ALMOST SYMMETRIC SUBMANIFOLDS

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Abstract. We introduce the class of almost symmetric submanifolds of Euclidean space, a close relative of symmetric submanifolds and (contact) sub-Riemannian symmetric spaces. More specifically:

- Homogeneous classification: We prove that every full irreducible almost symmetric submanifold of Euclidean space is either: a most singular orbit of an s-representation; or an almost singular orbit, which can be realized as a holonomy tube over a symmetric submanifold; or a codimension 3 submanifold. We include tables of all examples with Lie-theoretic data.
- Inhomogeneous structure: We prove that any inhomogeneous almost symmetric submanifold has cohomogeneity one and describe possible structures, including multiply-warped products.
- Connection with sub-Riemannian geometry: We interpret almost symmetric submanifolds as embeddings of sub-Riemannian symmetric spaces, highlighting the interplay between extrinsic and intrinsic symmetry.
- New invariant: We propose the co-index of extrinsic symmetry as a potential tool to study and hierarchize highly symmetric submanifolds.

1. Introduction

A submanifold of a Euclidean space is called *(extrinsically) symmetric* if it is reflectionally symmetric with respect to any affine normal space. Symmetric submanifolds were introduced and classified by Ferus [Fer80], who locally characterized them by the condition that the second fundamental form be parallel (see also [Str79, ET20]). In particular Ferus showed that symmetric submanifolds are homogeneous and arise as special orbits of the isotropy representation of a symmetric space, or *s-representation*, for short. Symmetric submanifolds split extrinsically, and the irreducible ones are classified by the work of Kobayashi and Nagano [KN64]. Symmetric submanifolds enjoy a rich interplay between extrinsic geometry and representation theory, as the geometry is closely tied to the structure of the isotropy representation.

The goal of this paper is to introduce a related class of Euclidean submanifolds and obtain similar structure and classification results. An *(extrinsic) almost symmetry* of a Euclidean submanifold M at a point p in M is an involutive ambient isometry that preserves M, fixes the affine normal space at p pointwise, and whose fixed point set in the tangent space has dimension one. A submanifold will be called *(extrinsically) almost symmetric* if it admits an almost symmetry at every point. This notion interpolates between fully symmetric submanifolds and certain contact sub-Riemannian symmetric spaces (see below).

Almost symmetric submanifolds are interesting because, despite relaxing full symmetry, they retain strong rigidity and homogeneity properties in many cases, and in general have at most cohomogeneity one. In particular, any homogeneous almost symmetric submanifold can be viewed as a compatible embedding of a sub-Riemannian symmetric space into Euclidean space, where the restriction of the almost symmetry gives a sub-symmetry in the sense of Strichartz [Str79]. Sub-Riemannian symmetric spaces have been classified in low dimensions [Str79, FG96, Alm99] and for

2020 *Mathematics Subject Classification.* 53C40, 53C42, 53C35.

Key words and phrases. Submanifolds with symmetries; s-representations; cohomogeneity one metrics; sub-Riemannian symmetric spaces.

The first named author acknowledges partial support from FAPESP Thematic Grant 22/16097-2 and CNPq Fellowship 304252/2021-2, Brazil. The second named author acknowledges partial support by FaMAF, Universidad Nacional de Córdoba and CIEM-CONICET, Argentina.

contact structures [FG95, BFG99]. Our work shows that most homogeneous examples of almost symmetric submanifolds arise from embeddings of contact sub-Riemannian symmetric spaces.

Recall that a *sub-Riemannian manifold* is a smooth manifold M carrying a distribution $\mathcal{D}$ (a subbundle of the tangent bundle TM) and a smooth Riemannian metric g defined only on vectors in $\mathcal{D}$. A sub-Riemannian symmetric space is a Sub-Riemannian manifold that admits an involutive isometry at each point, called a sub-symmetry, that preserves the $\mathcal{D}$ and whose differential is minus the identity on $\mathcal{D}$ [Str79]. When $\mathcal{D}$ is contact, homogeneity follows from the existence of sub-symmetries [FG95]. Homogeneous almost symmetric submanifolds provide concrete realizations of certain contact sub-Riemannian symmetric spaces inside Euclidean space.

More generally, one can define the *co-index of extrinsic symmetry* of M at p as the smallest integer k such that there is an involutive ambient isometry preserving M, fixing the affine normal space at p pointwise, and whose fixed point set in the tangent space has dimension k. An interesting problem is to hierarchize submanifolds in terms of this invariant.

The first author wishes to thank Wolfgang Ziller for useful discussions.

2. Main results

Next we state the main results in this paper. We first enunciate structure and classification theorems in the homogeneous case. Throughout the paper homogeneous submanifold always stands for extrinsically homogeneous submanifold (see section 3 for unexplained notation and terminology).

Theorem 2.1. *Let M be a full irreducible compact homogeneous submanifold of Euclidean space of codimension at least two. Assume that M is an almost symmetric submanifold. Then:*

(a) *If the codimension of M is greater that three, then M is an orbit of an irreducible s-representation.*

(b) *If M is an orbit of an s-representation, then M is either a most singular orbit or an almost most singular orbit.*

(c) *If M is an almost most singular orbit of an s-representation, then it is a partial holonomy tube $(\bar{M})_\xi$ of an symmetric submanifold $\bar{M}$. Moreover, the normal vector ξ belongs to a 2-dimensional irreducible factor of the normal holonomy of $\bar{M}$. Conversely, any holonomy tube over a symmetric submanifold through a sufficiently small vector in a two dimensional factor of the normal holonomy is an almost symmetric submanifold.*

(d) *If M is not an almost most singular orbit of an s-representation, then there exists a unique almost symmetry σ_p at p, for all $p \in M$. In particular σ_p is invariant under the isotropy group at p. Further, there are examples where the almost symmetry is not unique.*

Theorem 2.2. (a) *Every full irreducible almost symmetric orbit $M = Ka$ of the s-representation associated to a symmetric pair $(\mathfrak{g}, \mathfrak{k})$ is listed in the following table. Each M is a circle bundle over M/S^1.*

Nr.	M/S^1	$\mathfrak{g}$	$\mathfrak{k}$	$\mathfrak{k}_a$	*Conditions*
1	$\mathbb{C}P^{q-1}$	$\mathfrak{sp}(q)$	$\mathfrak{u}(q)$	$\mathfrak{u}(q-1)$	$q \geq 2$
2	$S^2 \times S^2$	$\mathfrak{g}_2$	$\mathfrak{so}(4)$	$\mathfrak{so}(2)$	
3	$Sp(3)/U(3) \times S^2$	$\mathfrak{f}_4$	$\mathfrak{sp}(3) \oplus \mathfrak{sp}(1)$	$\mathfrak{su}(3) \oplus \mathfrak{u}(1)$	
4	$Sp(4)/U(4)$	$\mathfrak{e}_6$	$\mathfrak{sp}(4)$	$\mathfrak{su}(4)$	
5	$SU(8)/S(U(4) \times U(4))$	$\mathfrak{e}_7$	$\mathfrak{su}(8)$	$\mathfrak{su}(4) \oplus \mathfrak{su}(4)$	
6	$SO(16)/U(8)$	$\mathfrak{e}_8$	$\mathfrak{so}(16)$	$\mathfrak{su}(8)$	
7	$G_2(\mathbb{R}^{q+1})$	$\mathfrak{su}(q+1)$	$\mathfrak{so}(q+1)$	$\mathfrak{so}(q-1)$	$q \geq 2$
8	$S^2 \times S^1$	$\mathfrak{so}(5)$	$\mathfrak{so}(3) \oplus \mathfrak{so}(2)$	$\{0\}$	
9	$\mathbb{C}P^{p-1} \times \mathbb{C}P^{q-1}$	$\mathfrak{su}(p+q)$	$\mathfrak{s}(\mathfrak{u}(p) \oplus \mathfrak{u}(q))$	$\mathfrak{s}(\mathfrak{u}(p-1) \oplus \mathfrak{u}(q-1) \oplus \mathfrak{u}(1))$	$p \geq q \geq 2$
10	$E_6/\{[Spin(10) \times U(1)]/\mathbb{Z}_4\}$	$\mathfrak{e}_7$	$\mathfrak{e}_6 \oplus \mathfrak{u}(1)$	$\mathfrak{so}(10) \oplus \mathfrak{u}(1)$	
11	$Spin(10)/U(5)$	$\mathfrak{e}_6$	$\mathfrak{so}(10) \oplus \mathfrak{u}(1)$	$\mathfrak{su}(5) \oplus \mathfrak{u}(1)$	
12	$G_2(\mathbb{R}^p) \times G_2(\mathbb{R}^q)$	$\mathfrak{so}(p+q)$	$\mathfrak{so}(p) \oplus \mathfrak{so}(q)$	$\mathfrak{so}(p-2) \oplus \mathfrak{so}(q-2) \oplus \mathfrak{so}(2)$	$p \geq q \geq 2$
13	$G_3(\mathbb{C}^6) \times \mathbb{C}P^1$	$\mathfrak{e}_6$	$\mathfrak{su}(6) \oplus \mathfrak{su}(2)$	$\mathfrak{s}(\mathfrak{u}(3) \oplus \mathfrak{u}(3))$	
14	$SO(12)/U(6) \times \mathbb{C}P^1$	$\mathfrak{e}_7$	$\mathfrak{spin}(12) \oplus \mathfrak{su}(2)$	$\mathfrak{u}(6)$	
15	$E_7/\{[E_6 \times U(1)]/\mathbb{Z}_3\} \times \mathbb{C}P^1$	$\mathfrak{e}_8$	$\mathfrak{e}_7 \oplus \mathfrak{su}(2)$	$\mathfrak{e}_6 \oplus \mathfrak{u}(1)$	

(b) *Every full irreducible compact almost symmetric homogeneous submanifold M of Euclidean space which is not an orbit of an s-representation is an arbitrary principal orbit of ρ, where*

ρ is one of the following three reducible representations of cohomogeneity three:

Nr.	*M*	*ρ*	*Conditions*
16	$V_2(\mathbb{R}^n)$	$(SO(n), \mathbb{R}^n \oplus \mathbb{R}^n)$	$n \geq 3$
17	$U(2)$	$(U(2), \mathbb{C}^2 \oplus \mathbb{R}^3)$	–
18	$T^2 \times S^3$	$(U(1) \times SU(2) \times U(1), \mathbb{C}^2 \oplus \mathbb{C}^2)$	–

We make a few comments about the above tables. Examples Nr. 7 and 12 provide different embeddings of the Stiefel manifold $V_2(\mathbb{R}^n)$ into Euclidean space as an almost symmetric submanifold.

All examples other than Nr. 8 are intrinsically contact sub-Riemannian symmetric spaces: the almost symmetry restricts to a sub-symmetry, and it passes to the circle quotient defining a structure of symmetric space in the quotient; further there is a contact CR structure on M that defines a Kähler structure on the quotient, which then becomes a compact Hermitian symmetric space. Example Nr. 8 is intrinsically an odd-contact sub-Riemannian symmetric space of dimension 4 (cf. [FG96]).

Finally, we consider the inhomogeneous case and prove that inhomogeneous almost symmetric submanifolds must be of cohomogeneity one, already in the intrinsic case (see section 9 for the definition).

Theorem 2.3. *Let M be a complete almost symmetric space, denote by K the closure of the group of isometries of M generated by the almost symmetries of M, and assume that K is not transitive on M. Then:*

- *(a) K acts on M with cohomogeneity 1.*
- *(b) In the open and dense subset Ω of M consisting of regular points of the K-action, the almost symmetry at a point is unique.*
- *(c) For a K^0-orbit N of codimension $k \geq 2$, there is a connected normal subgroup H of K^0 which acts trivially on N and whose isotropy representation at any $p \in N$ on $T_pM = T_pN \oplus \nu_pN$ is $\mathrm{Id} \times SO(\nu_pN)$. In particular, the isotropy algebra $\mathfrak{k}_p$ contains an ideal isomorphic to $\mathfrak{so}_k$.*
- *(d) The connected components of the singular K-orbits are totally geodesic.*
- *(e) The connected components of the K-orbits are (intrinsic) symmetric spaces.*

Theorem 2.4. *Let M be an inhomogeneous simply-connected complete almost symmetric space. Then M falls into one of the following classes:*

- *(a) M is a multiply warped product $\mathbb{R} \times_f N$, where N is a simply-connected symmetric space, and each component of the warping map $f(t)$, $t \in \mathbb{R}$, scales independently each irreducible factor of N.*
- *(b) M is a Riemannian product $\mathbb{R}^k \times N$, where N is a simply-connected symmetric space, and $\mathbb{R}^k$ carries a $SO(k)$-invariant metric.*
- *(c) M is a Riemannian product $S^k \times N$, where N is a simply-connected symmetric space, and S^k carries a $SO(r) \times SO(k-r)$-invariant metric.*

It is easy to see that the manifolds in Theorem 2.4 can be embedded as into Euclidean space as almost symmetric submanifolds if and only if each irreducible factor of N is an extrinsic symmetric space.

It is also clear that the Riemannian universal covering of an almost symmetric space is as well almost symmetric, but in principle it could be homogeneous even if the base manifold is of cohomogeneity one. So the classification of non-simply connected inhomogeneous complete almost symmetric spaces is a delicate problem in the intrinsic case. On the other hand, in the extrinsic case, it is not difficult to obtain a complete classification of inhomogeneous complete almost symmetric submanifolds of Euclidean space, but we leave this result to a forthcoming paper so as not to make this one much longer.

3. Preliminaries

Throughout this article, a homogeneous submanifold stands for an extrinsically homogeneous submanifold.

3.1. **Canonical connections on the normal bundle.** Let K be a not necessarily connected closed subgroup of $O(n)$, and let $M = K^0 \cdot p$, where $0 \neq p \in \mathbb{R}^n$, and $()^0$ denotes the identity component. We assume that $KM = M$. Otherwise, we replace K by the group $\{k \in K : kM = M\}$ which has the same identity component K^0. Assume that M is full and irreducible as a submanifold of $\mathbb{R}^n$; this is always the case if K^0 acts irreducibly (but there exist full and irreducible orbits of reducible representations). Further assume that M is not the sphere of radius $\|p\|$. Let us consider an $\mathrm{Ad}(K_v)$-invariant positive definite inner product b on $\mathfrak{k} := \mathrm{Lie}(K)$, and let $\mathfrak{m} := \mathfrak{k}^\perp$. Then $\mathfrak{m}$ induces a canonical connection on the principal fiber bundle

$$0 \to K_p \to K \to K/K_p = M \to 0.$$

Observe that neither K_p nor K are in general connected, but M is so. Consider the slice representation ρ of K_p on the normal space $\nu_p M$, and the associated metric vector bundle $K \underset{\rho}{\times} \nu_p M$ with the induced canonical connection $\tilde{\nabla}^\perp$, which is K-invariant.

We identify $\nu M \simeq K \times_\rho \nu_v M$ in the natural way: $[g, \xi] \simeq \mathrm{d}g_p(\xi)$, where $g \in K$, $\xi \in \nu_p M$. Such a canonical normal connection $\tilde{\nabla}^\perp$ has the property that given a piecewise differentiable curve $c : [0,1] \to M$, there exists an element of $g \in K$, non-unique if $\ker \rho$ is non-trivial, with $g(c(0)) = c(1)$ and such that $\mathrm{d}g_{c(0)}$ coincides with the $\tilde{\nabla}^\perp$-parallel transport $\tilde{\tau}_c^\perp$ along c. In fact, if $\tilde{c}$ is a horizontal lift of c to K^0, then

$$\tilde{\tau}_c^\perp : \nu_{c(0)} M \simeq [\tilde{c}(0), \nu_v M] \to \nu_{c(1)} M \simeq [\tilde{c}(1), \nu_v M]$$

is given by

$$\tilde{\tau}_c^\perp([\tilde{c}(0), \xi]) = [\tilde{c}(1), \xi] = \tilde{c}(1)\tilde{c}(0)^{-1}[\tilde{c}(0), \xi] = g[\tilde{c}(0), \xi].$$

Observe that, in our construction, $g = \tilde{c}(1)\tilde{c}(0)^{-1}$ belongs to K^0.

