The maximum sum of sizes of non-empty cross L-intersecting families

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Abstract

Let n, r, and k be positive integers such that $k, r \geq 2$, L a non-empty subset of $[k]$, and $\mathcal{F}_i \subseteq \binom{[n]}{k}$ for $1 \leq i \leq r$. We say that non-empty families $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_r$ are r-cross L-intersecting if $|\bigcap_{i=1}^{r} F_i| \in L$ for every choice of $F_i \in \mathcal{F}_i$ with $1 \leq i \leq r$. They are called pairwise cross L-intersecting if $|A \cap B| \in L$ for all $A \in \mathcal{F}_i$, $B \in \mathcal{F}_j$ with $i \neq j$. If $r = 2$, we simply say cross L-intersecting instead of 2-cross L-intersecting or pairwise cross L-intersecting. In this paper, we determine the maximum possible sum of sizes of non-empty cross L-intersecting families $\mathcal{F}_1$ and $\mathcal{F}_2$ for all admissible n, k, and L, and we characterize all the extremal structures. We also establish the maximum value of the sum of sizes of families $\mathcal{F}_1, \ldots, \mathcal{F}_r$ that are both pairwise cross L-intersecting and r-cross L-intersecting, provided n is sufficiently large and L satisfies certain conditions. Furthermore, we characterize all such families attaining the maximum total size.

Keywords: Extremal set theory; cross L-intersecting; pairwise cross L-intersecting; r-cross L-intersecting

1 Introduction

Set $\binom{X}{k}$ to denote the family of all k-subsets of X. In particular, for a positive integer n, let $[n]$ denote the set $\{1, 2, \ldots, n\}$, and $[k, n] = \{k, \ldots, n\}$ for $k \leq n$. For two positive integers a, b, we set $\binom{a}{b} = 0$ if $b > a$.

Let n, t, and k be positive integers such that $k \geq 2$ and $1 \leq t \leq k-1$. A family $\mathcal{A} \subseteq \binom{[n]}{k}$ is called *t-intersecting* if $|A \cap B| \geq t$ for all $A, B \in \mathcal{A}$. When $t = 1$, we say $\mathcal{A}$ is intersecting rather than t-intersecting. The celebrated Erdős–Ko–Rado theorem [2] states that for $n \geq 2k$, a full star is the unique intersecting k-uniform family of maximum size.

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Theorem 1.1 (**[2]**) *Let $n \geq 2k$, and let $\mathcal{A} \subseteq \binom{[n]}{k}$ be an intersecting family. Then*

$$|\mathcal{A}| \leq \binom{n-1}{k-1}.$$

Moreover, when $n > 2k$, equality holds if and only if $\mathcal{A} = \{A \in \binom{[n]}{k} : i \in A\}$ for some $i \in [n]$.

The Erdős–Ko–Rado theorem is widely regarded as a cornerstone of extremal combinatorics and has led to numerous generalizations and applications. More generally, a family $\mathcal{A}$ is said to be *t-intersecting* if the size of the intersection of any two distinct sets lies in the interval $[t, k]$. Viewing $[t, k]$ as a subset of $[k]$, one can naturally extend the notion of t-intersection to that of L-intersection.

Specifically, for a non-empty set $L \subseteq [k]$, a family $\mathcal{A}$ is called *L-intersecting* if $|A \cap B| \in L$ for all distinct $A, B \in \mathcal{A}$. The problem of determining the maximum size of an L-intersecting family has been extensively studied in extremal set theory; see, for example, [4]. The celebrated Deza–Erdős–Frankl theorem provides a general upper bound for such families and describes the rough structure of L-intersecting families that are nearly optimal in size.

Theorem 1.2 (**[1]**) *Let $k \geq 3$, $n \geq 2^k k^3$, and $L \subseteq [0, k]$. If a family $\mathcal{F} \subseteq \binom{[n]}{k}$ is L-intersecting, then $|\mathcal{F}| \leq \prod_{\ell \in L} \frac{n-\ell}{k-\ell}$. Moreover, there exists a constant $C = C(k, L)$ such that every L-intersecting family $\mathcal{F}' \subseteq \binom{[n]}{k}$ with $|\mathcal{F}'| \geq Cn^{|L|-1}$ satisfies $\left|\bigcap_{A \in \mathcal{F}'} A\right| \geq \min L$.*

For non-empty families $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_r \subseteq \binom{[n]}{k}$, they are said to be *pairwise cross t-intersecting* if $|A \cap B| \geq t$ for all $A \in \mathcal{A}_i$, $B \in \mathcal{A}_j$ with $i \neq j$. They are called *r-cross t-intersecting* if $|\bigcap_{i=1}^r A_i| \geq t$ for all $A_i \in \mathcal{A}_i$.

This paper generalizes the study of cross t-intersecting families to the cross L-intersecting families. For a non-empty set $L \subseteq [0, k]$, families $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_r \subseteq \binom{[n]}{k}$ are called *pairwise cross L-intersecting* if $|A \cap B| \in L$ for all $A \in \mathcal{A}_i$, $B \in \mathcal{A}_j$ with $i \neq j$. They are called *r-cross L-intersecting* if $|\bigcap_{i=1}^r A_i| \in L$ for all $A_i \in \mathcal{A}_i$. Note that when $L = [t, k]$, pairwise cross L-intersecting (resp. r-cross L-intersecting) reduces to pairwise cross t-intersecting (resp. r-cross t-intersecting). When $r = 2$, we simply say *cross L-intersecting* or *cross t-intersecting* instead of 2-cross L-intersecting or 2-cross t-intersecting. In what follows, let $L \subseteq [0, k]$ be a non-empty set, and define $k - L = \{k - i : i \in L\}$. For $\mathcal{A} \subseteq \binom{[n]}{k}$, let $\overline{\mathcal{A}} = \binom{[n]}{k} \setminus \mathcal{A}$.

Wang and Zhang [16] determined the maximum total size of two cross t-intersecting families of finite sets.

Theorem 1.3 (**[16]**) *Let n, a, b, t be positive integers such that $n \geq 4$, $a, b \geq 2$, $t < \min\{a, b\}$, $a + b < n + t$, $(n, t) \neq (a + b, 1)$, and $\binom{n}{a} \leq \binom{n}{b}$. If non-empty families $\mathcal{A} \subseteq \binom{[n]}{a}$ and $\mathcal{B} \subseteq \binom{[n]}{b}$ are cross t-intersecting, then*

$$|\mathcal{A}| + |\mathcal{B}| \leq \binom{n}{b} - \sum_{i=0}^{t-1} \binom{a}{i} \binom{n-a}{b-i} + 1.$$

The main method in the proof of the above theorem involves analyzing a bipartite graph defined in [16]. Using a similar approach, we generalize their result in a specific setting. Let $\mathcal{A}, \mathcal{B} \subseteq \binom{[n]}{k}$. Note that when $k \leq n < 2k$ and $[2k-n, k] \cap L = \emptyset$, any two k-sets in $\binom{[n]}{k}$ intersect in at least $2k - n$ elements, which forces one of $\mathcal{A}$ or $\mathcal{B}$ to be empty. Hence, we first make the following remark.

Remark. If $n < 2k$ and $[2k-n, k] \cap L = \emptyset$, then there exist no non-empty cross L-intersecting families $\mathcal{A}, \mathcal{B} \subseteq \binom{[n]}{k}$.

We now present our first result.

Theorem 1.4 *Let n, k be positive integers with $n \geq k \geq 2$, and $\emptyset \neq L \subseteq [0, k]$. Let $\mathcal{A}, \mathcal{B} \subseteq \binom{[n]}{k}$ be non-empty cross L-intersecting families.*

(i) *If $n \geq 2k$ and $L = [0, k]$, or if $n < 2k$ and $[2k-n, k] \subseteq L$, then*

$$|\mathcal{A}| + |\mathcal{B}| \leq 2\binom{n}{k},$$

and equality holds if and only if $\mathcal{A} = \mathcal{B} = \binom{[n]}{k}$.

(ii) *If $n > 2k$ and $L \neq [0, k]$, or $n = 2k$ and $L \notin \{[0, k], k - L\}$, or $n < 2k$ and $[2k-n, k] \cap L \notin \{\emptyset, [2k-n, k]\}$, then*

$$|\mathcal{A}| + |\mathcal{B}| \leq \sum_{i \in L} \binom{k}{i}\binom{n-k}{k-i} + 1,$$

and equality holds if and only if (up to isomorphism) one of the following holds:

1. $\mathcal{A} = \{[k]\}$, $\mathcal{B} = \left\{F \in \binom{[n]}{k} : |F \cap [k]| \in L\right\}$;
2. $n \geq 2k$ *and* $L = [0, k-1]$, *or* $n < 2k$ *and* $[2k-n, k-1] \subseteq L$, *and* $\mathcal{A} = \overline{\mathcal{B}}$;
3. $k = 2$, $L = \{1, 2\}$, *and* $\mathcal{A} = \mathcal{B} = \left\{F \in \binom{[n]}{2} : 1 \in F\right\}$;
4. $k = n-2$, $L \cap [k-2, k] = \{k-1, k\}$, *and* $\mathcal{A} = \mathcal{B} = \binom{[n-1]}{n-2}$.

(iii) *If $n = 2k$ and $L = k - L$, then*

$$|\mathcal{A}| + |\mathcal{B}| \leq \sum_{i \in L} \binom{k}{i}\binom{n-k}{k-i} + 2,$$

and equality holds if and only if (up to isomorphism) one of the following holds:

1. $\mathcal{A} = \{[k], [k+1, 2k]\}$, $\mathcal{B} = \left\{F \in \binom{[n]}{k} : |F \cap [k]| \in L\right\}$;
2. $L = [k-1]$, $\emptyset \neq \mathcal{A} = \overline{\mathcal{B}} \subseteq \binom{[n]}{k}$, *and $\mathcal{A}$ is closed under complementation, i.e., if $A \in \mathcal{A}$ then $[n] \setminus A \in \mathcal{A}$.*

We now turn to the pairwise cross L-intersecting case. Shi, Frankl, and Qian [13] determined the maximum total size of pairwise L-intersecting families when $L = [k]$, i.e., when every two sets from different families have non-empty intersection. When $L = [k]$, we call the pairwise cross L-intersecting families the *pairwise cross-intersecting families*.

Theorem 1.5 ([13]) *Let n, k, t be positive integers with $n \geq 2k$ and $r \geq 2$. If non-empty families $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_r \subseteq \binom{[n]}{k}$ are pairwise cross-intersecting, then*

$$\sum_{i=1}^{r} |\mathcal{A}_i| \leq \max\left\{ \binom{n}{k} - \binom{n-k}{k} + r - 1,\ r\binom{n-1}{k-1} \right\}.$$

Subsequently, Li and Zhang determined the maximum total size of pairwise cross t-intersecting families.

Theorem 1.6 ([11]) *Let n, k, t, r be positive integers with $n \geq 2k - t + 1$, $k > t \geq 1$, and $r \geq 2$. If non-empty families $\mathcal{A}_1, \ldots, \mathcal{A}_r \subseteq \binom{[n]}{k}$ are pairwise cross t-intersecting, then*

$$\sum_{i=1}^{r} |\mathcal{A}_i| \leq \max\left\{ \binom{n}{k} - \sum_{i=0}^{t-1} \binom{k}{i}\binom{n-k}{k-i} + r - 1,\ r \cdot M(n, k, t) \right\},$$

where $M(n, k, t)$ denotes the maximum size of a t-intersecting family in $\binom{[n]}{k}$.

Many related results can be found in [17, 8]. We generalize the above result to pairwise cross L-intersecting families.

Theorem 1.7 *Let n, k, r be positive integers with $k, r \geq 2$, and $\emptyset \neq L \subseteq [0, k]$. Let $\mathcal{A}_1, \ldots, \mathcal{A}_r \subseteq \binom{[n]}{k}$ be non-empty pairwise cross L-intersecting families with $|\mathcal{A}_1| \leq |\mathcal{A}_2| \leq \cdots \leq |\mathcal{A}_r|$. For sufficiently large n, the following results hold.*

(i) *If $L = [0, k]$, then*

$$\sum_{i=1}^{r} |\mathcal{A}_i| \leq r\binom{n}{k},$$

and equality holds if and only if $\mathcal{A}_1 = \cdots = \mathcal{A}_r = \binom{[n]}{k}$ (up to isomorphism).

(ii) *If $L = [t, k]$ for some $t \in [k]$, then*

$$\sum_{i=1}^{r} |\mathcal{A}_i| \leq \max\left\{ \binom{n}{k} - \sum_{i=0}^{t-1} \binom{k}{i}\binom{n-k}{k-i} + r - 1,\ r\binom{n-t}{k-t} \right\}.$$

(iii) *If $L = [0, k-1]$, then*

$$\sum_{i=1}^{r} |\mathcal{A}_i| \leq \binom{n}{k},$$

and equality holds if and only if $\mathcal{A}_1, \ldots, \mathcal{A}_r$ are disjoint and $\bigcup_{i=1}^{r} \mathcal{A}_i = \binom{[n]}{k}$.

(iv) *If $k \in L$ and $L \neq [t,k]$ for any $t \in [0,k]$, then*

$$\sum_{i=1}^{r} |\mathcal{A}_i| \leq \sum_{i \in L} \binom{k}{i}\binom{n-k}{k-i} + r - 1,$$

and equality holds if and only if $\mathcal{A}_1 = \cdots = \mathcal{A}_{r-1} = \{[k]\}$ and $\mathcal{A}_r = \left\{A \in \binom{[n]}{k} : |A \cap [k]| \in L\right\}$.

