SECANT SHEAVES ON ABELIAN n-FOLDS WITH REAL MULTIPLICATION AND WEIL CLASSES ON ABELIAN $2n$-FOLDS WITH COMPLEX MULTIPLICATION

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Abstract. Let K be a CM-field, i.e., a totally complex quadratic extension of a totally real field F. Let X be an abelian variety admitting an algebra embedding $\hat{\eta} : F \to \mathrm{End}_{\mathbb{Q}}(X)$, and let $\hat{X}$ be the dual abelian variety. We construct an embedding $\eta : K \to \mathrm{End}_{\mathbb{Q}}(X \times \hat{X})$ associated to a choice of a polarization Θ in $\wedge^2_F H^1(X, \mathbb{Q})$ and an element $q \in F$, such that $K = F(\sqrt{-q})$. We get the $[K : \mathbb{Q}]$-dimensional subspace $HW(X \times \hat{X}, \eta)$ of Hodge Weil classes in $H^d(X \times \hat{X}, \mathbb{Q})$, where $d := 4\dim(X)/[K : \mathbb{Q}]$.

The even cohomology $S^+_{\mathbb{C}} := H^{ev}(X, \mathbb{C})$ is the half-spin representation of the group $\mathrm{Spin}(H^1(X \times \hat{X}, \mathbb{C}))$ and so $\mathbb{P}(S^+_{\mathbb{C}})$ contains the even spinorial variety. The latter is a component of the Grassmannian of maximal isotropic subspaces of $H^1(X \times \hat{X}, \mathbb{C})$. We associate to (Θ, q) a rational $2^{[F:\mathbb{Q}]}$-dimensional subspace B of $S^+_{\mathbb{Q}}$ such that $\mathbb{P}(B)$ is secant to the spinorial variety. Associated to two coherent sheaves F_1 and F_2 on X with Chern characters in B we obtain the object $E := \Phi(F_1 \boxtimes F_2^\vee)$ in the derived category $D^b(X \times \hat{X})$, where $\Phi : D^b(X \times X) \to D^b(X \times \hat{X})$ is Orlov's equivalence. The flat deformations of the normalized Chern class $\kappa(E) := ch(E)\exp\left(-\frac{c_1(E)}{\mathrm{rank}(E)}\right)$ of E remain of Hodge type under every deformation of $(X \times \hat{X}, \eta)$ as an abelian variety (A', η') of Weil type.

We provide a criterion for $ch(F_1) \otimes ch(F_2)$ to belong to the open subset in $B \otimes B$ for which the algebraicity of the flat deformation of $\kappa(E)$ implies the algebraicity of all classes in $HW(A', \eta')$. The algebraicity would thus follow if E is semiregular in the appropriate sense. Examples of such F_1 and F_2 are provided for X the Jacobian with real multiplication by F of a genus 4 curve when $[F : \mathbb{Q}] = 2$ and $Gal(K/\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, but the semi-regularity of E has not been addressed yet.

Contents

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1. Introduction

1.1. **A summary of the construction.** Let K be a CM-field and F its totally real subfield. The Galois involution ι in $Gal(K/F)$ is a central element of $Gal(K/\mathbb{Q})$ and for every embedding $\sigma: K\rightarrow\mathbb{C}$ we have $\sigma(\iota(t))=\overline{\sigma(t)}$, for all $t\in K$, where the right hand side is complex conjugation [DM, Sec. 4]. Set

$$e:=\dim_{\mathbb{Q}}(K).$$

Let $\Sigma:=\mathrm{Hom}(K,\mathbb{C})$ be the set of complex embeddings. Note that the cardinality of Σ is e. The group $Gal(K/\mathbb{Q})$ acts on Σ by $g(\sigma)=\sigma\circ g^{-1}$, for all $g\in Gal(K/\mathbb{Q})$ and $\sigma\in\Sigma$. The latter action is transitive if K is a Galois extension of $\mathbb{Q}$.

Let A be an abelian variety and set $\mathrm{End}_{\mathbb{Q}}(A):=\mathrm{End}(A)\otimes_{\mathbb{Z}}\mathbb{Q}$. Given an embedding $\eta: K\rightarrow\mathrm{End}_{\mathbb{Q}}(A)$, we get a decomposition

$$H^1(A,\mathbb{C})=\oplus_{\sigma\in\Sigma}H^1_\sigma(A),$$

where $\eta(K)$ acts on $H^1_\sigma(A)$ via the embedding σ. The complex multiplication η corresponds to an embedding $\eta : K \to \mathrm{End}(H^1(A,\mathbb{Q}))$ preserving the Hodge structure. Set

$$d := \dim_K(H^1(A,\mathbb{Q})).$$

We have $\dim_{\mathbb{C}}(H^1_\sigma(A)) = d$, for all $\sigma \in \Sigma$. Set $H^{1,0}_\sigma(A) := H^1_\sigma(A) \cap H^{1,0}(A)$ and $H^{0,1}_\sigma(A) := H^1_\sigma(A) \cap H^{0,1}(A)$. Assume that for every $\sigma \in \Sigma$, we have

$$\dim(H^{1,0}_\sigma(A)) = \dim(H^{0,1}_\sigma(A)) = \frac{d}{2}. \tag{1.1.1}$$

Then $\wedge^d_K H^1(A,\mathbb{Q})$ is an e-dimensional subspace $HW(A,\eta)$ of $H^{\frac{d}{2},\frac{d}{2}}(A,\mathbb{Q})$, which is a one-dimensional K vector space [DM, Prop. 4.4]. The challenge is then to prove the algebraicity of the classes in $HW(A,\eta)$. It suffices to prove the algebraicity of one non-zero class in $HW(A,\eta)$, as K acts via algebraic correspondences. If $d = 2$, then $HW(A,\eta)$ consists of rational $(1,1)$ classes, which are algebraic. Hence, we may assume that $d > 2$.

A pair (A,η) satisfying (1.1.1) is called an abelian variety of *Weil type* and the classes in $HW(A,\eta)$ are called *Weil classes* in the literature (see [MZ]). When $F = \mathbb{Q}$, so that K is an imaginary quadratic number field, the Weil classes were introduced in [W]. They were proved algebraic for all abelian fourfolds of Weil type and for abelian sixfolds of split Weil type when $K = \mathbb{Q}(\sqrt{-3})$ in [S1, S2], when $K = \mathbb{Q}(\sqrt{-1})$ in [vG, K], and for all imaginary quadratic number fields in [M2]. Proofs of the algebraicity of the Weil classes on abelian fourfolds of split Weil type for all imaginary quadratic number fields were obtained using hyper-Kähler techniques in [FF, M1]. The current paper presents the natural generalization of the strategy of [M2] for more general CM-fields.

Let d be a positive even integer, let X be an abelian variety of dimension

$$n := de/4,$$

and set $A := X \times \hat{X}$. We will construct the embedding $\eta : K \to \mathrm{End}_{\mathbb{Q}}(A)$ assuming X admits an embedding $\hat{\eta} : F \to \mathrm{End}_{\mathbb{Q}}(X)$ of the totally real subfield F of K.