Remark 3.1. Assume that K acts on $\mathbb{R}^n$ as an irreducible s-representation, and that the orbit $M = K \cdot p$, $p \neq 0$, is not a sphere. Then $K = \bar{K} := \{g \in SO(n) : gM = M\}^0$. In fact, let R the Riemannian curvature tensor associated with the s-representation. Since $K \subset \bar{K}$, then $[\mathbb{R}^n, R, K]$ and $[\mathbb{R}^n, R, \hat{K}]$ are irreducible and non-transitive holonomy systems. Then, by Simons' holonomy theorem [Sim62], $K = \bar{K}$.

We denote by $\nabla^\perp$ the normal connection on νM induced by the connection of the ambient Euclidean space, by $D^\perp$ the mixed K-invariant tensor

$$D^\perp := \nabla^\perp - \tilde{\nabla}^\perp,$$

and by $\mathcal{D}^K$ the K-invariant distribution on M defined by

$$\mathcal{D}^K_q = \{z \in T_q M : D^\perp_z = 0\}$$

for $q \in M$. The distribution $\mathcal{D}^K$ depends on K and on b. In fact, if we replace K by K', where $K \cdot v = K' \cdot v$, then $\mathcal{D}^K$ and $\mathcal{D}^{K'}$ are not in general equal. For instance, the isotropy representation of the oriented Grassmannian $SO(2n)/SO(n) \times SO(n)$ has an extrinsic symmetric orbit $M = SO(n) \times SO(n) \cdot v = SO(n) \times SO(n)/\mathrm{diag}(SO(n))$. In this case the associated canonical normal connection coincides with $\nabla^\perp$, and thus $\mathcal{D}^{SO(n)\times SO(n)} = TM$. But $M = SO(n) \cdot v$ is also a principal orbit of $SO(n)$. In this case the canonical normal connection associated to $SO(n)$ is flat and hence different from $\nabla^\perp$. Thus, $\mathcal{D}^{SO(n)} \neq TM$.

Let ∇^c be the canonical tangent connection on M induced by the canonical connection $\mathcal{H}$ on the associate bundle $K \times_\chi T_v M \simeq TM$, where χ is the isotropy representation of K_v. The connection $\nabla^c \oplus \tilde{\nabla}^\perp$, on the bundle $TM \oplus \nu M$, has the property that any K^0-invariant mixed tensor, in particular the second fundamental form α, is $\nabla^c \oplus \tilde{\nabla}^\perp$-parallel. Then the main theorem in [OS91], and Remark 3.1 imply the following result:

Theorem 3.2. *Let $M = K^0 \cdot p$ be an irreducible and full homogeneous submanifold of $\mathbb{R}^N$, where K is a compact subgroup of $O(N)$, with $KM = M$. Assume, keeping the notation of this section, that $\mathcal{D}^K = TM$ (equivalently, $\tilde{\nabla}^\perp = \nabla^\perp$). Then there exists a compact subgroup $\hat{K} \subset SO(N)$, acting as an irreducible s-representation, such that $M = K^0 \cdot p = \hat{K} \cdot p$. Moreover, $K^0 \subset \hat{K}$ and $\mathcal{D}^{\hat{K}} = TM$.*

3.2. **Rank rigidity and holonomy tubes.** The *rank* of a homogeneous submanifold M of Euclidean space is the maximal number of linearly independent parallel normal vector fields. Equivalently, the rank is the dimension of the flat factor of the restricted normal holonomy group. If M is contained in a sphere, then the position vector provides a umbilical parallel normal vector field.

The following result can be found in [BCO03, Corollary 5.1.8]:

Theorem 3.3 ([Olm93, Olm94])**.** *Let M^n, $n \geq 2$, be a homogeneous full irreducible submanifold of $\mathbb{R}^N$ with* $\operatorname{rank}(M) \geq 1$. *Then M is contained in a sphere. In addition, if* $\operatorname{rank}(M) \geq 2$, *then M is an orbit of an s-representation.*

Next recall the following theorem (it holds without the compactness assumption).

Theorem 3.4 ([Olm94], Theorem C)**.** *Let $M^n = K \cdot p$, $n \geq 2$, be a full irreducible compact homogeneous submanifold of $\mathbb{R}^N$ and let $k \in K$, $q \in M$. Then there exists a piecewise differentiable curve $c : [0,1] \to M$ with $c(0) = p$, $c(1) = kp$ and such that $\mathrm{d}k|_{\nu_p M} = \tau_c^{\perp}$, where $\tau^{\perp}$ denotes the $\nabla^{\perp}$-parallel transport.*

We now turn to the definition of a holonomy tube (see [BCO03, sec. 3.4.3]). If N is an Euclidean submanifold, $p \in N$ and and $\eta_p \in \nu_p M$, the holonomy tube $(N)_{\eta_p}$ at p is defined by

$$(N)_{\eta_p} := \{\gamma(1) + \tau_\gamma^{\perp}(\eta_p)\},$$

where $\gamma : [0,1] \to N$ runs through all piecewise differentiable curves with $\gamma(0) = p$. If $\|\eta_p\|$ is less than the focal distance, then $(N)_{\eta_p}$ is a submanifold and $\dim(N)_{\eta_p} = \dim N + \dim(\Phi(p) \cdot \eta_p)$. If the connected component $\Phi^*(p)$ of $\Phi(p)$, the so-called restricted normal holonomy group, does not fix η_p, then $N = ((N)_{\eta_p})_{\bar\eta}$ is a parallel focal manifold of $(N)_{\eta_p}$, where $\bar\eta$ is the non-umbilical parallel normal vector field of $(N)_{\eta_p}$ with $\bar\eta(p + \eta_p) = -\eta_p$.

Observe that Theorem 3.4 implies that $\rho(K_p) \subset \Phi(p)$, where ρ is the slice representation and $\Phi(p)$ is the normal holonomy group of M at p. Therefore:

Corollary 3.5. *Let $M^n = K \cdot p$, $n \geq 2$, be a full irreducible compact homogeneous submanifold of $\mathbb{R}^N$ and let $\eta_p \in \nu_p M$. Then $K \cdot (M)_{\eta_p} = (M)_{\eta_p}$.*

Corollary 3.6. *Let $M^n = K \cdot p$, $n \geq 2$, be a full irreducible compact homogeneous submanifold of $\mathbb{R}^N$ and let $\eta_p \in \nu_p M$ be sufficiently small so that the parallel submanifold $(M)_{\eta_p}$ is an embedded submanifold. Assume that K acts transitively on $(M)_{\eta_p}$ and that η_p is not fixed by the restricted normal holonomy group $\Phi^*(p)$. Then M is an orbit of an s-representation.*

Proof. By assumption K acts transitively on $(M)_{\eta_p}$. This implies that the non-umbilical parallel normal vector field $\bar\eta$ with $\bar\eta(p + \eta_p) = -\eta_p$ yields $((M)_{\eta_p})_{\bar\eta} = M$. $(M)_{\eta_p}$ is a full irreducible compact homogeneous Euclidean submanifold of dimension and rank at least 2. Due to Theorem 3.3, $\hat K \cdot (p + \eta_p) = (M)_{\eta_p}$, where $\hat K$ acts as an s-representation. Finally, owing to Theorem 3.4, $\bar\eta$ is $\hat K$-invariant, so $M = \hat K \cdot p$, and this is also an orbit of an s-representation. □

Remark 3.7. From Corollary 3.5 it follows that the assumption of Corollary 3.6 can be replaced by: $(K_p)^0 \cdot \eta_p = \Phi^*(p) \cdot \eta_p$ and this orbit is not a point.

We finish this section with a result that will be used later.

Theorem 3.8. *Let $M^n = K \cdot v$, $n \geq 2$, be a full irreducible compact homogeneous submanifold of $\mathbb{R}^{n+k}$, where $k \geq 4$. Assume that $\rho(K_p)$ is a subgroup of codimension at most one of $\Phi(p)$. Then M is an orbit of an s-representation.*

Proof. If the codimension is zero, then $\rho((K_p)^0) = \Phi^*(p)$. Either we can find a non-trivial orbit of $\Phi^*(p)$ in $\nu_p M$ and we apply Corollary 3.6 and Remark 3.7), or there are no non-trivial orbits and we apply Theorem 3.3; in both cases we deduce that M is an orbit of an s-representation.

So let us assume that the codimension is one. The compactness of $\Phi^*(p)$ implies that $\rho((K_p)^0)$ is a normal subgroup. In view of the normal holonomy theorem [BCO03], $\Phi^*(p)$ acts on $\nu_p M$ as an s-representation, up to a component V_0 of fixed vectors. The case $\dim V_0 \geq 2$ is again covered

by Theorem 3.3, so we may assume that $V_0 = \mathbb{R}p$. Consider $\nu_p M = \mathbb{R}p \oplus V_1 \oplus \cdots \oplus V_r$, $\Phi^*(p)$-irreducible decomposition, where $\Phi^*(p) = \Phi_1 \times \cdots \times \Phi_r$, and Φ_i acts irreducibly on V_i and trivially on V_j, for $i \neq j$. Denote by $\rho_i((K_p)^0)) \subset \Phi_i$ the projection of $\rho((K_p)^0))$ to V_i for $i = 1, \dots, r$. Observe that $\rho((K_p)^0) \subset \rho_1((K_p)^0) \times \cdots \times \rho_r((K_p)^0)$.

Suppose first that $r \geq 2$. Since the codimension of $\rho((K_p)^0)$ in $\Phi^*(p)$ is one, there must exist an index i, which we may assume to be 1, such that $\rho_1((K_p)^0) = \Phi_1$. Let $\eta_p \in V_1$ be a sufficiently small nonzero vector such that $(M)_{\eta_p}$ is an embedded submanifold. Then the assumptions of Corollary 3.6 are satisfied so that M is an orbit of an s-representation (compare Remark 3.7).

It remains to analyse the case in which $r = 1$. Since $k \geq 4$, $\dim V_1 \geq 3$. Now $\rho_1((K_p)^0)$ has codimension 1 in Φ_1. Consider the case in which Φ_1 acts transitively on the sphere of V_1 of radius $\|\eta_p\|$. If $\rho_1((K_p)^0)$ also acts transitively on this sphere, then K acts transitively on the holonomy tube $(M)_{\eta_p}$, where $0 \neq \eta_p \in V_1$ is sufficiently small. Then M is an orbit of an s-representation by Corollary 3.6. If $\rho_1((K_p)^0)$ acts with cohomogeneity 1 on the sphere, then all orbits must have codimension 1, since $\rho_1((K_p)^0))$ is a normal subgroup of Φ_1. This is a contradiction because an action of cohomogeneity 1 on a sphere is polar and has thus singular orbits. Finally we assume that Φ_1 does not act transitively on the sphere. We may choose η_p such that its holonomy orbit is principal. By Corollary 3.6 we may assume that $\rho_1((K_p)^0)$ is not transitive on the principal orbit $\Phi_1 \cdot \eta_p$. Then, again by normality of $\rho_1((K_p)^0)$ in Φ_i, all orbits of $\rho_1((K_p)^0)$ in the irreducible isoparametric submanifold $\Phi_1 \cdot \eta_p \subset V_1$ have codimension 1. There must exist a curvature distribution E of $\Phi_1 \cdot \eta_p$ such that $E_{\eta_p} + T_{\eta_p}(\rho_1((K_p)^0) \cdot \eta_p) = T_{\eta_p}(\Phi_1 \cdot \eta_p)$. If we pass to a singular orbit $\Phi_1 \cdot \bar{\eta}$ that focalizes the eigendistribution E, then $\rho_1((K_p)^0) \cdot \bar{\eta} = \Phi_1 \cdot \bar{\eta}$. Due to Remark 3.7, M is an orbit of an s-representation. This finishes the proof. □

4. Involutions of orbits of s-representations

There are certain canonical extrinsic isometries of order 2 of orbits of s-representations, which were constructed in [BOR21]. Herein we recall their properties and present an alternative construction.

Proposition 4.1. *Let $X = G/K$ be a simply-connected symmetric space of compact type without Euclidean factor, where G is simply-connected and K is connected, write $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ for the decomposition of the Lie algebra of G into the eigenspaces of the involution, and consider the orbit $M = Ka_0$ of the isotropy representation of K through a point $a_0 \in \mathfrak{p}$. Then there exists a canonically defined involutive isometry f of X such that:*

- *(a) f fixes the basepoint o of X.*
- *(b) df_o preserves all K-orbits in $T_oX \cong \mathfrak{p}$, fixes a_0 and restricts to the identity along the normal space $\nu_{a_0}M$.*
- *(c) In case X is irreducible and of rank bigger than one, M is extrinsically symmetric if and only if the fixed point set of f is precisely $\nu_{a_0}M$.*

Proof. Let $\mathfrak{a}$ be a Cartan subspace of $\mathfrak{p}$ containing a_0 and consider the restricted root space decomposition

$$\mathfrak{k} = \mathfrak{k}_0 + \sum_{\alpha \in \Delta^+} \mathfrak{k}_\alpha, \quad \mathfrak{p} = \mathfrak{a} + \sum_{\alpha \in \Delta^+} \mathfrak{p}_\alpha.$$

Recall that the restricted roots are purely imaginary valued on $\mathfrak{a}$ since X is of compact type.

Denote by $\alpha_1, \dots, \alpha_q$ the simple restricted roots and note that

$$T_a(Ka) = \sum_{\alpha : \alpha(a) \neq 0} \mathfrak{p}_\alpha \tag{4.1}$$

and

$$\nu_a(Ka) = \mathfrak{a} + \sum_{\alpha : \alpha(a) = 0} \mathfrak{p}_\alpha \tag{4.2}$$

for all $a \in \mathfrak{a}$. Therefore, by replacing a_0 with another element of $\mathfrak{a}$ whose K-orbit has the same tangent space, we may assume that for each i, $\alpha_i(a_0) = 0$ equals either 0 or $\sqrt{-1}$. If we extend $\mathfrak{a}$

to a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, it follows that all roots of $\mathfrak{g}$ relative to $\mathfrak{h}$ take integer values on $\frac{1}{\sqrt{-1}}a_0$ and thus $\exp 2\pi a_0$ lies in the center $Z(G)$ of G. Hence

$$e^{2\pi \mathrm{ad}_{a_0}} = \mathrm{Ad}_{\exp 2\pi a_0} = 1. \tag{4.3}$$

For each $x \in \mathfrak{k}_\alpha$ we denote by $\hat{x} \in \mathfrak{p}_\alpha$ the vector related to x in the sense that

$$\mathrm{ad}_a x = -\sqrt{-1}\alpha(a)\hat{x}, \quad \mathrm{ad}_a \hat{x} = \sqrt{-1}\alpha(a)x$$

for all $a \in \mathfrak{a}$. Similarly, each $y \in \mathfrak{p}_\alpha$ we denote by $\check{y} \in \mathfrak{k}_\alpha$ the vector related to y in the sense that

$$\mathrm{ad}_a y = \sqrt{-1}\alpha(a)\check{y}, \quad \mathrm{ad}_a \check{y} = -\sqrt{-1}\alpha(a)y$$

for all $a \in \mathfrak{a}$. An easy calculation yields that

$$e^{t\mathrm{ad}_a} \cdot x = \cos(t|\alpha(a)|)x + \sin(t|\alpha(a)|)\hat{x} \tag{4.4}$$

for $x \in \mathfrak{k}_\alpha$, and

$$e^{t\mathrm{ad}_a} \cdot y = \cos(t|\alpha(a)|)y - \sin(t|\alpha(a)|)\check{y} \tag{4.5}$$

for $y \in \mathfrak{p}_\alpha$.

Consider the automorphism

$$\Phi = e^{\pi \mathrm{ad}_{a_0}} = \mathrm{Ad}_{\exp \pi a_0}$$

of $\mathfrak{g}$. It follows from (4.4) and (4.5) that Φ preserves $\mathfrak{k}$ and $\mathfrak{p}$. Also, $\Phi : \mathfrak{p} \to \mathfrak{p}$ is an isometry since the inner product is Ad-invariant. Since G is simply-connected, there is a unique automorphism φ of G such that $d\varphi = \Phi$. Define $f : X \to X$ to be $f(gK) = \varphi(g)K$. Then $f(o) = f(1K) = 1K = o$ and $df_o = \Phi$ is a linear isometry. Since $f(hgK) = \varphi(h)\varphi(g)K$ for g, $h \in G$, we now get that f is a global isometry. It also follows from (4.3) that Φ and f are involutions.