There are several partial results concerning the maximum sizes of r-cross t-intersecting families, starting with Hilton [6] and with Hilton and Milner [7]. Recently, Gupta, Mogge, Piga and Schülke [5] gave the maximum sum of sizes of r-cross t-intersecting families.

Theorem 1.8 ([5]) *Let $r \geq 2$, and $n, t \geq 1$ be integers, $k \in [n]$, and for $i \in [r]$ let $\mathcal{A}_i \subseteq \binom{[n]}{k}$. If $\mathcal{A}_1, \ldots, \mathcal{A}_r$ are non-empty r-cross t-intersecting families and $n \geq 2k - t$, then*

$$\sum_{i=1}^{r} |\mathcal{A}_i| \leq \max_{m \in [t,k]} \left\{ \sum_{i=t}^{k} \binom{m}{i}\binom{n-m}{k-i} + (r-1)\binom{n-m}{k-m} \right\},$$

and this bound is attained.

Let $L = [\ell, t]$. We generalize the result about r-cross t-intersecting families to r-cross L-intersecting families. By Theorem 1.8, we assume $t \neq k$ and obtain the following result.

Theorem 1.9 *Let n, k, r be positive integers with $k, r \geq 2$. Let $\mathcal{A}_1, \ldots, \mathcal{A}_r \subseteq \binom{[n]}{k}$ be non-empty r-cross $[\ell, s-1]$-intersecting families, where $\ell \in [0,k]$ and $s \in [k]$. When n is large enough, we have*

$$\sum_{i=1}^{r} |\mathcal{A}_i| \leq (r-1)\binom{n-\ell}{k-\ell} - \sum_{i=s-\ell}^{k-\ell} \binom{k-\ell}{i}\binom{n-k}{k-\ell-i} + 1,$$

and this bound is attained.

For a family $\mathcal{F} \subset \binom{[n]}{k}$ and $0 \leq i \leq n$, the *i-shadow* $\partial_i(\mathcal{F})$ is defined by $\partial_i(\mathcal{F}) = \left\{G \in \binom{[n]}{i} : G \subset F \text{ for some } F \in \mathcal{F}\right\}$. If $\mathcal{F} = \{F\}$, we simply write $\partial_i(F)$ for $\partial_i(\mathcal{F})$. For given positive integers m, k, i with $k > i$, the problem of determining the minimum size of $\partial_i(\mathcal{F})$ over all families $\mathcal{F} \subset \binom{[n]}{k}$ of size m was completely solved by Kruskal [10] and Katona [9]. This result is of fundamental importance in combinatorics and will be useful in our proofs. In fact, we only require a slightly weaker version due to Lovász [12]. For any real number $x > k-1$, define $\binom{x}{k} = x(x-1)\cdots(x-k+1)/k!$. Thus, for every positive integer m and fixed k, there exists a unique $x > k-1$ such that $\binom{x}{k} = m$.

Theorem 1.10 ([12]) *Let $1 \leq i < k \leq n$ and $\mathcal{F} \subset \binom{[n]}{k}$ with $|\mathcal{F}| \geq \binom{x}{k}$. Then*

$$|\partial_i(\mathcal{F})| \geq \binom{x}{i}.$$

The above theorem establishes that the size of a family $\mathcal{F}$ yields a lower bound on the size of its i-shadow. Conversely, the size of the i-shadow of $\mathcal{F}$ implies an upper bound on the size of $\mathcal{F}$ itself. This leads naturally to the following corollary.

Corollary 1.11 *Let $1 \leq i < k \leq n$ and $\mathcal{F} \subset \binom{[n]}{k}$ with $|\partial_i \mathcal{F}| \leq \binom{x}{i}$. Then*

$$|\mathcal{F}| \leq \binom{x}{k}.$$

Proof. Suppose $|\mathcal{F}| > \binom{x}{k}$, say $|\mathcal{F}| = \binom{x_0}{k}$ with $x_0 > x$. By Theorem 1.10, $|\partial_i(\mathcal{F})| \geq \binom{x_0}{i} > \binom{x}{i}$, a contradiction. $\square$

This paper is organised as follows. In Section 2, we will prove Theorem 1.4. In Section 3, we will prove Theorem 1.7, and in Section 4, we will give the proof of Theorem 1.9. Some open problems are given in Section 5.

2 Cross L-intersecting families

Let X be a finite set and Γ be a group transitively acting on X. We say that the action of Γ on X is **primitive**, if Γ preserves no nontrivial partition of X. In any other case, the action of Γ is **imprimitive**. It is easy to see that if the action of Γ on X is transitive and imprimitive, then there is a subset B of X such that $1 < |B| < |X|$ and $\gamma(B) \cap B = B$ or $\emptyset$ for every $\gamma \in \Gamma$. In this case, B is called an **imprimitive set** in X. Otherwise, B is called a **primitive set** in X. Furthermore, a subset B of X is said to be **semi-imprimitive** if $1 < |B| < |X|$ and $|\gamma(B) \cap B| = 0$, 1 or $|B|$ for each $\gamma \in \Gamma$.

Let $G(X,Y)$ be a non-complete bipartite graph and let $A \cup B$ be a nontrivial independent set of $G(X,Y)$ (that is, $A \cup B \not\subseteq X$ and $A \cup B \not\subseteq Y$), where $A \subseteq X$ and $B \subseteq Y$. If $A \subseteq X \setminus N(B)$ and $B \subseteq Y \setminus N(A)$, we have

$$|A| + |B| \leq \max\{|A| + |Y| - |N(A)|, |B| + |X| - |N(B)|\}.$$

We set

$$\epsilon(X) = \min\{|N(A)| - |A| : \ A \subseteq X \text{ and } N(A) \neq Y\},$$

and

$$\epsilon(Y) = \min\{|N(B)| - |B| : \ B \subseteq Y \text{ and } N(B) \neq X\}.$$

Let $\alpha(X,Y)$ denote the independent number of $G(X,Y)$. Then one sees that

$$\alpha(X,Y) = \max\{|Y| - \epsilon(X), |X| - \epsilon(Y)\}.$$

A subset A of X is called a **fragment** of $G(X,Y)$ in X if $N(A) \neq Y$ and $|N(A)| - |A| = \epsilon(X)$. We denote $\mathcal{F}(X)$ as the set of all fragments in X. Similarly we can define $\mathcal{F}(Y)$. Let $\mathcal{F}(X,Y) = \mathcal{F}(X) \cup \mathcal{F}(Y)$. $A \in \mathcal{F}(X,Y)$ is called a k-fragment if $|A| = k$. A bipartite graph $G(X,Y)$ is called **part-transitive** if there is a group Γ that transitively acts on X and Y

respectively and preserves the adjacent relationship. If $G(X,Y)$ is part-transitive, then every two vertices $u_1, u_2 \in X$ (resp. $v_1, v_2 \in Y$) has the same degree and we use $d(X)$ (resp. $d(Y)$) to denote the degree of vertices in X (resp. Y).

We need the following result about the part-transitive bipartite graphs which proved in [16].

Theorem 2.1 ([16]) *Let $G(X,Y)$ be a non-complete bipartite graph with $|X| \leq |Y|$. If $G(X,Y)$ is part-transitive and every fragment of $G(X,Y)$ is primitive under the action of a group Γ, then*

$$\alpha(X,Y) = |Y| - d(X) + 1.$$

Moreover,

1. *if $|X| < |Y|$, then X has only 1-fragments;*
2. *if $|X| = |Y|$, then each fragment in X has size 1 or $|X| - d(X)$ unless there is a semi-imprimitive fragment in X or Y.*

Theorem 2.1 completely solves the case where every fragment of $G(X,Y)$ is primitive under the action of the group Γ. However, we must also consider situations where some fragments are imprimitive.

A bijection $\phi : \mathcal{F}(X,Y) \to \mathcal{F}(X,Y)$ is defined by

$$\phi(A) = \begin{cases} Y \setminus N(A) & \text{if } A \in \mathcal{F}(X), \\ X \setminus N(A) & \text{if } A \in \mathcal{F}(Y). \end{cases}$$

The following properties are proposed in [16]. The mapping ϕ is an involution, satisfying $\phi(\phi(A)) = A$ for all $A \in \mathcal{F}(X,Y)$. Furthermore, it satisfies the size relation $|A| + |\phi(A)| = \alpha(X,Y)$. A fragment A is called **balanced** if $|A| = |\phi(A)|$. It follows immediately that all balanced fragments have size $\frac{1}{2}\alpha(X,Y)$.

Lemma 2.2 ([16]) *Let $G(X,Y)$ be a non-complete and part-transitive bipartite graph under the action of a group Γ. Suppose that $A \in \mathcal{F}(X,Y)$ such that $\emptyset \neq \gamma(A) \cap A \neq A$ for some $\gamma \in \Gamma$. If $|A| \leq |\phi(A)|$, then $A \cup \gamma(A)$ and $A \cap \gamma(A)$ are both in $\mathcal{F}(X,Y)$.*

Define $\mathcal{I}(X)$ (resp. $\mathcal{I}(Y)$) to be the collection of all the imprimitive fragments in X (resp. Y). Then we have the following result.

Theorem 2.3 *Let $G(X,Y)$ be a non-complete bipartite graph with $|X| = |Y|$. If $G(X,Y)$ is part-transitive under the action of a group Γ and $\alpha(X,Y) > |Y| - d(X) + 1$, then there exists an imprimitive fragment in X. Moreover, each fragment in X of size no more than $\frac{1}{2}\alpha(X,Y)$ is in $\mathcal{I}(X)$ unless there is a primitive fragment A of size no more than $\frac{1}{2}\alpha(X,Y)$ in X such that $A \cap \gamma(A) \in \mathcal{I}(X) \cup \{\emptyset\} \cup \{A\}$ for any $\gamma \in \Gamma$.*

Proof. By Theorem 2.1 and $\alpha(X,Y) > |Y| - d(X) + 1$, there exists an imprimitive fragment in X.

Assume there exists a fragment, say A, in X such that $|A| < \frac{1}{2}\alpha(X,Y)$ and $A \notin \mathcal{I}(X)$. We choose A such that $|A|$ as small as possible. Since $\alpha(X,Y) > |Y| - d(X) + 1$, $|A| \geq 2$. Since $A \notin \mathcal{I}(X)$, A is primitive. For any $\gamma \in \Gamma$ with $\emptyset \neq \gamma(A) \cap A \neq A$, since $|A| < \frac{1}{2}\alpha(X,Y)$, we have $|A| \leq |\phi(A)|$. By Lemma 2.2, $\gamma(A) \cap A$ is a fragment. From the minimality of A, $\gamma(A) \cap A \in \mathcal{I}(X)$, which ends the proof. $\square$

To determine the extremal cases, we also need the following properties.

Lemma 2.4 ([16]) *Let $G(X,Y)$ be a non-complete bipartite graph with $|X| = |Y|$ and $\epsilon(X) = d(X) - 1$, and let Γ be a group acting part-transitively on $G(X,Y)$. If there exists a 2-fragment in X, then either*

1. *there exists an imprimitive set $A \subseteq X$ satisfying $|N(A)| - |A| = d(X) - 1$, or*
2. *there exists a subset $A \subseteq X$ (where either $A = X$ or A is Γ-imprimitive with $|A| > 2$) such that the quotient group $\Gamma_A / \left(\bigcap_{a \in A} \Gamma_a\right)$ is isomorphic to a subgroup of the dihedral group $D_{|A|}$. Here, $\Gamma_A = \{\sigma \in \Gamma \mid \sigma(A) = A\}$ denotes the stabilizer of A.*

Lemma 2.5 ([16]) *Let $G(X,Y)$ satisfy the hypotheses of Lemma 2.4. If every fragment of $G(X,Y)$ is primitive and $\mathcal{F}(X,Y)$ contains no 2-fragments, then every nontrivial fragment $A \in \mathcal{F}(X)$ (if any exists) is balanced, and for each $a \in A$, there exists a unique nontrivial fragment B such that $A \cap B = \{a\}$.*

Now we are going to prove Theorem 1.4. The main idea comes from [16].

Proof of Theorem 1.4. Let $\mathcal{A}, \mathcal{B} \subseteq \binom{[n]}{k}$ be non-empty cross L-intersecting.

We construct a bipartite graph $G(X,Y)$, where $X = Y = \binom{[n]}{k}$ and for $A \in X$ and $B \in Y$, $AB \in E(G)$ if and only if $|A \cap B| \notin L$, meaning A and B are not L-intersecting.

Let S_n denote the symmetric group on $[n]$. The group S_n acts transitively on both X and Y while preserving the cross L-intersecting property. Consequently, for any $A \in X$, $|N_G(A)| = d(X)$ and similarly $|N_G(B)| = d(Y)$ for all $B \in Y$. Let $\overline{L} = \{0, 1, \ldots, k\} \setminus L$. For any $A \in X$ and $B \in Y$,

$$N_G(A) = \bigcup_{i \in \overline{L}} \left\{ B \in \binom{[n]}{k} : |A \cap B| = i \right\} \quad \text{and} \quad N_G(B) = \bigcup_{i \in \overline{L}} \left\{ A \in \binom{[n]}{k} : |B \cap A| = i \right\}.$$

Thus we have $d(X) = d(Y) = \sum_{i \in \overline{L}} \binom{k}{i}\binom{n-k}{k-i}$. Moreover, $|\mathcal{A}| + |\mathcal{B}| \leq \alpha(X,Y)$, and the equality holds if and only if $\mathcal{A} \in \mathcal{F}(X)$ and $\mathcal{B} = Y \setminus N(\mathcal{A})$. Consequently, we transform the original problem into determining $\alpha(X,Y)$ and $\mathcal{F}(X,Y)$.