The free abelian group $V := H^1(X,\mathbb{Z}) \oplus H^1(\hat{X},\mathbb{Z})$ admits a natural unimodular symmetric bilinear pairing. Using the natural isomorphism $H^1(\hat{X},\mathbb{Z}) \cong H^1(X,\mathbb{Z})^*$, the pairing is given by

$$((w_1,\theta_1),(w_2,\theta_2))_V := \theta_1(w_2) + \theta_2(w_1). \tag{1.1.2}$$

The K vector space $V_{\hat{\eta},K} := [H^1(X,\mathbb{Q}) \oplus H^1(\hat{X},\mathbb{Q})] \otimes_F K$ admits a natural K-valued non-degenerate symmetric bilinear form (5.1.1). Let W be a maximal isotropic subspace of $V_{\hat{\eta},K}$. Let $\bar{W}$ be the image of W via the non-trivial element ι in $Gal(K/F)$. Assume that $W \cap \bar{W} = (0)$. Then W is naturally isomorphic to $H^1(A,\mathbb{Q})$, as a vector space over $\mathbb{Q}$, and so determines a K-vector space structure on $H^1(A,\mathbb{Q})$ (Section 5). We construct such a W associated to a choice of a polarization $\Theta \in [H^{1,1}(X,\mathbb{Q}) \cap \wedge^2_F H^1(X,\mathbb{Q})]$ and a non-zero element $q \in K$, such that $\iota(q) = -q$, and show that scalar multiplication by elements of K preserves the Hodge structure of $H^1(A,\mathbb{Q})$, hence yields an embedding $\eta : K \to \mathrm{End}_{\mathbb{Q}}(A)$, such that (A,η) is of Weil type (Section 11.1)

The grassmannian $IGr^+(2n, V_{\mathbb{C}})$ of even maximal isotropic subspaces of $V_{\mathbb{C}} := V \otimes_{\mathbb{Z}} \mathbb{C}$ naturally embeds in the projectivization $\mathbb{P}(S^+_{\mathbb{C}})$ of the even half-spin representation

$S^+_{\mathbb{C}} := H^{ev}(X,\mathbb{C})$ of $\mathrm{Spin}(V_{\mathbb{C}})$. We refer to the image of $IGr^+(2n,V_{\mathbb{C}})$ in $\mathbb{P}(S^+_{\mathbb{C}})$ as the *even spinorial variety.* The point $\ell_W\in\mathbb{P}(S^+_{\mathbb{C}})$ associated to a an even maximal isotropic subspace W of $V_{\mathbb{C}}$ is called an *even pure spinor*, as are the classes in the corresponding line in $S^+_{\mathbb{C}}$ [Ch].

Let $\hat{\Sigma}:=\Sigma/\iota$ be the set of embeddings of F in $\mathbb{R}$. A *complex multiplication type* (CM-type) $T:\hat{\Sigma}\rightarrow\Sigma$ consists of a choice of an embedding $T(\hat{\sigma})\in\Sigma$ restricting to F as $\hat{\sigma}$, for each $\hat{\sigma}\in\hat{\Sigma}$ (see [DM, Sec. 5]). For each CM-type T we get an even maximal isotropic subspace W_T of $H^1(A,\mathbb{C})$ and so a pure spinor $\ell_{W_T}\in\mathbb{P}(S^+_{\mathbb{C}})$. We denote ℓ_{W_T} by ℓ_T for simplicity. The subspace of $S^+_{\mathbb{C}}$ spanned by the set $\{\ell_T\}$, as T varies over all CM-types, is defined over $\mathbb{Q}$ and corresponds to a subspace B of $S^+_{\mathbb{Q}}$ of dimension $2^{e/2}$ (Lemma 7.1.3). By definition, the linear subspace $\mathbb{P}(B)$ is secant to the even spinorial variety.

A *B-secant sheaf* is a coherent sheaf G on X with $ch(G)$ in B. Given two B-secant sheaves F_1 and F_2 we get the object $E:=\Phi(F_1\boxtimes F_2^\vee)$ in $D^b(X\times\hat{X})$ via Orlov's derived equivalence $\Phi:D^b(X\times X)\rightarrow D^b(X\times\hat{X})$ given in (10.1.3). Assume that $r:=\mathrm{rank}(E)\neq 0$. We prove that the characteristic class $\kappa(E):=ch(E)\exp(-c_1(E)/r)$ remains of Hodge type under every deformation of $(A,\eta:K\rightarrow \mathrm{End}_{\mathbb{Q}}(A))$ as an abelian variety of Weil type (Proposition 10.2.1(1)). Let $\mathcal{A}^{2k}$ be the subspace of $H^{k,k}(A,\mathbb{Q})$ consisting of classes which remain of Hodge type under every deformation of (A,η) as an abelian variety of Weil type. The subspace $\mathcal{A}^d$ is the direct sum

$$\mathcal{A}^d = Im(\mathrm{Sym}^{d/2}(\mathcal{A}^2))\oplus HW(A,\eta), \tag{1.1.3}$$

where $Im(\mathrm{Sym}^{d/2}(\mathcal{A}^2))$ is the image of $\mathrm{Sym}^{d/2}(\mathcal{A}^2)$ in $H^d(A,\mathbb{Q})$, by Proposition 8.0.1.

We provide a criterion for the graded summand $\kappa_{d/2}(E)\in\mathcal{A}^d$ to project to a non-zero class in the summand $HW(A,\eta)$ of (1.1.3) in terms of the position of $ch(F_1)\otimes ch(F_2)$ with respect to a natural grading on $B\otimes B$ introduced in the next paragraph. A generic class in $B\otimes B$ satisfies the criterion, which is an open condition. A refinement of the latter criterion, in terms of a grading of B, is obtained in Lemma 11.2.9, when the Galois group $Gal(\tilde{K}/\mathbb{Q})$ of the Galois closure $\tilde{K}$ of K splits as $\mathbb{Z}/2\mathbb{Z}\times Gal(\tilde{F}/\mathbb{Q})$, where $\tilde{F}$ is the Galois closure of F. In Example 11.2.7 we exhibit two B-secant sheaves F_1 and F_2, satisfying the criterion for $\kappa_{d/2}(E)$ to project to a non-zero class in $HW(A,\eta)$, when X is the Jacobian of a genus 4 curve with real multiplication $\hat{\eta}:F\rightarrow\mathrm{End}_{\mathbb{Q}}(X)$ by a real quadratic number field F, and $K=F(\sqrt{-q})$, where q is a positive integer (Lemma 11.2.8).

We state next the criterion for the graded summand $\kappa_{d/2}(E)\in H^d(A,\mathbb{Q})$ to project to a non-zero class in $HW(A,\eta)$. Given two CM-types T, T', let $|T\cap T'|$ be the cardinality of the subset $T(\hat{\Sigma})\cap T'(\hat{\Sigma})$ of Σ. Then $B\otimes B$ decomposes as a direct sum $\oplus_{k=0}^{e/2}BB_k$, where $BB_k:=\oplus_{\{(T,T')\ :\ |T\cap T'|=k\}}\ell_T\otimes\ell_{T'}$, and BB_k is defined over $\mathbb{Q}$. We get the decomposition $ch(F_1)\otimes ch(F_2)=\sum_{i=0}^{e/2}c_k$, where $c_k\in BB_k$. Let $\phi:H^*(X\times X)\rightarrow H^*(X\times\hat{X})$ be the isomorphism induces by Orlov's derive equivalence Φ and let τ act on $H^i(X)$ by multiplication by $(-1)^{i(i-1)/2}$. The image of BB_1 via $\phi\circ(id\otimes\tau)$ is contained in $\oplus_{i\geq d}H^i(X\times\hat{X},\mathbb{Q})$ and the image of the composition

$$BB_1 \stackrel{\phi\circ(id\otimes\tau)}{\rightarrow} \oplus_{i\geq d}H^i(X\times\hat{X},\mathbb{Q})\rightarrow H^d(X\times\hat{X},\mathbb{Q}) \tag{1.1.4}$$

is $HW(X \times \hat{X}, \eta)$, by Lemma 10.1.3. The above composition maps the line $\ell_T \otimes \ell_{T'}$ isomorphically onto the line $\wedge^d V_\sigma$, where $\{\sigma\} = T(\hat{\Sigma}) \cap T'(\hat{\Sigma})$. The dimension of BB_1 is[1] $e2^{\frac{e}{2}-1}$ and $\dim HW(A, \eta) = e$, so the above composition is an isomorphism if and only if $e := [K : \mathbb{Q}] = 2$. Write $c_1 = \sum_{\{(T,T') \ : \ |T \cap T'|=1\}} c_{T,T'}$, where $c_{T,T'}$ is the direct summand in $\ell_T \otimes \ell_{T'}$. Let Π be the composition (1.1.4).