Owing to the fact that $\mathfrak{a}$ is Abelian and from (4.2), we see that Φ restricts to the identity along $\nu_{a_0}M$. Further,

$$\begin{aligned}\Phi(\mathrm{Ad}_k b) &= \Phi\left(\frac{d}{dt}\Big|_{t=0} k \exp tbk^{-1}\right)\\ &= \frac{d}{dt}\Big|_{t=0} \varphi(k)\exp t\Phi(b)\varphi(k)^{-1}\\ &= \mathrm{Ad}_{\varphi(k)}\Phi(b)\end{aligned}$$

for all $k \in K$, $b \in \mathfrak{a}$, so Φ preserves all K-orbits in $\mathfrak{p}$.

Note that on $\mathfrak{p}_\alpha \subset T_{a_0}M$ we have that

$$\Phi|_{\mathfrak{p}_\alpha} = (-1)^{|\alpha(a_0)|}\mathrm{id}, \tag{4.6}$$

where $0 \neq |\alpha(a_0)| \in \mathbb{Z}$.

Finally, $M = Ka_0$ is extrinsically symmetric if and only if a_0 can be taken dual to a simple restricted root α_i with coefficient 1 in the highest restricted root [EH95, Remarks, p.523]. This implies that $\alpha(a_0) = 1$ for every positive restricted root α that does not kill a_0. Hence $\Phi|_{T_{a_0}M} = -\mathrm{id}$. □

Remark 4.2. Put $g = \exp \pi a_0$. Then $\sigma(g) = g^{-1} = g$, so $\exp \pi a_0$ normalizes K. This implies that the tangent spaces to M at o and $\exp \pi a_0 \cdot o$ are canonically isomorphic to $\mathfrak{p}$. Now we claim that $df_o = \Phi$ coincides with the parallel transport along $\gamma(t) = (\exp ta_0) \cdot o$ from $t = 0$ to $t = \pi$. Recall X is simply-connected and consider the covering $\pi : X \to \bar{X}$, where $\bar{X}$ is the bottom space. It is enough to show that $\Phi : \mathfrak{p} \cong T_{\pi(o)}\bar{X} \to \mathfrak{p} \cong T_{\pi(o)}\bar{X}$ is the parallel transport along $\bar{\gamma}(t) = \pi \circ \gamma(t)$ from $t = 0$ to $t = \pi$ (note that $\bar{\gamma}(\pi) = \bar{\gamma}(0) = \pi(o)$). In fact, g satisfies $(dL_g)_o = \mathrm{Ad}_g d(R_{g^{-1}})_o = \mathrm{Ad}_g = \Phi$, since $R_{g^{-1}} = \mathrm{id}$ on $\bar{X}$, and $t \mapsto \exp ta_0 \in \exp[\mathfrak{p}]$ is a one-parameter group of transvections of $\bar{X}$.

5. The proof of Theorem 2.1

Let $M^n = Kp$, $n \geq 2$, be a full irreducible compact homogeneous almost symmetric submanifold of Euclidean space V of codimension k. Let $\tilde{K}$ be the group of ambient isometries that preserve M. Then $\tilde{K} \supset K$.

Lemma 5.1. *If there exist two different almost symmetries at p, then M is an orbit of an s-representation.*

Proof. Denote by H the kernel of the slice representation $\rho : \tilde{K}_p \to O(\nu_p M)$. We claim that if H does not fix any non-zero vector of T_pM, then M coincides with an orbit of an s-representation. In fact, for arbitrary ξ, $\eta \in \nu_p M$, the linear functional on T_pM given by $\ell(w) = \langle D^\perp_w \xi, \eta\rangle$ is H-invariant. Our assumption says that the action of H on the dual space $(T_pM)^*$ does not fix any non-zero vector. This imples that $\ell = 0$ and proves that $D^\perp = 0$. Now we invoke Theorem 3.2 to get that M coincides with an orbit of an s-representation. We finish by noting that if σ, ν are two different almost symmetries of M at p, then they belong to H and their common fixed point set in T_pM is zero, so H does not fix any non-zero vector of T_pM, and the argument is complete. □

5.1. **Part (a).** Suppose, to the contrary, that M is not an s-orbit. It follows from Lemma 5.1 that the almost symmetry at any $q \in M$ is uniquely defined, and we shall denote it by σ_q. Define $\mathcal{L}_q$ to be the $(+1)$-eigenspace of σ_q, for all $q \in M$.

Lemma 5.2. *The subspaces $\mathcal{L}_q$ for $q \in M$ define a $\tilde{K}$-invariant autoparallel distribution $\mathcal{L}$ on M. Its integral manifold thorugh $q \in M$ is the component of the fixed point set of σ_q containing q. Finally $\mathcal{L}$ is invariant under all shape operators of M.*

Proof. By uniqueness of the almost symmetry, σ_q commutes with the isotropy action of $\tilde{K}_q$, so $\mathcal{L}$ is $\tilde{K}$-invariant. A standard argument shows that the integral manifold $L(q)$ of $\mathcal{L}$ through q coincides with the component of the fixed point set of σ_q containing q (cf. [BCO03], Lemma 9.1.2 and its proof). Thus $L(q)$ is totally geodesic. Since σ_q is a linear isometry of the ambient Euclidean space $\mathbb{R}^N$ for all $q \in M$, the differentials $\mathrm{d}(\sigma_q)$ at different points have constant eigenvalues (1 with multiplicity $N - n + 1$, and -1 with multiplicity $n - 1$). By a continuity argument, we must have that $\mathrm{d}(\sigma_q)_r|_{\nu_r M} = \mathrm{Id}_{\nu_r M}$ for all $r \in L(q)$. This implies that $\sigma_q = \sigma_r$ for all $r \in L(q)$. The last assertion follows from the fact that $d(\sigma_q)_q|_{\nu_q M} = \mathrm{Id}_{\nu_q M}$, so $d(\sigma_q)_q$ commutes the shape operators of M at q. □

Denote by $\mathcal{L}^\perp$ be distribution on M whose fiber at $q \in M$ is the orthogonal complement of $\mathcal{L}_q$ in T_qM. Recall that $\nabla^\perp$ denotes the normal connection on νM induced from the Levi-Civitá connection and $\tilde{\nabla}^\perp$ denotes the canonical connection on νM induced by K.

Lemma 5.3. $\nabla^\perp = \tilde{\nabla}^\perp$ *along* $\mathcal{L}^\perp$.

Proof. Of course the almost symmetries commute with $D^\perp = \nabla^\perp - \tilde{\nabla}^\perp$. Now the sign rule implies that $\mathcal{D}^K = \{v \in TM : D^\perp_v = 0\}$ contains $\mathcal{L}^\perp$ as a subdistribution. Our standing assumption that M is not an s-orbit precludes the possibility that $\mathcal{D}^K = TM$ via Theorem 3.2. Since $\mathcal{L}^\perp$ has codimension one in M, we deduce that $\mathcal{L}^\perp = \mathcal{D}^K$, as wished. □

Recall that $\rho : K_p \to O(\nu_p M)$ denotes the slice representation of M at p.

Lemma 5.4. *There exists a one-parameter subgroup S of $SO(\nu_p M)$ that normalizes $\rho(K_p)$ such that*

$$\rho(K_p) \subset \Phi(p) \subset S \cdot \rho(K_p).$$

Proof. Denote by L_q the leaf of $\mathcal{L}$ through $q \in M$. Since $\mathcal{L}$ is K-invariant, the stabilizer K_{L_p} acts transitively on L_p, so we can write $L_p = K_{L_p}/K_p$ as a homogeneous space. In particular $\dim K_{L_p} = \dim K_p + 1$. Owing to the fact that K_{L_p} is a compact Lie group, for the Lie algebras this yields that $\mathfrak{k}_{L_p} = \mathfrak{k}_p \oplus \mathbb{R}u$, direct sum of ideals, for some $u \in \mathfrak{k}_{L_p}$.

Let $c : [0, 1] \to M$ a smooth loop at p. By Lemma 5.2, the shape operators of M preserve the distribution $\mathcal{L}$. Due to the Ricci equation, $R^\perp(\mathcal{L}, \mathcal{L}^\perp) = 0$ for the curvature of the normal

connection. Now [Olm93, Lem., App.] can be applied to yield a factorization of the parallel transport in νM along c,

$$\tau_c^\perp = \tau_\beta^\perp \circ \tau_{\tilde{c}}^\perp, \tag{5.7}$$

where $\tilde{c}$ is a smooth curve defined on $[0,1]$, everywhere tangent to $\mathcal{L}^\perp$, originating at p and ending at a point q in L_p, and $\beta : [0,1] \to L_p$ is a smooth curve joining q to p. Since $L(v)$ is one-dimensional, we may assume that $\beta(t) = \exp(-t\lambda u)\cdot q$, where $\lambda \in \mathbb{R}$ is such that $\exp(-\lambda u)\cdot q = p$. The fact that $\tilde{c}$ is everywhere tangent to $\mathcal{L}^\perp$ implies that $\tau_{\tilde{c}}^\perp$ coincides with the parallel transport $\tilde{\tau}_{\tilde{c}}^\perp$ with respect to the canonical connection $\tilde{\nabla}^\perp$, and thus $\tau_{\tilde{c}}^\perp = dk|_{\nu_p M}$ for some $k \in K^0$, due to Theorem 3.4. Set $k' = \exp(-\lambda u)k \in K_p$.

The action of $u \in \mathfrak{k}_{L_p}$ on $\xi \in \nu_p M$ given by

$$\mathcal{A}_u \xi = \frac{d}{dt}\Big|_{t=0} d(\exp tu)_p \cdot \xi$$

defines a skew-symmetric map on $\nu_p M$ such that

$$d\exp(-\lambda u) \circ (\tau_\beta^\perp)^{-1} = e^{-\lambda \mathcal{A}_u}$$

(cf. [BCO03, § 5.2.2]). We substitute into (5.7) to get

$$\tau_c^\perp = e^{\lambda \mathcal{A}_u} \circ dk'|_{\nu_p M}.$$

This equation shows that we can take S to be the one-parameter subgroup of $SO(\nu_p M)$ generated by $\mathcal{A}_u \in \mathfrak{so}(\nu_p M)$. □

Now part (a) of Theorem 2.1 follows from Theorem 3.8.

5.2. **Part (b).** Next we assume that M an orbit of an s-representation, but not a most singular orbit, and we wish to prove that it must be an almost singular orbit.

Herein we work only at the point p. Fix an almost symmetry σ of M at p and denote the subspace of fixed points of $d\sigma_p$ in T_pM by $L(\sigma_p)$. Since $d\sigma_p$ is the identity on $\nu_p M$, it commutes with any shape operator at p and therefore any shape operator at p preserves $L(\sigma_p)$. The restricted normal holonomy $\Phi^*(p)$ coincides with $\rho((K_p)^0)$, since M is an s-orbit, and the dimension of the subspace $(\nu_0 M)_p$ of fixed points in $\nu_p M$ is the rank r of M. Note that $r \geq 2$ as we are assuming that M is not most singular.

The family of shape operators $\{A_\xi : \xi \in (\nu_0 M)\}$ is commutative, due to the Ricci equation, and hence simultanelously diagonalizable. Then there are g parallel normal vector fields $\eta_1, \dots, \eta_g$, the so-called curvature normals, with corresponding autoparallel distributions $E_1, \dots, E_g$ and an eigenbundle decomposition $TM = E_1 \oplus \cdots \oplus E_g$. Note that the curvature normals span $\nu_0 M$, for otherwise M would reduce codimension; hence $g \geq r \geq 2$.

We have that $L(\sigma_p)$ is contained in $E_{i_0}(p)$ for some i_0, due to its invariance under the shape operators at p. The isometry σ maps parallel manifolds to M to parallel manifolds (including focal manifolds). Since it is the identity on $\nu_p M$ and this subspace meets all K-orbits, σ preserves all K-orbits. Consider the K-orbit N through $q = p + \xi$, where ξ is a parallel section of $\nu_0 M$ satisfying $\langle \xi, \eta_i \rangle = 1$ if and only if $i = i_0$. Then N is a focal manifold to M, $\sigma(q) = q$ and $d\sigma_q|_{T_qN}$ is the restriction of $d\sigma_p$ to $T_pM \cap E_{i_0}^\perp$, and hence is $-\mathrm{Id}$. This forces $d\sigma_q$ to be the identity on the first normal space $\nu_q^1 N$. Now $\nu^1 N$ is $\nabla^\perp$-parallel, since parallel translation is given by the group, and hence $\nu_q^1 N = \nu_q N$, due to the fullness of N (since K acts irreducibly). This shows that N is extrinsically symmetric. Moreover $L(\sigma_p) = E_{i_0}(p)$ because $d\sigma_q$ does not have (-1)-eigenvalues on $\nu_q N$. Therefore $\dim N = \dim M - 1$. Finally, it is known that extrinsically symmetric submanifolds are most singular orbits of s-representations, hence M is almost singular. This completes the proof of part (b).

5.3. **Part (c).** For an almost most singular orbit M of an s-representation which is almost symmetric, let N be the focal manifold constructed in part (b). Thanks to [BCO16, Theorem 4.5.4], M is a holonomy tube over N, that is, $M = (N)_{\bar{\xi}}$ for $\bar{\xi} = -\xi \in \nu_q N$ (notation as in párt (b)). Since $\dim M = \dim N + 1$, ξ must lie in an irreducible 2-dimensional component of the normal

holonomy representation of N at q. Conversely, suppose now N is an extrinsic symmetric submanifold with symmetry at $q \in N$ given by σ. Take a sufficiently short non-zero $\xi \in \nu_q N$ and consider the holonomy tube $M = (N)_\xi$. Then σ preserves M, fixes $p = q + \xi$, coincides with the identity along $\nu_p M$, and its set of fixed vectors in T_pM is precisely $T_\xi(\Phi(q) \cdot \xi)$. We deduce that if ξ lies in a 2-dimensional irreducible component of $\Phi^*(q)$ then M is almost symmetric.

5.4. **Part (d).** For the non-uniqueness statement of part (d), see subsection 7.1.2.

If M is not an orbit of an s-representation, then the uniqueness result is contained in Lemma 5.1. In view of part (b) we may thus assume that M is a most singular orbit of an s-representation. For each $q \in N$, denote by $\mathcal{F}_q$ the family of almost symmetries at q. Suppose, to the contrary, that $\mathcal{F}_q$ contains at least two elements.

For each $\sigma \in \mathcal{F}_q$, denote by $L(\sigma)$ the one-dimensional subspace of T_qM of fixed points of $d\sigma_q$. Let also $V(q)$ denote the linear span in T_qM of $L(\sigma)$ for $\sigma \in \mathcal{F}_q$. We obtain an invariant distribution $\mathcal{V}$ on M, which is moreover invariant under the group $\tilde{K}$ of extrinsic isometries of M.

Lemma 5.5. *$V(q)$ is a proper subspace of T_qM for all $q \in M$.*

Proof. Since $L(\sigma)$ is invariant under the shape operator A_ξ for all $\xi \in \nu_q M$, all shape operators at q diagonalize simultaneously on $V(q)$. However, as a most singular orbit of an s-representation, M nas non-flat normal bundle, so $V(q)$ cannot coincide with T_qM. □

For $q \in M$, let $H(q)$ denote the closure of the group generated by even products of elements of $\mathcal{F}_q$. By or assumption, $H(q)$ is non-trivial. Let also $W(q)$ denote the subspace of fixed points of $H(q)$ in T_qM.