It is known that for $n \neq 2k$, the action of S_n on $\binom{[n]}{k}$ is primitive [14], making every fragment of $G(X,Y)$ primitive. For $n \geq 2k+1$ with $L \neq [0,k]$, or when $k \leq n < 2k$ and

$[2k-n,k]\cap L \notin \{\emptyset, [2k-n,k]\}$, $G(X,Y)$ is not complete bipartite. By Theorem 2.1, we obtain

$$\begin{aligned}\alpha(X,Y) = |Y| - d(X) + 1 &= \binom{n}{k} - \sum_{i\in\overline{L}} \binom{k}{i}\binom{n-k}{k-i} + 1 \\ &= \sum_{i\in L} \binom{k}{i}\binom{n-k}{k-i} + 1. \qquad (1)\end{aligned}$$

Now, we handle the case $n = 2k$. According to [14], the only imprimitive sets are complementary pairs $\{A, \overline{A}\}$ where $A \in \binom{[2k]}{k}$ and $\overline{A} = [n] \setminus A$. Let $A, \overline{A} \in X$. When $L = k - L$, for any such pair we have

$$\alpha(X,Y) \geq |Y| - |N(\{A, \overline{A}\})| + 2 = |Y| - d(X) + 2 > |Y| - d(X) + 1.$$

By Theorem 2.3, $\{A, \overline{A}\}$ forms an imprimitive fragment for some $A \in \binom{[n]}{k}$. So we have

$$\alpha(X,Y) = |Y| - d(X) + 2 = \sum_{i\in L} \binom{k}{i}\binom{n-k}{k-i} + 2. \qquad (2)$$

When $L \neq k - L$, we have that $|Y| - |N(\{A, \overline{A}\})| + 2 < |Y| - d(X) + 1$. So $\{A, \overline{A}\}$ cannot be a fragment which implies every fragment of $G(X,Y)$ is primitive under the action of S_n. By Theorem 2.1,

$$\alpha(X,Y) = |Y| - d(X) + 1 = \sum_{i\in L} \binom{k}{i}\binom{n-k}{k-i} + 1. \qquad (3)$$

We complete the proof by considering three cases.

Case 1. $n \geq 2k, L = [0,k]$ or $n < 2k, [2k-n,k] \subseteq L$.

In these cases, $G(X,Y)$ is an empty graph. Then $|\mathcal{A}| + |\mathcal{B}| \leq \alpha(X,Y) = 2\binom{n}{k}$, with equality when $\mathcal{A} = \mathcal{B} = \binom{[n]}{k}$.

Case 2. $n > 2k, L \neq [0,k]$ or $n = 2k, L \neq k - L$ or $n < 2k, [2k-n,k] \cap L \notin \{\emptyset, [2n-k,k]\}$.

In these cases, $|\mathcal{A}| + |\mathcal{B}| \leq \alpha(X,Y) = \sum_{i\in L}\binom{k}{i}\binom{n-k}{k-i} + 1$ by (1) and (3), and the equality holds if and only if $\mathcal{A} \in \mathcal{F}(X)$ and $\mathcal{B} = Y \setminus N(\mathcal{A})$. By Theorem 2.1, the equality holds if $|\mathcal{A}| = 1$ or $|\mathcal{B}| = 1$ unless there is a semi-imprimitive fragment in $\mathcal{F}(X)$. We now proceed to characterize all extremal cases by determining all semi-imprimitive fragment in $\mathcal{F}(X)$.

If $|\mathcal{A}| = 1$ or $|\mathcal{B}| = 1$, say $|\mathcal{A}| = 1$, then we let $\mathcal{A} = \{[k]\}, \mathcal{B} = \{F \in \binom{[n]}{k} : |F \cap [1,k]| \in L\}$ and we are done.

If $n \geq 2k, L = [0, k-1]$ or $n < 2k, L \cap [2k-n,k] = [2k-n,k-1]$, then for any $\emptyset \neq S \subsetneqq \binom{[n]}{k}$, we have $S \in \mathcal{F}(X,Y)$. Then we let $\mathcal{A} = \overline{\mathcal{B}}$ and we are done.

Now, we assume $n \geq 2k, L \neq [0,k-1]$ or $n < 2k, L \cap [2k-n,k] \neq [2k-n,k-1]$, and assume that $S \in \mathcal{F}(X)$ is semi-imprimitive. Then $2 \leq |S| < \binom{n}{k}$. We first have the following claim.

Claim 2.6 *There is a k-set $C \in S$, such that either $|C \cap B| = 1$ for every $B \in S \setminus \{C\}$, or $|C \cap B| = k-1$ for every $B \in S \setminus \{C\}$.*

Proof of Claim 2.6 All fragments in this case are primitive, and S_n is not isomorphic to a subgroup of $D_{|X|}$ for $n \geq 4$. By Lemma 2.4, there are no 2-fragments in $G(X,Y)$, which implies that S is balanced by Lemma 2.5.

For each subset $C \subseteq [n]$, let $S_C = \{\sigma \in S_n : \sigma \text{ fix all elements of } \overline{C}\}$. Given $C \in S$, let $\Gamma^C = S_C \times S_{\overline{C}}$, $\Gamma_S^C = \{\sigma \in \Gamma^C : \sigma(S) = S\}$, $\Gamma_S = \{\sigma \in S_n : \sigma(S) = S\}$. Then $C \in \sigma(S)$ for each $\sigma \in \Gamma_S^C$. Since $|S| \geq 2$, there exists some $C \in S$ for which $\Gamma_S^C \neq \Gamma^C$. (Otherwise, we have $S_C \times S_{\overline{C}} = \Gamma_S^C \subseteq \Gamma_S, \Gamma^B = S_B \times S_{\overline{B}} = \Gamma_S^B \subseteq \Gamma_S$ for any $B \in S \backslash \{C\}$. Since a complementary pair is not a fragment by directly calculation, assume $B \neq \overline{C}$. Then, $S_C \times S_{\overline{C}}$ and $S_B \times S_{\overline{B}}$ will generate S_n, which implies $\Gamma_S = S_n$ and $S = \binom{[n]}{k}$, a contradiction with S being a fragment.) Lemma 2.5 then shows that $[\Gamma^C : \Gamma_S^C]$, the index of Γ_S^C in Γ^C, equals 2.

Let $\Gamma_S[C]$ denote the projection of Γ_S^C onto S_C. Then $\Gamma_S[C]$ is a subgroup of S_C with index ≤ 2, meaning $\Gamma_S[C] = S_C$ or A_C, where A_C is an alternating group on C. Consequently, the only index-2 subgroups of Γ^C are $A_C \times S_{\overline{C}}$ and $S_C \times A_{\overline{C}}$. So Γ_S^C must be one of them.

For each $B \in S \backslash \{C\}$, we have $k = |B \cap C| + |B \cap \overline{C}|$. If $|B \cap C| > 1$ (resp. $|B \cap \overline{C}| > 1$), let (i,j) be a transposition with $i, j \in B \cap C$ (resp. $i, j \in B \cap \overline{C}$), then (i,j) fixes both B and C (resp. $\overline{C}$). By the semi-imprimitivity of S, we have $(i,j) \in \Gamma_S^C$, which forces $\Gamma_S^C = S_C \times A_{\overline{C}}$ (resp. $\Gamma_S^C = A_C \times S_{\overline{C}}$). This process reveals that for each $B \in S \setminus \{C\}$, at most one of $|B \cap C|$ and $|B \cap \overline{C}|$ exceeds 1.

If $B \subseteq \overline{C}$, then both S_C and S_B fix C and B, implying $S_C \times S_B \subseteq \Gamma_S^C$. However, neither $A_C \times S_{\overline{C}}$ nor $S_C \times A_{\overline{C}}$ contains $S_C \times S_B$, a contradiction. We thus conclude that either $|C \cap B| = 1$ for every $B \in S \setminus \{C\}$, or $|C \cap B| = k-1$ for every $B \in S \setminus \{C\}$. ■

We now characterize all nontrivial fragments of $G(X,Y)$ and the corresponding set L. By Claim 2.6, let $C \in S$ such that either $|C \cap B| = 1$ for every $B \in S \setminus \{C\}$, or $|C \cap B| = k-1$ for every $B \in S \setminus \{C\}$.

Assume $|C \cap B| = 1$ for every $B \in S \setminus \{C\}$. Without loss of generality, assume $C \cap B = \{1\}$ for some $B \in S$. If $k > 2$, then $|B \cap \overline{C}| \geq 2$, so $\Gamma_S^C = A_C \times S_{\overline{C}}$. However, we can find distinct $i, j \in C \setminus \{1\}$ such that $(1ij)(B) = (B \setminus \{1\}) \cup \{i\} \in S$ since $(1ij) \in A_C$. Consequently, $(1i)(S)$ contains more than one element of S, so $(1i) \in \Gamma_S^C$, leading to a contradiction. This proves $k = 2$, meaning S consists of all 2-subsets $\{1, i\}$ for $i \in [2, n]$. Since $L \subseteq [0, k]$, L must be one of $\{0\}, \{1\}, \{2\}, \{0,1\}, \{0,2\}, \{1,2\}$. Computing $d(X)$ and $N_G(S)$ while satisfying $|N_G(S)| - |S| = d(X) - 1$ reveals that $L = \{1, 2\}$. In this case, we let $\mathcal{A} = \mathcal{B} = \{F \in \binom{[n]}{2} : 1 \in F\}$ and we are done.

Next, assume $|C \cap B| = k - 1 > 1$ for every $B \in S \setminus \{C\}$. Similarly, we can show that $n - k = 2$, $L \cap [k-2, k] = \{k-1, k\}$, and $\Gamma_S^C = S_C \times A_{\overline{C}}$. Thus, $S = \binom{A}{n-2}$, where A is an $(n-1)$-subset of $[n]$. Therefore, $\mathcal{A} = \mathcal{B} = \binom{A}{n-2}$ is the extremal structure, where A is an $(n-1)$-subset of $[n]$.

Case 3. $n = 2k$, $L = k - L \neq [0, k]$.

In this case, $|\mathcal{A}|+|\mathcal{B}| \leq \alpha(X,Y) = \sum_{i\in L}\binom{k}{i}\binom{n-k}{k-i}+2$ by (2). If $\mathcal{A}=\{A,\overline{A}\}$ or $\mathcal{B}=\{B,\overline{B}\}$ (i.e., a complementary pair), the equality holds by the discussion above, where $A,B\in\binom{[n]}{k}$. Thus complementary pairs are in $\mathcal{F}(X)=\mathcal{F}(Y)$. Let $S\in\mathcal{F}(X)$. Then $|S|\leq\frac{1}{2}\alpha(X,Y)$.

Since $\alpha(X,Y)=\sum_{i\in L}\binom{k}{i}\binom{n-k}{k-i}+2$, by the discussion above, $|S|\geq 2$. If $|S|=2$ for any $S\in\mathcal{F}(X)$, then S is a complementary pair. Let $\mathcal{A}=\{[k],[k+1,2k]\},\mathcal{B}=\{F\in\binom{[n]}{k}:|F\cap[k]|\in L\}$ and we are done.

Now we assume there is $S\in\mathcal{F}(X)$ such that $|\phi(S)|\geq|S|\geq 3$. Choose S such that $|S|$ as smaller as possible. Note that for any $\gamma\in S_n$, $\gamma(S)\cap S$ is either empty, equal to S, or a complementary pair (otherwise, $\gamma(S)\cap S$ would be a fragment smaller than S by Lemma 2.2, a contradiction with the choice of S). The following properties hold for S.

Claim 2.7 (i) For any $A\in S$, $\overline{A}\in S$.

(ii) S consists of 2 or 3 different pairs of complementary k-sets. Moreover, if S consists of 3 such pairs, it must have the form $\{A_1\cup A_2, A_1\cup A_3, A_1\cup A_4, A_2\cup A_3, A_2\cup A_4, A_3\cup A_4\}$, where $\{A_i\}_{i=1}^4$ is a balanced partition of $[2k]$ with $|A_i|=\frac{k}{2}$.

Proof of Claim 2.7 (i) Suppose there exist a complementary pair $\{B,\overline{B}\}\subseteq S$ and a set $A\in S$ but $\overline{A}\notin S$. Choose $\gamma\in S_n$ such that $\gamma(A)=B$. Then $B\in\gamma(S)$, but $\overline{B}\notin\gamma(S)$, meaning $\gamma(S)\cap S$ is neither empty, nor S, nor a complementary pair, a contradiction.

Suppose for any $B\in S$, we have $\overline{B}\notin S$. Then $\gamma(S)\cap S$ can never be a complementary pair for any $\gamma\in S_n$ which implies S is primitive. However, all primitive sets are complementary pairs, a contradiction.

(ii) By (i) and $|S|\geq 3$, S consists of complementary k-set pairs. Assume S contains at least 3 complementary k-set pairs. Let $\{A,B,\overline{A},\overline{B}\}\subsetneq S$ and $\Gamma_S=\{\gamma\in S_n:\gamma(S)=S\}$.