Proposition 1.1.1. *(Proposition 10.2.1(3)). Assume that $d > 2$, the rank of $\Phi(F_1 \boxtimes F_2^\vee)$ is non-zero, and* $\sum_{\{(T,T') \ : \ |T\cap T'|=\{\sigma\}\}} \Pi(c_{T,T'}) \neq 0$, *for some $\sigma \in \Sigma$. Then $\kappa_{d/2}(E)$ projects to a non-zero class in the summand $HW(A, \eta)$ of $\mathcal{A}^d$ in (1.1.3).*

In Example 11.2.7 and Lemma 11.2.9, where the above criterion is verified, we choose the sheaves F_1 and F_2 so that all but one summand $c_{T,T'}$ vanish, for each $\sigma \in \Sigma$, and so the non-vanishing of the sum in the above proposition, for some $\sigma \in \Sigma$, is equivalent to the non-vanishing of the summand c_1 in BB_1.

We summarize the above results as follows. Let F_1 and F_2 be B-secant sheaves satisfying the genericity criterion of Proposition 1.1.1.

Theorem 1.1.2. *(Corollary 10.2.3) A flat deformation of the class $\kappa(\Phi(F_1 \boxtimes F_2^\vee))$ remains of Hodge type, on every abelian variety of Weil type (A', η') deformation equivalent to $(X \times \hat{X}, \eta)$. If the flat deformation of the class $\kappa(\Phi(F_1 \boxtimes F_2^\vee))$ remains furthermore algebraic, then every Weil class in $HW(A', \eta')$ is algebraic.*

In [M2, Theorem 1.5.1] the flat deformations of the class $\kappa(\Phi(F_1 \boxtimes F_2^\vee))$ were shown to remain algebraic, when $F = \mathbb{Q}$, X is the Jacobian of a genus 3 curve, and the B-secant sheaves F_1 and F_2 on X were chosen to be equivariantly semi-regular in the sense of Buchweitz-Flenner [BF]. We postpone for future work the search for semiregular B-secant sheaves for CM-fields with $[K : \mathbb{Q}] > 2$.

1.2. **Organization of the paper.** A table of notation is provided in Section 2. Throughout the paper F is a totally real field and X is an abelian variety endowed with an embedding $\hat{\eta} : F \to \mathrm{End}_{\mathbb{Q}}(X)$. In Section 3 we review the exterior subalgebra $\wedge^*_F H^1(X, \mathbb{Q})$ of $\wedge^*_{\mathbb{Q}} H^1(X, \mathbb{Q})$. In addition, we review the relation between the $\mathbb{Q}$-valued and F-valued quadratic forms on $V_{\mathbb{Q}} := H^1(X, \mathbb{Q}) \oplus H^1(\hat{X}, \mathbb{Q})$. We describe the tensor product factorization of even pure spinors in $S^+_{\mathbb{C}}$ associated to the direct sum decomposition $V_{\mathbb{C}} = \oplus_{\hat{\sigma} \in \hat{\Sigma}} V_{\hat{\sigma}, \mathbb{C}}$.

We denote by $V_{\hat{\eta}}$ the space $V_{\mathbb{Q}}$ considered as a vector space over F with its natural F-valued quadratic form. In Section 4 we construct a natural homomorphism from $\mathrm{Spin}(V_{\hat{\eta}})$ to $\mathrm{Spin}(V_{\mathbb{Q}})$, which is injective if $[F : \mathbb{Q}]$ is odd, and has kernel $\{\pm 1\}$ if $[F : \mathbb{Q}]$ is even. We denote by $\mathrm{Spin}(V_{\mathbb{Q}})_{\hat{\eta}}$ the image of $\mathrm{Spin}(V_{\hat{\eta}})$ in $\mathrm{Spin}(V_{\mathbb{Q}})$.

In Section 5 we construct an embedding $\eta : K \to \mathrm{End}(V_{\mathbb{Q}})$ of the CM-field K associated to a maximal isotropic subspace W of $V_{\hat{\eta}} \otimes_F K$. We construct a homomorphism $\Xi : K_- \to \wedge^2_F V_{\hat{\eta}}$ from the -1-eigenspace K_- of the Galois involution $\iota \in Gal(K/F)$.

[1] In contrast, $\dim(B \otimes B) = 2^e$. When $[K : \mathbb{Q}] = 4$, for example, $\dim HW(A, \eta) = 4$, $\dim(BB_1) = 8$, and $\dim(B \otimes B) = 16$.

We construct a maximal isotropic subspace W_T of $V_{\mathbb{C}} := V \otimes_{\mathbb{Z}} \mathbb{C}$ associated to W and a choice of a CM-type T.

Let $\tilde{K}$ be the Galois closure of the extension K of $\mathbb{Q}$. In Section 6 we describe the $Gal(\tilde{K}/\mathbb{Q})$-action on the set $\{W_T \ : \ T \text{ a CM-type}\}$ and the associated set of pure spinors $\{\ell_T\}$. We conclude that the subspace of $S^+_{\mathbb{C}}$ spanned by the lines ℓ_T is defined over $\mathbb{Q}$ and corresponds to a subspace B of $S^+_{\mathbb{Q}}$.

In Section 7 we consider the subgroup $\mathrm{Spin}(V_{\mathbb{Q}})_\eta$ of $\mathrm{Spin}(V_{\mathbb{Q}})_{\hat{\eta}}$ respecting the direct sum decomposition $V_{\tilde{K}} := V_{\mathbb{Q}} \otimes_{\mathbb{Q}} \tilde{K} = \oplus_{\sigma\in\Sigma} V_\sigma$. We denote by $\mathrm{Spin}(V_{\mathbb{Q}})_{\eta,B}$ the subgroup of $\mathrm{Spin}(V_{\mathbb{Q}})_\eta$ fixing all points in B. The character $\ell_{T_1} \otimes \ell_{T_2}$ of $\mathrm{Spin}(V_{\mathbb{Q}})_\eta$, associated to two CM-types T_1 and T_2, is proved to be the tensor product of the characters $\wedge^d V_\sigma$, as σ varies in the subset $T_1(\hat{\Sigma}) \cap T_2(\hat{\Sigma})$ of Σ. We conclude that the set of lines $\{\ell_T \ : \ T \text{ a CM-type}\}$ is linearly independent and $\dim(B) = 2^{e/2}$. We construct a $\mathrm{Spin}(V_{\mathbb{Q}})_\eta$-invariant K-valued hermitian form $H_t : V_{\mathbb{Q}} \times V_{\mathbb{Q}} \to K$, associated to non-zero $t \in K_-$. Finally we prove that if B is spanned by Hodge classes, then so does the space $HW(X \times \hat{X}, \eta)$ of Weil classes. The group $\mathrm{Spin}(V_{\mathbb{Q}})_{\eta,B}$ is the special Mumford-Tate group of the generic abelian variety of Weil type deformation equivalent to $(X \times \hat{X}, \eta)$.