Lemma 5.6. *$W(q) = V(q)^\perp$.*

Proof. If $v \in V(q)^\perp$ then $v \in L(\sigma)^\perp$ and thus $d\sigma_q(v) = -v$ for all $\sigma \in \mathcal{F}_q$. In particular v is fixed by $H(q)$, which checks the inclusion $V(q)^\perp \subset W(q)$. Let us prove the reverse inclusion. By our assumption, there exist two different elements σ, σ' in $\mathcal{F}_q$. The composition $\sigma\sigma'$ is a rotation in $L(\sigma) \oplus L(\sigma') \subset V(q)$, whose fixed point set $(L(\sigma) \oplus L(\sigma'))^\perp$ contains $W(q)$. Since these 2-planes generate $V(q)$ for such pairs σ, σ', this proves that $V(q)^\perp \supset W(q)$, as desired. □

Lemma 5.7. *The subspaces $V(q)$ for $q \in M$ define a smooth distribution $\mathcal{V}$ on M. Further, $\mathcal{V}$ is contained in the relative nullity distribution $\mathcal{N}$ of M as a submanifold of the sphere of radius $||p||$.*

Proof. It is obvious that $k\mathcal{F}_q k^{-1} = \mathcal{F}_q$ for all $k \in \tilde{K}_q$. It follows that $H(q)$ is a normal subgroup of $\tilde{K}_q$ for all $q \in M$. Therefore the subspaces $W(q)$ give rise to a $\tilde{K}$-invariant autoparallel distribution $\mathcal{W}$ of M. The first statement now follows from $\mathcal{V} = \mathcal{W}^\perp$ (Lemma 5.6).

Recall that the shape operators of M are simultaneously diagonalized along $\mathcal{V}$. This means there exist $\tilde{K}$-invariant normal vector fields $\eta_1, \dots, \eta_g$ on M and a a common eigenbundle decomposition $\mathcal{V} = E_1 \oplus \cdots \oplus E_g$ such that $A_\xi|_{E_i} = \langle \xi, \eta_i \rangle \, \mathrm{id}_{E_i}$. Since $\tilde{K}^0$ acts as an s-representation, $\tilde{K}^0$-invariant normal vector fields must be parallel. This implies that the η_i are parallel. However M is a most singular orbit, so $g = 1$ and η_1 is the position vector. Now $A_\xi|_{\mathcal{V}} = 0$ for ξ tangent to the sphere that contains M, that is, $\mathcal{V} \subset \mathcal{N}$. □

Now we can finish the proof of part (d). By Lemma 5.7, the nullity $\mathcal{N}$ of the second fundamental form of M as a submanifold of the sphere is non-trivial. Since M is homogeneous, $\mathcal{N}$ has constant rank. By a remarkable result of F. Vittone [Vit12, Theorem 1], the distribution $\mathcal{N}^\perp$ is completely non-integrable, that is, any two points in M can be joined by a piecewise smooth curve tangent to $\mathcal{N}^\perp$. On the other hand, the distribution $\mathcal{N}^\perp$ is contained in the integrable (autoparallel) distribution $\mathcal{W} = \mathcal{V}^\perp \subsetneq TM$. This contradiction shows that there can be only one almost symmetry at any point of M. □

6. Reduction to automorphisms

We now initiate the proof of Theorem 2.2, namely the classification of almost symmetric homogeneous submanifolds of Euclidean space. In this section we treat the case of orbits of s-representations that are almost symmetric and relate the almost symmetry to an automorphism of the group of isometries.

Lemma 6.1. *Suppose $X = G/K$ is a simply-connected symmetric space of compact type and rank bigger than one, where G is the transvection group. Write $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ as usual and the denote the basepoint by o. If $\Phi : \mathfrak{p} \to \mathfrak{p}$ is an isometry that preserves the orbit $M = Ka_0$ for some $a_0 \in \mathfrak{p}$, then $\Phi = df_o$ for a unique isometry f of X with $f(o) = o$.*

Proof. We know K is connected. Φ maps holonomy tubes of M onto holonomy tubes of M, and these are the K-orbits in $\mathfrak{p}$. It follows that K and $\Phi K \Phi^{-1}$ have the same orbits in $\mathfrak{p}$. We deduce from this and from [BCO16, Remark 8.3.5] (or [BCO03, Proposition 4.3.9]) that Φ normalizes K. We deduce from [BCO16, Lemma 8.3.2] that $\Phi(R)$ and R are multiples, but they have the same scalar curvature (since the curvature operators are conjugate, and the scalar curvature is twice the trace of the curvature operator). It follows that Φ preserves R. By the Cartan-Ambrose theorem, Φ is induced by a global isometry of X. Uniqueness is clear since f is determined by df_o. □

Lemma 6.2. *Suppose X is as in Lemma 6.1. If $\Phi : \mathfrak{p} \to \mathfrak{p}$ is an isometry (resp. an involutive isometry) that preserves $M = Ka_0$ for some $a_0 \in \mathfrak{p}$, then $\Phi = \tau|_{\mathfrak{p}}$ for a unique (resp. involutive) automorphism of τ of $\mathfrak{g}$ that commutes with σ, that is, $\tau \in \mathrm{Aut}(\mathfrak{g})^\sigma$.*

Proof. Due to Lemma 6.1, we can find an isometry f of X such that $\Phi = df_o$. In view of [Loo69, Proposition 4.1(a), §4, Ch. 7], we can write $df_o = \tau|_{\mathfrak{p}}$ for a unique $\tau \in \mathrm{Aut}(\mathfrak{g})^\sigma$. □

Fix a Cartan subspace (that is, a maximal Abelian subspace) $\mathfrak{a}$ of $\mathfrak{p}$. Then we have the (restricted) root space decomposition of $\mathfrak{g}$ with respect to $\mathfrak{a}$:

$$\mathfrak{g} = \mathfrak{g}_0 + \sum_{\alpha \in \Delta} \mathfrak{g}_\alpha,$$

where $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{a}$, $\mathfrak{k}_0$ is the centralizer $Z_{\mathfrak{k}}(\mathfrak{a})$ of $\mathfrak{a}$ in $\mathfrak{k}$, and Δ denotes the root system.

Extend $\mathfrak{a}$ to a σ-invariant Cartan subalgebra $\mathfrak{h} = \mathfrak{t} + \mathfrak{a}$, where $\mathfrak{t}$ is a CSA of $\mathfrak{k}_0$. We consider the root system of the complexification $\mathfrak{g}^c$ with respect to $\mathfrak{h}^c$; these roots are real-valued on $\sqrt{-1}\mathfrak{t} + \mathfrak{a}$. Then the elements of Δ can be seen as restrictions along $\mathfrak{a}$ of roots of $\mathfrak{g}^c$ with respect to $\mathfrak{h}^c$. If we introduce an ordering of the roots that takes $\mathfrak{a}$ before $\sqrt{-1}\mathfrak{t}$, as we now do, the restriction map $\mathfrak{h}^c \to \mathfrak{a}$ maps positive roots to positive roots, and simple roots to simple roots. Write as usual

$$\mathfrak{k} = \mathfrak{k}_0 + \sum_{\alpha \in \Delta^+} \mathfrak{k}_\alpha \quad \text{and} \quad \mathfrak{p} = \mathfrak{a} + \sum_{\alpha \in \Delta^+} \mathfrak{p}_\alpha,$$

with

$$\mathfrak{k}_\alpha + \mathfrak{p}_\alpha = \mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha}$$

for $\alpha \in \Delta^+$.

Recall that the Satake diagram of the symmetric space associated to a symmetric pair (G, K) is obtained from the Dynkin diagram of $\mathfrak{g}$ by painting black the nodes associated to simple roots of $(\mathfrak{g}^c, \mathfrak{h}^c)$ that vanish along $\mathfrak{a}$; and by joining by a curved arrow two white nodes which are associated to simpĺe roots of $(\mathfrak{g}^c, \mathfrak{h}^c)$ that have the same restriction along $\mathfrak{a}$. The Satake diagram compĺetely determines the symmetric space, up to duality.

Also recall that the group of outer automorphisms of a compact semisimple Lie algebra $\mathfrak{g}$ is canonically isomorphic to the group of automorphisms of its Dynkin diagram. In fact, every automorphism of $\mathfrak{g}$ can be assumed to fix a given CSA and a given Weyl chamber, up to inner automorphism, and then it permutes the simple roots.

Lemma 6.3. *Let $\tau \in \mathrm{Aut}(\mathfrak{g})^\sigma$ of order 2 be such that $\tau|_{\mathfrak{a}} = \mathrm{id}$. Then τ induces an automorphism of order 2 of the Satake diagram that maps each white node either to itself or to another white node joined to itself by a curved arrow.*

Proof. Since $\tau|_{\mathfrak{a}} = \mathrm{id}$, we have $\tau(\mathfrak{k}_0) = \mathfrak{k}_0$. Now $\tau(\mathfrak{t})$ is a CSA of $\mathfrak{k}_0$, so by the conjugacy of CSA there is an inner automorphism of $\mathfrak{k}_0$ mapping $\tau(\mathfrak{t})$ to $\mathfrak{t}$, say $\mathfrak{t} = \mathrm{Ad}_k \tau(\mathfrak{t})$ for some $k \in K_0$, where K_0 is the centralizer of $\mathfrak{a}$ in K. Further, by conjugacy of Weyl chambers, we may assume that $\tilde{\tau} := \mathrm{Ad}_k \tau$ fixes the chosen Weyl chamber of $\mathfrak{t}$.

Now $\tilde{\tau} \in \mathrm{Aut}(\mathfrak{g})^\sigma$, $\tilde{\tau}$ preserves the CSA $\mathfrak{h}$ and its Weyl chamber, so it maps simple roots to simple roots. Also, $\tilde{\tau}|_{\mathfrak{a}} = \mathrm{id}$, so if α is a restricted root, then $\tilde{\tau}\alpha = \alpha \circ \tilde{\tau}$ and α have the same restriction along $\mathfrak{a}$. The desired result follows. □

We shall use Lemma 6.3 as follows. Many Satake diagrams do not admit a non-trivial automorphim that fixes every white node or maps it to another white node joined to the first by a curved arrow. For the corresponding symmetric space, this means that Φ must be given as the inner automorphism induced by an element of G.

7. The classification of almost symmetric s-orbits

In this section we run through the cases of simply-connected irreducible symmetric spaces of compact type and use the results of section 6 to determine the orbits of s-representations that are almost symmetric.

Denote the simple restricted roots by $\alpha_1, \dots, \alpha_q$, where q is the rank of the symmetric space, and by δ the highest restricted root. The fundamental restricted weights $\lambda_1, \dots, \lambda_q$ as usual are defined by $2\langle \lambda_i, \alpha_j\rangle / ||\alpha_j||^2 = \delta_{ij}$ for all i, j (Kronecker delta).

7.1. **Spaces of maximal rank.** These are characterized by anyone of the following equivalent conditions: $\operatorname{rk} M = \operatorname{rk} G$; or $\mathfrak{k}_0 = 0$; or uniform multiplicity 1. In the table, q denotes the rank of M.

AI	$SU(q+1)/SO(q+1)$
BI	$SO(2q+1)/S(O(q+1)\times O(q))$
CI	$Sp(q)/U(q)$
DI	$SO(2q)/S(O(q)\times O(q))$
G	$G_2/SO(4)$
FI	$F_4/Sp(3)Sp(1)$
EI	$E_6/[Sp(4)/\mathbb{Z}_2]$
EV	$E_7/[SU(8)/\mathbb{Z}_2]$
$EVIII$	$E_8/Spin(16)$

All nodes of the Satake diagram are white and there are no curved arrows, so the Satake diagram coincides with the Dynkin diagram. Therefore τ is of inner type, $\tau = \mathrm{Ad}_g$ for some $g \in N_G(K)$. Going through the detailed description of bottom spaces for the spaces of maximal rank in [Wol64, p. 314-5] and noting that $g \in Z_G(\mathfrak{a})$, we deduce that $g \in K_0$. Here $K_0 = \mathbb{Z}_2^q$, so g is completely determined by its sign on $\mathfrak{p}_{\alpha_i} \cong \mathbb{R}$ for $i = 1, \dots, q$.

For each $\alpha \in \Delta^+$, write $\alpha = \sum_{i=1}^q n_i(\alpha)\alpha_i$, where the $n_i(\alpha)$ are non-negative integers.

7.1.1. *M is a most singular orbit.* Without loss of generality, we may pass to another s-orbit with the same tangent space $T_{a_0}M$ and assume that $\alpha_{i_1}(a_0) = \sqrt{-1}$ for some i_1, and $\alpha_i(a_0) = 0$ for $i \neq i_1$. Now $\mathfrak{p}_\alpha \subset T_{a_0}M$ holds for some $\alpha \in \Delta^+$ if and only if $n_{i_1}(\alpha) > 0$. In particular $\mathfrak{p}_{\alpha_i} \subset \nu_{a_0}M$ for $i \neq i_1$, so $\Phi|_{\mathfrak{p}_{\alpha_i}} = \mathrm{id}$ for $i \neq i_1$, and, since Φ is assumed non-trivial, $\Phi|_{\mathfrak{p}_{\alpha_{i_1}}} = -\mathrm{id}$. It follows that $\Phi|_{\mathfrak{p}_\alpha} = (-1)^{n_{i_1}(\alpha)}$ for all $\alpha \in \Delta^+$.

There is exactly one $\beta \in \Delta^+$ such that $n_{i_1}(\beta)$ is nonzero and even (as this implies that $\mathfrak{p}_\beta \subset T_{a_0}M$ and $\Phi|_{\mathfrak{p}_\beta} = 1$), and it must be that $n_{i_1}(\beta) = 2$ (for $n_{i_1}(\beta) \geq 4$ would entail the existence of roots α with $n_{i_1}(\alpha) = 2$). We come to two cases.

• $\beta = \delta$. Then $\delta - \alpha_i \notin \Delta$ for $i \neq i_1$ (because $n_{i_1}(\delta - \alpha_i) = 2$), so $\langle \delta, \alpha_i\rangle = 0$ for all $i \neq i_1$ (since the α_i-chain through β consists of β only), and this implies that δ is a multiple of the fundamental weight λ_{i_1}. By inspection of the highest root in each case (see e.g. [Wol84, p. 273]), we see that that this happens in all cases in the table, but A_q.

A_q	$\delta = \lambda_1 + \lambda_q$
B_q	$\delta = \lambda_2$
C_q	$\delta = 2\lambda_1$
D_q	$\delta = \lambda_2$
G_2	$\delta = \lambda_2$
F_4	$\delta = \lambda_1$
E_6	$\delta = \lambda_2$
E_7	$\delta = \lambda_1$
E_8	$\delta = \lambda_8$

For such orbits, we have seen in section 4 that $\tau = e^{\pi \mathrm{ad}_{a_0}}$ induces an almost symmetry of $M = Ka_0$ (cf. eqn. (4.6)). This gives exactly 8 examples.

• $\beta \neq \delta$. Here $n_{i_1}(\delta) = 3$. By inspection, the highest root having a coefficient equal to 3 implies that $\mathfrak{g}$ is of exceptional type (cf. [Wol84, p. 270]). In the sequel, we use Bourbaki's convention for the numbering of the simple roots (cf. [Bou81, pp. 251-275]).

In the case of G_2, we have $\delta = 3\alpha_1 + 2\alpha_2$, so $i_1 = 1$ and $\beta = 2\alpha_1 + \alpha_2$ (α_1 is the short root). The other roots are $3\alpha_1 + \alpha_2$, $\alpha_1 + \alpha_2$, α_2, α_1 and all have $n_1(\alpha) \neq 2$, so $\alpha_1(a_0) = \sqrt{-1}$, $\alpha_2(a_0) = 0$ yields an example.

In the case of F_4, we have $i_1 = 2$, and $\delta - \alpha_1 - \alpha_2$ and $\delta - \alpha_1 - \alpha_2 - \alpha_3$ are two roots with $n_2 = 2$, so there are no examples.

In the case of E_6, we have $i_1 = 4$, and $\delta - \alpha_2 - \alpha_4$ and $\delta - \alpha_2 - \alpha_3 - \alpha_4$ are two roots with $n_4 = 2$, so there are no examples.

