U-generating system dynamics with a product state using local operations

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A quantum system may undergo undesired dynamics if it is already correlated with its environment before global unitary evolution takes place. To mitigate these effects, we apply local operations on the system of interest prior to the global evolution. This ensures that a specific dynamics-matching condition is satisfied—namely, that the system's dynamics can be U-generated starting from an initial product state. We review two strategies for achieving this: local measurements and local unitaries. For each, we outline their respective advantages and disadvantages and provide examples illustrating when the dynamics-matching condition can be satisfied. Since most operations inevitably alter the system's initial state, we optimize the fidelity between the original system state and the post-operation state, ensuring the condition can be met with minimal perturbation. Local measurements always guarantee that the system dynamics can be U-generated from a product state, but this comes at the cost of reduced fidelity. By contrast, local unitary transformations typically avoid this loss of fidelity: in our numerical study of 402 cases, they succeeded in nearly all instances, with a minimum fidelity of about 94.2%. We further strengthen these results by showing that a single unitary operation can prevent all non–completely positive quantum dynamics for a system and environment undergoing time-dependent global evolution. Moreover, we demonstrate that a general two-term Kraus channel—implementable with the addition of just one qubit—can raise all fidelities to 1 across the 402 cases, while still ensuring that the dynamics-matching condition is satisfied.

I. INTRODUCTION

The dynamics of a quantum system in an open setting are strongly influenced by correlations with its surrounding environment, often referred to as the bath. A standard simplifying assumption in much of the open quantum systems literature is that the system and environment begin in an uncorrelated state, i.e., a product state [1, 2]. This assumption is particularly convenient because it guarantees that the resulting reduced dynamics of the system can be described by completely positive (CP) maps. Such maps admit a Kraus operator representation with non-negative coefficients, providing a well-defined and widely used mathematical framework for analyzing open system evolution.

In practice, however, perfectly isolating the system or perfectly preparing it in a pure state is challenging. Even when projective measurements are employed to prepare the system, residual correlations with the environment often remain [3–6]. These correlations complicate the picture: while certain correlated initial states still lead to CP dynamics [3–5], this is not universally true. In many cases, correlated initial conditions can give rise to dynamics that cannot be captured by CP maps, raising fundamental questions about the limits of product-state modeling in open quantum systems.

In this work, we investigate conditions under which subsystem dynamics can nevertheless be faithfully represented as though they originated from an initial product state. Specifically, we study a system obtained as the partial trace of a two-qubit system undergoing global unitary evolution. By focusing on this well-defined scenario, we are able to probe the extent to which local operations—performed before the onset of global evolution—can remove the influence of unwanted initial correlations.

We examine two classes of local operations on the system: measurements and unitaries. For local measurements, we characterize the minimal measurement strength required to reproduce system dynamics consistent with a product-state assumption. Unlike the work of Chitambar et al. [7], which considered global unitary dynamics under the simplifying assumption of an initially uncorrelated state, our framework explicitly inserts a local operation—measurement or unitary—between state preparation and global evolution. This distinction allows us to quantify how much "work" is needed locally to enforce the product-state description. Importantly, we also clarify our terminology: we use evolution to denote the global unitary acting jointly on the system and environment, and dynamics to denote the induced changes in the system's reduced state.

A central question we ask is: what is the weakest local operation that ensures the same system dynamics as those obtained from a product initial state? Here, "weakest" does not refer to the degree of projectivity in the case of measurements, but rather to the operation that optimizes fidelity between the system's pre-operation and post-operation states. Our analysis reveals

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that in some parameter regimes, only low-fidelity transformations are possible, establishing fundamental limits on how well one can model correlated dynamics through the product-state assumption.

We then turn to local unitary operations as an alternative. Our results demonstrate that, for arbitrary initial states and for one- and two-parameter families of global unitaries, it is always possible to model the reduced system dynamics as arising from a product state. Numerical simulations further indicate that this finding extends to general three-parameter two-qubit global unitaries. Remarkably, we find that unitary operations can often preserve the state of the system with high fidelity, avoiding the tradeoff between enforcing product-state dynamics and disturbing the system that arises with measurements. In fact, in nearly all of the 402 cases we studied numerically, a single local unitary transformation sufficed to reproduce the product-state dynamics while maintaining fidelity above 94.2%.

Finally, we broaden our perspective by considering whether additional quantum resources can further enhance this approach. We show that a general two-term Kraus channel—implementable with the aid of just a single ancillary qubit—can boost all fidelities to unity across our tested cases, while still ensuring that the dynamics-matching condition is satisfied. This result demonstrates that modest extensions of the system provide a universal strategy for reconciling correlated dynamics with product-state modeling.

All numerical data and code supporting our findings are publicly available in the associated GitHub repository [8], ensuring transparency and reproducibility. Taken together, our results clarify the interplay between system–environment correlations, local operations, and the validity of product-state assumptions. They also establish concrete strategies—based on measurements, unitaries, and ancillary-assisted channels—for mitigating the effects of initial correlations in open quantum system dynamics.

II. BACKGROUND

In quantum mechanics, the evolution of a quantum system is often described by completely positive (CP) maps [9, 10]. A CP quantum map acting on a system can be expressed as:

$$\mathcal{E}(\rho^S) \triangleq \mathrm{Tr}_E\left(U\rho^{SE}U^\dagger\right), \tag{1}$$

where U is a global unitary operator acting on the combined system-environment state, ρ^{SE}. The symbol "$\triangleq$" represents "is defined as" rather than "is equal to." The reduced system state, ρ^S, is obtained by tracing out the environment, $\rho^S = \mathrm{Tr}_E(\rho^{SE})$. This formalism ensures the physicality of the quantum operation, maintaining the complete positivity of the map. However, when the initial system-environment state is correlated ($\rho^{SE} \neq \rho^S \otimes \rho^E$), the system's evolution may involve non-completely positive (NCP) maps:

$$\mathcal{E}(\rho) \triangleq \sum_i \eta_i E_i \rho E_i^\dagger, \quad \eta_i = \pm 1, \tag{2}$$

as discussed in works such as [11–13]. While CP evolution is desirable for its well-defined conditions of reversibility [9, 14], understanding the interplay between CP and NCP maps is critical for advancing quantum technologies, particularly in noisy intermediate-scale quantum (NISQ) devices.

When correlations between the system and environment are present, the resulting dynamics may deviate from completely positive (CP) behavior [11, 15]. This highlights the importance of explicitly accounting for initial correlations in system-environment studies. For a given global evolution U , we say that the system dynamics, from $\rho^S \triangleq \mathrm{tr}_E(\rho^{SE})$ to $\rho^S_U \triangleq \mathrm{tr}_E(U\rho^{SE}U^\dagger)$, can be U-generated by a product state if there exists a valid ζ^E such that

$$\mathrm{tr}_E(U\rho^{SE}U^\dagger) = \mathrm{tr}_E(U\rho^S \otimes \zeta^E U^\dagger). \tag{3}$$

We say that the dynamics-matching condition is satisfied if Eq. 3 is true. However, this may not always be the case.

One important topic to cover in this section is that of completely positive trace-preserving (CPTP) maps. Any CPTP map can be expressed in the form $\sum_i K_i \rho K_i^\dagger$ for Kraus operators that satisfy $\sum_i K_i^\dagger K_i \leq \mathbb{I}$. In our problem, if we assume that the equality

$$\mathrm{tr}_E(U\rho^{SE}U^\dagger) \triangleq \mathrm{tr}_E(U\rho^S \otimes \zeta^E U^\dagger) \tag{4}$$

is true for some valid state ζ^E, we can show exactly what the Kraus representation will be to obtain the same dynamics. The right-hand side of Eq. (4) can be expressed as

$$\begin{aligned} &\sum_{j,\eta} \langle\eta|^E U(\rho^S \otimes \sum_j p_j |\zeta_j\rangle\langle\zeta_j|^E)U^\dagger |\eta\rangle^E \\ &= \sum_{j,\eta} p_j \langle\eta|^E U |\zeta_j\rangle^E \rho^S \langle\zeta_j|^E U^\dagger |\eta\rangle^E \\ &= \sum_{j,\eta} p_j K_{j\eta} \rho^S K_{j\eta}^\dagger \end{aligned} \tag{5}$$

for the Kraus operators $K_{j\eta} \triangleq \langle\eta|^E U |\zeta_j\rangle^E$, given an orthonormal basis set $\{|\eta\rangle^E\}$ on the environment. If ζ^E is a valid state, then all the probabilities p_j are nonnegative, and Eq. (5) is a valid Kraus representation for completely positive quantum dynamics on the system. If there exists a valid environment state ζ^E such that Eq. 3 holds, then the reduced dynamics of the actual experiment is indistinguishable (at its single admissible input) from the action of the CPTP map ζ^E; meaning that the actual experiment happens to sit on the trajectory of the CPTP map in Eq. (5). In that sense, the observed transformation admits a CP representation. However, the existence

of ζ^E does not enlarge the set of physical inputs; it only shows that the real input-output pair can be embedded into a CP channel.