In Section 8 we show that the subalgebra $\mathcal{A} := (\wedge^* V_{\mathbb{Q}})^{\mathrm{Spin}(V_{\mathbb{Q}})_{\eta,B}}$ of the exterior algebra $\wedge^* V_{\mathbb{Q}}$ of $\mathrm{Spin}(V_{\mathbb{Q}})_{\eta,B}$-invariant classes is generated by the $(e/2)$-dimensional $(\wedge^2 V_{\mathbb{Q}})^{\mathrm{Spin}(V_{\mathbb{Q}})_{\eta,B}}$ and the e-dimensional $HW(X \times \hat{X}, \eta)$.

In Section 9 we show that the subgroup $\mathrm{Spin}(V_{\mathbb{R}})_B$ of $\mathrm{Spin}(V_{\mathbb{R}})$, fixing all points of B, is the product of the special unitary groups $SU(V_{\hat{\sigma},\mathbb{R}})$, $\hat{\sigma} \in \hat{\Sigma}$ (see Remark 9.1.2 for the definition of $SU(V_{\hat{\sigma},\mathbb{R}})$). The group $\mathrm{Spin}(V_{\mathbb{R}})_B$ maps injectively into the image $SO_+(V_{\mathbb{R}})$ of $\mathrm{Spin}(V_{\mathbb{R}})$ in $SO(V_{\mathbb{R}})$. We show that a finite union Ω_B of adjoint orbits in $\mathrm{Spin}(V_{\mathbb{R}})_B$ of complex structures on $V_{\mathbb{R}}$ is a period domain of polarized abelian varieties of Weil type with complex multiplication by K. The dimension of each adjoint orbit in Ω_B is $d^2e/2$, so each is a connected component of Ω_B. The set of connected components is in bijection with the set of CM-types.

In Section 10 we introduce the grading $B \otimes B = \oplus_{i=0}^{e/2} BB_i$, where $BB_i \otimes_{\mathbb{Q}} \tilde{K}$ is the direct sum of $\ell_T \otimes \ell_{T'}$, where the ordered pairs (T, T') consist of CM-types $T, T' : \hat{\Sigma} \to \Sigma$ having precisely i common values. We show that $HW(X \times \hat{X}, \eta)$ is the image of a natural homomorphism $BB_1 \to \wedge^d V_{\mathbb{Q}}$, given in (10.1.6), and describe the kernel. Finally we prove Theorem 1.1.2.

Section 11 consists of Examples. We construct a maximal isotropic subspace W of $[H^1(X, \mathbb{Q}) \oplus H^1(\hat{X}, \mathbb{Q})] \otimes_F K$ as the image of $H^1(\hat{X}, \mathbb{Q}) \otimes_F K$ under the element of $\mathrm{Spin}(V_{\hat{\eta}} \otimes_F K)$ of cup product with $\exp(\sqrt{-q}\Theta)$, where Θ is a polarization of X in $\wedge^2_F H^1(X, \mathbb{Q})$, and $q \in F$ satisfies $K = F(\sqrt{-q})$. We get the embedding $\eta : K \to \mathrm{End}_{\mathbb{Q}}(X \times \hat{X})$ and the rational subspace B of $S^+_{\mathbb{Q}} := H^{ev}(X, \mathbb{Q})$ described above. We show that the adjoint orbit in $\mathrm{Spin}(V_{\mathbb{R}})_B$ of the complex structure $I_{X\times\hat{X}}$ of $X \times \hat{X}$ is a period domain of abelian varieties of split Weil type (Definition 7.3.4) deformation equivalent to $(X \times \hat{X}, \eta)$ (Lemma 11.1.2).

In Section 11.2.1 we consider the case, where $[F : \mathbb{Q}] = 2$ and $K = F(\sqrt{-q})$, where q is a positive integer. We choose X to be a Jacobian of a genus 4 curve with an embedding $\hat{\eta} : F \to \mathrm{End}_{\mathbb{Q}}(X)$. We give an example of two coherent sheaves F_1 and F_2 on X

satisfying the hypothesis of Theorem 1.1.2, so that flat deformations of $\kappa(\Phi(F_1 \boxtimes F_2^\vee))$ remain of Hodge type under all deformations of $(X \times \hat{X}, \eta)$ in the connected component of the period domain Ω_B, but we do not know if the class $\kappa(\Phi(F_1 \boxtimes F_2^\vee))$ remains algebraic (Lemma 11.2.8). If it does remain algebraic, then so do the Weil classes, by Lemma 11.2.8.

In Section 11.2.2 we consider more general totally real fields F, but specialize to the case where $Gal(\tilde{K}/\mathbb{Q})$ splits as the direct product of $Gal(\tilde{F}/\mathbb{Q})$ and $\{id, \iota\}$, where ι is the extension to $\tilde{K}$ of the involution in $Gal(K/F)$. We prove a genericity criterion for the Chern characters of coherent sheaves F_1 and F_2, so that $\kappa_{d/2}(\Phi(F_1 \boxtimes F_2^\vee)$ is the sum of a non-zero element of $HW(X \times \hat{X}, \eta)$ and a polynomial in Hodge (necessarily algebraic) classes in $H^2(X \times \hat{X}, \mathbb{Q})^{\mathrm{Spin}(V_\mathbb{Q})_{\eta,B}}$ (Lemma 11.2.9). We do not provide examples of such F_1 and F_2, when $[F:\mathbb{Q}] > 2$. Example of semiregular such F_1 and F_2 are likely to yield the algebraicity of the Weil classes of all abelian varieties of Weil type deformation equivalent to $(X \times \hat{X}, \eta)$.

Remark 1.2.1. We have $2^{[F:\mathbb{Q}]}$ maximal isotropic subspaces $W_{T,\mathbb{C}}$ in the quadratic vector space $V_\mathbb{C} := H^1(X \times \hat{X}, \mathbb{C})$, one for each CM-type T. Nevertheless, each direct summand $V_{\hat{\sigma},\mathbb{C}}$, $\hat{\sigma} \in \hat{\Sigma}$, of $V_\mathbb{C}$ decompose as the direct sum of two maximal isotropic subspaces $V_{\hat{\sigma},\mathbb{C}} = V_{\sigma,\mathbb{C}} \oplus V_{\bar{\sigma},\mathbb{C}}$, where σ and $\bar{\sigma}$ are the two embeddings of K in $\mathbb{C}$ restricting to the embedding $\hat{\sigma}$ of F. As a result, many of the proofs in Sections 7 to 10 involve a reduction of a statement to a direct summand $V_{\hat{\sigma},\mathbb{C}}$ in $V_\mathbb{C}$ or to a pure spinor in $S^+_{\hat{\sigma}}$, which is a tensor factor of a pure spinor ℓ_T, and then invoke an argument from [M2] in the quadratic field extension case.