In the case of E_7, we can take $i_1 = 3$ or 5. In the first case $\delta - \alpha_1 - \alpha_3$ and $\delta - \alpha_1 - \alpha_3 - \alpha_4$ are two roots with $n_3 = 2$, so we do not get an example. In the second case, $\delta - \alpha_1 - \alpha_3 - \alpha_4 - \alpha_5$ and $\delta - \alpha_1 - \alpha_3 - \alpha_4 - \alpha_5 - \alpha_6$ are two roots with $n_5 = 2$, so we do not get an example either.

In the case of E_8, we can take $i_1 = 7$ or 2. In the first case $\delta - \alpha_8 - \alpha_7$ and $\delta - \alpha_8 - \alpha_7 - \alpha_6$ are two roots with $n_7 = 2$, so we do not get an example. In the second case, $\delta - \alpha_8 - \alpha_7 - \alpha_6 - \alpha_5 - \alpha_4 - \alpha_2$ and $\delta - \alpha_8 - \alpha_7 - \alpha_6 - \alpha_5 - \alpha_4 - \alpha_2 - \alpha_3$ are two roots with $n_2 = 2$, so we do not get an example either.

7.1.2. *M is an almost singular orbit.* Without loss of generality, we may assume that there are i_1, i_2 such that $\alpha_i(a_0) = \sqrt{-1}$ for $i = i_1$ or i_2 and $\alpha_i(a_0) = 0$ otherwise. In particular $\mathfrak{p}_{\alpha_{i_1}} \oplus \mathfrak{p}_{\alpha_{\alpha_{i_2}}} \subset T_{a_0}M$. Here $\Phi|_{\mathfrak{p}_{\alpha_i}} = \mathrm{id}$ for $i \neq i_1, i_2$, and there are two subcases.

• $\Phi|_{\mathfrak{p}_{\alpha_{i_1}}} = \mathrm{id}$, $\Phi|_{\mathfrak{p}_{\alpha_{i_2}}} = -\mathrm{id}$. Note that α_{i_1} cannot be adjacent to α_i for $i \neq i_2$ in the Dynkin diagram. In fact, otherwise $\alpha_{i_1} + \alpha_i \in \Delta^+$ for some $i \neq i_2$ with $\mathfrak{p}_{\alpha_{i_1}+\alpha_i} \subset T_{a_0}M$ and $\Phi|_{\alpha_{i_1}+\alpha_i} = \mathrm{id}$, yielding coindex at least 2, which is a contradiction to almost symmetry of M. This implies that α_{i_1} is an extremal node of the Dynkin diagram and α_{i_2} is adjacent to α_{i_1}. Again by almost symmetry of M, we must have $n_{i_2}(\alpha)$ odd for all $\alpha \in \Delta^+ \setminus \{\alpha_{i_1}\}$. In particular, $n_{i_2}(\delta) = 1$. Glancing at the coefficients of δ (recall that α_{i_2} is adjacent to an extremal node), we obtain only the following two possibilities.

– A_q. $i_1 = 1$, $i_2 = 2$ and M is a principal orbit and a circle bundle over one of the extrinsic symmetric focal orbits $G_2(\mathbb{R}^{q+1})$; the almost symmetry of M coincides with the extrinsic symmetry of the focal orbit. Since there are two focal orbits, we obtain two different almost symmetries for M.

– B_2. $i_1 = 1$, $i_2 = 2$ and M is a principal orbit $[SO(3) \times SO(2)]/\mathbb{Z}_2$.

Since we can freely choose the signs of the action of k on each $\mathfrak{p}_{\alpha_i}$, this yields exactly two examples.

• $\Phi|_{\mathfrak{p}_{\alpha_{i_1}}} = -\mathrm{id}$, $\Phi|_{\mathfrak{p}_{\alpha_{i_2}}} = -\mathrm{id}$. Since Φ fixes a unique tangent direction, there is a unique $\beta \in \Delta^+$ such that $n_{i_1}(\beta) + n_{i_2}(\beta)$ is positive and even, thus, it is equal to 2. We consider two possibilities.

$-\ \beta = \delta$. Then $\delta - \alpha_i \notin \Delta$ if $i \neq i_1,\, i_2$. This implies that $\langle \delta, \alpha_i \rangle = 0$ for all $i \neq i_1,\, i_2$, hence δ is a linear combination of two fundamental weights $\lambda_{i_1},\, \lambda_{i_2}$. The only posibility is A_q and

$$a_0 = \begin{pmatrix} 1 & & & & \\ & 0 & & & \\ & & \ddots & & \\ & & & 0 & \\ & & & & -1 \end{pmatrix}.$$

The (uniform) multiplicity must be one. This example has already appeared above, a circle bundle over $G_2(\mathbb{R}^{q+1})$.

$-\ \beta \neq \delta$. Since $\beta \pm \alpha_i \notin \Delta$ for $i \neq i_1,\, i_2$, β and α_i are strongly orthogonal for all $i \neq i_1,\, i_2$, so β lies in the linear span of $\lambda_{i_1},\, \lambda_{i_2}$. Also, $n_{i_1}(\delta) + n_{i_2}(\delta)$ is odd, hence equals 3. We run through the cases.

For A_q, G_2, F_4 and E_8, there are no distinct indices $i_1,\, i_2$ such that $n_{i_1}(\delta) + n_{i_2}(\delta) = 3$.

For E_6, we can have $(i_1, i_2) = (1, 2)$ or $(1, 3)$, up to symmetry of the Dynkin diagram, but then $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$ and $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5$ are two roots with $n_{i_1} + n_{i_2} = 2$.

For E_7, we can have $(i_1, i_2) = (1, 7)$ or $(2, 7)$, but $\alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7$, $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7$, $\alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7$ are roots; the first two have $n_1 + n_7 = 2$, and the last two have $n_2 + n_7 = 2$.

For B_q, $i_1 = 1$ and $i_2 > 1$. Now θ_1 always has $n_{i_1} + n_{i_2} = 2$, $\theta_1 + \theta_q$ has $n_{i_1} + n_{i_2} = 2$ if $i_2 < q$, and $\theta_{q-1} + \theta_q$ has $n_{i_1} + n_{i_2} = 2$ if $i_2 = q$.

For C_q, $i_1 < q$ and $i_2 = q$. Now $\theta_1 + \theta_q$ always has $n_{i_1} + n_{i_2} = 2$, $\theta_2 + \theta_q$ has $n_{i_1} + n_{i_2} = 2$ if $i_1 > 1$, and $\theta_1 + \theta_{q-1}$ has $n_{i_1} + n_{i_2} = 2$ if $i_1 = 1$.

For D_q, $i_1 = 1$ and $i_2 < q - 1$ or $1 < i_1$ and $i_2 = q - 1$, up to symmetry of the Dynkin diagram. Now $\theta_1 + \theta_{q-1}$ always has $n_{i_1} + n_{i_2} = 2$, $\theta_1 - \theta_{q-1}$ has $n_{i_1} + n_{i_2} = 2$ if $i_1 = 1$ and $i_2 < q - 1$, and $\theta_2 + \theta_{q-1}$ has $n_{i_1} + n_{i_2} = 2$ if $i_1 > 1$ and $i_2 = q - 1$.

Hence no new examples appear here.

7.2. **The group manifold case.** These are the symmetric spaces of the form $X = L \times L/\Delta_L$, where L is a compact connected simple Lie group. They can be equivalently characterized as having uniform multiplicity 2.

The s-orbital foliation of $\mathfrak{p}$ is identified with the adjoint foliation of $\mathfrak{l}$. Consider an orbit $M = La_0$ and assume $\Phi : \mathfrak{l} \to \mathfrak{l}$ is an almost symmetry for M. In particular Φ restricts to the identity along the Cartan subalgebra of $\mathfrak{l}$, so it is of inner type, $\Phi = \mathrm{Ad}_g$ for some $g \in L$ (in fact, g lies in the maximal torus of L). Since L is connected, $\det \mathrm{Ad}_g = 1$. On the other hand, the (-1)-eigenspace of Φ on $T_{a_0}M$ has codimension 1 in $T_{a_0}M$ and M is even dimensional, which forces $\det \Phi = -1$, a contradiction. So there are no examples in the group case.

We can push this idea further. Recall that an irreducible symmetric space G/K is called *of splitting rank* if $\operatorname{rk} G = \operatorname{rk} K + \operatorname{rk} M$. They can be equivalently characterized as having all multiplicities even.

Proposition 7.1. *Suppose $X = G/K$ is a simply-connected irreducible symmetric space of compact type, rank bigger than one, and splitting rank. Then no K-orbit in $\mathfrak{p}$ is almost symmetric with respect to an inner automorphism of G defined by an element of K.*

Proof. Consider an orbit $M = Ka_0$. Then $T_{a_0}M = \sum_{\alpha \in \Delta^+ : \alpha(a_0) \neq 0} \mathfrak{p}_\alpha$ is even dimensional (since all multiplicities are even). If $\Phi : \mathfrak{p} \to \mathfrak{p}$ is an almost symmetry for M, then $\det \Phi = (-1)^{\dim M - 1} = -1$. If $\Phi = \mathrm{Ad}_k$ is an almost symmetry of M, where $k \in K$, then $\det \mathrm{Ad}_k = 1$ (since K is connected), a contradiction. □

7.3. **The A_q-case.** There are four irreducible symmetric spaces of compact type and Dynkin diagram of type A_q, each with uniform multiplicity m, where $m = 1,\, 2,\, 4,\, 8$, respectively, and $q = 2$ if $m = 8$. The case $m = 1$ is the space of maximal rank AI, and the case $m = 2$ is the group case of type A, which have both been dealt with already. There remains two spaces.

7.3.1. *AII.* The space is $SU(2q+2)/Sp(q+1)$ and $m=4$. The effective space is

$$[SU(2q+2)/\mathbb{Z}_2]/[Sp(q+1)/\mathbb{Z}_2].$$

From the Satake diagram of AII, we invoke Lemmata 6.2 and 6.3 to see that $\Phi = \mathrm{Ad}_g|_{\mathfrak{p}}$ for some $g \in G$. Now the conjugation $\mathrm{Inn}_g|_K$ is an automorphism of K, which must be inner because K has no outer automorphisms, that is, we can find $k \in K$ such that $gk^{-1} \in Z_G(K)$. But in this case the latter group coincides with the center $Z(G)$ (since he bottom space of X is $\frac{[SU(2q+2)/\mathbb{Z}_2]/\mathbb{Z}_{q+1}}{Sp(q+1)/\mathbb{Z}_2}$, where $\mathbb{Z}_{q+1}$ is the center of $SU(2q+2)/\mathbb{Z}_2$, cf. [Wol64, p. 315]), so $\Phi = \mathrm{Ad}_k|_{\mathfrak{p}}$. In view of Proposition 7.1 no orbit is almost symmetric.

7.3.2. *EIV.* The space is E_6/F_4 and $m=8$. This is an effective presentation and the bottom space is

$$\frac{E_6/\mathbb{Z}_3}{F_4\mathbb{Z}_3/\mathbb{Z}_3} = \frac{E_6}{F_4\mathbb{Z}_3} = \frac{E_6/\mathbb{Z}_3}{F_4},$$

so $Z_{E_6}(F_4) = Z(E_6) = \mathbb{Z}_3$. The same argument involving Lemmata 6.2 and 6.3 and Proposition 7.1 as in case AII implies that no orbit is almost symmetric.

7.4. **Hermitian symmetric spaces.**

7.4.1. *AIII.* The space is $SU(p+q)/S(U(p)\times U(q))$. Suppose $M = Ka_0$ is almost symmetric, with almost symmetry Φ. Then $\Phi|_{\mathfrak{a}} = \mathrm{id}$. In view of Lemma 6.3 and the Satake diagram, there are two cases to be considered.

• Φ is inner. Here $\Phi = \mathrm{Ad}_g$ for some g in the normalizer $N_G(K)$ of K in G. Consider first the case $g = k \in K$, so that $k \in K_0 \cong S(U(1)^q \times U(p-q))$. Then Φ commutes with the complex structure J on $\mathfrak{p}$, therefore $\Phi|_{J\mathfrak{a}} = \mathrm{id}$, that is, $\Phi|_{\mathfrak{p}_{2\theta_i}} = \mathrm{id}$ for $i = 1, \dots, q$. Since we are assuming that the rank $q \geq 2$, by the almost-symmetry assumption, at most one index i has $2\theta_i(a_0) \neq 0$, in fact, exactly one, say, $\theta_1(a_0) = \sqrt{-1}$ and $\theta_i(a_0) = 0$ for $i > 1$ (since the orthogonal complement of $J\mathfrak{a}$ in $T_{a_0}M$ is a complex subspace). Now $\alpha_1(a_0) = \sqrt{-1}$ and $\alpha_2(a_0) = \cdots = \alpha_q(a_0) = 0$. The tangent space

$$T_{a_0}M = \mathfrak{p}_{\theta_1} + \mathfrak{p}_{2\theta_1} + \sum_{j=2}^{q} \mathfrak{p}_{\theta_1+\theta_j} + \sum_{j=2}^{q} \mathfrak{p}_{\theta_1-\theta_j}.$$

We need $\Phi = \mathrm{Ad}_k$ to be $-\mathrm{id}$ on all summands of $T_{a_0}M$, but the second one, and to be id on $\mathfrak{a}$ and the other restricted root spaces. Suppose

$$\mathfrak{k} = \left\{ \begin{pmatrix} A & \\ & B \end{pmatrix} \ : \ A \in \mathfrak{u}(q), B \in \mathfrak{u}(p), \mathrm{tr}(A+B) = 0 \right\}$$

and

$$\mathfrak{p} = \left\{ \begin{pmatrix} & Z \\ -Z^* & \end{pmatrix} \ : \ Z \in \mathfrak{gl}(q, \mathbb{C}) \right\}. \tag{7.8}$$

Take

$$\mathfrak{a} = \sum_{i=1}^{q} \mathbb{R}(E_{i,q+i} - E_{q+i,i}).$$

Then

$$k = \begin{pmatrix} I_{1,q-1} & & \\ & I_{1,q-1} & \\ & & I_{p-q} \end{pmatrix} \in K_0 \tag{7.9}$$

does the job, where

$$I_{1,q-1} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

has size q, and I_{p-q} denotes the identity matrix of size $p-q$.

Next we observe that $N_G(K) \neq K$ occurs only if $p = q$, in which case $N_G(K)/K = \mathbb{Z}_2$ generated by the class of the map $\beta : x \mapsto x^\perp$ that takes a q-plane to its orthogonal complement in $2q$-space. We can represent β by right-multiplication by the matrix

$$g_0 = \begin{pmatrix} & I \\ -I & \end{pmatrix}. \tag{7.10}$$

If Φ involves g_0, this means that $\Phi = \mathrm{Ad}_{g_0 k}$ for some $k \in K_0 \cong S(U(1)^q)$. We have $\mathrm{Ad}_{g_0}(Z) = Z^*$ in regard to (7.8). The Dynkin diagram is C_q, where $\theta_i \pm \theta_j$ has multiplicity 2 and $2\theta_i$ has multiplicity 1. The group K_0 is connected, so it acts trivially on $\mathfrak{p}_{2\theta_i}$. Now $\Phi|_{\mathfrak{p}_{2\theta_i}} = -\mathrm{id}$, so $\mathfrak{p}_{2\theta_i} \subset T_{a_0}M$ for all i. Also $\Phi|_{\mathfrak{p}_{\theta_i \pm \theta_j}} = \mp \mathrm{Ad}_k|_{\mathfrak{p}_{\theta_i \pm \theta_j}}$, but $\dim \mathfrak{p}_{\theta_i \pm \theta_j} = 2$, so it is impossible to have an almost symmetric $M = Ka_0$.

• Φ is not inner. Suppose $\Phi = \sigma_0 \mathrm{Ad}_k$ for some $k \in K_0$, where $\sigma_0 X = \bar{X}$ for $X \in \mathfrak{g}$. Then Φ anticommutes with the complex structure J on $\mathfrak{p}$, therefore $\Phi|_{J\mathfrak{a}} = -\mathrm{id}$, so that $\Phi|_{\mathfrak{p}_{2\theta_i}} = -\mathrm{id}$ for $i = 1, \dots, q$. The restricted root spaces

$$\mathfrak{p}_{\theta_i \pm \theta_j} \text{ and } \mathfrak{p}_{\theta_i}$$

are complex and Φ-invariant, with multiplicities 2 and $2(p-q)$. In particular Φ has 1 and -1 as eigenvalues on $\mathfrak{p}_{\theta_1 \pm \theta_2}$, each with multiplicity 1 (recall that $q \geq 2$). This implies that both $\mathfrak{p}_{\theta_1 \pm \theta_2}$ must be in $T_{a_0}M$, but then the codimension of the fixed point set of Φ in $T_{a_0}M$ is at least 2, a contradiction. Hence there are no examples in this case.