Note that $\{A,B,\overline{A},\overline{B}\}\subseteq\gamma(S)\cap S$ for any $\gamma\in S_{A\cap B}\cup S_{A\cap\overline{B}}\cup S_{\overline{A}\cap B}\cup S_{\overline{A}\cap\overline{B}}$. Thus $S_{A\cap B}\cup S_{A\cap\overline{B}}\cup S_{\overline{A}\cap B}\cup S_{\overline{A}\cap\overline{B}}\subseteq\Gamma_S$. For any $\{C,\overline{C}\}\subseteq S\setminus\{A,B,\overline{A},\overline{B}\}$, we similarly have $S_{A\cap C}\cup S_{A\cap\overline{C}}\cup S_{\overline{A}\cap C}\cup S_{\overline{A}\cap\overline{C}}\subseteq\Gamma_S$.

If $\{A\cap C,A\cap\overline{C},\overline{A}\cap C,\overline{A}\cap\overline{C}\}\neq\{A\cap B,A\cap\overline{B},\overline{A}\cap B,\overline{A}\cap\overline{B}\}$, then $S_{A\cap B}$, $S_{A\cap\overline{B}}$, $S_{\overline{A}\cap B}$, $S_{\overline{A}\cap\overline{B}}$, $S_{A\cap C}$, $S_{A\cap\overline{C}}$, $S_{\overline{A}\cap C}$, $S_{\overline{A}\cap\overline{C}}$ generate S_n, implying $S=\binom{[2k]}{k}$, a contradiction. Thus, $\{A\cap C,A\cap\overline{C},\overline{A}\cap C,\overline{A}\cap\overline{C}\}=\{A\cap B,A\cap\overline{B},\overline{A}\cap B,\overline{A}\cap\overline{B}\}$, and the only possible configuration is $\{A_1\cup A_2, A_1\cup A_3, A_1\cup A_4, A_2\cup A_3, A_2\cup A_4, A_3\cup A_4\}$, where $\{A_i\}_{i=1}^4$ partitions $[2k]$ with $|A_i|=\frac{k}{2}$. ■

Claim 2.7 characterize the structure of S, and we now determine all possible L. Since $L=k-L$, we have $x\in L$ if and only if $k-x\in L$. We complete the proof by considering two subcases.

Subcase 3.1 $S=\{A,B,\overline{A},\overline{B}\}$.

Since $S\in\mathcal{F}(X)$ and $N(\{A,\overline{A}\})=N(A)$, we have

$$\begin{aligned} d(X)-2 &= |N(S)|-|S| \\ &= |N(\{A,\overline{A}\})|+|N(\{B,\overline{B}\})\setminus N(\{A,\overline{A}\})|-4 \end{aligned}$$

$$= d(X) - 4 + |N(\{B\}) \setminus N(\{A\})|. \tag{4}$$

Thus, $|N(\{B\}) \setminus N(\{A\})| = 2$.

From $|N(S)| - |S| = |N(\{A, \overline{A}\})| - |\{A, \overline{A}\}|$, we derive $|N(\{B, \overline{B}\}) \setminus N(\{A, \overline{A}\})| = 2$. Assume $|A \cap B| = a \leq \frac{k}{2}$ (otherwise, replace B by $\overline{B}$). Let

$$F_{w,x,y,z} = \left\{ F \in \binom{[2k]}{k} : |F \cap A \cap B| = w, |F \cap A \cap \overline{B}| = x, |F \cap \overline{A} \cap B| = y, |F \cap \overline{A} \cap \overline{B}| = z \right\}.$$

So $|F_{w,x,y,z}| = \mathbf{1}_{w+x+y+z=k}(w,x,y,z)\binom{a}{w}\binom{a}{z}\binom{k-a}{x}\binom{k-a}{y}$. For fixed w, x, y, z, either $F_{w,x,y,z} \subseteq N(\{B, \overline{B}\}) \setminus N(\{A, \overline{A}\})$ or $F_{w,x,y,z} \cap (N(\{B, \overline{B}\}) \setminus N(\{A, \overline{A}\})) = \emptyset$. Denote

$$g_L(w,x,y,z) = \mathbf{1}_{w+x+y+z=k}(w,x,y,z)\mathbf{1}_{w+x\in L}(w,x)\mathbf{1}_{w+y\notin L}(w,y)\binom{a}{w}\binom{a}{z}\binom{k-a}{x}\binom{k-a}{y}$$

for $0 \leq w, x, y, z \leq k$. Then

$$2 = |N(\{B, \overline{B}\}) \setminus N(\{A, \overline{A}\})| = \sum_{0\leq w,z\leq a, 0\leq x,y\leq k-a} g_L(w,x,y,z). \tag{5}$$

Suppose $\{0, k\} \subseteq L$. There exists at least one (w, x, y, z) with $g_L(w, x, y, z) > 0$. Fix such a tuple. If any of $\binom{a}{w}, \binom{a}{z}, \binom{k-a}{x}, \binom{k-a}{y}$ exceeds 1, by $w + x + y + z = k$, we have either at least two of them do, or $(a - w, k - a - x, k - a - y, a - z) \neq (w, x, y, z)$ with $g(a - w, k - a - x, k - a - y, a - z) = g_L(w, x, y, z)$. But in those two cases, we have $|N(\{B, \overline{B}\}) \setminus N(\{A, \overline{A}\})| \geq \sum_{0\leq w,x,y,z\leq k} g_L(w, x, y, z) \geq 4$, a contradiction with (5). Thus we have $(w, x, y, z) = (0, 0, a, k - a)$ or $(a, k - a, 0, 0)$, and then $a, k - a \notin L$. If $\{0, k\} \subsetneq L$, there exists $\{b, k - b\} \subseteq L$ with $b \leq \frac{k}{2}$. Then $(w, x, y, z) = (a, k - 2a, b - a, 2a - b)$ (if $b > a$) or $(a, 0, k - b - a, b)$ (if $a > b$) would satisfy $g_L(w, x, y, z) \geq 4$, a contradiction with (5). Thus, $L = \{0, k\}$ is the only possible solution of (5).

Suppose $\{0, k\} \subseteq \overline{L}$. By (5), we have

$$\begin{aligned} 2 = |N(\{B, \overline{B}\}) \setminus N(\{A, \overline{A}\})| &= \sum_{0\leq w,z\leq a, 0\leq x,y\leq k-a} g_L(w,x,y,z) \\ &= \sum_{0\leq w,z\leq a, 0\leq x,y\leq k-a} g_L(a-w, k-a-x, k-a-y, a-z) \\ &= \sum_{0\leq w,z\leq a, 0\leq x,y\leq k-a} g_{\overline{L}}(w,x,y,z). \end{aligned} \tag{6}$$

By (6) and the same discussion above, we have derive a contradiction if $\{0, k\} \subsetneq \overline{L}$. Thus, $\overline{L} = \{0, k\}$ and $L = [k - 1]$.

From the analysis above, L must be $\{0, k\}$ or $[k-1]$. If $L = \{0, k\}$, we have $N(S) = Y$, which implies S is not a fragment, a contradiction. If $L = [k - 1]$, easily verify that any nontrivial family $\mathcal{A}$ of k-sets closed under complementation is a fragment. Let $\emptyset \neq \mathcal{A} = \overline{\mathcal{B}} \subseteq \binom{[n]}{k}$ and we are done.

Subcase 3.2 $S = \{A_1 \cup A_2, A_1 \cup A_3, A_1 \cup A_4, A_2 \cup A_3, A_2 \cup A_4, A_3 \cup A_4\}$, where $\{A_i\}_{i=1}^4$ partitions $[2k]$ with $|A_i| = \frac{k}{2}$.

Let $A = A_1 \cup A_2$, $B = A_1 \cup A_3$, $C = A_1 \cup A_4$. We have

$$\begin{aligned}
d(X) - 2 &= |N(S)| - |S| \\
&= |N(\{A, \overline{A}\})| + |N(\{B, \overline{B}\}) \setminus N(\{A, \overline{A}\})| \\
&+ |N(\{C, \overline{C}\}) \setminus (N(\{A, \overline{A}\}) \cup N(\{B, \overline{B}\}))| - 6 \\
&\geq d(X) + |N(\{B\}) \setminus N(\{A\})| - 6.
\end{aligned}$$

Thus, $|N(\{B\}) \setminus N(\{A\})| \leq 4$. On the other hand,

$$\begin{aligned}
d(X) - 2 &= |N(S)| - |S| \\
&\leq |N(\{A\})| + |N(\{B\}) \setminus N(\{A\})| + |N(\{C\}) \setminus N(\{A\}))| - 6.
\end{aligned}$$

Thus, $|N(\{B, \overline{B}\}) \setminus N(\{A, \overline{A}\})| + |N(\{C, \overline{C}\}) \setminus N(\{A, \overline{A}\}))| \geq 4$. Without loss of generality, assume $|N(\{B\}) \setminus N(\{A\})| > 0$.

Let

$$F_{w,x,y,z} = \left\{ F \in \binom{[2k]}{k} : |F \cap A_1| = w, |F \cap A_2| = x, |F \cap A_3| = y, |F \cap A_4| = z \right\}.$$

So $|F_{w,x,y,z}| = \mathbf{1}_{w+x+y+z=k}(w, x, y, z)\binom{k/2}{w}\binom{k/2}{z}\binom{k/2}{x}\binom{k/2}{y}$. Write

$$g_L(w, x, y, z) = \mathbf{1}_{w+x+y+z=k}(w, x, y, z)\mathbf{1}_{w+x \in L}(w, x)\mathbf{1}_{w+y \notin L}(w, y)\binom{k/2}{w}\binom{k/2}{z}\binom{k/2}{x}\binom{k/2}{y}$$

for $0 \leq w, x, y, z \leq \frac{k}{2}$. Therefore

$$4 \geq |N(\{B\}) \setminus N(\{A\})| = \sum_{0 \leq w,x,y,z \leq k/2} g_L(w, x, y, z). \tag{7}$$

If $g_L(w, x, y, z) \leq 1$ for any $0 \leq w, x, y, z \leq k/2$, then the only possible solution such that $g_L(w, x, y, z) > 0$ is $(w, x, y, z) = (0, 0, a, k - a)$ or $(a, k - a, 0, 0)$. Thus, $|N(\{B, \overline{B}\}) \setminus N(\{A, \overline{A}\})| = 2$, meaning $\{A, B, \overline{A}, \overline{B}\}$ is a fragment, a contradiction to the minimality of S.

If there is a (w, x, y, z) satisfies $g_L(w, x, y, z) > 1$, then $g_L(w, x, y, z) \geq 4$ (since at least two binomial coefficients exceed 1). The equality holds only when $k = 4$ and $(w, x, y, z) = (2, 1, 0, 1)$, $(1, 1, 0, 2)$, $(2, 0, 1, 1)$, or $(1, 0, 1, 2)$. In each case, $g(2 - w, 2 - x, 2 - y, 2 - z) = 4$. So $|N(\{B\}) \setminus N(\{A\})| \geq 8$, a contradiction with (7). □

3 Pairwise cross L-intersecting

For any $S \subseteq [n]$ and $\mathcal{F} \subseteq \binom{[n]}{k}$, define $\mathcal{F}(S) = \{F \in \mathcal{F} : S \subseteq F\}$ and $\hat{\mathcal{F}}(S) = \{F \setminus S : F \in \mathcal{F}(S)\}$. Let L be a set of integers with $L \subseteq [n]$ and $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_r \subseteq \binom{[n]}{k}$ be pairwise non-empty cross-intersecting families. When $L=[0,k]$, it's trivial to have $\sum_{i=1}^{r}|\mathcal{A}_i|\le r\binom{n}{k}$. Let $L=[0,k-1]$. For every $A\in\mathcal{A}_i$ and $B\in\mathcal{A}_j$ with $i\neq j$, we have $A\neq B$. So $\mathcal{A}_1,\dots,\mathcal{A}_r$ are disjoint and then $\sum_{i=1}^{r}|\mathcal{A}_i|\le\binom{n}{k}$. When $L=[t,k]$ for some $t\in[k]$, it has been done by Theorem 1.6. Then we left the case (iv) in Theorem 1.7. Recall the conditions in Theorem 1.7(iv) are $k\in L$ and $L\neq[t,k]$ for any $t\in[0,k]$.

Let $\mathcal{A}_1,\dots,\mathcal{A}_r\subseteq\binom{[n]}{k}$ be the non-empty pairwise cross L-intersecting families, which obtain the maximum $\sum_{i=1}^{r}|\mathcal{A}_i|$. Set $\mathcal{B}_1=\mathcal{B}_2=\cdots=\mathcal{B}_{r-1}=\{[k]\}$ and $\mathcal{B}_r=\{A\in\binom{[n]}{k}:\ |A\cap[k]|\in L\}$. Since $k\in L$, $\mathcal{B}_1,\dots,\mathcal{B}_r$ are pairwise cross L-intersecting. Then

$$\sum_{i=1}^{r}|\mathcal{A}_i|\ge\sum_{i\in L}\binom{k}{i}\binom{n-k}{k-i}+r-1=\binom{n}{k}-\binom{k}{s}\binom{n-s}{k-s}-o(n^{k-s}).\tag{8}$$

We first give some lemmas which will be used in the proof of our result by considering whether $0\in L$. First, we deal with the case where $0\in L$.

