berlin, 2016).
- [25] M. L. Lewis, Groups where the centers of the irreducible characters form a chain, *Monatsh. Math.* **192**(2) (2020), 371–399.
- [26] M. Murai, Characterizations of p -nilpotent groups, *Osaka J. Math.* **31**(1) (1994), 1–8.
- [27] A. Nenciu, Isomorphic character tables of nested GVZ -groups, *J. Algebra Appl.* **11**(2) (2012), 1250033, 12 pp.
- [28] A. Nenciu, Character tables of 2-generator p -groups of class two, *Comm. Algebra* **41**(7) (2013), 2598–2613.
- [29] A. Nenciu, Nested GVZ -groups, *J. Group Theory* **19**(4) (2016), 693–704.
- [30] G. Olteanu, Computing the Wedderburn decomposition of group algebras by the Brauer-Witt theorem, *Math. Comp.* **76**(258) (2007), 1073–1087.
- [31] T. Ono, A note on the Artin map, *Proc. Japan Acad. Ser. A Math. Sci.* **65**(8) (1989), 304–306.
- [32] S. Perlis and G. L. Walker, Abelian group algebras of finite order, *Trans. Amer. Math. Soc.*, **68**(3) (1950), 420–426.
- [33] S. K. Prajapati, M. R. Darafsheh and M. Ghorbany, Irreducible characters of p -group of order $\leq p^5$, *Algebr. Represent. Theory* **20**(5) (2017), 1289–1303.
- [34] I. Reiner, The Schur index in the theory of group representations, *Michigan Math. J.* **8** (1961), 39–47.
- [35] J. Ritter and S. K. Sehgal, Construction of units in integral group rings of finite nilpotent groups, *Trans. Amer. Math. Soc.* **324**(2) (1991), 603–621.
- [36] M. Tărnăuceanu, An arithmetic method of counting the subgroups of a finite abelian group, *Bull. Math. Soc. Sci. Math. Roumanie (N.S.)* **53**(101)(4) (2010), 373–386.
- [37] T. Yamada, *The Schur Subgroup of the Brauer Group, Lecture Notes in Mathematics*, **397** (Springer Verlag, Berlin, 1974).
- [38] Y. Yeh, On prime power Abelian groups, *Bull. Amer. Math. Soc.* **54** (1948), 323–327.
- [39] The GAP Group, *GAP— Groups, Algorithms, and Programming*, Version 4.13.0, 2024, <https://www.gap-system.org>

INDIAN INSTITUTE OF TECHNOLOGY, BHUBANESWAR, ARUGUL CAMPUS, JATNI, KHURDA-752050, INDIA.
Email address: ramkchoudhary1997@gmail.com

INDIAN INSTITUTE OF TECHNOLOGY, BHUBANESWAR, ARUGUL CAMPUS, JATNI, KHURDA-752050, INDIA.
Email address: skprajapati@iitbbs.ac.in