The other case is $p = q$ and $\Phi = \sigma_0 \mathrm{Ad}_{g_0 k}$, g_0 as in (7.10), for some $k \in K_0$. Then Φ commutes with the complex structure on $\mathfrak{p}$ and this forces a_0 to be given by $\theta_1(a_0) = \sqrt{-1}$ and $\theta_i(a_0) = 0$ for $i > 1$, up to renaming the indices, by the same argument as in the case Φ is inner. But we saw above that this s-orbit is almost symmetric with respect to Φ of the form Ad_k where k is given by (7.9).

7.4.2. *EVII.* The space is $X = E_7/\{[E_6 \times U(1)]/\mathbb{Z}_3\}$. Suppose $M = Ka_0$ is almost symmetric, with almost symmetry Φ. Then $\Phi|_{\mathfrak{a}} = \mathrm{id}$. According to [Wol64, p. 314], the bottom space is the quotient by a $\mathbb{Z}_2$-group which is not contained in E_7. Therefore $N_G(K) = K$. In view of Lemma 6.3 and the Satake diagram, $\Phi = \mathrm{Ad}_k$ for some $k \in K_0 \cong Spin(8)$.

Now Φ commutes with the complex structure J on $\mathfrak{p}$. The Dynkin diagram is C_3 and $J(\mathfrak{a})$ is spanned by $\mathfrak{p}_{\theta_i}$ for $i = 1, 2, 3$ (each one of dimension 1). Therefore $\mathfrak{p}_{\theta_i} \subset T_{a_0}M$ for at most one index, say, 1. Thus we can write

$$T_{a_0}M = \mathfrak{p}_{\theta_1+\theta_2} + \mathfrak{p}_{\theta_1-\theta_2} + \mathfrak{p}_{\theta_1+\theta_3} + \mathfrak{p}_{\theta_1-\theta_3} + \mathfrak{p}_{\theta_1},$$

where $\Phi = -\mathrm{id}$ on each summand, but the last one where it is id.

It is known that K_0 acts on $\mathfrak{p}_{\theta_i+\theta_j}$ and $\mathfrak{p}_{\theta_i-\theta_j}$ by the same one of the 8-dimensional irreducible representations of $Spin(8)$, and in fact, it acts on the triple $(\mathfrak{p}_{\theta_1+\theta_2}, \mathfrak{p}_{\theta_2+\theta_3}, \mathfrak{p}_{\theta_1+\theta_3})$ by three inequivalent 8-dimensional irreducible representations of $Spin(8)$, say $(\rho_8, \Delta_8^+, \Delta_8^-)$. Let k be the nontrivial element in $\ker \Delta_8^+ \cong \mathbb{Z}_2$. Then $\rho_8(k) = \Delta_8^-(k) = -\mathrm{id}$ acts on $\mathfrak{p}$ as wished. This produces the example of almost symmetric submanifold $M = Ka_0 \cong E_6/Spin(10)$. It is also a homogeneous CR-manifold.

7.4.3. *EIII.* The space is $X = E_6/\{[Spin(10) \times U(1)]/\mathbb{Z}_4\}$ and this is also the bottom space [Wol64, p. 314]. In particular $N_G(K) = K$. Suppose $M = Ka_0$ is almost symmetric, with almost symmetry Φ. Then $\Phi|_{\mathfrak{a}} = \mathrm{id}$. In view of Lemma 6.3 and the Satake diagram, there are two cases, namely, $\Phi = \mathrm{Ad}_k$ or $\Phi = \tau \mathrm{Ad}_k$, for some $k \in K_0 \cong U(4)$, where τ is the outer involution of E_6 with fixed point set F_4. It follows from [Kol09, Example 3.5] that τ has eigenvalues ± 1 on $\mathfrak{p}$ with multiplicities 16 and 16. In particular, τ preserves the orientation of X. Since K_0 is connected, in any case Φ is orientation-preserving. It follows that the dimension of M must be odd. But X is a rank 2 symmetric space, and the s-orbits have dimensions 30, 24 and 21, respectively. Therefore M must be the 21-dimensional singular orbit $Spin(10)/SU(5)$ (cf. [Kon03, p. 77]).

We next show that there is $k \in K_0$ such that $\Phi = \mathrm{Ad}_k$ induces an almost symmetry on that singular orbit. The Dynkin diagram of X is BC_2, and $M = Ka_0$, where we may assume

$\theta_1(a_0) = \sqrt{-1}$, $\theta_2(a_0) = 0$. Now

$$T_{a_0}M = \mathfrak{p}_{\theta_1+\theta_2} + \mathfrak{p}_{\theta_1-\theta_2} + \mathfrak{p}_{\theta_2} + \mathfrak{p}_{2\theta_2},$$

where the summands have dimensions 6, 6, 8 and 1, respectively, and $\Phi = -\mathrm{id}$ on each summand, but the last one where $\Phi = \mathrm{id}$. It is known that $K_0 \cong U(4)$ acts as $(SU(4), \mathbb{C}^4)$ on $\mathfrak{p}_{\theta_2}$, as $(SO(6), \mathbb{R}^6)$ on $\mathfrak{p}_{\theta_1\pm\theta_2}$, and trivially on $\mathfrak{p}_{2\theta_2}$. Let $k = (\sqrt{-1}I_4, \sqrt{-1}I_4) \in U(1) \times SU(4)$. Since $\sqrt{-1}I_4$ maps to I_6 under the double covering $SU(4) \to SO(6)$, k acts as -1 on $\mathfrak{p}_{\theta_1\pm\theta_2} \cong \mathbb{R}^6$ and on $\mathfrak{p}_{\theta_2} \cong \mathbb{C}^4$, as wished.

7.4.4. *BDI.* The space is $X = SO(p+2)/SO(2) \times SO(p)$ and the bottom space is $SO(p+2)/S(O(2) \times O(p))$ [Wol64, p. 314], that is $N_G(K) = S(O(2) \times O(p))$, where BI (resp. DI) corresponds to p odd (resp. even). Suppose $M = Ka_0$ is almost symmetric, with almost symmetry Φ. Then $\Phi|_{\mathfrak{a}} = \mathrm{id}$.

• p is odd. In view of Lemma 6.3 and the Satake diagram, $\Phi = \mathrm{Ad}_g$ for some $g \in Z_{N_G(K)}(\mathfrak{a}) \cong \mathbb{Z}_2^2 \times SO(p-2)$. Suppose first that $g = k \in K_0 = Z_K(\mathfrak{a}) \cong \mathbb{Z}_2 \times SO(p-2)$. Then Φ commutes with the complex structure on $\mathfrak{p}$, so $\Phi|_{\mathfrak{p}_{\theta_1\pm\theta_2}} = \mathrm{id}$. It follows that exactly one of $\theta_1 \pm \theta_2$ in nonzero on a_0, say $\theta_1(a_0) = \theta_2(a_0) = \sqrt{-1}$. Now

$$T_{a_0}M = \mathfrak{p}_{\theta_1} + \mathfrak{p}_{\theta_2} + \mathfrak{p}_{\theta_1+\theta_2},$$

where $\Phi = -\mathrm{id}$ on the first two summands, and $\Phi = \mathrm{id}$ on the last summand.

Suppose

$$\mathfrak{k} = \left\{ \begin{pmatrix} A & \\ & B \end{pmatrix} \ : \ A \in \mathfrak{so}(2), B \in \mathfrak{so}(p), \right\}.$$

Take

$$\mathfrak{a} = \sum_{i=1}^{2} \mathbb{R}(E_{i,2+i} - E_{2+i,i}).$$

Then

$$k = I_{4,p-2} = \begin{pmatrix} -I_4 & \\ & I_{p-2} \end{pmatrix} \in K_0 \tag{7.11}$$

does the job and yields an example of almost symmetric submanifold (this element generates the $\mathbb{Z}_2$-factor of K_0). In fact

$$\mathfrak{p}_{\theta_i} = \sum_{j=1}^{p-2} \mathbb{R}(E_{i,4+j} - E_{4+j,i}),$$

and

$$\mathfrak{p}_{\theta_1\pm\theta_2} = \mathbb{R}(E_{14} \mp E_{23}) - (E_{41} \mp E_{32}).$$

Next we consider the case in which $g \in N_G(K) \setminus K$; we may assume $g = g_0 k$, where

$$g_0 = \left(\begin{pmatrix} -1 & \\ & 1 \end{pmatrix}, \begin{pmatrix} -1 & \\ & I_{p-1} \end{pmatrix} \right) \in S(O(2) \times O(p))$$

and $k \in K_0$. In this case Φ anticommutes with the complex structure on $\mathfrak{p}$, so $\Phi|_{\mathfrak{p}_{\theta_1\pm\theta_2}} = -\mathrm{id}$. It follows that $\mathfrak{p}_{\theta_1\pm\theta_2} \subset T_{a_0}M$. If one of $\mathfrak{p}_{\theta_1}$, $\mathfrak{p}_{\theta_2}$ is contained in $\nu_{a_0}M$, say, $\mathfrak{p}_{\theta_1} \subset \nu_{a_0}M$, then $\Phi|_{\mathfrak{p}_{\theta_1}} = \mathrm{id}$. Since $J\mathfrak{p}_{\theta_1} = \mathfrak{p}_{\theta_2}$, this implies that $\Phi|_{\mathfrak{p}_{\theta_2}} = -\mathrm{id}$ and $\mathfrak{p}_{\theta_2} \subset T_{a_0}M$. This gives $\Phi = -\mathrm{id}$ on $T_{a_0}M$, a contradiction. We deduce that M is a principal orbit. But we saw above that $\Phi|_{\mathfrak{p}_{\theta_1\pm\theta_2}} = -\mathrm{id}$, and Φ has opposite signs on $\mathfrak{p}_{\theta_1}$ and $\mathfrak{p}_{\theta_2}$ (both of dimension $p-2$). The only possibility is $p = 3$, which gives a space of maximal rank, already studied in subsection 7.1.

• p is even. Here $\Phi = \mathrm{Ad}_g$ or $\Phi = \tau\mathrm{Ad}_g$ for some $g \in N_G(K)$, where τ is the outer automorphism of $\mathfrak{g}$ given by conjugation by the matrix $g_0 = I_{1,p+1}$. Since p is even, $K_0 \cong \mathbb{Z}_2 \times SO(p-2)$ acts on $\mathfrak{p}$ preserving the orientation, and $\det(\tau : \mathfrak{p} \to \mathfrak{p}) = \det(\mathrm{Ad}_{g_0} : \mathfrak{p} \to \mathfrak{p}) = (-1)^p = 1$. Therefore Φ is orientation preserving. It follows that $\dim M$ is odd. Since X has dimension $2p$ and rank 2, with multiplicities 1 and $p-2$, we deduce that again M must the singular orbit Ka_0 with $\theta_1(a_0) - \theta_2(a_0) = \alpha_1(a_0) = 0$. And in fact $\Phi = \mathrm{Ad}_k$ with the same $k \in K_0$ given as in (7.11) defines an almost symmetry for this M.

We deduce that, for all $p \geq 3$, the singular orbit (Stiefel manifold) $SO(p)/SO(p-2) = V_2(\mathbb{R}^p)$ is an almost symmetric submanifold.

7.4.5. *DIII.* The space is $X = SO(2n)/U(n)$, where $n = 2q$ or $2q+1$. Suppose $M = Ka_0$ is almost symmetric, with almost symmetry Φ.

The Dynkin diagram is C_q or BC_q according to whether n is even or odd, and the Satake diagram also differs. According to Lemma 6.3 and the Satake diagram, if n is even then $\Phi = \mathrm{Ad}_g$ for some $g \in N_G(K)$. Here $N_G(K)$ generated by $g_0 = I_{n,n}$ and $K_0 \cong SU(2)^q$. On the other hand, if n is odd then $N_G(K) = K$, but there is an admissible symmetry of the Satake diagram, so $\Phi = \mathrm{Ad}_k$ or $\Phi = \tau\mathrm{Ad}_k$, for some $k \in K_0 \cong SU(2)^q \cdot U(1)$, where $\tau = \mathrm{Ad}_{I_{n,n}}$ is an outer automorphism of $\mathfrak{so}(2n)$. Thus in either case $\Phi = \mathrm{Ad}_k$ or $\Phi = \tau\mathrm{Ad}_k$, where $k \in K_0$ and $\tau = \mathrm{Ad}_{I_{n,n}}$. Note that τ preserves the restricted root spaces.

• $\Phi = \mathrm{Ad}_k$. Then Φ commutes with the complex structure J on $\mathfrak{p}$, so $\Phi = \mathrm{id}$ on $J(\mathfrak{a}) = \sum_{i=1}^q \mathfrak{p}_{2\theta_i}$. Therefore for exactly one index i we have $\mathfrak{p}_{2\theta_i} \subset T_{a_0}M$ (the multiplicity of all $2\theta_i$ is 1). Recall that

$$\alpha_1 = \theta_1 - \theta_2, \ldots, \alpha_{q-1} = \theta_{q-1} - \theta_q, \ \alpha_q = \begin{cases} 2\theta_q, & \text{if } q \text{ is even,} \\ \theta_q, & \text{if } q \text{ is odd.} \end{cases}$$

Without loss of generality, we may assume that $\theta_1(a_0) = \sqrt{-1}$ and $\theta_2(a_0) = \cdots = \theta_q(a_0) = 0$, that is, $\alpha_1(a_0) = 1$ and $\alpha_2(a_0) = \cdots = \alpha(a_0) = 0$. Now

$$T_{a_0}M = \mathfrak{p}_{2\theta_1} + \sum_{j=2}^q \mathfrak{p}_{\theta_1+\theta_j} + \sum_{j=2}^q \mathfrak{p}_{\theta_1-\theta_j},$$

if n is even, and

$$T_{a_0}M = \mathfrak{p}_{2\theta_1} + \mathfrak{p}_{\theta_1} + \sum_{j=2}^q \mathfrak{p}_{\theta_1+\theta_j} + \sum_{j=2}^q \mathfrak{p}_{\theta_1-\theta_j},$$

if n is odd, where in each case $\Phi = \mathrm{id}$ on the first summand, and $\Phi = -\mathrm{id}$ on the other summands.

Now we take $k = (-1, 1, \ldots, 1) \in K_0 \cong Sp(1)^q$ (resp. $k = (-1, 1, \ldots, 1; 1) \in Sp(1)^q \cdot U(1)$) and note that Ad_k acts on $\mathfrak{p}$ as wished, since $(k_1, \ldots, k_q) \in K_0 \cong Sp(1)^q$ (resp. $(k_1, \ldots, k_q, t) \in K_0 \cong Sp(1)^q \cdot U(1)$) acts on $\mathfrak{p}_{2\theta_i} \cong \mathbb{R}$ trivially and on $x \in \mathfrak{p}_{\theta_i \pm \theta_j} \cong \mathbb{H}$ as $k_i x k_j^{-1}$ (quaternionic multiplication) in both cases; and on $y \in \mathfrak{p}_{\theta_i} \cong \mathbb{C}^2$ as $k_i y t^{-1}$ in case n is odd. Therefore $M \cong U(2q)/SU(2) \times U(2q-2)$ (resp. $U(2q+1)/SU(2) \times U(2q-1)$) is an almost symmetric submanifold.