3.1 The case when $0\in L$

In this subsection, let n,k,r be positive integers with $n\ge 2k$ and $k,r\ge 2$. Let $0,k\in L$ and $L\neq[t,k]$ for any $t\in[0,k]$. Let s be the minimum non-negative integer such that $s\notin L$, then $1\le s\le k-1$. Assume $|\mathcal{A}_1|\le|\mathcal{A}_2|\le\cdots\le|\mathcal{A}_r|$. We have the following results.

Lemma 3.1 *For any $S\in\binom{[n]}{s}$, one of the following results holds:*

1. $\sum_{i=1}^{r}|\mathcal{A}_i(S)|\le\max\left\{\binom{n-s}{k-s}-\binom{n-k}{k-s}+r-1,r\binom{n-s-1}{k-s-1}\right\}$.
2. *There are exactly $r-1$ families in $\mathcal{A}_1(S),\mathcal{A}_2(S),\dots,\mathcal{A}_r(S)$ are empty.*

Proof. For any $S\in\binom{[n]}{s}$, if all families $\mathcal{A}_1(S),\mathcal{A}_2(S),\dots,\mathcal{A}_r(S)$ are empty, then the conclusion holds trivially. Therefore, we only need to consider the case where at least two families of $\mathcal{A}_1(S),\mathcal{A}_2(S),\dots,\mathcal{A}_r(S)$ are non-empty. Assume $\mathcal{A}_{i_1}(S),\mathcal{A}_{i_2}(S),\dots,\mathcal{A}_{i_{r'}}(S)$ are the non-empty families, where $2\le r'\le r$. We have the following claim.

Claim 3.2 *The families $\hat{\mathcal{A}}_{i_1}(S),\hat{\mathcal{A}}_{i_2}(S),\dots,\hat{\mathcal{A}}_{i_{r'}}(S)$ are pairwise intersecting.*

Proof of Claim 3.2 By contradiction. Suppose, without loss of generality, there are $F_1\in\hat{\mathcal{A}}_{i_1}(S)$ and $F_2\in\hat{\mathcal{A}}_{i_2}(S)$ with $F_1\cap F_2=\emptyset$. Then $F_1\cup S\in\mathcal{A}_{i_1}$ and $F_2\cup S\in\mathcal{A}_{i_2}$ and $|(F_1\cup S)\cap(F_2\cup S)|=s\notin L$, a contradiction. ■

To complete the proof of Lemma 3.1, we proceed by case analysis on s.

Case 1: $1\le s\le k-2$.

Since $\hat{\mathcal{A}}_{i_1}(S),\dots,\hat{\mathcal{A}}_{i_{r'}}(S)\subseteq\binom{[n]\setminus S}{k-s}$, by Claim 3.2 and Theorem 1.6, we have

$$\sum_{i=1}^{r}|\mathcal{A}_i(S)|=\sum_{j=1}^{r'}|\hat{\mathcal{A}}_{i_j}(S)|\le\max\left\{\binom{n-s}{k-s}-\binom{n-k}{k-s}+r'-1,r'\binom{n-s-1}{k-s-1}\right\}$$

$$\le \max\left\{\binom{n-s}{k-s}-\binom{n-k}{k-s}+r-1, r\binom{n-s-1}{k-s-1}\right\}.$$

Case 2: $s=k-1$.

In this case, the 1-uniform families $\hat{\mathcal{A}}_{i_j}(S)$ must be identical singletons by Claim 3.2, that is, there exists $x\in[n]$ with $\hat{\mathcal{A}}_{i_j}=\{x\}$ for all $j\in[r']$. Thus

$$\sum_{i=1}^{r}|\mathcal{A}_i(S)|=r'\le r=\max\left\{\binom{n-s}{k-s}-\binom{n-k}{k-s}+r-1, r\binom{n-s-1}{k-s-1}\right\},$$

where the equality holds because $\binom{n-(k-1)}{1}-\binom{n-k}{1}+r-1=r-1$ and $r\binom{n-(k-1)-1}{0}=r$. $\square$

For $\mathcal{F}\subseteq\binom{[n]}{k}$, we define

$$\mathcal{S}_s(\mathcal{F})=\left\{S\in\binom{[n]}{s}:|\mathcal{F}(S)|>\frac{3}{2}\binom{n-s}{k-s}-\binom{n-k}{k-s}\right\}.$$

Lemma 3.3 *The families $\mathcal{S}_s(\mathcal{A}_1),\dots,\mathcal{S}_s(\mathcal{A}_r)\subseteq\binom{[n]}{s}$ are pairwise cross $[0,\max\{0,2s-k-1\}]$-intersecting.*

Proof. To the contrary, without loss of generality, suppose that $|S_1\cap S_2|>\max\{0,2s-k-1\}$, where $S_1\in\mathcal{S}_s(\mathcal{A}_1)$ and $S_2\in\mathcal{S}_s(\mathcal{A}_2)$. Since $2s-k-1<|S_1\cap S_2|\le s$, we have $|S_1\cup S_2|=2s-|S_1\cap S_2|\le k$.

Define $\mathcal{G}_1=\{F\in\mathcal{A}_1(S_1):F\cap S_2=S_1\cap S_2\}$, $\mathcal{G}_2=\{F\in\mathcal{A}_2(S_2):F\cap S_1=S_1\cap S_2\}$ and define $\mathcal{G}_1'=\{G\setminus S_1:G\in\mathcal{G}_1\}$, $\mathcal{G}_1'=\{G\setminus S_2:G\in\mathcal{G}_2\}$. For $\{i,j\}=\{1,2\}$, we have

$$\begin{aligned}|\mathcal{G}_i'|&=|\mathcal{A}_i(S_i)|-|\{F\in\mathcal{A}_i(S_i):(F\setminus S_i)\cap S_j\neq\emptyset\}|\\&>\frac{3}{2}\binom{n-s}{k-s}-\binom{n-k}{k-s}-\left(\binom{n-s}{k-s}-\binom{n-k}{k-s}\right)\\&=\frac{1}{2}\binom{n-s}{k-s}.\end{aligned}$$

Therefore,

$$|\mathcal{G}_1'|+|\mathcal{G}_2'|\ge\binom{n-s}{k-s}+1.$$

Noticing that $\mathcal{G}_1',\mathcal{G}_2'\subseteq\binom{[n]\setminus(S_1\cup S_2)}{k-s}$ and

$$\binom{n-s}{k-s}>\binom{n-|S_1\cup S_2|}{k-s}-\binom{k-s}{s-|S_1\cap S_2|}\binom{n-|S_1\cup S_2|-k+s}{k-|S_1\cap S_2|},$$

by Theorem 1.4 (ii), $\mathcal{G}_1',\mathcal{G}_2'$ can not be cross $([0,k-s]\setminus\{s-|S_1\cap S_2|\})$-intersecting (note that $0\le s-|S_1\cap S_2|\le k-s$). Then, there exist $G_1\in\mathcal{G}_1'$ and $G_2\in\mathcal{G}_2'$ such that $|G_1\cap G_2|=s-|S_1\cap S_2|$. Therefore, $|(G_1\cup S_1)\cap(G_2\cup S_2)|=s\notin L$, where $G_1\cup S_1\in\mathcal{A}_1$ and $G_2\cup S_2\in\mathcal{A}_2$, a contradiction. $\square$

Lemma 3.4 *Let n be sufficiently large (relative to r) and $r \geq 2$. Then $\sum_{i=1}^{r} |\mathcal{S}_s(\mathcal{A}_i)| \geq \binom{n}{s} - 3\binom{k}{s}^2$.*

Proof. Let $\sum_{i=1}^{r} |\mathcal{S}_s(\mathcal{A}_i)| = \binom{n}{s} - a$. It remains to show that $a \leq 3\binom{k}{s}^2$. If there are at least $r-1$ families in $\{\mathcal{A}_i(S); 1 \leq i \leq s\}$ are empty, then we have

$$\sum_{i=1}^{r} |\mathcal{A}_i(S)| \leq \begin{cases} \binom{n-s}{k-s} & \text{for every } S \in \bigcup_{i=1}^{r} \mathcal{S}_s(\mathcal{A}_i) \\ \frac{3}{2}\binom{n-s}{k-s} - \binom{n-k}{k-s} & \text{for every } S \in \binom{[n]}{s} \setminus \bigcup_{i=1}^{r} \mathcal{S}_s(\mathcal{A}_i). \end{cases} \tag{9}$$

Assume there are at least two families in $\{\mathcal{A}_i(S); 1 \leq i \leq s\}$ are non-empty. By Lemma 3.1, $\sum_{i=1}^{r} |\mathcal{A}_i(S)| \leq \max\left\{\binom{n-s}{k-s} - \binom{n-k}{k-s} + r - 1, r\binom{n-s-1}{k-s-1}\right\} = O(n^{k-s-1}) < \min\{\binom{n-s}{k-s}, \frac{3}{2}\binom{n-s}{k-s} - \binom{n-k}{k-s}\}$ for every $S \in \binom{[n]}{s}$ which implies (9) always holds. Now we estimate the size of $\sum_{i=1}^{r} |\mathcal{A}_r|$.

$$\begin{aligned}
\sum_{i=1}^{r} |\mathcal{A}_r| &= \binom{k}{s}^{-1} \sum_{S \in \binom{[n]}{s}} \sum_{i=1}^{r} |\mathcal{A}_i(S)| \\
&= \binom{k}{s}^{-1} \left(\sum_{S \in \bigcup_{i=1}^{r} \mathcal{S}_s(\mathcal{A}_i)} \sum_{i=1}^{r} |\mathcal{A}_i(S)| + \sum_{S \in \binom{[n]}{s} \setminus (\bigcup_{i=1}^{r} \mathcal{S}_s(\mathcal{A}_i))} \sum_{i=1}^{r} |\mathcal{A}_i(S)| \right) \\
&\leq \binom{k}{s}^{-1} |\bigcup_{i=1}^{r} \mathcal{S}_s(\mathcal{A}_i)| \binom{n-s}{k-s} \\
&+ \binom{k}{s}^{-1} \left(\binom{n}{s} - |\bigcup_{i=1}^{r} \mathcal{S}_s(\mathcal{A}_i)| \right) \left(\frac{3}{2}\binom{n-s}{k-s} - \binom{n-k}{k-s} \right) \\
&= \binom{k}{s}^{-1} \left(\binom{n-k}{k-s} - \frac{1}{2}\binom{n-s}{k-s} \right) \left(|\bigcup_{i=1}^{r} \mathcal{S}_s(\mathcal{A}_i)| \right) \\
&+ \binom{k}{s}^{-1} \left(\frac{3}{2}\binom{n-s}{k-s} - \binom{n-k}{k-s} \right) \binom{n}{s}. \\
&\leq \binom{k}{s}^{-1} \left(\binom{n-k}{k-s} - \frac{1}{2}\binom{n-s}{k-s} \right) \left(\sum_{i=1}^{r} |\mathcal{S}_s(\mathcal{A}_i)| \right) \\
&+ \binom{k}{s}^{-1} \left(\frac{3}{2}\binom{n-s}{k-s} - \binom{n-k}{k-s} \right) \binom{n}{s}.
\end{aligned}$$

Noticing that $\binom{n}{k}\binom{k}{s} = \binom{n-s}{k-s}\binom{n}{s}$, and combining with (8), we have

$$\begin{aligned}
&\binom{k}{s}\binom{n}{k} - \binom{k}{s}^2 \binom{n-k}{k-s} + o(n^{k-s}) \\
&\leq \left(\binom{n-k}{k-s} - \frac{1}{2}\binom{n-s}{k-s} \right) \left(\binom{n}{s} - a \right) + \left(\frac{3}{2}\binom{n-s}{k-s} - \binom{n-k}{k-s} \right) \binom{n}{s} \\
&= \binom{k}{s}\binom{n}{k} - \left(\binom{n-k}{k-s} - \frac{1}{2}\binom{n-s}{k-s} \right) a.
\end{aligned}$$

Then we have

$$a \le \frac{\binom{k}{s}^2\binom{n-k}{k-s} + o(n^{k-s})}{\binom{n-k}{k-s} - \frac{1}{2}\binom{n-s}{k-s}} \le 3\binom{k}{s}^2$$

when n is large enough. $\square$

Lemma 3.5 *Let n be sufficiently large (relative to r) and $r \ge 2$. If there are exactly $r-1$ families of $\mathcal{S}_s(\mathcal{A}_1), \ldots, \mathcal{S}_s(\mathcal{A}_r)$ (i.e. $\mathcal{S}_s(\mathcal{A}_1), \ldots, \mathcal{S}_s(\mathcal{A}_{r-1})$) are empty, then $|\bigcup_{i=1}^{r-1} \mathcal{A}_i| = 1$.*