In general, the set of input states possible on the system is bound due to the correlations with the environment, and our global evolution can be time-dependent. We will first assume a fixed initial state and a fixed global evolution for the sake of simplicity in demonstrating when we can U-generate the system dynamics with an initial product state. Note that this does not imply that the system dynamics us CP since we only have access to a single input/single output experiments with a fixed global unitary evolution. We later relax these assumptions and provide an example illustrating how our framework can be leveraged to prevent non-completely positive dynamics in Section III G.

Inspired by Brun's framework [16], we explore a more generalized measurement formalism characterized by the Hermitian operators

$$M_{\pm}(\epsilon,\hat{n}) \triangleq \frac{1}{2}\Big(\varepsilon_{+}\mathbb{I} \pm \varepsilon_{-}\hat{n}\cdot\vec{\sigma}\Big), \tag{6}$$

such that we satisfy the completion relation $[M_{+}(\epsilon,\hat{n})]^2 + [M_{-}(\epsilon,\hat{n})]^2 = \mathbb{I}$ if the coefficients are defined as

$$\varepsilon_{\pm} \triangleq \sqrt{\frac{1+\epsilon}{2}} \pm \sqrt{\frac{1-\epsilon}{2}}, \quad \hat{n} \triangleq \{n_x, n_y, n_z\}. \tag{7}$$

Here, both ϵ $(0 \leq \epsilon \leq 1)$ and the unit vector $\hat{n}$ control the measurement strength, depending on the state of the system S. For instance, when the Bloch vector of S is $\{0,0,0\}$, the measurement strength is directly proportional to the size of ϵ; that is, a larger ϵ-value will give a state further away from maximally mixed. On the other hand, if the state is already close to being pure, $\mathrm{tr}\big([\rho^S]^2\big) \approx 1$, then a larger ϵ-value and a particular direction η is required to have minimal impact on the state ρ^S. However, if our system is entangled with any data qubit, we would like to prevent any destruction of the correlations with it. From here on out, we will always assume that the measurements depend on ϵ and $\hat{n}$ and just use $M_{\pm}$ to represent them.

The initial state of a composite system and environment can be parameterized as

$$\rho^{SE} \triangleq \frac{1}{4}\Big(\mathbb{I}\otimes\mathbb{I} + \vec{a}\cdot\vec{\sigma}\otimes\mathbb{I} + \mathbb{I}\otimes\vec{b}\cdot\vec{\sigma} + \sum_{i,j=1}^{3} t_{ij}\hat{\sigma}_i\otimes\hat{\sigma}_j\Big), \tag{8}$$

where $\vec{a}$ and $\vec{b}$ are the Bloch vectors of the system and the environment, respectively, and t_{ij} represents the elements of the correlation matrix. Here, $\vec{\sigma} \triangleq (\hat{\sigma}_1, \hat{\sigma}_2, \hat{\sigma}_3)$ is the vector of Pauli matrices. A global unitary transformation U acts on the combined system-environment state.

III. RESULTS

In this section, we present three different strategies that enable us to U-generate system dynamics with an initial product state. We begin with the assumption that the pair (U, ρ^{SE}) cannot by itself produce such dynamics, but that this becomes possible once we allow an additional local operation $\mathcal{G}$. We therefore redefine our experiment in terms of the pair $(U, (\mathcal{G}\otimes\mathbb{I})\rho^{SE}(\mathcal{G}\otimes\mathbb{I})^\dagger)$. This construction gives us a way to modify system-environment correlations without requiring any direct control over the environment itself. In general, the form of $\mathcal{G}$ depends on both the global evolution U and the initial joint state ρ^{SE}. While this makes any general statement difficult to extrapolate, we can still provide results for common situations and show the benefits of adding a single ancilla qubit when implementing particular strategies.

The first strategy is to take $\mathcal{G}$ as a local measurement. We use this example to illustrate the idea because, although measurements introduce uncertainty and may disrupt the system's state, they can always be tuned to perturb correlations just enough to ensure the system dynamics can always be generated using a product state. In the worst-case scenario, a projective measurement will destroy all correlations, leaving the system in a product state with the environment. However, this may be detrimental to the fidelity of the initial state of the system.

The second strategy, applying a local unitary operation, is deterministic and often more effective at controlling correlations, though it still cannot avoid fidelity loss in every case. This brings us to our third strategy of using two-term Kraus operators, which can overcome the limitations of both local unitary operations and measurements. We restrict attention to having two terms in the Kraus representation because only one additional ancilla is necessary to implement them experimentally. Below, we describe how correlations are transformed under the first two strategies.

A. Transformations to Correlations

The way correlations evolve between the system and environment depends on whether we apply a measurement or a unitary to the local system. For simplicity, we consider only the application of the measurement M_{+}, which changes the local Bloch vectors and two-qubit correlation matrix as follows [17]:

$$\begin{aligned}
\vec{b}' &= \frac{\vec{b} + \epsilon\,\mathcal{T}_\rho^T\hat{n}}{1+\epsilon\,\hat{n}\cdot\vec{a}}, \\
\vec{a}' &= \frac{\sqrt{1-\epsilon^2}\,\vec{a} + (1-\sqrt{1-\epsilon^2})(\hat{n}\hat{n}^T)\cdot\vec{a} + \epsilon\,\hat{n}}{1+\epsilon\,\hat{n}\cdot\vec{a}}, \\
\mathcal{T}_\rho' &= \frac{\sqrt{1-\epsilon^2}\mathcal{T}_\rho + (1-\sqrt{1-\epsilon^2})(\hat{n}\hat{n}^T)\cdot\mathcal{T}_\rho + \epsilon\,\hat{n}\vec{b}^T}{1+\epsilon\,\hat{n}\cdot\vec{a}}.
\end{aligned} \tag{9}$$

Here, $\mathcal{T}_\rho$ denotes the initial correlation matrix, and the probability of obtaining this result is given by $(1/2)(1+\epsilon\ \hat{n}\cdot\vec{a})$. The probability of the "$-$" outcome is given by flipping the sign on ϵ, so we can easily determine the trade-off between the parameters and the likely outcome. In practice, we discard any event that is not associated with the chosen measurement output.

As a useful check, we see that the projective measurement ($\epsilon = 1$) completely destroys the correlations with the environment, leaving the overall state in product form; i.e., $\mathcal{T}'_\rho = \vec{a}' \cdot (\vec{b}')^T$. However, this may come at the cost of fidelity, since we assume initial correlations between the system and environment, which restricts the local state to being mixed. For example, if the initial Bloch vector of the system is $\{0,0,0\}$, the resulting fidelity is only one-half in the worst-case scenario. This guarantees a product form in the end, but with minimal preservation of the state of the system, as we show in Section III B. In practice, we ideally want the state to be pure and completely separate from the environment. However, a projective measurement will also eliminate correlations with data qubits that are entangled with the system.

When we instead apply a local unitary $\mathcal{L}$, the effect on the correlations is much simpler and does not destroy correlations with other data qubits. The frame just becomes altered some, since the basis is changed locally. By taking $\mathcal{G}$ as a local unitary, we get the transformations

$$\vec{b}' = \vec{b}, \quad \vec{a}' = \mathcal{O}_\mathcal{L} \cdot \vec{a}, \text{ and } \mathcal{T}'_\rho = \mathcal{O}_\mathcal{L} \cdot \mathcal{T}_\rho. \tag{10}$$

Here, the operation $\mathcal{O}_\mathcal{L}$ is the orthogonal matrix induced by $\mathcal{L}$, with elements given by

$$\mathcal{O}_{ij} = \frac{1}{2}\operatorname{Tr}(\hat{\sigma}_i \mathcal{L}\hat{\sigma}_j \mathcal{L}^\dagger). \tag{11}$$

In this framework, there are two unitaries that map to the same orthogonal rotation on the original Bloch vector due to SU(2) being able to double-cover SO(3). The restriction of only being able to access the system prevents us from being able to diagonalize the correlation matrix, which we see from Corollary 1 would allow us to always U-generate the system dynamics with an initial product state. Next, we examine how these transformations affect fidelity.