• $\Phi = \tau\mathrm{Ad}_k$. Recall that

$$\mathfrak{k} = \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} : A \in \mathfrak{so}(n), \ B^t = B \right\},$$

and

$$\mathfrak{p} = \left\{ \begin{pmatrix} A & B \\ B & -A \end{pmatrix} : A, \ B \in \mathfrak{so}(n) \right\}.$$

The complex structure on $\mathfrak{p}$ is given by

$$J \begin{pmatrix} A & B \\ B & -A \end{pmatrix} = \begin{pmatrix} -B & A \\ A & B \end{pmatrix}.$$

It follows that τ anti-commutes with J, and so does Φ. Therefore $\Phi|_{\mathfrak{p}_{2\theta_i}} = -\mathrm{id}$, so $\sum_{i=1}^q \mathfrak{p}_{\theta_{2i}} + \sum_{i=1}^q \mathfrak{p}_{\theta_i} \subset T_{a_0}M$. This implies that for each pair i, j at least one of $\mathfrak{p}_{\theta_i \pm \theta_j}$ is contained in $T_{a_0}M$.

The description above of $\mathfrak{p}_{\theta_i \pm \theta_j}$ as the K_0-representation of real type

$$(SO(4) = Sp(1)_{(i)}Sp(1)_{(j)}, \mathbb{R}^4 = \mathbb{H} \otimes_{\mathbb{H}} \mathbb{H}^*)$$

shows that J must map $\mathfrak{p}_{\theta_i\pm\theta_j}$ to $\mathfrak{p}_{\theta_i\mp\theta_j}$ (this can also be checked directly from the formulae in [Loo69, p. 116]).

Fix i, j and note that $\mathfrak{p}_{\theta_i+\theta_j} + \mathfrak{p}_{\theta_i-\theta_j}$ is a Φ-invariant complex subspace of real dimension 8. Therefore Φ has 4 eigenvalues $+1$ and 4 eigenvalues -1 there. Thus it must be that one of $\mathfrak{p}_{\theta_i\pm\theta_j}$ is contained in $\nu_{a_0}M$ and the other one in $T_{a_0}M$. Recall that Φ anticommutes with J, so if n is even this means that $\Phi = -\mathrm{id}$ on $T_{a_0}M$ which is a contradition. If n is odd, this means that Φ can still have a $(+1)$-eigenvalue on $\mathfrak{p}_{i_0}$ ($\subset T_{a_0}M$) for some i_0. But $\mathfrak{p}_{i_0}$ is a Φ-invariant complex subspace of real dimension 4, therefore Φ has 2 eigenvalues $+1$ and 2 eigenvalues -1 there, which is also a contradiction. Hence we get no other examples in this case.

7.5. **Real Grassmannians (BDI).** The space is the Grassmannian of oriented planes $X = SO(p+q)/SO(p)\times SO(q)$, and the bottom space is the Grassmannian of unoriented planes $SO(p+q)/S(O(p)\times O(q))$. The transvection group is $SO(p+q)$ if p or q is odd, and $SO(p+q)/\mathbb{Z}_2$ if p and q are even. We may assume $p \geq q \geq 3$ ($q = 2$ yields a Hermitian symmetric space), and $p \neq q$, $q+1$ (for otherwise X is a space of maximal rank). Then the Dynkin diagram is B_q. We suppose

$$\mathfrak{k} = \left\{ \begin{pmatrix} A & \\ & B \end{pmatrix} \ : \ A \in \mathfrak{so}(q), B \in \mathfrak{so}(p), \right\},$$

and take

$$\mathfrak{a} = \sum_{i=1}^{q} \mathbb{R}(E_{i,q+i} - E_{q+i,i}).$$

Assume $M = Ka_0$ is almost symmetric, with almost symmetry Φ. Then $\Phi|_{\mathfrak{a}} = \mathrm{id}$.

• Φ is inner. In view of Lemma 6.3 and the Satake diagram, this must be the case if $n = p+q$ is odd. So $\Phi = \mathrm{Ad}_g$ for some $g \in Z_{N_G(K)}(\mathfrak{a}) \cong \mathbb{Z}_2^q \times SO(p-q)$. By relabeling the entries, we may assume that

$$g = \begin{pmatrix} I_{r,s} & & \\ & I_{r,s} & \\ & & A \end{pmatrix},$$

where $A \in SO(p-q)$ (A has order 2, so it can be further assumed to have only diagonal entries ± 1, up to conjugation). It follows that $\Phi|_{\mathfrak{p}_{\theta_i\pm\theta_j}} = \mathrm{id}$ if $1 < i < j \leq r$ or $r+1 \leq i < j \leq q$, and $\Phi|_{\mathfrak{p}_{\theta_i\pm\theta_j}} = -\mathrm{id}$ otherwise. In particular, at most one of $\mathfrak{p}_{\theta_i+\theta_j}$ or $\mathfrak{p}_{\theta_i-\theta_j}$ for some $i < j \leq r$ or $r+1 \leq i < j$ is contained in $T_{a_0}M$. It is clear that at least one of them must be contained in $T_{a_0}M$ (otherwise all $\theta_i(a_0) = 0$). By relabeling the indices, we may assume $\theta_1 + \theta_2(a_0) \neq 0$, $\theta_1 - \theta_2(a_0) = 0$ and $\theta_i \pm \theta_j(a_0) = 0$ if $i < j \leq r$ and $(i,j) \neq (1,2)$, or $r+1 \leq i < j$. And $\theta_i \pm \theta_j(a_0) \neq 0$ if $i \leq r < j$. The only solution, up to scaling, is

$$r = 2,\ \theta_1(a_0) = \theta_2(a_0) = \sqrt{-1},\ \theta_3(a_0) = \cdots \theta_q(a_0) = 0.$$

Now

$$T_{a_0}M = \mathfrak{p}_{\theta_1+\theta_2} + \mathfrak{p}_{\theta_1} + \mathfrak{p}_{\theta_2} + \sum_{j=3}^{q} \mathfrak{p}_{\theta_1+\theta_j} + \sum_{j=3}^{q} \mathfrak{p}_{\theta_1-\theta_j} + \sum_{j=3}^{q} \mathfrak{p}_{\theta_2+\theta_j} + \sum_{j=3}^{q} \mathfrak{p}_{\theta_2-\theta_j},$$

where $\Phi = \mathrm{id}$ on the first summand, and $\Phi = -\mathrm{id}$ on the other summands.

Note that

$$g = \begin{pmatrix} I_{2,q-2} & & \\ & I_{2,q-2} & \\ & & I_{p-q} \end{pmatrix} \tag{7.12}$$

indeed does the job, so we get an almost symmetric submanifold $M \cong S(O(p)\times O(q))/S(O(2)\times O(p-2)\times O(q-2))$.

• Φ is not inner. This can happen if n is even. The only difference from above is that $A \in O(p-q)$, but this does not affect the above argument.

7.6. **Quaternion-Kähler symmetric spaces.** We prove the next proposition just in the degree of generality needed for the sequel.

Proposition 7.2. *Let X' be a reflective submanifold of maximal rank of a symmetric space X of exceptional type, with $o \in X'$. If M is an almost symmetric s-orbit in T_oX, then $M' = M \cap T_oX'$ is either an almost symmetric or a symmetric s-orbit in T_oX'.*

Proof. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the involutive decomposition under σ of the Lie algebra $\mathfrak{g}$ of the transvection group of X. The reflective property of X' implies that there is an involutive automorphism τ of $\mathfrak{g}$ that commutes with σ such that the fixed point set $\mathfrak{p}^\tau = T_oX'$, and $\mathfrak{g}^\tau = \mathfrak{k}^\tau + \mathfrak{p}^\tau$ is the involutive Lie algebra associated to the symmetric space X'. Since X' has maximal rank, we can take a common Cartan subspace $\mathfrak{a} \subset \mathfrak{p}^\tau \subset \mathfrak{p}$. Write as usual

$$\mathfrak{k} = \mathfrak{k}_0 + \sum_{\alpha\in\Delta^+} \mathfrak{k}_\alpha,\ \mathfrak{p} = \mathfrak{a} + \sum_{\alpha\in\Delta^+} \mathfrak{p}_\alpha.$$

Consider the s-orbit M of X through a given point $a \in \mathfrak{a}$, and a non-zero tangent vector $v = [x, a] \in \mathfrak{p}_\alpha$ at a, where $x \in \mathfrak{k}_\alpha$. In particular, $\alpha(a) \neq 0$. Assume that $v \in T_oX'$. Then $\tau v = v$, so $\tau x - x$ lies in the isotropy algebra $\mathfrak{k}_a$. Since $\tau|_{\mathfrak{a}} = \mathrm{id}$, we have $\tau(\mathfrak{k}_\alpha) = \mathfrak{k}_\alpha$, so in fact $\tau x - x \in (\mathfrak{k}_\alpha)_a$. Owing to $\alpha(a) \neq 0$, the latter isotropy algebra is zero, which yields $\tau x = x$. Hence $x \in \mathfrak{k}^\tau$ and v is tangent to the the s-orbit of X' through a. This shows that M' is the s-orbit of X' through a.

Recall that

$$T_aM = \sum_{\alpha:\alpha(a)\neq 0} \mathfrak{p}_\alpha,\ \nu_aM = \mathfrak{a} + \sum_{\alpha:\alpha(a)=0} \mathfrak{p}_\alpha.$$

Since $\tau|_{\mathfrak{a}} = \mathrm{id}$, we have $\tau(\mathfrak{p}_\alpha) = \mathfrak{p}_\alpha$ for all α. In particular

$$T_aM = \underbrace{T_aM^\tau}_{=T_aM'} \oplus T_aM^{-\tau},\ \nu_aM = \nu_aM^\tau \oplus \nu_aM^{-\tau}.$$

Let Φ be an almost symmetry for M. Then $\Phi = \nu|_{\mathfrak{p}}$ for some involutive automorphism ν of $\mathfrak{g}$. Since X is of exceptional type, we may conjugate ν and assume that ν and τ commute [Con69]. It follows that the ± 1-eigenspaces of ν on $\mathfrak{p}$ decompose with respect to $\mathfrak{p} = \mathfrak{p}^\tau + \mathfrak{p}^{-\tau}$. Note that $\nu = \mathrm{id}$ on $\nu_aM^\tau = \nu_aM' \subset \mathfrak{p}^\tau$. Next there are two cases: either $\nu = -\mathrm{id}$ on T_aM' or ν has an eigenvalue 1 of multiplicity 1 in T_aM'; these two cases corresponding accordingly to M' being symmetric or almost symmetric. This completes the proof of the proposition. □

7.6.1. *Exceptional spaces.* There is one irreducible qK symmetric space for each simple Lie group. We first consider the spaces of exceptional type. Note that $G_2/SO(4)$ and $F_4/[Sp(3)Sp(1)]$ are spaces of maximal rank and thus they have already been dealt with; in particular, $F_4/[Sp(3) \cdot Sp(1)]$ gives rise to exactly one almost symmetric submanifold. Next, there is a chain of reflective submanifolds:

$$F_4/[Sp(3) \cdot Sp(1)] \subset E_6/[SU(6) \cdot SU(2)] \subset E_7/[Spin(12) \cdot SU(2)] \subset E_8/[E_7 \cdot SU(2)],$$

or $FI \subset EII \subset EVI \subset EIX$. All spaces have rank 4 and indeed Dynkin diagram F_4. The almost symmetric s-orbit of FI is the s-orbit through $a_0 \in \mathfrak{a}$ defined by

$$\alpha_1(a_0) = \sqrt{-1},\ \alpha_2(a_0) = \alpha_3(a_0) = \alpha_4(a_0) = 0.$$

And there are no symmetric s-orbits in FI (cf. [ET20]). It follows from Proposition 7.2 that, in the other three cases, it is only the s-orbit through the same point $a_0 \in \mathfrak{a}$ that has a chance of being almost symmetric. That orbit is indeed almost symmetric in those three cases, because the almost symmetry for FI is given by $\Phi = e^{\pi \mathrm{ad}_{a_0}}|_{\mathfrak{p}}$, and this clearly extends to an almost symmetry in the other three spaces (as in all cases $\Phi|_{\mathfrak{p}_\alpha} = (-1)^{n_1(\alpha)}$, the root system is F_4, and the only $\alpha \in \Delta^+$ for which $n_1(\alpha)$ is even and nonzero is the highest root δ, which has multiplicity one).

7.6.2. *Quaternionic Grassmannians.* We consider the case CII, that is, $X = Sp(p+q)/[Sp(p) \times Sp(q)]$, where $p \geq q \geq 2$. This is also the bottom space, except in the case $p = q$ in which the bottom space is the $\mathbb{Z}_2$-quotient by the map $\beta : x \mapsto x^\perp$ that takes a q-plane to its orthogonal complement in $2q$-space. There are no symmetries of the Satake diagram, so we need to consider the following two cases only.

• Φ does not involve β. Here $\Phi = \mathrm{Ad}_k$ for some $k \in K_0 \cong Sp(1)^q \times Sp(p-q)$ of order 2. It follows that $k = (\pm 1, \dots, \pm 1, A)$ for some $A \in Sp(p-q)$. In particular $\Phi|_{\mathfrak{p}_{2\theta_i}} = \mathrm{id}$ for all $i = 1, \dots, q$. Now for at least one index i we must have $\mathfrak{p}_{2\theta_i} \subset T_{a_0}M$, and $\mathfrak{p}_{2\theta_i}$ has dimension 3, so there cannot exist an almost symmetric $M = Ka_0$.
• Φ involves β. Here $p = q$ and $\Phi = \mathrm{Ad}_{g_0 k}$ for some $k \in K_0$ and g_0 as in (7.10). We suppose

$$\mathfrak{k} = \left\{ \begin{pmatrix} A & \\ & B \end{pmatrix} \ : \ A \in \mathfrak{sp}(q), B \in \mathfrak{sp}(q), \right\},$$

$$\mathfrak{p} = \left\{ \begin{pmatrix} & Z \\ -Z^* & \end{pmatrix} \ : \ Z \in \mathfrak{gl}(q, \mathbb{H}), \right\},$$

where $Z^* = \bar{Z}^t$, and take

$$\mathfrak{a} = \sum_{i=1}^{q} \mathbb{R}(E_{i,q+i} - E_{q+i,i}).$$

We have $\mathrm{Ad}_{g_0}(Z) = Z^*$.

The Dynkin diagram is C_q, where $\theta_i \pm \theta_j$ has multiplicity 4 and $2\theta_i$ has multiplicity 3. Now $\Phi|_{\mathfrak{p}_{2\theta_i}} = -\mathrm{id}$, so $\mathfrak{p}_{2\theta_i} \subset T_{a_0}M$ for all i. Also $\Phi|_{\mathfrak{p}_{\theta_i \pm \theta_j}} = \mp \mathrm{Ad}_k|_{\mathfrak{p}_{\theta_i \pm \theta_j}}$, but $\dim \mathfrak{p}_{\theta_i \pm \theta_j} = 4$, so it is impossible to have an almost symmetric $M = Ka_0$.

This completes the case-by-case analysis of irreducible symmetric spaces of compact type and the proof of part (a) of Theorem 2.2.

8. Classification in the case of non-s-orbits

Herein we deal with almost symmetric homogeneous submanifolds of Euclidean space that are not orbits of s-representations.

Let M be a full irreducible compact almost symmetric homogeneous submanifold of an Euclidean space V which is not an s-orbit. Owing to Theorem 2.1(a), M has codimension 3. Denote by G the maximal connected subgroup of $SO(V)$ that preserves M. Since representations of cohomogeneity at most two are always polar, and polar representations have the same orbits as an s-representation [Dad85], the cohomogeneity of (G, V) must be 3. Now M is a principal orbit of (G, V) and G is the maximal connected group in its orbit-equivalence class.

Fix $p \in M$ and let σ denote the (unique) almost symmetry at p. Write $\hat{G}$ for the identity component of the group generated by G and σ. By construction of G, we have $\hat{G} = G$, so σ normalizes G. Now conjugation by σ defines an involutive automorphism of G and hence specifies a possibly non-effective symmetric pair (G, G^σ). By uniqueness of σ we have that $k\sigma k^{-1} = \sigma$ for all $k \in G_p$, so the fixed point group G^σ contains the isotropy group G_p, and indeed with codimension one, as the $(+1)$-eigenspace of σ on T_pM is 1-dimensional.