Proof. Recall $\partial_s(\mathcal{F}) = \left\{G \in \binom{[n]}{s} : G \subset F \text{ for some } F \in \mathcal{F}\right\}$. Note that $\mathcal{S}_s(\mathcal{A}_r) \neq \emptyset$. Then $\mathcal{A}_r(S) \neq \emptyset$ if $S \in \mathcal{S}_s(\mathcal{A}_r)$. Let $S \in \mathcal{S}_s(\mathcal{A}_r)$. Since

$$\sum_{i=1}^{r} |\mathcal{A}_i(S)| \ge |\mathcal{A}_r(S)| > \frac{3}{2}\binom{n-s}{k-s} - \binom{n-k}{k-s},$$

by Lemma 3.1 and $\mathcal{A}_r(S) \neq \emptyset$, we have $\mathcal{A}_i(S) = \emptyset$ for $i \in [r-1]$. Hence $\mathcal{S}_s(\mathcal{A}_r) \cap \partial_s(\mathcal{A}_i) = \emptyset$ for any $i \in [r-1]$ which implies $\binom{n}{s} - \sum_{i=1}^{r-1} |\mathcal{S}_s(\mathcal{A}_r)| = \binom{n}{s} - |\mathcal{S}_s(\mathcal{A}_r)| \ge |\bigcup_{i=1}^{r-1} \partial_s(\mathcal{A}_i)|$. Assume $|\partial_s(\bigcup_{i=1}^{r-1} \mathcal{A}_i)| = \binom{x}{s}$, where x is a real number. By Lemma 3.4, we have $\binom{n}{s} - \sum_{i=1}^{r-1} |\mathcal{S}_s(\mathcal{A}_r)| \le 3\binom{k}{s}^2 = O(1)$. Hence, $\binom{x}{s} \le 3\binom{k}{s}^2 = O(1)$. By Corollary 1.11, we have $|\bigcup_{i=1}^{r-1} \mathcal{A}_i| \le \binom{x}{k} = O(1)$. For any $S \in \bigcup_{i=1}^{r-1} \partial_s(\mathcal{A}_i)$, say $S \in \partial_s(\mathcal{A}_1)$ and $S \subseteq K \in \mathcal{A}_1$, we have $\hat{\mathcal{A}}_1(S) \neq \emptyset$. Assume, without loss of generality, that $\hat{\mathcal{A}}_r(S) \neq \emptyset$. By the proof of Claim 3.2, we have $\hat{\mathcal{A}}_1(S), \hat{\mathcal{A}}_r(S)$ are cross-intersecting. Then $|\mathcal{A}_r(S)| = |\hat{\mathcal{A}}_r(S)| \le |\{F \in \binom{[n]\setminus S}{k-s} : F \cap K \neq \emptyset\}| \le \binom{n-s}{k-s} - \binom{n-k}{k-s}$. Thus $\sum_{i=1}^{r} |\mathcal{A}_i(S)| \le \sum_{i=1}^{r-1} |\mathcal{A}_i(S)| + |\mathcal{A}_r(S)| \le (r-1)\binom{x}{k} + \binom{n-s}{k-s} - \binom{n-k}{k-s} = O(n^{k-s-1})$. Then

$$\begin{aligned}
\sum_{i=1}^{r} |\mathcal{A}_i| &\le \sum_{i=1}^{r-1} |\mathcal{A}_i| + \left|\left\{F \in \mathcal{A}_r : \partial_s(F) \cap (\bigcup_{i=1}^{r-1} \partial_s(\mathcal{A}_i)) = \emptyset\right\}\right| + \sum_{S \in \bigcup_{i=1}^{r-1} \partial_s(\mathcal{A}_i)} |\mathcal{A}_r(S)| \\
&\le (r-1)\binom{x}{k} + \left(\binom{n}{k} - (1-o(1))\binom{x}{s}\binom{n-s}{k-s}\right) + \binom{x}{s} O(n^{k-s-1}) \\
&= \binom{n}{k} - (1-o(1))\binom{n-s}{k-s}\binom{x}{s}.
\end{aligned}$$

Combining with (8), we have $x \le k$ which implies $|\bigcup_{i=1}^{r-1} \mathcal{A}_i| \le \binom{x}{k} = 1$. Since $\mathcal{A}_i \neq \emptyset$ for all $1 \le i \le r-1$, we have $|\bigcup_{i=1}^{r-1} \mathcal{A}_i| = 1$. $\square$

Lemma 3.6 *Let n, r be positive integers with n sufficiently large (relative to r) and $r \ge 2$. Let $L = \{0, 2\}$. If non-empty families $\mathcal{A}_1, \ldots, \mathcal{A}_r \subseteq \binom{[n]}{2}$ are pairwise cross L-intersecting, then*

$$\sum_{i=1}^{r} |\mathcal{A}_i| \le \frac{(n-2)(n-3)}{2} + r,$$

the equality holds only when $\mathcal{A}_1 = \mathcal{A}_2 = \cdots = \mathcal{A}_{r-1} = \{[2]\}$ and $\mathcal{A}_r = \{A \in \binom{[n]}{2} : |A \cap [2]| \in L\}$ (up to isomorphism).

Proof. Assume $\sum_{i=1}^{r}|\mathcal{A}_i|$ is maximized. By (8), we have $\sum_{i=1}^{r}|\mathcal{A}_r| \geq \frac{(n-2)(n-3)}{2}+r$. If $\mathcal{S}_1(\mathcal{A}_1), \mathcal{S}_1(\mathcal{A}_2), \dots, \mathcal{S}_1(\mathcal{A}_r)$ are all empty, by (9),

$$\sum_{i=1}^{r}|\mathcal{A}_r| = \frac{1}{2}\sum_{i\in[n]}|\mathcal{A}_r(\{i\})| \leq \frac{1}{2}\sum_{i\in[n]}(\frac{3}{2}(n-1)-(n-2)) \leq \frac{1}{4}n^2+\frac{1}{4}n < \frac{(n-2)(n-3)}{2}+r,$$

yielding a contradiction. Thus, at least one of $\mathcal{S}_1(\mathcal{A}_1), \mathcal{S}_1(\mathcal{A}_2), \dots, \mathcal{S}_1(\mathcal{A}_r)$ is non-empty.

We first show that there are exactly $r-1$ empty families in $\mathcal{S}_1(\mathcal{A}_1), \mathcal{S}_1(\mathcal{A}_2), \dots, \mathcal{S}_1(\mathcal{A}_r)$. Otherwise, at least two of them are non-empty. Without loss of generality, assume $\mathcal{S}_1(\mathcal{A}_1)$, $\mathcal{S}_1(\mathcal{A}_2)$ are non-empty and $\{i\} \in \mathcal{S}_1(\mathcal{A}_1)$, $\{j\} \in \mathcal{S}_1(\mathcal{A}_2)$. Then, $i \neq j$ follows from Lemma 3.3. Define $\mathcal{B}_i = \{x \in [n] \setminus \{i,j\} : \{i,x\} \in \mathcal{A}_1\}$ and $\mathcal{B}_j = \{x \in [n] \setminus \{i,j\} : \{j,x\} \in \mathcal{A}_2\}$. And we have $|\mathcal{B}_i| + |\mathcal{B}_j| > 2(\frac{1}{2}(n+1)-1) = n-1$. Thus, we can select $x \in \mathcal{B}_i \cap \mathcal{B}_j$ such that $|\{x,i\} \cap \{x,j\}| = 1 \notin L$ where $\{x,i\} \in \mathcal{A}_1$, $\{x,j\} \in \mathcal{A}_2$, a contradiction.

By Lemma 3.5, when n is large enough, we have $|\bigcup_{i=1}^{r-1}\mathcal{A}_i| = 1$. It is immediately obtained that $\mathcal{A}_i$ are identical singletons for $i \in [1, r-1]$, which ends the proof. □

3.2 The case when $0 \notin L$

By Theorem 1.6, we only need to consider the case $L = \{\ell_1, \dots, \ell_t\} \neq [\ell_1, k], \ell_1 < \ell_2 < \cdots < \ell_t$ and $\ell_1 \neq 0$.

First, we present a result which is helpful in the proof.

Theorem 3.7 ([3]) *For positive integers s and k, there exist $n_0 = n_0(s,k)$, such that when $n \geq n_0$, if a family $\mathcal{F} \subseteq \binom{[n]}{k}$ with $|\mathcal{F}| \geq \binom{n}{k} - \binom{n-s}{k} = D(s,k)n^{k-1} + O(n^{k-2})$ for some $D(s,k)$, then there exist at least s disjoint sets in $\mathcal{F}$.*

In the following discussion, we set $M = M(k,L) = \max\{C(k,[\ell_1,k]), D(k+1,k-\ell_1)+1\}$, where $C(k,[\ell_1,k])$ is described in Theorem 1.2, and $D(k+1,k-\ell_1)$ is described in Theorem 3.7.

Lemma 3.8 *Let $\mathcal{A}_1, \dots, \mathcal{A}_r \subseteq \binom{[n]}{k}$ be pairwise cross L-intersecting non-empty families, where $L = \{\ell_1, \dots, \ell_t\} \neq [\ell_1, k], \ell_1 < \ell_2 < \cdots < \ell_t$ and $\ell_1 \neq 0$. Assume $|\mathcal{A}_1| \leq |\mathcal{A}_2| \leq \cdots \leq |\mathcal{A}_r|$. If n is large enough and $\sum_{i=1}^{r}|\mathcal{A}_r| \geq \sum_{i\in L}\binom{k}{i}\binom{n-k}{k-i} + r - 1$, we have the following properties:*

(i) *$\mathcal{A}_i$ is ℓ_1-intersecting for any $1 \leq i \leq r-1$.*

(ii) *If $|\mathcal{A}_i| \geq Mn^{k-\ell_1-1}$ for some $1 \leq i \leq r-1$, then there exists a set X with size ℓ_1 such that $X \subseteq A$ for every $A \in \bigcup_{j=1}^{r}\mathcal{A}_j$.*

Proof (i) Suppose there is $i \in [r-1]$, say $i = 1$ and $A_1, A_2 \in \mathcal{A}_1$ such that $|A_1 \cap A_2| \leq \ell_1 - 1$. Then we have $|F \cap (A_1 \cup A_2)| \geq \ell_1 + 1$ for every $F \in \mathcal{A}_r$. Hence

$$|\mathcal{A}_r| \leq \sum_{i=\ell_1+1}^{k}\binom{|A_1 \cup A_2|}{i}\binom{n-|A_1 \cup A_2|}{k-i} = O(n^{k-\ell_1-1}),$$

and then $\sum_{i=1}^r |\mathcal{A}_i| = O(n^{k-\ell_1-1})$, a contradiction with the assumption.
(ii) Assume there is $i \in [r-1]$ such that $|\mathcal{A}_i| \geq Mn^{k-\ell_1-1}$. By (i) and Theorem 1.2, there exist a set X with size ℓ_1 such that $X \subseteq \cap_{A\in\mathcal{A}_i} A$. Then $|\hat{\mathcal{A}}_i(X)| \geq (D(k+1, k-\ell_1)+1)n^{k-\ell_1-1}$. By Theorem 3.7, there exist $k+1$ disjoint sets (denoted as $B_1, \ldots, B_{k+1}$) in $\hat{\mathcal{A}}_i(X)$ when n is large enough.

Suppose there is $A \in \mathcal{A}_j$ with $j \in [r] \setminus \{i\}$ such that $A \cap X \neq X$. Then there is $t \in [k+1]$ such that $|A \cap (X \cup B_t)| < \ell_1$. Since $X \cup B_t \in \mathcal{A}_i$, a contradiction with (i). $\square$

3.3 Proof of Theorem 1.7

Now we complete the proof of Theorem 1.7.

Proof of Theorem 1.7 Let $\mathcal{A}_1, \ldots, \mathcal{A}_r$ be pairwise cross L-intersecting families with $k \in L$ which obtain the maximum $\sum_{i=1}^r |\mathcal{A}_i|$, where $L = \{\ell_1, \ldots, \ell_t\} \neq [\ell_1, k]$ with $\ell_1 < \ell_2 < \cdots < \ell_t$. We complete the proof by consider two cases.

Case 1. $\ell_1 = 0$.

In this case, we proceed by induction on k. By Lemma 3.6, together with the trivial case when $L = [1]$ and $L = [2]$, the conclusion holds for $k = 2$. Assume $k \geq 3$. Let s be the minimum non-negative integer such that $s \notin L$. Then $1 \leq s \leq k-1$. By (8), we have $\sum_{i=1}^r |\mathcal{A}_i| \geq \sum_{i\in L} \binom{k}{i}\binom{n-k}{k-i} + r - 1$.

We claim that there are exactly $r-1$ empty families in $\mathcal{S}_s(\mathcal{A}_1), \ldots, \mathcal{S}_s(\mathcal{A}_r)$.

If $\mathcal{S}_s(\mathcal{A}_1), \ldots, \mathcal{S}_s(\mathcal{A}_r)$ are all empty, by (9)

$$\begin{aligned}\sum_{i=1}^r |\mathcal{A}_i| &= \binom{k}{s}^{-1} \sum_{S\in\binom{[n]}{s}} |\sum_{i=1}^r |\mathcal{A}_i(S)|| \leq \binom{k}{s}^{-1}\binom{n}{s}\left(\frac{3}{2}\binom{n-s}{k-s} - \binom{n-k}{k-s}\right)\\ &< \sum_{i\in L}\binom{k}{i}\binom{n-k}{k-i} + r - 1,\end{aligned}$$

yielding a contradiction.

Now, consider that there are at least two families in $\mathcal{S}_s(\mathcal{A}_1), \mathcal{S}_s(\mathcal{A}_2), \ldots, \mathcal{S}_s(\mathcal{A}_r)$ are non-empty. By Lemma 3.3, $\mathcal{S}_s(\mathcal{A}_1)$,$\mathcal{S}_s(\mathcal{A}_2)$, $\ldots$, $\mathcal{S}_s(\mathcal{A}_r)$ are pairwise cross-$[0, \max\{0, 2s-k-1\}]$-intersecting, and also $[0, \max\{0, 2s-k-1\}] \cup \{s\}$-intersecting. Set $L' = [0, \max\{0, 2s-k-1\}] \cup \{s\}$. Note that $s-1 \notin L'$ by $s \leq k-1$. Then $L' \subseteq [0, s-2] \cup \{s\} \subseteq [0, s]$. By induction hypothesis, we have

$$\sum_{i=1}^r |\mathcal{S}_s(\mathcal{A}_i)| \leq \binom{n}{s} - \binom{s}{s-1}\binom{n-s}{1} + r - 1 = \binom{n}{s} - s(n-s) + r - 1. \tag{10}$$

Set $\sum_{i=1}^r |\mathcal{S}_s(\mathcal{A}_i)| = \binom{n}{s} - y$, then we have $y \geq s(n-s) - r + 1$ by (10), a contradiction with Lemma 3.4 when n is large enough. Hence there are exactly $r-1$ empty families in $\mathcal{S}_s(\mathcal{A}_1), \ldots, \mathcal{S}_s(\mathcal{A}_r)$.