B. Fidelities of Different Strategies

Suppose we begin with an experiment described by the pair (U, ρ^{SE}), where the system dynamics cannot be U-generated by a product state input. By applying a measurement before the global unitary, we can try to enforce the product form. Note that if we find a solution for the measurement M_-, we know that there also exists a solution for M_+, and vice versa. So, the task is to find a measurement, M_+, and a valid environmental Bloch vector, $\vec{\zeta}^E$, that we satisfy the relationship

$$\begin{aligned} &\operatorname{tr}_E\left[U(M_+\otimes\mathbb{I})\rho^{SE}(M_+\otimes\mathbb{I})U^\dagger\right] \\ &= \operatorname{tr}_E\left[(UM_+\rho^S M_+)\otimes\zeta^E U^\dagger\right]. \end{aligned} \tag{12}$$

To ensure that we preserve the initial state as much as we can, we optimize the fidelity between ρ^S and $\rho^S_M = M_+\rho^S M_+^\dagger$. This value is given by

$$F(\rho,\rho_\pm) = 1 - \frac{[1-(\vec{a}\cdot\hat{n})^2][1-\sqrt{1-\epsilon^2}]}{2(1\pm\epsilon\vec{a}\cdot\hat{n})}, \tag{13}$$

where $\rho_\pm \triangleq M_\pm\rho^S M_\pm / \operatorname{tr}\big(M_\pm\rho^S M_\pm\big)$. We obtained this exact solution by using the explicit single-qubit fidelity equation given by

$$F(\rho,\sigma) = \operatorname{tr}(\rho\cdot\sigma) + 2\sqrt{\det(\rho)\cdot\det(\sigma)} \tag{14}$$

and simplifying. Notice that the fidelity is affected by the magnitude of the local Bloch vector $\vec{a}$ and the direction $\hat{n}$. If its initial magnitude is 1, notice that $\hat{n} = \hat{a}$ for any ϵ-value will result in a maximum fidelity of 1. Therefore, ϵ does not really symbolize how strong the measurement is for all $\hat{a}$. On the other hand, if the local Bloch vector $\hat{a} = \{0,0,0\}$, then the fidelity monotonically decreases with an increase of ϵ, which we use to optimize in some of our examples.

Now we consider the case when a local unitary transformation, $\mathcal{L}$, is used to U-generate system dynamics with an initial product state. The fidelity between the state of the system ρ^S and its rotated version $\mathcal{L}\rho^S\mathcal{L}$ is given by

$$F_\mathcal{L} = 1 - \frac{1}{2}\left(\vec{a} - [\mathcal{O}_\mathcal{L}\cdot\vec{a}]\right)\cdot\vec{a}. \tag{15}$$

This equation gives us great insight into how fidelity changes, as it achieves maximum fidelity ($F = 1$) when $\vec{a}$ is left invariant by $\mathcal{O}_L$ and minimum fidelity ($F = 1-\vec{a}\cdot\vec{a}$) when it rotates from $\vec{a}$ to $-\vec{a}$. So we would like to rotate about the Bloch vector $\vec{a}$ whenever possible to perfectly preserve the fidelity of the initial system while ensuring CP dynamics. This is possible in many experiments, as we later show in Sec. III K, since the local unitary on the system also induces the same orthogonal rotation to the left side of the correlation matrix $\mathcal{T}_\rho$.

C. Application of a Measurement

We now give an example where the use of a local measurement makes it possible for the system evolution to be U-generated by a product state. To illustrate, consider the initial Bell state $|\Phi^+\rangle \triangleq (1/\sqrt{2})(|00\rangle + |11\rangle)$. We apply the measurement given in Eq. (6) to the system, evolve the joint state under the global unitary $U = \text{CNOT}$, and then trace out the environment to obtain the reduced dynamics of the system. Before the application of the CNOT gate, the unnormalized reduced

state of the system has the form

$$\rho_{\pm}^{S} = \mathrm{tr}_E\left[(\mathcal{M}_{\pm}\otimes\mathbb{I})\left|\Phi^+\right\rangle\!\left\langle\Phi^+\right|(\mathcal{M}_{\pm}\otimes\mathbb{I})\right] = \frac{1}{2}\big(\mathbb{I}\pm\epsilon\,\hat{n}\cdot\vec{\sigma}\big). \tag{16}$$

After the global CNOT evolution acts across the two qubits, the unnormalized reduced state of the system has the form

$$\rho_{U}^{S} = \mathrm{tr}_E\left[U(\mathcal{M}_{\pm}\otimes\mathbb{I})\left|\Phi^+\right\rangle\!\left\langle\Phi^+\right|(\mathcal{M}_{\pm}\otimes\mathbb{I})U^{\dagger}\right]. \tag{17}$$

We seek to determine when

$$\rho_{U}^{S} = \mathrm{tr}_E[U(\mathcal{M}_{\pm}\rho^{S}\mathcal{M}_{\pm}\otimes\zeta^{E})U^{\dagger}] \tag{18}$$

for some valid environmental state ζ^E, such that the fidelity between ρ^S and ρ_U^S is optimal. In other words, under what conditions does there exist an environmental state ζ^E such that the observed dynamics are exactly those of a product state, namely $\rho^S\otimes\zeta^E$, passed through U?

In order to determine the "weakest" disturbance that allows the system evolution to be U-generated by a product state, we must minimize the ϵ-value since the initial local Bloch vector is $\{0,0,0\}$. By setting the state of the systems from Eqs. (16) and (18) equal to each other, for only the "+"-outcome, we must satisfy the two relationships

$$n_y[n_x(\sqrt{1-\epsilon^2}-1)+\epsilon\,\zeta_x] = 0 \tag{19}$$

$$\text{and } (1-n_x^2)\sqrt{1-\epsilon^2}+n_x(n_x-\epsilon\,\zeta^x) = 0 \tag{20}$$

for the smallest possible ϵ. We define the local Bloch vector of the environment to be $\vec{\zeta}\triangleq\{\zeta_x,\zeta_y,\zeta_z\}$ from here on out. Since both equalities must be satisfied and the second one does not depend on n_y, we can set $n_y=0$. Then we only have to minimize ϵ such that the second equality is satisfied. The optimal solution that ensures the equality between Eqs. (17) and (18) is given by $\epsilon=\frac{2\sqrt{2}}{3}$ and $\vec{n}=\left(\frac{1}{\sqrt{2}},0,\frac{-1}{\sqrt{2}}\right)$. The valid density operator on the environment that results in the system dynamics being U-generated by a product state is given by $\zeta^E=\frac{1}{2}(\mathbb{I}+\hat{\sigma}_x)=|+\rangle\langle+|$. A natural question arises: what happens if we apply a sequence of such measurements before the global evolution?

Since measurements are repeatable, one might hope that multiple rounds could be compressed into a single, stronger measurement with a different parameter choice for ϵ. However, this is not the case. As discussed previously, the measurement in Eq. 6 is related to the weak measurement $A_{0|1}$ in [16] by a unitary rotation. To demonstrate that the two are not equal, we examine the effect of multiple rounds of measurements on an arbitrary initial state $|\psi\rangle$. First, we observe that $M_+M_-\propto\mathbb{I}$, so that

$$M_+^m M_-^n = \begin{cases} M_+^{m-n} \text{ if } m>n, \\ M_-^{n-m} \text{ if } n>m, \end{cases} \tag{21}$$

up to a factor that does not affect the state of the system. This is because each measurement operator acts as the inverse of the other after renormalization.

Next, we consider both measurement operators raised to the k^{th} power. We find that

$$M_{\pm}^{k} = \frac{1}{2}\left\{\tilde{\epsilon}_{+}^{k}\mathbb{I}\pm\tilde{\epsilon}_{-}^{k}\,\hat{n}\cdot\vec{\sigma}\right\}, \tag{22}$$

where the new coefficients are given by

$$\tilde{\varepsilon}_{\pm}^{k} = \left(\frac{1+\epsilon'}{2}\right)^{k/2}\pm\left(\frac{1-\epsilon'}{2}\right)^{k/2}. \tag{23}$$

These coefficients resemble the original ones in Eq. (7), and it will become clear that whenever $\tilde{\varepsilon}_+^k=\varepsilon_+$ for some k, we cannot have $\tilde{\varepsilon}_-^k=\varepsilon_-$. The reason we include the primes in Eq. 23 is that the repeated measurements will satisfy $\epsilon'\le\epsilon$, with equality only occurring when $k=1$. The exact solution for when $\tilde{\epsilon}_+^k=\epsilon_+$ is given by $\epsilon=\tilde{\epsilon}_+^k\sqrt{2-(\tilde{\epsilon}_+^k)^2}$. On the other hand, the exact solution for when $\tilde{\epsilon}_-^k=\epsilon_-$ is given by $\epsilon=\tilde{\epsilon}_-^k\sqrt{2-(\tilde{\epsilon}_-^k)^2}$. Define the function $g(q)=q\sqrt{2-q^2}$. Note that $g(\sqrt{2-q^2})=g(q)$. Hence, $g(\tilde{\epsilon}_+^k)=g(\tilde{\epsilon}_-^k)$ *iff* $\sqrt{2-(\tilde{\epsilon}_+^k)^2}=\tilde{\epsilon}_-^k$. So, if $\tilde{\epsilon}_+^k\sqrt{2-(\tilde{\epsilon}_+^k)^2}=\tilde{\epsilon}_-^k\sqrt{2-(\tilde{\epsilon}_-^k)^2}$, we must satisfy $(\tilde{\epsilon}_+^k)^2+(\tilde{\epsilon}_+^k)^2=2$.