2. Notation

X	a complex abelian variety	$\dim_\mathbb{C}(X) = n = de/4$
V	the lattice $H^1(X, \mathbb{Z}) \oplus H^1(\hat{X}, \mathbb{Z})$	$\mathrm{rank}(V) = 4n = de$
K	a CM field	$[K:\mathbb{Q}] = e$
F	the totally real subfield of K	$[F:\mathbb{Q}] = e/2$
$\hat{\eta}$	an algebra homomorphism $F \to \mathrm{End}_{Hdg}(V_\mathbb{Q})$	(3.1.1)
$V_{\hat{\eta}}$	$V_\mathbb{Q}$ considered as a vector space over F	$\dim_F(V_{\hat{\eta}}) = 2d$
$V_{\hat{\eta},K}$	$V_{\hat{\eta}} \otimes_F K$	Sec. 5
$(\bullet,\bullet)_V$	the bilinear pairing on the lattice V	(1.1.2)
$(\bullet,\bullet)_{V_{\hat{\eta}}}$	F-valued F-bilinear pairing	(3.1.2)
$(\bullet,\bullet)_{\hat{\eta},K}$	the K-valued pairing on $V_{\hat{\eta},K}$	(5.1.1)
$C(V)$	the Clifford algebra	Sec. 3.2
ρ	the vector representation $\mathrm{Spin}(V) \to SO^+(V)$	(3.2.1)
S	$H^*(X, \mathbb{Z})$ considered as the spin representation	Sec. 3.2
S^+	the half spin representation $H^{ev}(X, \mathbb{Z})$	Sec. 3.2
S^-	the half spin representation $H^{odd}(X, \mathbb{Z})$	Sec. 3.2
m	the spin representation $m : C(V) \to \mathrm{End}(S)$	(3.2.2)
m	the spin representation $m : \mathrm{Spin}(V) \to GL(S)$	(3.2.3)

η	an algebra homomorphism $K\rightarrow \mathrm{End}_{Hdg}(V_{\mathbb{Q}})$	(5.1.7)
V_η	$V_{\mathbb{Q}}$ considered as a vector space over K	$\dim_K(V_\eta)=d$
$\tilde{F}$	the Galois closure of F over $\mathbb{Q}$	
Σ	set of embeddings of the CM-field K in $\mathbb{C}$	
$\hat{\Sigma}$	set of embeddings of the field F in $\mathbb{R}$	
$\hat{\sigma}$	an embedding $F\rightarrow\mathbb{R}$	
$V_{\hat{\sigma}}$	the direct summand of $V_{\hat{\eta}}\otimes_F\tilde{F}\cong\oplus_{\hat{\sigma}\in\hat{\Sigma}}V_{\hat{\sigma}}$	$\dim_{\tilde{F}}(V_{\hat{\sigma}})=2d$
$V_{\hat{\sigma},\mathbb{R}}$		(3.1.3)
$\tilde{K}$	the Galois closure of K over $\mathbb{Q}$	
σ	an embedding $K\rightarrow\mathbb{C}$	
V_σ	the direct summand of $V_\eta\otimes_K\tilde{K}\cong\oplus_{\sigma\in\Sigma}V_\sigma$	$\dim_{\tilde{K}}(V_\sigma)=d$
$V_\sigma^{1,0}$, $V_\sigma^{0,1}$	the Hodge direct summands of $V_{\sigma,\mathbb{C}}$	$\dim_{\mathbb{C}}(V_\sigma^{1,0})=d/2$
W	a maximal isotropic subspace of $V_{\hat{\eta}}\otimes_F K$	$\dim_K(W)=d$
T	a CM-type for the CM-field K	Sec. 1
$\mathcal{T}_K$	The set of CM-types for the CM-field K	Sec. 6
e_T	embedding $V_{\hat{\eta}}\otimes_F K\rightarrow V_{\tilde{K}}$ determined by T	(6.0.1)
$W_T\subset V_{\tilde{K}}$	$\mathrm{span}_{\tilde{K}}(e_T(W))$	$\dim_{\tilde{K}}(W_T)=2n=\frac{de}{2}$
$W_{T,\hat{\sigma}}$	the direct summand of W_T	$\dim_{\tilde{K}}(W_{T,\hat{\sigma}})=d$
$\mathrm{Spin}(V_\bullet)$	Spin group of $V\otimes_{\mathbb{Z}}\bullet$ for a field $\bullet$	
$SO_+(V_\bullet)$	the image of $\mathrm{Spin}(V_\bullet)$ in $SO(V_\bullet)$	
$\mathrm{Spin}(V)$	the arithmetic subgroup of $\mathrm{Spin}(V_{\mathbb{Q}})$	
$\mathrm{Spin}(V_{\hat{\eta}})$	spin group of $V_{\mathbb{Q}}$ as an F vector space	
$\mathrm{Spin}(V_\eta)$	spin group of $V_{\mathbb{Q}}$ as a K vector space	
$\mathrm{Spin}(V_{\mathbb{Q}})_{\hat{\eta}}$	subgroup of $\mathrm{Spin}(V_{\mathbb{Q}})$ defined in (4.2.1)	
$\mathrm{Spin}(V)_\eta$	subgroup of $\mathrm{Spin}(V)_{\hat{\eta}}$ defined in (7.2.1)	
$\mathrm{Spin}(V_{\tilde{K}})_\eta$		(7.2.2)
ζ	homomorphism $\zeta:\mathrm{Spin}(V_{\hat{\eta}})\rightarrow\mathrm{Spin}(V_{\mathbb{Q}})_{\hat{\eta}}$	(4.2.3)
ℓ_T	the pure spinor of W_T in $S^+_{\tilde{K}}$	Sec. 6
B	subspace of $H^{ev}(X,\mathbb{Q})$ spanned by the ℓ_T's	$\dim_{\mathbb{Q}}(B)=2^{e/2}$
$B_{\sqrt{-q}\Theta}$	B associated to $\Theta\in\wedge^2_F H^1(X,\mathbb{Q})$ and $\sqrt{-q}\in K$	Sec. 11.2.1
K_-	subspace of K of purely imaginary elements	Sec. 5.2, 7.3
$\tilde{\Xi}_t$	an F-valued 2-form in $\wedge^2_F V^*_{\hat{\eta}}$, $t\in K_-$	(5.2.1)
Ξ_t	the 2-form $tr_{F/\mathbb{Q}}\circ\Xi_t$ in $\wedge^2_{\mathbb{Q}}V^*_{\mathbb{Q}}$, $t\in K_-$	(5.2.1)
H_t	a K-valued hermitian form on V_η, $t\in K_-$	(7.3.1)
$\mathrm{Spin}(V)_{\eta,B}$	subgroup of $\mathrm{Spin}(V)_\eta$ fixing all points of B	(7.2.3)
$\det_\sigma$	character of $\mathrm{Spin}(V_{\tilde{K}})_\eta$, $\sigma\in\Sigma$	(7.2.7)
$\mathrm{Spin}(V_{\mathbb{R}})_B$	subgroup of $\mathrm{Spin}(V_{\mathbb{R}})$ fixing all points of B	Sec. 9.1
$\Omega_{B,t}$	period domain of Weil type abelian varieties	(9.2.2)
$\Omega_{B,T}$	$\Omega_{B,t}$, T is the CM-type associated to $t\in K_-$	Sec. 9.2
Ω_B	$\cup_{T\in\mathcal{T}_K}\Omega_{B,T}$	(9.2.3)
$S^+_{\hat{\sigma}}$	the half-spin representation of $\mathrm{Spin}(V_{\hat{\sigma}})$	
ℓ_σ	the pure spinor of V_σ in $S^+_{\hat{\sigma}}$	
$P_{\hat{\sigma}}$	the plane $\mathrm{span}\{\ell_\sigma,\ell_{\hat{\sigma}}\}$ in $S^+_{\hat{\sigma}}$	
$HW(X\times\hat{X},\eta)$	$\wedge^d_{\eta(K)}H^1(X\times\hat{X},\mathbb{Q})$	(7.0.1)