Assume $\mathcal{S}_s(\mathcal{A}_i) = \emptyset$ for any $i \in [r-1]$. By Lemma 3.5, $|\bigcup_{i=1}^{r-1} \mathcal{A}_i| = 1$, it is immediately obtained that $\mathcal{A}_i$ are identical singletons for $i \in [r-1]$ and we are done.

Case 2. $\ell_1 \neq 0$.

We first show that $|\mathcal{A}_i| = 1$ for any $i \in [r-1]$. Suppose there is $i \in [r-1]$, say $i = r-1$, such that $|\mathcal{A}_{r-1}| \geq 2$. Let $A_1, A_2 \in \mathcal{A}_{r-1}$. Then we have

$$\begin{aligned}
|\mathcal{A}_r| &= \sum_{i=\ell_1}^{k} \left| \left\{ F \in \binom{[n]}{k} : |F \cap A_1|, |F \cap A_2| \in L, |F \cap (A_1 \cup A_2)| = i \right\} \right| \\
&= \left| \left\{ F \in \binom{[n]}{k} : |F \cap A_1|, |F \cap A_2| \in L, |F \cap (A_1 \cup A_2)| = \ell_1 \right\} \right| + O(n^{k-\ell_1-1}) \\
&= \binom{|A_1 \cap A_2|}{\ell_1} \binom{n - |A_1 \cup A_2|}{k - \ell_1} + O(n^{k-\ell_1-1}) \\
&\leq \binom{k-1}{\ell_1} \binom{n-(k+1)}{k-\ell_1} + O(n^{k-\ell_1-1}).
\end{aligned}$$

Since $\sum_{i=1}^{r} |\mathcal{A}_i| \geq \sum_{i \in L} \binom{k}{i}\binom{n-k}{k-i} + r - 1 = \binom{k}{\ell_1}\binom{n-k}{k-\ell_1} - O(n^{k-\ell_1-1})$ by (8), we have

$$\begin{aligned}
\sum_{i=1}^{r-1} |\mathcal{A}_i| &\geq \left(\binom{k}{\ell_1} - \binom{k-1}{\ell_1} \right) \binom{n-(k+1)}{k-\ell_1} - O(n^{k-\ell_1-1}) \\
&= \binom{k-1}{\ell_1 - 1} \binom{n-(k+1)}{k-\ell_1} - O(n^{k-\ell_1-1}).
\end{aligned}$$

For a fixed constant $M \geq \max\{C(k, [\ell_1, k]), D(k+1, k-\ell_1) + 1\}$, where $C(k, [\ell_1, k])$ is described in Theorem 1.2 and $D(k+1, k-\ell_1)$ is described in Theorem 3.7, $|\mathcal{A}_i| \geq Mn^{k-\ell_1-1}$ holds for some $i \in [1, r-1]$. By Lemma 3.8 (ii), there exists a set X with size ℓ_1 such that $X \subseteq A$ for every $A \in \bigcup_{j=1}^{r} \mathcal{A}_j$. Define $\mathcal{A}_i^0 = \{F \setminus X : F \in \mathcal{A}_i\}$ and $L' = \{x - \ell_1 : x \in L\}$. Then $\mathcal{A}_1^0, \ldots, \mathcal{A}_r^0$ are $(k-\ell_1)$-uniform pairwise cross L'-intersecting families and $0 \in L'$. By Case 1, we have

$$\begin{aligned}
\sum_{i=1}^{r} |\mathcal{A}_i| = \sum_{i=1}^{r} |\mathcal{A}_i^0| &\leq \sum_{i \in L'} \binom{k-\ell_1}{i} \binom{n-k}{k-\ell_1-i} + (r-1) \\
&< \sum_{i \in L} \binom{k}{i} \binom{n-k}{k-i} + r - 1,
\end{aligned}$$

a contradiction with (8).

Thus we have $|\mathcal{A}_i| = 1$ for $i \in [r-1]$. When $r \geq 3$, we may assume $\mathcal{A}_1 = \cdots = \mathcal{A}_{r-1}$, otherwise we can similarly have $|\mathcal{A}_r| \leq \binom{k-1}{\ell_1}\binom{n-(k+1)}{k-\ell_1} + O(n^{k-\ell_1-1})$ and then

$$\sum_{i=1}^{r} |\mathcal{A}_i| \leq \binom{k-1}{\ell_1} \binom{n-(k+1)}{k-\ell_1} + O(n^{k-\ell_1-1}) < \binom{k}{\ell_1} \binom{n-k}{k-\ell_1} - O(n^{k-\ell_1-1}),$$

a contradiction with (8).

Hence, we obtain that $\mathcal{A}_1 = \cdots = \mathcal{A}_{r-1} = \{A\}$, which completes the proof. $\square$

4 r-cross L-intersecting families

Proof of Theorem 1.9 Let $\mathcal{A}_1,\dots,\mathcal{A}_r \subseteq \binom{[n]}{k}$ be a non-empty r-cross $[\ell, s-1]$-intersecting families that attain the maximum value of $\sum_{i=1}^r |\mathcal{A}_i|$, where $s \in [k]$. Let $\mathcal{A}_1^0 = \{[k]\}$, $\mathcal{A}_2^0 = \{A \in \binom{[n]}{k} : |A \cap [k]| \leq s-1, [\ell] \subseteq A\}$ and $\mathcal{A}_3^0 = \cdots = \mathcal{A}_r^0 = \{A \in \binom{[n]}{k} : [\ell] \subseteq A\}$ (if $\ell = 0$, let $[\ell] = \emptyset$). It is easy to check $\mathcal{A}_1^0,\dots,\mathcal{A}_r^0$ are r-cross $[\ell, s-1]$-intersecting. Thus we have

$$\sum_{i=1}^r |\mathcal{A}_i| \geq \sum_{i=1}^r |\mathcal{A}_i^0| = (r-1)\binom{n-\ell}{k-\ell} - \sum_{i=s-\ell}^{k-\ell} \binom{k-\ell}{i}\binom{n-\ell-k}{k-\ell-i} + 1. \tag{11}$$

We will divide the proof into two cases, when $\ell = 0$ and $\ell > 0$. Let $L = [\ell, s-1]$.

Case 1. $\ell = 0$.

In this case, we will prove

$$\sum_{i=1}^r |\mathcal{A}_i| \leq (r-1)\binom{n}{k} - \sum_{i=s}^{k} \binom{k}{i}\binom{n-k}{k-i} + 1.$$

We prove it by induction on k. When $k=1$, we have $L = \{0\}$. Since every $i \in [n]$ is contained in at most $r-1$ families of $\mathcal{A}_1,\dots,\mathcal{A}_r$, we have

$$\sum_{i=1}^r |\mathcal{A}_i| \leq (r-1)n = (r-1)\binom{n}{1} - \binom{1}{1}\binom{n-1}{1-1} + 1,$$

and we are done. Then we assume $k \geq 2$, and the result holds for every $\ell \leq k-1$.

Let $t \in [k]$, $S \in \binom{[n]}{t}$ and $\mathcal{F} \subseteq \binom{[n]}{k}$. Recall $\mathcal{F}(S) = \{F \in \mathcal{F} : S \subseteq F\}$ and $\hat{\mathcal{F}}(S) = \{F \setminus S : F \in \mathcal{F}(S)\}$. By the definition of r-cross $[0, s-1]$-intersecting, for every $S \in \binom{[n]}{s}$, at least one of $\mathcal{A}_1(S),\dots,\mathcal{A}_r(S)$ is empty which implies $\sum_{i=1}^r |\mathcal{A}_i(S)| \leq (r-1)\binom{n-s}{k-s}$. We set

$$\mathcal{T}_s(\mathcal{F}) = \left\{T \in \binom{[n]}{s} : |\mathcal{F}(T)| \geq \binom{n-s}{k-s} - \binom{n-k}{k-s}\right\},$$

and $\mathcal{S} = \left\{S \in \binom{[n]}{s} : \sum_{i=1}^r |\mathcal{A}_i(S)| \geq (r-1)\binom{n-s}{k-s} - \binom{n-k}{k-s}\right\}$. Then we have the following result.

Claim 4.1 *At least one of $\mathcal{T}_s(\mathcal{A}_1),\dots,\mathcal{T}_s(\mathcal{A}_r)$ is empty, and $|\mathcal{S}| \geq \binom{n}{s} - \binom{k}{s}^2$.*

Proof of Claim 4.1 First, we have the following inequality.

$$\begin{aligned}\sum_{i=1}^r |\mathcal{A}_i| &\leq \binom{k}{s}^{-1} \sum_{S\in\mathcal{S}}\sum_{i=1}^r |\mathcal{A}_i(S)| + \binom{k}{s}^{-1} \sum_{S \in \binom{[n]}{s}\setminus\mathcal{S}} \sum_{i=1}^r |\mathcal{A}_i(S)| \\ &\leq (r-1)\binom{k}{s}^{-1}\binom{n-s}{k-s}|\mathcal{S}| \\ &\quad + \binom{k}{s}^{-1}\left((r-1)\binom{n-s}{k-s} - \binom{n-k}{k-s}\right)\left(\binom{n}{s} - |\mathcal{S}|\right).\end{aligned}$$

Combining with (11), we have

$$\binom{n-k}{k-s}|\mathcal{S}| \geq \binom{n}{s}\binom{n-k}{k-s} - \binom{k}{s}^2\binom{n-k}{k-s},$$

which implies $|\mathcal{S}| \geq \binom{n}{s} - \binom{k}{s}^2$.

Let $S \in \mathcal{S}$. Then at least one of $\mathcal{T}_s(\mathcal{A}_1), \ldots, \mathcal{T}_s(A_r)$ does not contain S by $\mathcal{A}_1, \ldots, \mathcal{A}_r$ being r-cross $[0, s-1]$-intersecting. Suppose S is contained in at most $r-2$ families of $\mathcal{T}_s(\mathcal{A}_1), \ldots, \mathcal{T}_s(A_r)$. Then we have

$$\sum_{i=1}^{r} |\mathcal{A}_i(S)| \leq (r-2)\binom{n-s}{k-s} + \binom{n-s}{k-s} - \binom{n-k}{k-s} - 1 < (r-1)\binom{n-s}{k-s} - \binom{n-k}{k-s},$$

a contradiction with $S \in \mathcal{S}$. Hence for every $S \in \mathcal{S}$, S is contained in exactly $r-1$ families of $\mathcal{T}_s(\mathcal{A}_1), \ldots, \mathcal{T}_s(A_r)$. Thus we have

$$\sum_{i=1}^{r} |\mathcal{T}_s(A_i)| \geq (r-1)|\mathcal{S}| \geq (r-1)\binom{n}{s} - (r-1)\binom{k}{s}^2. \tag{12}$$

Note that $\mathcal{T}_s(\mathcal{A}_1), \ldots, \mathcal{T}_s(\mathcal{A}_r) \subseteq \binom{[n]}{s}$ are r-cross $[0, s-1]$-intersecting. If $\mathcal{T}_s(\mathcal{A}_i) \neq \emptyset$ for any $i \in [r]$, then by induction hypothesis, we have

$$\sum_{i=1}^{r} |\mathcal{T}_s(\mathcal{A}_i)| \leq (r-1)\binom{n}{s} - \binom{k}{s}\binom{n-k}{k-s} + 1 < (r-1)\binom{n}{s} - (r-1)\binom{k}{s}^2,$$

a contradiction with (12). Thus at least one of $\mathcal{T}_s(\mathcal{A}_1), \ldots, \mathcal{T}_s(\mathcal{A}_r)$ is empty. ■

By Claim 4.1, we may assume $\mathcal{T}_s(\mathcal{A}_1)$ is empty. Then for any $S \in \binom{[n]}{s}$, $|\mathcal{A}_1(S)| \leq \binom{n-s}{k-s} - \binom{n-k}{k-s} - 1$.

Claim 4.2 $\mathcal{S} \cap \partial_s(\mathcal{A}_1) = \emptyset$.