From the previous equality, we calculate an equivalent expression given by

$$h(\epsilon)\triangleq(1+\epsilon')^k+(1-\epsilon')^k=2^k. \tag{24}$$

The derivative is given by

$$\frac{\partial h(\epsilon')}{\partial k} = k[(1+\epsilon')^{k-1}-(1-\epsilon')^{k-1}], \tag{25}$$

which is strictly positive for $k\ge2$ and $0<\epsilon<1$. Therefore, repeated measurements always cause the function $h(\epsilon')$ to increase for all ϵ'-values. We also calculate h at the end points to obtain $h(0)=2$ and $h(1)=2^k$, which means that $2=h(0)<h(\epsilon')<h(1)=2^k$ for $0<\epsilon'<1$. We conclude our proof with the last case of $k=1$, and show that $h(\epsilon')=2$ for all ϵ'. This satisfies Eq. 24 as expected. Therefore, repeated measurements cannot be simulated by a single measurement unless they represent a series of the same projective measurement. For the purposes of this paper, we restrict our analysis to single measurements.

D. Werner States

To broaden the discussion, we now examine a class of mixed states that is $U\otimes U$-invariant, called Werner states. The Werner state $W(\lambda)$ is defined as a convex

combination of the antisymmetric Bell state $|\Psi^-\rangle\langle\Psi^-|$ and the maximally mixed state on two qubits. Explicitly,

$$W(\lambda) \triangleq \lambda \left|\Psi^-\right\rangle\!\left\langle\Psi^-\right| + (1-\lambda)\frac{\mathbb{I}\otimes\mathbb{I}}{4}, \tag{26}$$

where the parameter λ lies between 0 and 1. Importantly, Werner states have vanishing local Bloch vectors, since the local states are maximally mixed. If we assume that the global evolution is given by the CNOT gate, then the local measurement must have $n_y = 0$ for the second component of the local Bloch vectors to be equal; otherwise, we can never satisfy the dynamics-matching condition. If the environmental part of the product state has a Bloch vector given by $\vec{\zeta} \triangleq \{\zeta_x, \zeta_y, \zeta_z\}$, then the only condition for equality is

$$\zeta_x = -\left[\sqrt{1-\epsilon^2}\cdot\frac{1}{n_x} + (1-\sqrt{1-\epsilon^2})\cdot n_x\right]\cdot\frac{\lambda}{\epsilon}, \tag{27}$$

where we need the optimal $\hat{n}_x$-value that allows us to minimize ϵ for a valid Bloch vector $\vec{\zeta}$ constrained to $\zeta_x \in [-1, 1]$. This will ensure that we are also maximizing the fidelity.

Since the parameter $\epsilon \in [0, 1]$, the term $-\sqrt{1-\epsilon^2}$ is less than or equal to $-(1-\sqrt{1-\epsilon^2})$ for $\epsilon \in [0, \sqrt{3}/2]$, which divides our optimization into two parts. The minimum ϵ must occur when $n_x = 1$ to ensure there is a valid environmental Bloch vector on the first interval. The reason is that the $1/n_x$ term needs to be at its smallest for $\epsilon \in [0, \sqrt{3}/2]$ to prevent ζ_x from becoming less than -1 as ϵ becomes smaller. Since $n_x \in [-1, 1]$ and the absolute value of the left-hand term is always larger than or equal to the absolute value of the right-hand term for all valid n_x-values, then ζ_x must be determined at the end points for n_x; let us say 1 since -1 gives us the same bound by symmetry. This enforces $\zeta_x = -\lambda/\epsilon$ which tells us that $\epsilon_{min} = \lambda$ since $\epsilon \geq 0$. In other words, when λ is between the values 0 and $\sqrt{3}/2$, the minimum ϵ needed to U-generate the system dynamics with a product state is simply equal to λ. Thus, $\epsilon_{min} = \lambda$ for $\lambda \in [0, \sqrt{3}/2]$.

We now consider the interval in which $\epsilon \in [\sqrt{3}/2, 1]$ so that the term $-(1-\sqrt{1-\epsilon^2})$ is less than or equal to $-\sqrt{1-\epsilon^2}$. In this case, we would like n_x to be in the interval $[-1, 0]$ since we cannot simply minimize the right term in Eq. (27) because the terms on the left blow up to infinity. When our solution for ζ_x is unit ($\zeta_x \to \pm\infty$ in the limit as $\epsilon \to 0$, then by continuity, ζ_x must be one or minus one when ϵ is minimum), we get the equality

$$\epsilon = -\frac{n_x^3 - (1-n_x^2)\sqrt{\lambda^2 + n_x^2(1-2\lambda^2)}}{n_x^2 + (1-n_x^2)^2\lambda^2}\cdot\lambda. \tag{28}$$

Taking the derivative of ϵ with respect to n_x and setting it equal to zero, and applying the boundary conditions on n_x and ϵ, yields the solutions

$$n_x = -\frac{1}{\sqrt{4\lambda^2-2}} \text{ and } n_z = \pm\sqrt{1-\frac{1}{\sqrt{4\lambda^2-2}}}. \tag{29}$$

Plugging n_x into our definition of ϵ yields the final solution of

$$\epsilon_{min} = \begin{cases} \lambda \;\text{ if } \lambda \in [0, \sqrt{3}/2] \\ \dfrac{2\lambda\sqrt{4\lambda^2-2}}{4\lambda^2-1} \;\text{ if } \lambda \in [\sqrt{3}/2, 1] \end{cases} \tag{30}$$

for the minimum ϵ (maximum fidelity) that ensures the system dynamics is U-generated by a product state. As expected, the minimum value of ϵ depends on the parameter λ that defines the Werner state. In Figure 1, we plot the optimal fidelities for the entire family of Werner states given in Eq. (26). The plot monotonically decreases with an increase in λ, which means that it is harder to preserve fidelity with Werner states nearest to the singlet state. For the maximally mixed state, $\lambda = 0$, we can do this without any measurement, since it is already in the product form.

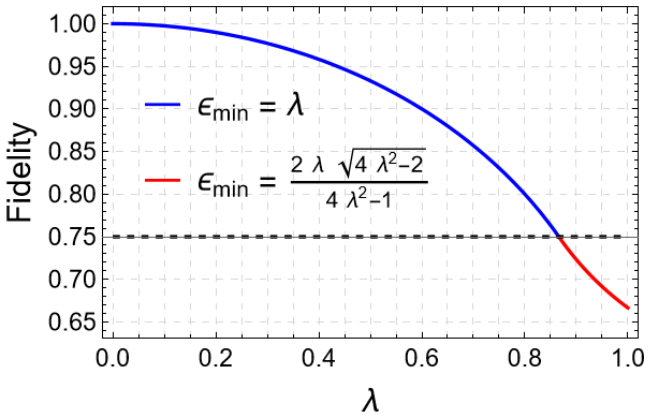

FIG. 1. Plot of optimal fidelity of U-generating the system dynamics using a product state over all family of Werner states $W(\lambda)$.

E. Optimal Measurement Can Be Projective

When optimizing the fidelity in order to reproduce the system dynamics from a product state, one may ask whether it is necessary to perform a projective measurement that completely destroys all correlations between the system and environment. This would represent one of the worst-case scenarios if the system is entangled with other data qubits, and we show that it can happen with this next example. To see this, consider when the initial correlations with the environment are given by the Bell state $\rho^{SE} = |\Phi^+\rangle\langle\Phi^+|$. We will use a composition of the swap with the CNOT gate to define our global evolution

$U = SWAP \circ CNOT$. Then the system evolution is given by

$$\rho^S = \frac{1}{2}\mathbb{I} \quad \overset{U}{\to} \quad \rho_U^S = |0\rangle\langle 0|\,, \tag{31}$$

which cannot be U-generated by a product state.

If a local measurement is applied before U, then the solution for the environmental Bloch vector is given by

$$\zeta_x = \epsilon n_x,\ \zeta_y = \frac{n_y(\sqrt{1-\epsilon^2}-1)}{\epsilon}, \tag{32}$$

$$\text{and } \zeta_z = \frac{n_z^2 + (1-n_z^2)\sqrt{1-\epsilon^2}}{\epsilon n_z}. \tag{33}$$

This ensures that we can generate the same system dynamics with an initial product state. Notice that the solution is only valid whenever $\epsilon = 1$ since we must satisfy the normalization condition for $\hat{n} \triangleq \{n_x, n_y, n_z\}$. The solution for the environmental Bloch vector then becomes $\vec{\zeta} = \{n_x, -n_y, n_z\}$. This example illustrates a fundamental limitation of weak measurements: in certain settings, a partial disturbance of the system–environment correlations is insufficient. A full projective measurement is required to make the product state description possible, even though it erases all correlations between the system and the environment. We now turn our attention to local unitaries and illustrate their power over measurements.

F. Application of Local Unitaries

We now revisit the same example as in Section III E, but instead of measurements, we allow arbitrary local unitaries on the system before global evolution. Once again, we take $U = SWAP.CNOT$, but this time the initial joint state is rotated by a local unitary,

$$\rho^{SE}(\theta) = (R_y(\theta) \otimes \mathbb{I})\Phi^+(R_y(\theta)^\dagger \otimes \mathbb{I}), \tag{34}$$

where $R_y(\theta) = e^{-i\theta/2\sigma_y}$. Since the local system is maximally mixed, any local unitary operator will preserve the initial state of the system. This is in contrast to using local measurements in this scenario. For example, Figure 2 shows the minimum ϵ-values that are required to U-generate the dynamics of the system using a product state for $\theta \in [0, \pi/2]$. These data points illustrate how optimal fidelity always decreases as the strength of the measurement increases. As would be expected, local measurements can never completely preserve fidelity. On the other hand, when $\theta = \pi/2$, no measurement is necessary and we can U-generate the system dynamics with a product state. This shows that local rotations have power in guiding system dynamics while perfectly preserving the initial state.