$\mathcal{A}$	$\mathrm{Spin}(V)_{\eta,B}$-invariant subalgebra of $\wedge^* V_{\mathbb{Q}}$	Sec. 8
$\mathcal{R}$	$\mathrm{Spin}(V)_{\eta}$-invariant subalgebra of $\wedge^* V_{\mathbb{Q}}$	Sec. 8
BB_i	a graded summand of $B \otimes_{\mathbb{Q}} B$	Sec. 10.1
KB_1	The kernel of $BB_1 \to HW(X \times \hat{X}, \eta)$	(10.1.6)
$F^k(\wedge^* V)$	the increasing filtration $\oplus_{i \leq k} \wedge^i V$ of $\wedge^* V$	Sec. 10.1
$F_k(\wedge^* V)$	the decreasing filtration $\oplus_{i \geq k} \wedge^i V$ of $\wedge^* V$	Sec. 10.1
τ	multiplication of $H^i(X, \mathbb{Z})$ by $(-1)^{i(i-1)/2}$	(10.1.5)

3. Field embeddings and their associated decompositions

We have the decomposition

$$H^1(X, \mathbb{R}) = H^1(X, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{R} \cong \oplus_{\hat{\sigma} \in \hat{\Sigma}} H^1_{\hat{\sigma}}(X),$$

since $F \otimes_{\mathbb{Q}} \mathbb{R} \cong \prod_{\hat{\sigma} \in \hat{\Sigma}} \mathbb{R}$. The dimensions $\dim_{\mathbb{R}} H^1_{\hat{\sigma}}(X)$ are the same for all $\hat{\sigma} \in \hat{\Sigma}$ and are equal to $d = \dim_F H^1(X, \mathbb{Q})$. We have the equality

$$\dim(H^{1,0}_{\hat{\sigma}}(X)) = \dim(H^{0,1}_{\hat{\sigma}}(X)) = d,$$

for all $\hat{\sigma} \in \hat{\Sigma}$, since the $\hat{\eta}(F)$-action commutes with the complex structure of X, by assumption, and with complex conjugation, as F is totally real.

3.1. **Trace forms.** Denote by $H^1(X, \mathbb{Q})_{\hat{\eta}}$ the vector space $H^1(X, \mathbb{Q})$ considered as an F vector space. We have the isomorphism of vector spaces over $\mathbb{Q}$

$$\mathrm{Hom}_F(H^1(X, \mathbb{Q})_{\hat{\eta}}, F) \to \mathrm{Hom}_{\mathbb{Q}}(H^1(X, \mathbb{Q}), \mathbb{Q})$$

sending $h \in \mathrm{Hom}_F(H^1(X, \mathbb{Q})_{\hat{\eta}}, F)$ to $tr_{F/\mathbb{Q}} \circ h$. We denote by

$$\hat{\eta} : F \to \mathrm{End}(V_{\mathbb{Q}}) \tag{3.1.1}$$

the scalar multiplication action of F on $H^1(X, \mathbb{Q}) \oplus \mathrm{Hom}_F(H^1(X, \mathbb{Q})_{\hat{\eta}}, F)$. Explicitly, $\hat{\eta}(a)$ acts on $\mathrm{Hom}_F(H^1(X, \mathbb{Q})_{\hat{\eta}}, F)$ by $\hat{\eta}_a(\theta) = \theta \circ \hat{\eta}(a)$. We defined $V_{\hat{\eta}}$ as $V_{\mathbb{Q}}$ with its F-vector space structure. Define the $\hat{\eta}(F)$-bilinear pairing

$$(\bullet, \bullet)_{V_{\hat{\eta}}} : V_{\mathbb{Q}} \otimes_{\hat{\eta}(F)} V_{\mathbb{Q}} \to F, \tag{3.1.2}$$

by $((w_1, \theta_1), (w_2, \theta_2))_{V_{\hat{\eta}}} = \theta_1(w_2) + \theta_2(w_1)$, where θ_1 and θ_2 are regarded as a elements of $\mathrm{Hom}_F(H^1(X, \mathbb{Q})_{\hat{\eta}}, F)$. The $\mathbb{Q}$-valued pairing $(\bullet, \bullet)_{V_{\mathbb{Q}}}$ is, by definition, the composition $tr_{F/\mathbb{Q}} \circ (\bullet, \bullet)_{V_{\hat{\eta}}}$. The $\hat{\eta}(F)$-bilinearity of $(\bullet, \bullet)_{V_{\hat{\eta}}}$ implies that $\hat{\eta}(a)$ is self-dual with respect to the pairing $(\bullet, \bullet)_{V_{\mathbb{Q}}}$.

Consider the $\mathbb{R}$-bilinear $\mathbb{R}$-valued pairing on $F \otimes_{\mathbb{Q}} \mathbb{R}$ given by

$$(f_1 \otimes r_1, f_2 \otimes r_2) = r_1 r_2 tr_{F/\mathbb{Q}}(f_1 f_2),$$

for $f_i \in F$ and $r_i \in \mathbb{R}$, $i = 1, 2$. The decomposition $F \otimes_{\mathbb{Q}} \mathbb{R} = \oplus_{\hat{\sigma} \in \hat{\Sigma}} \hat{\sigma}$ consists of pairwise orthogonal lines with respect to the above pairing. Set

$$V_{\hat{\sigma}, \mathbb{R}} \ := \ H^1_{\hat{\sigma}}(X) \oplus H^1_{\hat{\sigma}}(\hat{X}). \tag{3.1.3}$$

Lemma 3.1.1. *The decomposition $V \otimes_{\mathbb{Z}} \mathbb{R} = \oplus_{\hat{\sigma} \in \hat{\Sigma}} V_{\hat{\sigma}, \mathbb{R}}$ consists of pairwise orthogonal subspaces with respect to $(\bullet, \bullet)_{V_{\mathbb{R}}}$.*

Proof. This follows from the self-duality of $\hat{\eta}(a)$, for $a \in F$. Explicitly, the pairing $(\bullet,\bullet)_{V_{\mathbb{R}}} : V_{\mathbb{R}} \otimes V_{\mathbb{R}} \to \mathbb{R}$ factors as an $\hat{\eta}(F)$-bilinear pairing $(\bullet,\bullet)_{V_{\hat{\eta}}}$ with values in $F\otimes_{\mathbb{Q}}\mathbb{R}$ composed with $tr_{F/\mathbb{Q}}\otimes id_{\mathbb{R}} : F\otimes_{\mathbb{Q}}\mathbb{R}\to\mathbb{R}$. Now, given $x\in V_{\hat{\sigma}_1,\mathbb{R}}$, $y\in V_{\hat{\sigma}_2,\mathbb{R}}$, and $q\in F$ a primitive element, we have

$$\begin{aligned} q(x,y)_{V_{\hat{\eta}}} &= (\hat{\eta}_q x,y)_{V_{\hat{\eta}}} = (\hat{\sigma}_1(q)x,y)_{V_{\hat{\eta}}} = \hat{\sigma}_1(q)(x,y)_{V_{\hat{\eta}}} \\ q(x,y)_{V_{\hat{\eta}}} &= (x,\hat{\eta}_q y)_{V_{\hat{\eta}}} = (x,\hat{\sigma}_2(q)y)_{V_{\hat{\eta}}} = \hat{\sigma}_2(q)(x,y)_{V_{\hat{\eta}}}, \end{aligned}$$

where multiplication by q on the left above corresponds to multiplication in the left tensor factor of $F\otimes_{\mathbb{Q}}\mathbb{R}$ and multiplication by $\hat{\sigma}_i(q)$ on the right above corresponds to multiplication in the right tensor factor of $F\otimes_{\mathbb{Q}}\mathbb{R}$. Hence, if $\hat{\sigma}_1\neq\hat{\sigma}_2$, then $(x,y)_{V_{\hat{\eta}}}=0$. Consequently, its trace $(x,y)_{V_{\mathbb{R}}}$ vanishes as well. □