Proof of Claim 4.2 Suppose there is $S \in \mathcal{S} \cap \partial_s(\mathcal{A}_1)$. Then $\mathcal{A}_1(S) \neq \emptyset$. Since at least one of $\mathcal{A}_1(S), \ldots, \mathcal{A}_r(S)$ is empty, we have

$$\sum_{i=1}^{r} |\mathcal{A}_i(S)| \leq (r-2)\binom{n-s}{k-s} + \binom{n-s}{k-s} - \binom{n-k}{k-s} - 1 < (r-1)\binom{n-s}{k-s} - \binom{n-k}{k-s},$$

a contradiction with $S \in \mathcal{S}$. ■

By Claims 4.2 and 4.1, $|\partial_s(\mathcal{A}_1)| \leq \binom{k}{s}^2$. Assume $|\partial_s(\mathcal{A}_1)| = \binom{x}{s}$, where x is a real number. Then $x = O(1)$ and by Corollary 1.11, we have $|\mathcal{A}_1| \leq \binom{x}{k}$. For $i = 2, \ldots, r$, set $\mathcal{B}_i = \{F \in \mathcal{A}_i : \partial_s(F) \cap \partial_s(\mathcal{A}_1) = \emptyset\}$. Since $|\partial_s(\mathcal{A}_1)| = O(1)$, we have $|\partial_1(\mathcal{A}_1)| = O(1)$. Let $A_1' = [n] \setminus \partial_1(\mathcal{A}_1)$. Then $|A_1'| = n - O(1)$. For every $S \in \partial_s(\mathcal{A}_1)$, choose $B \in \binom{A_1'}{k-s}$, we have $S \cup B \in \overline{\mathcal{B}_i}$ for all $i = 2, \ldots, r$, where $\overline{\mathcal{B}_i} = \binom{[n]}{k} \setminus \mathcal{B}_i$. Thus for $i = 2, \ldots, r$,

$$|\overline{\mathcal{B}_i}| \geq |\partial_s(\mathcal{A}_1)|\binom{n - |\partial_1(\mathcal{A}_1)|}{k-s} = (1 - o(1))\binom{x}{s}\binom{n-s}{k-s}.$$

Then for $i = 2, \dots, r$, we have

$$|\mathcal{B}_i| \leq \binom{n}{k} - (1 - o(1))\binom{x}{s}\binom{n-s}{k-s}.$$

And for every $S \in \partial_s(\mathcal{A}_1)$, at least one of $\mathcal{A}_2(S), \dots, \mathcal{A}_r(S)$ is empty. Hence $\sum_{i=2}^{r} \mathcal{A}_i(S) \leq (r-2)\binom{n-s}{k-s}$. Thus

$$\begin{aligned}
\sum_{i=1}^{r} |\mathcal{A}_i| &\leq |\mathcal{A}_1| + \sum_{i=2}^{r} |\mathcal{B}_i| + \sum_{S \in \partial_s(\mathcal{A}_1)} \sum_{i=2}^{r} |\mathcal{A}_r(S)| \\
&\leq \binom{x}{k} + (r-1)\left(\binom{n}{k} - (1-o(1))\binom{x}{s}\binom{n-s}{k-s}\right) + (r-2)\binom{x}{s}\binom{n-s}{k-s} \\
&= (r-1)\binom{n}{k} - (1-o(1))\binom{x}{s}\binom{n-s}{k-s} + O(1).
\end{aligned}$$

By (11) and $\ell = 0$, $\sum_{i=1}^{r} |\mathcal{A}_i| \geq (r-1)\binom{n}{k} - \binom{k}{s}\binom{n-k}{k-s} - o(n^{k-s})$. Then we have $x \leq k$ when n is large enough. Since $\mathcal{A}_1$ is not empty, it's immediately obtained that $\mathcal{A}_1$ is identical singletons. We may assume $\mathcal{A}_1 = \{[k]\}$. For $i = 2, \dots, r$, we set

$$\mathcal{C}_i^{s-1} = \{A \in \mathcal{A}_i : |A \cap [k]| \leq s-1\},$$

and for every integer $t \in [s, k]$, set

$$\mathcal{C}_i^{t} = \{A \in \mathcal{A}_i : |A \cap [k]| = t\}.$$

Then $\mathcal{A}_i = \bigcup_{t=s-1}^{k} \mathcal{C}_i^t$ for $i = 2, \dots r$. And we have

$$\sum_{i=1}^{r} |\mathcal{C}_i^{s-1}| \leq (r-1)\left(\binom{n}{k} - \sum_{i=s}^{k}\binom{k}{i}\binom{n-k}{k-i}\right).$$

Since for every set $T \subseteq [k]$ with $|T| = t \geq s$, at least one of $\mathcal{A}_2(T), \dots, \mathcal{A}_r(T)$ is empty, we have

$$\sum_{t=s}^{k}\sum_{i=2}^{r} |\mathcal{C}_i^t| \leq (r-2)\sum_{i=s}^{k}\binom{k}{i}\binom{n-k}{k-i}.$$

Finally, we have

$$\begin{aligned}
\sum_{i=1}^{r} |\mathcal{A}_i| &= 1 + \sum_{i=2}^{r} |\mathcal{C}_i^{s-1}| + \sum_{i=2}^{r}\sum_{t=s}^{k} |\mathcal{C}_i^t| \\
&\leq 1 + (r-1)\left(\binom{n}{k} - \sum_{i=s}^{k}\binom{k}{i}\binom{n-k}{k-i}\right) + (r-2)\sum_{i=s}^{k}\binom{k}{i}\binom{n-k}{k-i} \\
&= (r-1)\binom{n}{k} - \sum_{i=s}^{k}\binom{k}{i}\binom{n-k}{k-i} + 1,
\end{aligned}$$

and we are done.

Case 2. $\ell > 0$.

In this case, we will prove

$$\sum_{i=1}^{r} |\mathcal{A}_i| \le (r-1)\binom{n-\ell}{k-\ell} - \sum_{i=s-\ell}^{k-\ell} \binom{k-\ell}{i}\binom{n-k}{k-\ell-i} + 1.$$

We may assume $|\mathcal{A}_1| \le \cdots \le |\mathcal{A}_r|$. We need the following results.

Claim 4.3 *$\mathcal{A}_i$ is ℓ-intersecting for all $i \in [r-1]$, and the families $\mathcal{A}_1, \ldots, \mathcal{A}_{r-1}$ are pairwise cross ℓ-intersecting.*

Proof of Claim 4.3 Suppose there is $i \in [r-1]$ with $A_1, A_2 \in \mathcal{A}_i$ or there are $i, j \in [r-1]$ with $i \neq j$ and $A_1 \in \mathcal{A}_i$ and $A_2 \in \mathcal{A}_j$ such that $|A_1 \cap A_2| \le \ell - 1$. Since for every set $A \in \mathcal{A}_r$, $|A \cap A_1 \cap A_2| \ge \ell$, we have $|A \cap (A_1 \cup A_2)| \ge \ell + 1$. Then

$$|\mathcal{A}_r| = \sum_{i=\ell+1}^{k} \binom{|A_1 \cup A_2|}{i}\binom{n - |A_1 \cup A_2|}{k-i} = O(n^{k-\ell-1}).$$

Then

$$\sum_{i=1}^{r} |\mathcal{A}_i| \le r|\mathcal{A}_r| = O(n^{k-\ell-1}),$$

a contradiction with (11). ■

Claim 4.4 *There exists a set $X \in \binom{[n]}{\ell}$ such that $X \subseteq \bigcap_{A \in \mathcal{A}_i} A$ for every $i \in [r]$.*

Proof of Claim 4.4 By Claim 4.3, for every two sets $A_1, A_2 \in \bigcup_{i=1}^{r-1} \mathcal{A}_i$, we have $|A_1 \cap A_2| \ge \ell$. And for every $A \in \mathcal{A}_r$, $|A \cap (A_1 \cup A_2)| \ge \ell$. So we have $|\mathcal{A}_r| \le \sum_{i=\ell}^{k} \binom{|A_1 \cup A_2|}{i}\binom{n}{k-i} = \binom{n}{k-\ell} + O(n^{k-\ell-1})$. Thus, by (11),

$$\sum_{i=1}^{r-1} |\mathcal{A}_i| \ge (r-2)\binom{n-\ell}{k-\ell} - O(n^{k-\ell-1}).$$

Then, for every fixed constant $M \ge \max\{C(k, L), D(k+1, k-\ell) + 1\}$, where $C(k, L)$ is the constant in Theorem 1.2 and $D(k+1, k-\ell)$ is described in Theorem 3.7, when n is large enough, there exists $i \in [r-1]$ with $|\mathcal{A}_i| \ge Mn^{k-\ell-1}$. According to Claim 4.3, $\mathcal{A}_i$ is ℓ-intersecting. By Theorem 1.2, there exists X with $|X| = \ell$ such that $X \subseteq \bigcap_{A \in \mathcal{A}_i} A$. By Theorem 3.7, we can find $k+1$ disjoint sets, denote as $B_1, B_2, \ldots, B_{k+1}$, in $\hat{A}_i(X)$.

Suppose there is $A \in \mathcal{A}_j$ with $j \neq i$ such that $A \cap X \neq X$. Then there is $B_t \in \{B_1, B_2, \ldots, B_{k+1}\}$ that is disjoint with A such that $B_t \cup X \in \mathcal{A}_i$ and $|(B_t \cup X) \cap A| = |X \cap A| \le \ell - 1$, a contradiction. ■

By Claim 4.4, there exist $X \in \binom{[n]}{\ell}$ and $X \subseteq A$ for all $A \in \mathcal{A}_i$ and all $i \in [r]$. For $i \in [r]$, we set $\mathcal{D}_i = \{A \setminus X : A \in \mathcal{A}_i\}$. Then $\mathcal{D}_1, \ldots, \mathcal{D}_r \subseteq \binom{[n]\setminus X}{k-\ell}$ are r-cross $[0, s-\ell-1]$-intersecting. By Case 1, we have

$$\sum_{i=1}^{r} |\mathcal{A}_i| = \sum_{i=1}^{r} |\mathcal{D}_i| \leq (r-1)\binom{n-\ell}{k-\ell} - \sum_{i=s-\ell}^{k-\ell} \binom{k-\ell}{i}\binom{n-k}{k-\ell-i},$$

and we are done. □

5 Open problems

In this paper, we completely resolved the maximum sum problem for cross L-intersecting families. However, for pairwise cross L-intersecting families, we have only addressed the case where n is sufficiently large and either $k \in L$ or $L = [0, k-1]$. Naturally, this leads to the following two open questions.

Problem 1: Let $r \geq 3$ and $\mathcal{A}_1, \ldots, \mathcal{A}_r \subseteq \binom{[n]}{k}$ be non-empty pairwise cross L-intersecting families. For $k \in L \neq [t, k]$ and $n \geq k$, determine $\max \sum_{i=1}^{r} |\mathcal{A}_i|$ and characterize the extremal structure.

Problem 2: Let $r \geq 3$ and $\mathcal{A}_1, \ldots, \mathcal{A}_r \subseteq \binom{[n]}{k}$ be non-empty pairwise cross L-intersecting families. For $k \notin L \neq [0, k-1]$ and sufficiently large n, determine $\max \sum_{i=1}^{r} |\mathcal{A}_i|$ and characterize the extremal structure.

Regarding r-cross L-intersecting families, much less is known. We thus pose the following additional question.

Problem 3: Let $r \geq 3$ and $\mathcal{A}_1, \ldots, \mathcal{A}_r \subseteq \binom{[n]}{k}$ be non-empty r-cross L-intersecting families. For $L \subseteq [0, k]$ where L is not an interval, determine $\max \sum_{i=1}^{r} |\mathcal{A}_i|$ and characterize the extremal structure.

One may also consider the maximum product problem for cross L-intersecting families. When the L-intersecting condition corresponds to $L = [t, k]$, this is a classical problem known as a conjecture of Tokushige. The latest results are shown below.

Theorem 5.1 ([15]) *Let n, k, t be positive integers such that $3 \leq t \leq k \leq n$ and $n \geq (t+1)(k-t+1)$. If $\mathcal{A} \subseteq \binom{[n]}{k}$ and $\mathcal{B} \subseteq \binom{[n]}{k}$ are cross-t-intersecting, then*

$$|\mathcal{A}||\mathcal{B}| \leq \binom{n-t}{k-t}^2.$$

Moreover, equality holds if and only if $\mathcal{A} = \mathcal{B}$ is a maximum t-intersecting subfamily of $\binom{[n]}{k}$.

A natural question is to generalize cross t-intersecting to cross L-intersecting.

Problem 4: Let $\mathcal{A}, \mathcal{B} \subseteq \binom{[n]}{k}$ be cross L-intersecting families. Determine $|\mathcal{A}||\mathcal{B}|$ and characterize the extremal structure.

It is worth noting that if $L = [t, k]$ in Problem 3, the extremal configuration consists of two identical maximum L-intersecting families. However, for general L, $\mathcal{A}$ and $\mathcal{B}$ are not necessarily equal to obtain the maximum. For example, assume $L = \{\ell_1, \ell_2, \ldots, \ell_m\} \subseteq [0, k]$. If $\mathcal{A} = \mathcal{B}$ are cross L-intersecting, $\mathcal{A}, \mathcal{B}$ must be L-intersecting. Thus by Theorem 1.2, $|\mathcal{A}||\mathcal{B}| \leq (\prod_{1 \leq i \leq m} \frac{n-\ell_i}{k-\ell_i})^2 \leq O(n^{2m})$. However, consider the cross L-intersecitng families $\mathcal{A}' = \{[1, k]\}$ and $\mathcal{B}' = \{F \in \binom{[n]}{k} : |F \cap [1, k]| = \ell_1\}$, we have $|\mathcal{A}'||\mathcal{B}'| = O(n^{k-\ell_1})$. If $k - \ell_1 > 2m$, $|\mathcal{A}'||\mathcal{B}'| > O(n^{2m})$. Thus, $\mathcal{A} = \mathcal{B}$ can not be extremal structure. Therefore, this problem presents significant challenges in conjecturing the extremal structures, but we believe it is a highly interesting and worthwhile question.

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Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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No data was used for the research described in the article.

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