This raises the question of whether a local unitary on the system can always be used to U-generate the dynamics on the system with a product state. For instance, the local unitary for the Werner state $W(\lambda)$ is the real rotation $e^{-i\pi/4\sigma_y}$ when the global evolution is the CNOT gate. This is true for any λ. If we only have access to the local system of the joint state ρ^{SE}, is it always possible to perform a local unitary to U-generate the system dynamics with an initial product state? This avoids the destructive process of performing measurements since it only induces orthogonal rotations on the correlation matrix between the system and other data qubits. No correlations are essentially destroyed, but they are altered in a predictable manner.

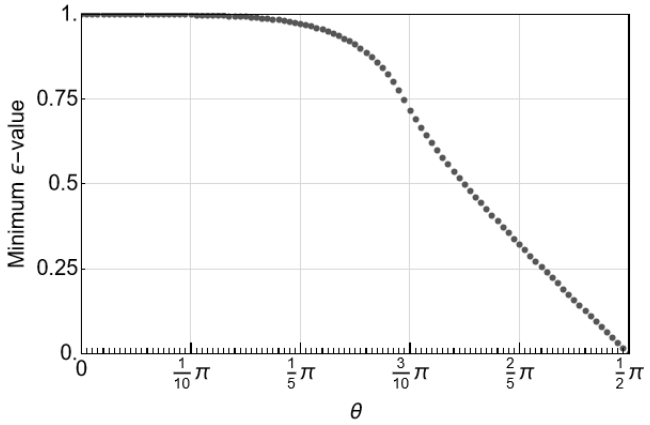

FIG. 2. Plot of the minimum ϵ-value needed to U-generate the system dynamics, $\rho^S(\theta) \xrightarrow{U} \mathrm{tr}_E(U\rho^{SE}(\theta)U^\dagger)$, using a product state. The rotation angle θ for the local unitary is varied between 0 and $\pi/2$.

G. Preventing NCP dynamics

In this section, we provide an example for when a local unitary can be used to completely prevent non-completely positive quantum dynamics on the system. We first need to define the NCP map, which requires a time-dependent global evolution $U(t)$ and a compatibility domain for the state of the system. Instead of using the initial experiment described by the pair $\{CNOT, \Phi^+\}$, we will consider the time-dependent experiment given by the pair $\{e^{-i\,tCNOT}, \Lambda_p^{SE}\}$ for the initial correlated state

$$\Lambda_p(\tau^S) \triangleq \tau^S \otimes \mathbb{I}/2 + (1-p)\left(\Phi^+ - \mathbb{I}/2 \otimes \mathbb{I}/2\right), \tag{35}$$

where $\mathrm{tr}_E[\Lambda_p(\tau^S)] = \tau^S$. The compatibility domain is the set of valid input states, τ^S, given by the local Bloch vector $\vec{\tau} = p\vec{\mu}$, where $\vec{\mu}$ is real and $||\vec{\mu}|| \leq 1$. For a fixed p-value, the correlations between the system and environment are fixed, as well as the local environmental state. We know that when $p = 1$, we can U-generate the dynamics of the system with a product state, but we cannot assume anything about the dynamical map of a single input and single output experiment.

The dynamical matrix that describes the subsystem

dynamics for this experiment is given by

$$A_S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ a & e^{-it}\cos(t) & 0 & a \\ a^* & 0 & e^{it}\cos(t) & a^* \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{36}$$

for $a \triangleq (1 - e^{-2it})(1 - p)/4$. This matrix is derived by vectorizing the state $\mathrm{tr}_E[\Lambda^{SE}(p)]$ and multiplying it by the general single-qubit trace-preserving map given by Eq. 59 of [18]. We define the parameters of the A_S-matrix such that the dynamics leads to the resultant state ρ_U^S for *all* inputs. This is because our map cannot depend on what state we possess in the laboratory. The matrix A_S has an associated realigned B_S-matrix [19] that has the negative eigenvalue

$$\sin(t/2)\left(\sin(t/2) - \frac{1}{2}\sqrt{1 + 3\sin(t/2)^2}\right) \tag{37}$$

for $0 < p < 1$ and for all times $t \in (0, \pi/2]$. This means that it is non-completely positive right after the global evolution kicks off, even when the state is in the separable regime. This makes sense when $t = 0$ because the map is identity. On the other hand, the choice of $(U(t), \rho)$ leads to a completely positive map for $p = 1$ at all times t.

We note that the experiment fixed at $p = 1$ and $t = \pi/2$ is described by a CP map, but the associated system dynamics cannot be U-generated by an initial product state. This is the reason why we cannot call the subsystem dynamics non-completely positive if it cannot be U-generated by a product state. However, we can say that the dynamics is NCP if the realigned B_S-matrix does not have all non-negative eigenvalues. To accomplish this task, we must have a nonstationary experiment with a time-dependent unitary and a subset of inputs/outputs, rather than a fixed single-input and single-output demonstration. For a general ρ^S and any time t, the environmental state that allows us to U_t-generate the system dynamics for all inputs is any valid state whose Bloch vector lies solely in the yz-plane.

We illustrate how powerful our technique can be in preventing NCP dynamics by showing how a single local unitary can cause the new B_S-matrix to become non-negative for all parameters p and t. For instance, assume that we apply the local rotation $R_Y(\pi/2) \triangleq e^{-i(\pi/4)\hat{\sigma}_y}$ to the system of Φ_p^+ prior to global evolution. The dynamical matrix that dictates the subsystem dynamics is now given by

$$A_S \triangleq \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{-it}\cos(t) & 0 & 0 \\ 0 & 0 & e^{it}\cos(t) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{38}$$

which will always have an associated realigned B_S-matrix that is non-negative. Therefore, a single local rotation on only the system was able to prevent all NCP dynamics during this experiment. So, the basis in which we prepare our state of our system makes a significant difference on how the environment and global evolution dictate the type of subsystem dynamics.

H. Different Classes of Global Unitaries

We can consider several different classes of global unitary evolutions when determining if it is possible to U-generate the system dynamics using a product state. Any two-qubit unitary can be described by 3 nonlocal parameters and 4 local rotations, given by Cartan's KAK decomposition [20–22]. If we set the nonlocal parameters to α_i for $i = 1, 2, 3$, then the different unitary classes that we define are given by $\alpha_i \in \mathcal{S} \;||\; \alpha_i \notin \mathcal{S}$ for $\mathcal{S} = \{n\pi/2 : n \in \mathbb{W}\}$. Before presenting our first theorem, we introduce a rotation operator that will serve as a useful tool throughout the analysis. This operator acts on the correlation components of a two-qubit state (namely, the $\mathcal{O}_\mathcal{L}$ in Eq. 10) and can always be implemented by means of local unitaries on the system.

Definition 1. *A Givens rotation matrix g_{ab} has nonzero elements satisfying:*

- $g_{aa} = 1$ *for* $a \neq i, j$,
- $g_{aa} = \cos(\theta)$ *for* $a = i, j$,
- $g_{ji} = -g_{ij} = -\sin(\theta)$ *(with fixed $i > j$).*

This rotation applies a counterclockwise rotation in the (i, j) plane by an angle θ We now provide our first theorem.

Theorem 1. *Define the nonlocal part of the global evolution as*

$$\Omega \triangleq e^{-i(\alpha_1\hat{\sigma}_1\otimes\hat{\sigma}_1 + \alpha_2\hat{\sigma}_2\otimes\hat{\sigma}_2 + \alpha_3\hat{\sigma}_3\otimes\hat{\sigma}_3)}. \tag{39}$$

We say that U is part of a two-parameter family if only one of the parameters α_1, α_2, or α_3 is in $\mathcal{S}$. Then Givens rotations can always be used on the subsystem of any two-qubit state, undergoing any two-parameter global evolution, to U-generate the system dynamics with an initial product state.

Proof. We want the following equality to hold for some local unitary, V, to the system and for some valid environmental state ζ^E:

$$tr_E(UV^S\rho^{SE}V^{S^\dagger}U^\dagger) = \mathrm{tr}_E(UV^S\rho^S V^{S^\dagger} \otimes \zeta^E U^\dagger). \tag{40}$$

We can define the global evolution as $U = (\mathcal{L}_1 \otimes \mathcal{L}_2)\Omega(\mathcal{R}_1 \otimes \mathcal{R}_2)$ [23–25] and eliminate the local unitaries on the left, $\mathcal{L}_1$ and $\mathcal{L}_2$, since they make no difference to the equality in Eq. 40. We also observe that $R_1V^S = R_1V^SR_1^\dagger R_1 \to V^SR_1^\dagger$, since V^S is chosen over all local unitaries. The same is true for the local environmental state ζ^E, where $R_2\zeta^E R_2^\dagger \to \zeta^E$ can be any state.