The j-th cartesian power $(F^\times)^j$ of the multiplicative group $F^\times$ acts on the j-th tensor power over $\mathbb{Q}$ of $H^1(X,\mathbb{Q})$ and hence on $\wedge^j_{\mathbb{Q}}H^1(X,\mathbb{Q})$ and on $\wedge^j_{\mathbb{Q}}H^1(X,\mathbb{Q})\otimes_{\mathbb{Q}}\mathbb{R}\cong \wedge^j_{\mathbb{R}}H^1(X,\mathbb{R})$. The latter decomposes as a direct sum of characters subspaces

$$\wedge^j_{\mathbb{R}}H^1(X,\mathbb{R}) = \oplus_{(\hat{\sigma}_1,\dots,\hat{\sigma}_j)\in(\hat{\Sigma})^j}\left(\wedge^j_{\mathbb{R}}H^1(X,\mathbb{R})\right)_{(\hat{\sigma}_1,\dots,\hat{\sigma}_j)}$$

The vector space $\wedge^j_{\mathbb{Q}}H^1(X,\mathbb{Q})$ decomposes into subrepresentations of $(F^\times)^j$ defined over $\mathbb{Q}$. The subrepresentation $\wedge^j_F H^1(X,\mathbb{Q})$ of $\wedge^j_{\mathbb{Q}}H^1(X,\mathbb{Q})$ corresponds to the subrepresentation

$$\oplus_{\hat{\sigma}\in\hat{\Sigma}}\wedge^j_{\mathbb{R}} H^1_{\hat{\sigma}}(X) = \oplus_{\hat{\sigma}\in\hat{\Sigma}}\left(\wedge^j_{\mathbb{R}}H^1(X,\mathbb{R})\right)_{(\hat{\sigma},\dots,\hat{\sigma})}.$$

of $\wedge^j_{\mathbb{R}}H^1(X,\mathbb{R})$. Given $\alpha_i\in\wedge^{j_i}_F H^1(X,\mathbb{Q})$, $i=1,2$, then $\alpha_1\wedge_F\alpha_2$ is the projection of $\alpha_1\wedge_{\mathbb{Q}}\alpha_2$ to the direct summand $\wedge^{j_1+j_2}_F H^1(X,\mathbb{Q})$.

The above action of $(F^\times)^j$ on $\wedge^j_F H^1(X,\mathbb{Q})$ factors through the natural action of F on it as an F-vector space via the product homomorphisn $(F^\times)^j\to F^\times\subset F$, $(f_1,\dots,f_j)\mapsto \prod_{i=1}^j f_j$.

3.2. **Tensor product factorizations of pure spinors.** Set $V := H^1(X,\mathbb{Z})\oplus H^1(\hat{X},\mathbb{Z})$ endowed with the unimodular even symmetric bilinear pairing (1.1.2). Let $C(V)$ be the Clifford algebra $\oplus_{k=0}^\infty V^{\otimes k}/\langle v_1v_2+v_2v_1-(v_1,v_2)\ :\ v_1,v_2\in V\rangle$, where the integer (v_1,v_2) is in $V^{\otimes 0}=\mathbb{Z}$. $C(V)$ is $\mathbb{Z}/2\mathbb{Z}$ graded, $C(V)=C(V)^{ev}\oplus C(V)^{odd}$. Set $S:=H^*(X,\mathbb{Z})$, $S^+ := H^{ev}(X,\mathbb{Z})$, and $S^- := H^{odd}(X,\mathbb{Z})$. We identify V with the image of $V^{\otimes 1}$ in $C(V)$. The Cliford group is the subgroup $G(V)$ of the group of invertible elements g in $C(V)^{ev}$ satisfying $gVg^{-1}=V$. By definition, $\mathrm{Spin}(V)$ is an index 2 subgroup of $G(V)\cap C(V)^{ev}$ (kernel of the norm character) [GLO]. We get the vector representation $\rho : G(V)\to O(V)$, which restricts to

$$\rho : \mathrm{Spin}(V)\to O(V) \tag{3.2.1}$$

and we denote its image by $SO^+(V)$. The kernel of ρ is of order 2 generated by -1 (the element $-1\in C(V)$). We have the isomorphism

$$m : C(V)\to \mathrm{End}(S), \tag{3.2.2}$$

by [GLO, Prop. 3.2.1(e)], which restricts to $m : V \rightarrow \mathrm{End}(S)$ and to the spin representation

$$m : \mathrm{Spin}(V) \rightarrow GL(S). \tag{3.2.3}$$

Set $S_{\hat{\sigma}} := \wedge^* H^1_{\hat{\sigma}}(X)$. Note the isomorphisms $S_{\mathbb{R}} := \wedge^* H^1(X, \mathbb{R}) \cong \otimes_{\hat{\sigma}\in\hat{\Sigma}} S_{\hat{\sigma}}$ and

$$S_{\mathbb{R}} \otimes S_{\mathbb{R}} \cong \otimes_{\hat{\sigma}\in\hat{\Sigma}}(S_{\hat{\sigma}} \otimes_{\mathbb{R}} S_{\hat{\sigma}}),$$

both in the categories of $\mathbb{Z}/2\mathbb{Z}$ graded algebras. The tensor product $V_{\hat{\eta}} \otimes_{\mathbb{Q}} \mathbb{R}$, with the induced bilinear pairing, decomposes as an orthogonal direct sum $\oplus_{\hat{\sigma}\in\hat{\Sigma}} V_{\hat{\sigma},\mathbb{R}}$, by Lemma 3.1.1. Hence, $C(V_{\mathbb{Q}}) \otimes_{\mathbb{Q}} \mathbb{R}$ is isomorphic to the tensor product $\otimes_{\hat{\sigma}\in\hat{\Sigma}} C(V_{\hat{\sigma},\mathbb{R}})$, where $V_{\hat{\sigma},\mathbb{R}}$ is endowed with the restriction of the bilinear pairing of $V_{\mathbb{R}}$ and the tensor product is over $\mathbb{R}$ in the category of $\mathbb{Z}_2$-graded algebras [Lam, Lemma V.1.7]. In the current section 3.2 we will denote $V_{\hat{\sigma},\mathbb{R}}$ by $V_{\hat{\sigma}}$. The isomorphism $\varphi : S_{\mathbb{R}} \otimes S_{\mathbb{R}} \rightarrow C(V_{\mathbb{R}})$, given in [Ch, III.3.1], factors as the tensor product of

$$\varphi_{\hat{\sigma}} : S_{\hat{\sigma}} \otimes S_{\hat{\sigma}} \rightarrow C(V_{\hat{\sigma}}).$$

Given maximal isotropic subspaces $W_{\hat{\sigma}}$ of $V_{\hat{\sigma}}$, for all $\hat{\sigma} \in \hat{\Sigma}$, set $W := \oplus_{\hat{\sigma}\in\hat{\Sigma}} W_{\hat{\sigma}}$. Let $\ell_{\hat{\sigma}} \subset S^+_{\hat{\sigma}}$ be the line spanned by a pure spinor of $W_{\hat{\sigma}}$ and by $\ell \subset S^+_{\mathbb{R}}$ the line spanned by a pure spinor of W. Then

$$\ell = \bigotimes_{\hat{\sigma}\in\hat{\Sigma}} \ell_{\hat{\sigma}}. \tag{3.2.4}$$

Given another set of maximal isotropic subspaces $W'_{\hat{\sigma}}$ of $V_{\hat{\sigma}}$, for all $\hat{\sigma} \in \hat{\Sigma}$, with pure spinors $\ell'_{\hat{\sigma}}$, we have $\varphi(\ell \otimes \ell') = \otimes_{\hat{\sigma}\in\hat{\Sigma}} \varphi_{\hat{\sigma}}(\ell_{\hat{\sigma}} \otimes \ell'_{\hat{\sigma}})$.