We run through the list of representations of cohomogeneity 3; the maximal connected Lie groups in their equivalence class are listed in [Str94, Table II]. All such representations have *copolarity* one (see [GOT04] for the irreducible case and [PP18] for the reducible case), namely, there is a 4-dimensional subspace Σ of the representation space meeting all orbits and containing the normal spaces to the principal orbits it meets. The reflection on Σ is a candidate for the almost symmetry of a principal orbit, but it preserves the G-orbits precisely in the three cases listed in the theorem, as we now explain.

8.1. $(SO(n), \mathbb{R}^n \oplus \mathbb{R}^n)$, $n \geq 3$. A principal isotropy group is the lower block $SO(n-2)$-subgroup, and its fixed point set is the upper $\mathbb{R}^2 \oplus \mathbb{R}^2$ and can be taken as Σ. The reflection on Σ is given by $\mathrm{diag}(1, 1, -1, \dots, -1)$ and normalizes the group, hence it preserves the orbits (since it is the identity along Σ and Σ meets all orbits). It follows that every principal orbit is almost symmetric.

8.2. $(U(2), V = \mathbb{C}^2 \oplus \mathbb{R}^3)$**.** It is convenient to replace $U(2)$ by $G = U(1) \times Sp(1)$ and describe this action in terms of quaternions as

$$(e^{i\theta}, q) \cdot (x, y) = (qxe^{-i\theta}, qyq^{-1}),$$

where $x \in \mathbb{H}$, $y \in \Im\mathbb{H}$, $q \in Sp(1)$, $\theta \in \mathbb{R}$. We identify $\mathbb{C}^2 \cong \mathbb{H}$ by mapping (α, β) to $\alpha + j\beta$ so that complex conjugation corresponds to conjugation by j.

The slice representation at $(x, y) = (1, 0)$ is the circle group $\{(e^{i\theta}, e^{i\theta})\} \subset G$ acting on $\Im H$ as rotation on the jk-plane, so it is clear that every G-orbit has a representative of the form (x, y) with $x \in \mathbb{R}$ and $y \in \mathbb{R}i \oplus \mathbb{R}k$.

Define

$$\tau(x, y) = (jxj^{-1}, -jyj^{-1}) = (-jxj, jyj).$$

Then τ is an involutive isometry of V that normalizes G and whose fixed point set

$$V^\tau = \{(x, y) \in V \mid x \in \mathbb{R} \oplus \mathbb{R}j,\ y \in \mathbb{R}i \oplus \mathbb{R}k\}.$$

It follows that τ preserves the orbits. Now we can take $\Sigma = V^\tau$ and every principal orbit is almost symmetric.

8.3. $(G = U(1) \times SU(n) \times U(1), V = \mathbb{C}^n \oplus \mathbb{C}^n)$**,** $n \geq 2$**.** . We describe the action as

$$(\lambda, A, \mu) \cdot (x, y) = (\lambda Ax, \mu Ay),$$

where λ, $\mu \in U(1)$, $A \in SU(n)$, x, $y \in \mathbb{C}^n$. We have the isotropy group

$$G_{(e_1,0)} = \{\lambda^{-1}, \begin{pmatrix} \lambda & \\ & B \end{pmatrix}, \mu) \mid B \in U(n-1), \lambda^{-1} = \det B, \mu \in U(1)\},$$

so that the iterated isotropy group

$$\begin{aligned} (G_{(e_1,0)})_{(0,e_1+e_2)} &= \{(\lambda^{-1}, \begin{pmatrix} \lambda & & \\ & \lambda & \\ & & C \end{pmatrix}, \lambda^{-1}) \mid C \in U(n-2), \lambda^{-2} = \det C\}, \\ &\cong \mathbb{Z}_2 \times U(n-2) \end{aligned}$$

is a principal isotropy group for $n \geq 3$, but the principal isotropy group is trivial for $n = 2$.

Any symmetric subalgebra of $\mathfrak{g} = \mathfrak{u}(1) \oplus \mathfrak{su}(n) \oplus \mathfrak{u}(1)$ has to split into symmetric subalgebras of the factors. For $n \geq 3$, it is clear that $\mathbb{R} \oplus \mathfrak{g}_{(e_1,e_1+e_2)} \cong \mathbb{R} \oplus \mathfrak{u}(n-2)$ cannot be a symmetric subalgebra of $\mathfrak{g}$.

On the other hand, for $n = 2$ we consider the compĺex conjugation of $\mathbb{C}^2 \oplus \mathbb{C}^2$ over $\mathbb{R}^2 \oplus \mathbb{R}^2$, denoted by σ. It is clear that σ normalizes G. Since $\Sigma = \mathbb{R}^2 \oplus \mathbb{R}^2$ meets every G-orbit, σ preserves the G-orbits and hence every principal orbit is almost symmetric.

8.4. **The other cases.** These are ruled out case-by-case by checking that there is no symmetric subalgebra of the form $\mathfrak{g}_p \oplus \mathbb{R}$ of $\mathfrak{g}$.

9. Inhomogeneous almost symmetric submanifolds

In this section we discuss some results about the inhomogeneous case. In particular we show that a inhomogeneous almost symmetric submanifold is of cohomogeneity one, and we construct examples of compact full irreducible almost symmetric submanifolds of Euclidean space with cohomogeneity one. A fuller investigation will be the subject of future work.

The simplest example of compact full irreducible almost symmetric submanifold of Euclidean space of cohomogeneity one is as follows.

Example 9.1. Let M be an ellipsoid of $SO(n)$-revolution in $\mathbb{R}^n \times \mathbb{R} \cong \mathbb{R}^{n+1}$, which is not a sphere. Then M is foliated by $n-1$-spheres, each such $n-1$-sphere Σ spans an affine subspace $\langle\Sigma\rangle$ parallel to $\mathbb{R}^n \times \{0\}$, and the almost symmetry at a point in Σ, different from the two vertices, is the reflection on the affine 2-plane $\langle\Sigma\rangle^\perp$. Note that M contains singular orbits of the isometry group $SO(n)$, namely, the vertices. Of course, at the vertices almost symmetries exist and are not unique. If we delete the vertices from M, we get an example of incomplete almost symmetric submanifold.

A more general construction is given by the following family of examples.

Example 9.2. Contemplate an irreducible s-representation of a compact connected Lie group K on $\mathbb{R}^n$, and suppose that $K \cdot w$ is a symmetric orbit. Consider also an action of $G = SO(k_1) \times SO(k_2) \times K$ on $\mathbb{R}^{k_1} \oplus \mathbb{R}^{k_2} \oplus \mathbb{R}^n$ in the standard way, and fix $v = e_1 + e_2 + w$, where where $e_i \in \mathbb{R}^{k_i}$ is of unit length for $i = 1, 2$. Finally, define the curve $\gamma(t) = a\cos(t)e_1 + b\sin(t)e_2 + (\cos(2t)+2)w$, with arbitrary a, $b > 0$. Then the image of $\gamma(t)$ is invariant under the reflection on $e_i^\perp$ for $i = 1, 2$ (in the space $\mathbb{R}e_1 \oplus \mathbb{R}e_2 \oplus \mathbb{R}w$), and it is easy to see that $G \cdot \gamma(\mathbb{R})$ is an extrinsic almost symmetric submanifold diffeomorphic to $S^{k_1+k_2-1} \times K \cdot w$.

Finally, we discuss why the cohomogeneity one property of the above examples is not unexpected. Indeed below we shall consider a more general situation of an *intrisically almost symmetric space* M, namely, for each $p \in M$ there exists an (intrinsic) isometry of M of order 2 fixing p and having a fixed point set of dimension 1 in T_pM.

We start with a complete Riemannian manifold M, and let K be a (possibly disconnected) closed subgroup of its isometry group. Then K acts properly on M, and the K-orbits in M are properly embedded submanifolds. Assume that K does not act transitively on M, and fix a connected component N of a K-orbit, so that $N = K^0p$ for some $p \in M$. Let $\mathcal{T}$ be an open tubular neighborhood of N in M. We can and will assume that $\mathcal{T}$ has radius $\epsilon > 0$ smaller than half of the distance from N to any other connected component of Kp. Denote by $\pi : \mathcal{T} \to N$ the map that assigns a point in $\mathcal{T}$ to its closest point in N.

Lemma 9.3. *The isotropy groups $K_z \subset K_{\pi(z)}$ for all $z \in \mathcal{T}$.*

Proof. For $z \in \mathcal{T}$, there is a unique shortest geodesic segment γ from z to $\pi(z)$. Let $k \in K_z$. Then $k \circ \gamma$ is a geodesic segment of the same length from z to $Kp = K \cdot N$. By our choice of ϵ, we must have $k \cdot N = N$. Due to the uniqueness above, $k \circ \gamma = \gamma$. In particular, $k\pi(z) = \pi(z)$, as wished. □

Remark 9.4. If M as above is in addition almost symmetric, and we take K to contain the almost symmetries at all points, then we see that the almost symmetry σ_z at $z \in \mathcal{T} \setminus N$ is unique. In fact it must fix the geodesic γ through $z = \gamma(0)$ and $q = \pi(z) = \gamma(1)$. It also follows that $d\sigma_z(\gamma'(1)) = \gamma'(1)$, and that σ_z is an almost symmetry at q. This construction indeed shows that for all $q \in N$ and all $\xi \in \nu_qN$, there exists an almost symmetry of M at q fixing ξ. It follows that almost symmetries along points of N are not unique if the codimension of N in M is bigger than one.

We have come to the proof of of the next main result of the paper.

Proof of Theorem 2.3. Let $p \in \Omega$. Then Kp is a principal orbit. It follows that the slice representation at p is trivial. If σ is an almost symmetry at p, then $\sigma \in K_p$, so $d\sigma_p$ is the identity on $\nu_p(Kp) \subset T_pM$. But the fixed point set of $d\sigma_p$ in T_pM is one-dimensional, therefore we must have $\dim \nu_p(Kp) = 1$. Since Kp is a principal orbit, this proves (a). Also the fix point set of $d\sigma_p$ must be $\nu_p(Kp)$, so this completly determines σ and proves (b).

Let N be a K^0-orbit of the codimension $k \geq 2$. Owing to Remark 9.4, for all $p \in N$ and $\xi \in \nu_pN$, there exists a unique almost symmetry σ_ξ at p such that $d(\sigma_\xi)_p(\xi) = \xi$; then we necessarily have $d(\sigma_\xi)_p|_{T_pN} = -\mathrm{Id}_{T_pN}$. If ξ, $\eta \in \nu_pN$ are linearly independent, then $\sigma_\xi \circ \sigma_\eta \in K_p$, and $d(\sigma_\xi \circ \sigma_\eta)_p$ is a rotation in the 2 -plane Π spanned by ξ, η, and the identity on the orthogonal complement $\Pi^\perp$ in T_pM. If we take ξ and η forming an angle incommensurate with π, the cyclic group generated by $\sigma_\xi \circ \sigma_\eta$ acting on Π is dense in $SO(\Pi) \cong SO(2)$. Let H denote the closure of the subgroup of K_p generated by products $\sigma_\xi \circ \sigma_\eta$, where ξ, $\eta \in \nu_pN$ are linearly independent. It is clear that H is a connected subgroup of K. Since Π can be any 2-plane in ν_pN, the above discussion shows H acts on $T_pM = T_pN \oplus \nu_pN$ as $\mathrm{Id} \times SO(\nu_pN)$. Since the isotropy representation is faithful, the Lie algebra of H is isomorphic to $\mathfrak{so}_k$. Note that H is a normal subgroup of K^0 (and more generally of the K-stabilizer of N), since it precisely consists of those elements that act trivially on N. This proves (c) and (d), because N is the connected component through p of the fixed point set of H. Finally, part (e) is immediate from the existence of the almost symmetry σ_ξ of M fixing any given $\xi \in \nu_q(Kq)$ and restricting to minus identity along $T_q(Kq)$, where $q \in M$ is arbitrary. □

We will use the following lemmata in the proof of Theorem 2.4, the proof of the first one being strightforward.

Lemma 9.5. *Let G, H, K be Lie groups such that K is a compact subgroup of G and H is compact, and let $\rho : K \times H \to O(V)$ be a representation such that $\rho(K)$ is trivial. Then the map*

$$\Phi : (G \times H)_{K\times H} V \to G/K \times V,\ \Phi[(g,h),v] = (gK, hv)$$

between homogeneous vector bundles over G/K defines a G-equivariant diffeomorphism.

Lemma 9.6. *(*[AA93, GWZ08],[Bre72, Ch. IV, Thms. 8.1 and 8.2]*) A cohomogeneity* 1 *Riemannian manifold is determined by its isotropy groups.*

Proof of Theorem 2.4. Let M be a simply-connected complete almost symmetric space. Denote by K its group of isometries. Suppose that M is inhomogeneous. Due to Theorem 2.3, K acts on M with cohomogeneity 1 and has principal orbits that are symmetric spaces. It follows that the principal orbit K/H is the product of certain irreducible symmetric spaces, and the regular set M_{reg} is a multiply warped product $(0,1) \times_f K/H$ where the metric on N is scaled independently on each factor.

We also know that there are no exceptional orbits, and that K and K^0 have the same orbits [GWZ08, pp. 41-42]. For the sake of simplicity, in the sequel we work with K^0 and drop the superscript from the notation.

According to [Mos57], the orbit space M/K is one of the following:

(a) $(0,1)$;
(b) S^1 ;
(c) $[0,1)$;
(d) $[0,1]$.

In case (a) there are no singular orbits and $M = (0,1) \times_f K/H$. Case (b) cannot occur due to the assumption that M is simply-connected.

The following remark will be useful. Suppose $q \in M$ is a singular point, and denote by $k \geq 2$ the codimension of the singular orbit $N = Kq$ in M. Then N is a symmetric space and $K = L \times K_1$, where $K_1 \cong SO(k)$. Also, the isotropy group $K_q = L_q \times K_1$, and a tubular neighborhood of N in M is diffeomorphic to

$$K \times_{K_q} \mathbb{R}^k = L/L_q \times \mathbb{R}^k = N \times \mathbb{R}^k,$$

namely, a product, by Lemma 9.5.

In case (c) there is only one singular orbit, so we deduce that M is a Riemannian product $L/H \times \mathbb{R}^k$, where L/H is a simply-connected symmetric space and $\mathbb{R}^k$ carries a rotationally invariant metric.

In case (d) there are two singular orbits, and there are two subcases. In the first one, $K = K_1 \times K_2 \times L$, where $K_i \cong SO(k_i)$, $k_i \geq 2$ and $i = 1, 2$. The principal orbits are symmetric spaces $K_1/H_1 \times K_2/H_2 \times L/H$, and the singular orbits are $K_1/H_1 \times L/H$ and $K_2/H_2 \times L/H$, where $H_i \cong SO(k_i - 1)$ for $i = 1, 2$.

The action of $K_1 \times K_2 \cong SO(k_1) \times SO(k_2)$ on $S^{k_1+k_2-1} \subset \mathbb{R}^{k_1} \times \mathbb{R}^{k_2}$ has cohomogeneity 1 with singular isotropy groups $H_1 \times K_2$ and $K_1 \times H_2$, and principal isotropy group $H_1 \times H_2$. Let now $(K_1 \times K_2) \times L$ act on $S^{k_1+k_2-1} \times L/H$ componentwise. This is an action of cohomogeneity 1 with isotropy groups as above and hence M is diffeomorphic to $S^{k_1+k_2-1} \times L/H$ by Lemma 9.5. The principal orbits are $S^{k_1} \times S^{k_2} \times L/H$. The admissible metrics scale the two spheres and each irreducible factor of L/H independently as a function of t, the arc-length parameter along a normal geodesic.

The second subcase is $K = K_1 \times L$, where $K_1 \cong SO(k_1)$ for $k_1 \geq 2$. The principal orbits are symmetric spaces $K_1/H_1 \times L/H$, and the two singular orbits are $K_1/H_1 \times L/H$, where $H_1 \cong SO(k_1 - 1)$. Here M is diffeomorphic to $S^{k_1} \times L/H$. The principal orbits are $S^{k_1-1} \times L/H$, and the admissible metrics scale the sphere and each irreducible factor of L/H independently as a function of t, the arc-length parameter along the normal geodesic.

Hence in either subcase M is as stated in the theorem. □

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