This means that if a solution exists for the equality

$$tr_E[\Omega V^S(\mathcal{R}_1\otimes\mathcal{R}_2)\rho^{SE}(\mathcal{R}_1^\dagger\otimes\mathcal{R}_2^\dagger)V^{S^\dagger}\Omega^\dagger] \quad (41)$$
$$=\mathrm{tr}_E[\Omega(V^S\mathcal{R}_1\rho^S\mathcal{R}_1^\dagger V^{S^\dagger})\otimes(\mathcal{R}_2\zeta^E\mathcal{R}_2^\dagger)\Omega^\dagger], \quad (42)$$

then it also exists for Eq. 40.

Since we want to prove Theorem 1 for all two-qubit states, equality must hold over any local unitaries $\mathcal{R}_1$ and $\mathcal{R}_2$. This will result in a generic state that is captured by any ρ^{SE}, so we can throw away these right local unitaries and only ensure that

$$tr_E(\Omega V^S\rho^{SE}V^{S^\dagger}\Omega^\dagger)=\mathrm{tr}_E(\Omega V^S\rho^S V^{S^\dagger}\otimes\zeta^E\Omega^\dagger) \quad (43)$$

can hold for any two-parameter family of Ω and any ρ^{SE}, for some local unitary V acting on the system, and some environmental state ζ^E. When we get a solution for Eq. (43), we can also obtain a solution for Eq. 40 by conjugating V^S by $R_1^\dagger$ and ζ^E by $R_2^\dagger$. Using the reduced form in Eq. 43, we must show that the three equalities

$$(t_{32}-a_3\zeta_2)\cos(2\alpha_3)\sin(2\alpha_2)+[(-t_{23}+a_2\zeta_3)\cos(2\alpha_2)+(b_1-\zeta_1)\sin(2\alpha_2)]\sin(2\alpha_3)=0, \quad (44)$$
$$(-t_{31}+a_3\zeta_1)\cos(2\alpha_3)\sin(2\alpha_1)+[(t_{13}-a_1\zeta_3)\cos(2\alpha_1)+(b_2-\zeta_2)\sin(2\alpha_1)]\sin(2\alpha_3)=0, \quad (45)$$
$$\text{and } (t_{21}-a_2\zeta_1)\cos(2\alpha_2)\sin(2\alpha_1)+[(-t_{12}+a_1\zeta_2)\cos(2\alpha_1)+(b_3-\zeta_3)\sin(2\alpha_1)]\sin(2\alpha_2)=0 \quad (46)$$

can always be satisfied for some local unitary inserted between U and ρ^{SE} $(\rho^S\otimes\zeta^E)$ on system S. We derived these equations by solving Eq. 43 exactly. The vectors $\vec{a}\triangleq\{a_1,a_2,a_3\}$, $\vec{b}\triangleq\{b_1,b_2,b_3\}$, and $\vec{\zeta}\triangleq\{\zeta_1,\zeta_2,\zeta_3\}$ are the local Bloch vectors for $V^S\rho^SV^{S^\dagger}$, ρ^E, and ζ^E, respectively. The correlation matrix of $V^S\rho^{SE}V^{S^\dagger}$ is given by $\{t_{ij}\}_{i,j=1}^3$.

In this section, we will review two cases: the one-parameter family and the two-parameter family of global unitaries. In the one-parameter family, we have that $\alpha_i\notin\mathcal{S}$ for only one of $i\in\{1,2,3\}$. Let us assume, without loss of generality, that $\alpha_2\in\mathcal{S}$ and $\alpha_3\in\mathcal{S}$. Then we must satisfy the relations

$$(a_3\zeta_1-t_{31})\sin(2\alpha_1)=(a_2\zeta_1-t_{21})\sin(2\alpha_1)=0 \quad (47)$$

for all α and some choice of ζ_1. We have the freedom to rotate the correlation matrix and the local Bloch vector $\vec{a}$. We can simply rotate the correlation matrix on the left side using Givens rotations to make $t_{21}=0$ and $t_{31}=0$, and then set $\sigma_1=0$ to satisfy these equalities for some local unitary acting on the system. This is always possible because we can apply whatever Givens rotations we want on the left of the correlation matrix by acting on the system with the appropriate local unitary. The same argument can also be made in the case when $\alpha_1\in\mathcal{S}$ and $\alpha_2\in\mathcal{S}$ or $\alpha_1\in\mathcal{S}$ and $\alpha_3\in\mathcal{S}$.

We now look at what occurs with the two-parameter family when only $\alpha_3\in\mathcal{S}$ and $\zeta_3=b_3$. By symmetry, we can come to the same conclusion for the other two-parameter families. Therefore, it suffices to show only what happens in this case. We obtain the conditions

$$(t_{32}-a_3\zeta_2)\sin(2\alpha_2)=0, \quad (48)$$
$$(-t_{31}+a_3\zeta_1)\sin(2\alpha_1)=0, \quad (49)$$
$$\text{and } (t_{21}-a_2\zeta_1)\cos(2\alpha_2)\sin(2\alpha_1)+(-t_{12}+a_1\zeta_2)\cos(2\alpha_1)\sin(2\alpha_2)=0, \quad (50)$$

in which we set $\zeta_1=0$ and $\zeta_2=0$. If we apply the Givens rotations

$$\begin{pmatrix}0&0&1\\ \cos(x)&-\sin(x)&0\\ \sin(x)&\cos(x)&0\end{pmatrix} \text{ and } \begin{pmatrix}\cos(y)&0&-\sin(y)\\ 0&1&0\\ \sin(y)&0&\cos(y)\end{pmatrix} \quad (51)$$

we can solve for x and y to always make $t_{32}=0$ and $t_{31}=0$. This covers the first and second equalities, so that we now only have to zero out the third component given by

$$t_{21}\cos(2\alpha_2)\sin(2\alpha_1)-t_{12}\cos(2\alpha_1)\sin(2\alpha_2). \quad (52)$$

To do this, we apply a final Givens rotation given by

$$\begin{pmatrix}\cos(z)&-\sin(z)&0\\ \sin(z)&\cos(z)&0\\ 0&0&1\end{pmatrix} \quad (53)$$

and solve for the variable z that makes the expression in Eq 52 equal to 0. We get the transformations $t_{21}\to t_{21}\cos(z)+t_{11}\sin(z)$ and $t_{12}\to t_{12}\cos(z)-t_{22}\sin(z)$ for the current correlation terms after the first two Givens rotations. We can always find some z-value that makes the value in Eq. (52) zero while leaving the first two equalities the same. □

To determine the unitaries needed to apply these Givens rotations, one can use the explicit formula in Ref. [26] to alter the correlation matrix. For instance, if $0<x<\pi/2$, then the first local unitary we apply is given by the matrix

$$U_1\triangleq\frac{1}{2}\begin{pmatrix}a-i\,b&-a-i\,b\\ a-i\,b&a+i\,b\end{pmatrix} \quad (54)$$

for $a\triangleq\sqrt{1-\sin(x)}$ and $b\triangleq\sqrt{1+\sin(x)}$, which will induce the first Givens rotation on the left side of the correlation matrix of ρ^{SE}. In Section III K, we numerically

test the most general case of three-parameter unitaries where $\alpha_i \notin \mathcal{S}$ for all $i \in \{1,2,3\}$.

I. Diagonally Correlated States

In this section, we study what happens when the global evolution has the form of Ω in Eq. 39 and the state ρ_d^{SE} has a diagonal correlation matrix. Although this may seem very selective, it covers every problem that is equivalent to up to local unitaries; that is, everything in the tuple $\{(\mathcal{L}_1 \otimes \mathcal{L}_2)\Omega(\mathcal{R}_1 \otimes \mathcal{R}_2), (\mathcal{R}_1 \otimes \mathcal{R}_2)^\dagger \rho_d^{SE} (\mathcal{R}_1 \otimes \mathcal{R}_2)\}$ will have the same properties for what we are considering. Thus, we are technically saying that the right local unitaries of the global evolution U are equal to the conjugate transpose of the local unitaries acting on the state ρ_d^{SE} with a diagonal correlation matrix. This brings us to our next theorem.

Theorem 2. *For all nonlocal unitaries Ω, given in Eq. (39), and two-qubit states ρ_D^{SE}, whose correlation matrix is diagonal, the system dynamics can always be U-generated by a product state.*

Proof. We begin the proof by noting that the conditions in Eqs. (44) - (46) can always be satisfied whenever one of the angles has the form $k\pi$ for $k \in \mathbb{W}$, while the other angles can be anything. For example, the solution for $\alpha_i = 0$ is given by $\zeta_j = 0$ when $i \neq j$ and $\zeta_j = b_j$ when $i = j$. Now we assume that none of the angles are equal to $k\pi$ for the next part of the proof to complete generality. The conditions can now be made into a compact expression given by

$$(S + M) \cdot \vec{\zeta} = S \cdot \vec{b} \tag{55}$$

for the matrices $S = \text{diag}(s_2 s_3, s_1 s_3, s_2 s_3)$ and

$$M \triangleq \begin{pmatrix} 0 & a_3 c_3 s_2 & -a_2 c_2 s_3 \\ -a_3 c_3 s_1 & 0 & a_1 c_1 s_3 \\ a_2 c_2 s_1 & -a_1 c_1 s_2 & 0 \end{pmatrix} \tag{56}$$

if we define $s_i \triangleq \sin(2\alpha_i)$ and $c_i \triangleq \cos(2\alpha_i)$. We can further reduce the expression to

$$(\mathbb{I} + S^{-1} \cdot M) \cdot \vec{\zeta} = \vec{b}. \tag{57}$$

if we use the fact that no $\alpha_i = k\pi$ to ensure that S is full rank. After some simplifications, we calculate the magnitude for the solution to be

$$\zeta^2 = \frac{b^2 + (\vec{b} \cdot \vec{a_t})^2}{1 + \vec{a_t} \cdot \vec{a_t}} = \frac{1 + \cos(\kappa) a_t^2}{1 + a_t^2} b^2 \tag{58}$$

for $\vec{a_t} = \{a_1 \cot(2\alpha_1), a_2 \cot(2\alpha_2), a_3 \cot(2\alpha_3)\}$. We need this magnitude to always be less than or equal to 1 to ensure that a solution exists for $\vec{\zeta}$. We see that

$$\frac{1 + \cos(\kappa) a_t^2}{1 + a_t^2} b^2 \Rightarrow b^2 \leq \frac{1 + a_t^2}{1 + \cos(\kappa) a_t^2} \in [1, 1 + a_t^2]. \tag{59}$$

This will always be true for any choice of non-zero angles and set of local Bloch vectors. □

1. Access to Both Qubits

Now imagine if we can apply local unitaries to both qubits in $\rho^{S_1 S_2}$ instead of treating it as a system and environment. One may be interested in preventing certain dynamics between two accessible data qubits that are negatively affected by their local environment, causing the two data qubits to interact in a non-standard way. The following corollary holds if we can apply local unitary operations to both qubits.

Corollary 1. *For any global two-qubit unitary, U, acting on a system qubit and an environment qubit, the resulting single-qubit dynamics can always be U-generated from a product state, as long as local operations are permitted on both qubits.*

Proof. The proof follows directly from Theorem 2 with the correct choice of local unitaries on the qubits. If global evolution has the form $U = (\mathcal{L}_1 \otimes \mathcal{L}_2) U (\mathcal{R}_1 \otimes \mathcal{R}_2)$ and the two-qubit state has the form $\rho^{S_1 S_2} = (\mathcal{V}_1 \otimes \mathcal{V}_2) \rho_D^{S_1 S_2} (\mathcal{V}_1^\dagger \otimes \mathcal{V}_2^\dagger)$, then the local unitaries $(\mathcal{R}_1^\dagger \mathcal{V}_1^\dagger \otimes \mathcal{R}_2^\dagger \mathcal{V}_2^\dagger)$ will cancel out the right local unitaries of the global evolution as well as diagonalizing the correlation matrix of the state. The local unitaries to the left of the global evolution, U, do not change the equality in Eq. (12). □

When we have access to both qubits, we may want to preserve or increase entanglement while ensuring the dynamics on the local systems can be U-generated by an initial product state. Surprisingly, post-selection of our measurement outcomes prior to global evolution can result in a larger concurrence [27], which may allow us to find a suitable measurement that prevents these undesired dynamics while *increasing* entanglement. This is something local unitaries are not capable of doing, since they simply preserve the entanglement. So, depending on the task, one method may be more preferential than the other.

J. Non-entangling Global Evolutions

The next particular case we consider provides another instance in which the system dynamics can always be U-generated from a product state. Specifically, we assume that the global evolution lacks the ability to generate entanglement from any input state. At first glance, the implications of this restriction are not obvious, since even separable states can sometimes give rise to non-completely positive dynamics that cannot be reproduced using a product state. However, in this special setting, the absence of entangling power guarantees that, regardless of the initial state prepared in the laboratory, the dynamics can always be modeled using a product state. We formalize this observation in the following theorem.

Theorem 3. *If for any two-qubit state, ρ^{SE}, the nonlocal*

part of the unitary in Eq. (39) satisfies the condition

$$\frac{3 - c_1(c_2 + c_3) - c_2 c_3}{4} = 0 \tag{60}$$

for $c_i \triangleq cos(4\alpha_i)$*, then the dynamics on the system can be* U*-generated by a product state.*

Proof. The condition in Eq. (60) is the explicit form for the normalized entangling power [28, 29] of a global two-qubit unitary with the nonlocal portion given in Eq. (39). It is defined as the average entanglement generated by the global unitary U when acting on the set of all two-qubit pure states. For all two-qubit unitaries, the entangling power is also equal to the disentangling power. Thus, when the entangling power is equal to 0, it has no capabilities of changing the entanglement of ρ^{SE}. If the entanglement does not change, then the dynamics on the system can always be U-generated by a product state. It is easy to see this because the only nonlocal two-qubit gates that are not entangling are locally equivalent to SWAP [30]. □

One natural question is whether these results can be extended to systems beyond two qubits, or even to general multipartite unitaries. Prior work on the entangling power of broader classes of unitary evolutions can be found in Refs. [31, 32], but a systematic extension of our framework to these settings remains open for future research. Another intriguing direction concerns the role of local unitaries in shaping the system–environment correlations. Specifically, consider the transformation of the joint state under

$$U(V \otimes \mathbb{I})\rho^{SE}(V^\dagger \otimes \mathbb{I})U^\dagger, \tag{61}$$

where V is a local unitary acting only on the system. If it were always possible to choose such a V so that the entanglement of ρ^{SE} is preserved under the subsequent global evolution by U, then the dynamics could always be U-generated by a product state. The reason is that the final state would remain equivalent up to local unitaries—or to a transformation locally equivalent to a swap—which we have already shown admits a product-state description of the system dynamics. This observation suggests a broader principle that connects local control to the ability to enforce the product-state prescription.

In the next section, we turn to a related but more practical question: how well can we preserve the initial system state while still ensuring that the dynamics are U-generated by a product state? Our numerical results indicate that, in most cases, this can be achieved with perfect fidelity. The strategy is straightforward: for each pair $\{U, \rho^{SE}\}$, we allow local unitary rotations about the system's Bloch vector. When such a rotation does not yield the desired result, an arbitrary local rotation can often suffice. These findings highlight the surprising effectiveness of simple local control in maintaining both the integrity of the system's initial state and the feasibility of modeling its dynamics through a product-state prescription.

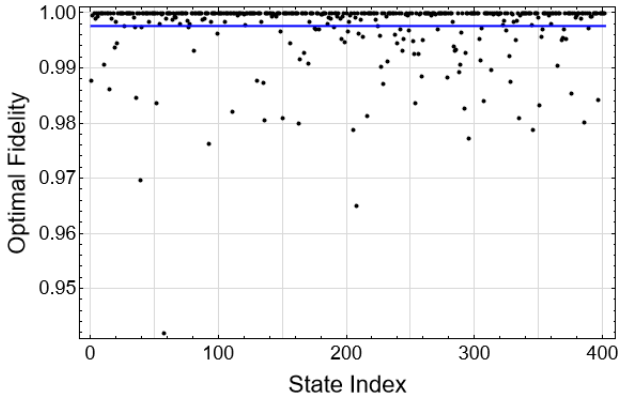

FIG. 3. Plot of the optimal fidelity for each of the 402 experiments (U_i, ρ_i^{SE}). For each case, a local unitary V acting only on the system is applied so that the dynamics generated by U_i can be reproduced using a product state, and the resulting fidelity between ρ^S and $V\rho^S V^\dagger$ is shown.

K. Numerical Analysis for 3-parameter Unitaries

We generated 1,000 two-qubit states and global unitaries that satisfy a number of conditions to increase our chances of obtaining cases that do not satisfy Eq. (12). The constraints are given by

$$t_{ij} \to \text{sgn}[\cos(2\alpha_i)\sin(2\alpha_j)]|t_{ij}|, \tag{62}$$

with $|t_{ij}| \in [1/4, 1]$ and $a_i, b_i \in [-1/2, 1/2]$ with

$$b_k \to \frac{1}{2}\sum_{i,j} |\epsilon_{ijk}| \ \text{sgn}[\sin(2\alpha_i)\sin(2\alpha_j)]|b_k| \tag{63}$$

for the Levi-Cevita tensor ϵ_{ijk}. From Eqs. (44) - (45), we see that these additional conditions will make it "harder" for the left-hand-sides to equal 0. Therefore, we increase the probability that we will run into instances of Eq. (3) not holding for a valid ζ^E. We also let each $\alpha_i \in [0, 2\pi]$ and $t_{ii} =$ for all i since they only reduce the correlations between the system and the environment and do not appear in Eqs. (44) - (45). When the Bloch vector of ζ^E has a magnitude greater than 1 in order for Eq. (12) to be true, we hold onto it for further processing. Otherwise, we throw it away since the condition already holds trivially. We retained 402 of the cases, which ranged in magnitudes, $|\vec{\zeta}|$, of slightly over 1 to over 20 in some cases.

The goal is to show if there exists a case where a general local unitary on the system cannot allow us to U-generate the system dynamics with a product state. In the cases where we can U-generate the system dynamics with a local unitary, we optimize the fidelity between the local state of the system before and after the unitary is applied. Throughout the optimization, we noticed that not only was it possible to always satisfy the dynamics matching condition in Eq. (12) with the application of a single local unitary on the system, we were able to do it with fidelity

equal to 1 in most cases; meaning we simply rotated the local state of the system about its own Bloch vector to change only the nonlocal correlations between the system and environment.

Our results lead us to the following conjecture:

Conjecture 1. *It is always possible to U-generate the system dynamics of the two-qubit state ρ^{SE} if a local unitary is applied to the system prior to the global evolution U.*

This would imply that a simple change of basis on the local system will lead to the product state prescription. The initially correlated states and the nonlocal parameters for the global evolutions, given in Eq. (39), can be found on GitHub [8]. We also provide the Mathematica notebooks that performed some of the calculations we see throughout the paper.

L. Closing Fidelity Gap with Quantum Channels

Since we are not always able to achieve perfect fidelity for every state in Figure 3, we must check to see if a more generalized Kraus map can achieve equality in Eq. 3 while preserving the fidelity entirely. We first use maps of the form

$$\mathcal{E}(\rho) = \left(1 - \sum_{j=1}^{3} p_j\right)\rho + \sum_{j=1}^{3} p_j (V_j \otimes \mathbb{I})\rho(V_j^\dagger \otimes \mathbb{I}) \quad (64)$$

for local unitary operations V_j that rotate about the Bloch vector on the system so that we preserve fidelity. The nice property of these Kraus operators is the ability to implement without any additional ancilla qubits since all the operations are probabilistic instances of local unitaries which can be applied directly to the circuit. However, simply looking at the worst case scenario, where the optimal unitary resulted in a fidelity between 94% and 95%, we were only able to slightly increase the fidelity to roughly 96% using these types of maps. This forced us to look into other fidelity-preserving Kraus maps to see if we can achieve unit fidelity for each case of Fig. (3).

It turns out that implementing quantum channels of the form

$$\mathcal{E}(\rho) = \sum_{j=1}^{2} (V_j \otimes \mathbb{I})\rho(V_j^\dagger \otimes \mathbb{I}), \quad (65)$$

for only two general Kraus operators, V_1 and V_2, that satisfy the completion relations, allows us to U-generate the system dynamics using a product state with perfect fidelity. This is true for all 402 cases. The reason we want only two terms in the Kraus representation is so that we can implement the quantum channel on the system with only one additional ancilla via the Stinespring dilation [33]. Therefore, the system dynamics, $\mathcal{E}(\rho^S)$, can now be defined as

$$\mathrm{tr}_{AE}[(\mathbb{I} \otimes U)(W \otimes \mathbb{I})(|0\rangle\langle 0|^A \otimes \rho^{SE})(W^\dagger \otimes \mathbb{I})(\mathbb{I} \otimes U^\dagger)] \quad (66)$$

where W is a two-qubit unitary acting across systems A and S used to induce the Kraus channel in Eq. 65 on S. We can easily derive W using the corresponding Kraus operators and the Gram-Schmidt process [34] to ensure that it is unitary.

Now the question is whether we can always find a generalized depolarizer, with only two terms in its representation, such that we can always make equality true in Eq. 3 with unit fidelity. In most of the 402 cases, it was simply possible by applying a rotation about the local Bloch vector on S.

IV. CONCLUSION

In this paper, we address a concrete and operational problem: given a fixed global unitary U acting on a system–environment pair and an initially correlated joint state ρ^{SE}, when can the observed reduced dynamics of the system be reproduced as the partial trace of U acting on a product input $\rho^S \otimes \zeta^E$? Our emphasis is on developing constructive, experimentally relevant procedures for achieving such "U-generation by a product state," together with quantitative criteria that minimize disturbance to the system.

The main contributions of the work are threefold. First, we developed a measurement-based prescription that, by design, can always be used to alter system-environment correlations so that the dynamics condition can always be satisfied. This approach provides a clear, analytic route to build the required environmental state ζ^E and to compute the minimal measurement strength needed in examples such as Werner and Bell-derived families. Importantly, the measurement method is a guaranteed construction, but it is destructive in the sense that it will almost always reduces fidelity between the pre- and post-preparation system states.

Second, and central to the paper, we showed that local unitary operations on the system provide a much less invasive and often sufficient alternative. For any two-qubit correlated state and any two-parameter family of global unitaries we gave a formal argument that a properly chosen local unitary on the system always suffices to U-generate the system dynamics from a product state. For more general three-parameter nonlocal two-qubit unitaries our numerical searches produced consistent evidence that the same conclusion holds, and in most of our numerical cases the optimal local operation was a rotation about the system's Bloch vector - a simple operation that preserves the fidelity of the system exactly.

These results establish an algorithmic pathway: given an experiment described by the pair $\{U, \rho^{SE}\}$, one can (i) compute or search if there exists a local rotation about

the Bloch vector of the system that makes the dynamics-matching condition true, and (ii) when necessary, supplement the rotation with a single ancilla two-term Kraus implementation to realize transformations that unitaries alone cannot achieve with unit fidelity. Thus far, and even in very extreme instances, this strategy has not failed to U-generate system dynamics using a product state with perfect fidelity.

Third, we opened the pathway to avoiding non-completely positive dynamics of the system using a single rotation. This was true for not only the whole family of states we defined, but also for all time t, given by our global evolution. In this example, preparing the initial state of the system in the basis $\{|\tilde{+}\rangle, |\tilde{-}\rangle\}$ will avoid all NCP dynamics without even having to apply the local unitary since it is a rotation about the y-axis. Learning more about how local unitary operations impact the overall dynamics of the system proves to be a challenging, but important task for avoiding such undesired maps. More research must be performed to understand the role of local unitaries—a simple change of basis—in guiding state dynamics in the presence of a time-dependent evolution with its environment.

We also emphasized the practical limitations and information requirements of these methods. All of the constructive procedures we present rely on knowledge of the joint correlations $\mathcal{T}_\rho$ and of the local Bloch vector of each qubits, as well as the global evolution acting across both the system and environment. In practice, obtaining this knowledge can be quite costly. For this reason, a major direction for future work is to remove or greatly weaken the assumption of what knowledge we have access to. As an example, we did not need to know anything about the input state for the experiment where we avoided NCP dynamics. We did however, need to know the interactions between the system and the environment, which may be available in some quantum architectures. This means we have information about the noise interacting with our data qubits. This provides us with explicit knowledge of the interaction Hamiltonian and permits the implementation of our methods without full tomographic knowledge of the environment.

One promising direction includes using a more realistic interaction with the environment, where we bound the norm of the χ-matrix describing the correlations between the system and environment. For instance, if $\|\mathcal{T}_\rho - \vec{a}\vec{b}^T\| \leq \kappa$, can one guarantee that we can always U-generate the dynamics of the system using a product state with a wider set of local unitary operations if κ is small enough? We observed that mitigating NCP dynamics requires comparatively less information than ensuring that we satisfy the dynamics-matching condition. This naturally raises the broader question of what minimal knowledge is necessary to suppress or circumvent non-Markovian behavior in open quantum dynamics. What other types of detrimental dynamics can be avoided using local unitaries or state preparation in a particular basis?

Looking ahead, several concrete next steps emerge from this work. A first priority is to establish a formal classification of two-qubit three-parameter nonlocal unitaries, thereby settling the empirical conjecture that local unitaries always suffice in these cases. Such a result would transform our numerical observations into a definitive theorem and sharpen our understanding of the role played by simple local rotations. We also must determine if only one additional ancilla qubit and two-term Kraus operators are needed to satisfy the dynamics-matching condition with perfect fidelity in all two-qubit experiments.

Equally important is the extension of our framework to higher-dimensional systems and multipartite environments, where questions of scaling and complexity will determine whether the single-qubit intuition—such as rotation about the Bloch vector—admits a faithful generalization. Finally, a systematic analysis of robustness to realistic imperfections such as gate errors, finite sampling, and temporal drift will be essential. Embedding our constructive search for local unitaries within an error-aware optimization framework could make the methods viable on near-term platforms.

Taken together, these future directions highlight both the conceptual and practical impact of our results. They suggest a new control-centric view of open-system dynamics, where modest local interventions—such as rotations, simple ancilla constructions, or targeted measurements—can effectively recast correlated dynamics as if they arose from product-state inputs. Ultimately, determining the scope of these methods in scalable quantum technologies will require both theoretical advances and practical demonstrations, but the pathway they open is clear: with carefully chosen local controls, we can begin to tame the complexity of correlated system–environment evolution in a way that is both operational and experimentally meaningful.

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