Similarly, the isomorphism $\psi : C(V_{\mathbb{R}}) \rightarrow \wedge^* V_{\mathbb{R}}$, given in [M2, Eq. (2.3.1)], decomposes as a tensor product of $\psi_{\hat{\sigma}} : C(V_{\hat{\sigma}}) \rightarrow \wedge^* V_{\hat{\sigma}}$ and so

$$\tilde{\varphi} := \psi \circ \varphi \tag{3.2.5}$$

decomposes as the tensor product of

$$\tilde{\varphi}_{\hat{\sigma}} := \psi_{\hat{\sigma}} \circ \varphi_{\hat{\sigma}} : S_{\hat{\sigma}} \otimes S_{\hat{\sigma}} \rightarrow \wedge^* V_{\hat{\sigma}}. \tag{3.2.6}$$

4. Spin groups over F and real embeddings of F

We define

$$\mathrm{Spin}(V_{\hat{\eta}})$$

as the Spin group associated to the F-valued F-bilinear pairing $(\bullet, \bullet)_{V_{\hat{\eta}}}$.

4.1. **A homomorphism from** $\mathrm{Spin}(V_{\hat{\eta}})$ **to** $\mathrm{Spin}(V_{\mathbb{R}})$**.**

In the current section 4.1 we continue to denote $V_{\hat{\sigma},\mathbb{R}}$ by $V_{\hat{\sigma}}$. The composition

$$C(V_{\mathbb{Q}}) \rightarrow C(V_{\mathbb{R}}) \stackrel{\cong}{\rightarrow} \otimes_{\hat{\sigma}\in\hat{\Sigma}} C(V_{\hat{\sigma}})$$

is a $\mathbb{Q}$-algebra homomorphism.

Consider the Clifford algebra

$$C(V_{\hat{\eta}}) := \oplus_{j\geq 0} V_{\hat{\eta}}^{\otimes_F j} / \langle u \otimes_F v + v \otimes_F u - (u, v)_{V_{\hat{\eta}}} \rangle,$$

where the scalar $(u,v)_{V_{\hat{\eta}}}$ is in F. The embedding

$$(\hat{\sigma})_{\hat{\sigma}\in\hat{\Sigma}} : V_{\hat{\eta}} \to V_{\mathbb{R}} = \oplus_{\hat{\sigma}\in\hat{\Sigma}} V_{\hat{\sigma}}$$

is isometric, if the pairing on the right hand side is given by $(u,v) \mapsto ((u_{\hat{\sigma}}, v_{\hat{\sigma}})_{V_{\hat{\sigma}}})_{\hat{\sigma}\in\hat{\Sigma}}$ as an element in $F\otimes_{\mathbb{Q}}\mathbb{R} \cong \oplus_{\hat{\sigma}\in\hat{\Sigma}}\mathbb{R}$. The diagonal embedding of $V_{\mathbb{Q}}$ in $V_{\mathbb{R}}$ sends $v\in V_{\mathbb{Q}}\subset C(V_{\mathbb{Q}})$ to $\sum_{\hat{\sigma}\in\hat{\Sigma}} \hat{\sigma}(v) \in \otimes_{\hat{\sigma}\in\hat{\Sigma}} C(V_{\hat{\sigma}})$ (see the proof of [Lam, Lemma V.1.7]). It does not extends to an algebra homomorphism $C(V_{\hat{\eta}}) \to C(V_{\mathbb{R}})$.

We construct next a diagonal multiplicative group homomorphism from the group $C(V_{\hat{\eta}})^{even,\times}$, of invertible even[2] elements in $C(V_{\hat{\eta}})$, into $C(V_{\mathbb{R}})^{even,\times}$. We have the F-algebra homomorphism

$$C(V_{\hat{\eta}}) \to C(V_{\hat{\eta}})\otimes_{\mathbb{Q}}\mathbb{R} \cong C(V_{\hat{\eta}})\otimes_F (F\otimes_{\mathbb{Q}}\mathbb{R}) \cong \prod_{\hat{\sigma}\in\hat{\Sigma}} C(V_{\hat{\sigma}}), \tag{4.1.1}$$

where the left homomorphism sends x to $x\otimes 1$. For each $\hat{\sigma}\in\hat{\Sigma}$ we have an embedding $\hat{\sigma} : C(V_{\hat{\eta}}) \to C(V_{\hat{\sigma}})$ obtained by composing the above homomorphism with projection to the $\hat{\sigma}$-factor. We get the diagram

$$\begin{array}{ccccccc} C(V_{\hat{\eta}})^{even,\times} & \xrightarrow{(4.1.1)} & \prod_{\hat{\sigma}\in\hat{\Sigma}} C(V_{\hat{\sigma}})^{even,\times} & \longrightarrow & \left[\otimes_{\hat{\sigma}\in\hat{\Sigma}} C(V_{\hat{\sigma}})\right]^{even,\times} & \xrightarrow{\cong} & C(V_{\mathbb{R}})^{even,\times} \\ \cup\uparrow & & \cup\uparrow & & & & \cup\uparrow \\ \mathrm{Spin}(V_{\hat{\eta}}) & \longrightarrow & \prod_{\hat{\sigma}\in\hat{\Sigma}} \mathrm{Spin}(V_{\hat{\sigma}}) & & \longrightarrow & & \mathrm{Spin}(V_{\mathbb{R}}) \end{array} \tag{4.1.2}$$

where the top middle arrow sends $(g_{\hat{\sigma}})_{\hat{\sigma}\in\hat{\Sigma}}$ to $\otimes_{\hat{\sigma}\in\hat{\Sigma}} g_{\hat{\sigma}}$. The composition

$$C(V_{\hat{\eta}})^{even,\times} \to C(V_{\mathbb{R}})^{even,\times}, \tag{4.1.3}$$

of the top horizontal homomorphisms above, restricts to $Nm_{F/\mathbb{Q}} : F^\times \to \mathbb{Q}^\times \subset \mathbb{R}^\times$. In particular, it is not injective.

4.2. **A homomorphism from** $\mathrm{Spin}(V_{\hat{\eta}})$ **to** $\mathrm{Spin}(V_{\mathbb{Q}})$**.** Let $SO(V_{\hat{\eta}})$ be the subgroup of $SO(V_{\mathbb{Q}})$ preserving the F-valued pairing $(\bullet,\bullet)_{V_{\hat{\eta}}}$. Let $SO_+(V_{\hat{\eta}})$ be the image of $\mathrm{Spin}(V_{\hat{\eta}})$ in $SO(V_{\hat{\eta}})$. Let

$$\mathrm{Spin}(V_{\mathbb{Q}})_{\hat{\eta}} \tag{4.2.1}$$

be the image of $\mathrm{Spin}(V_{\hat{\eta}})$ in $\mathrm{Spin}(V_{\mathbb{Q}})$ via the homomorphism (4.1.3). Lemma 4.2.1 below states that $\mathrm{Spin}(V_{\mathbb{Q}})_{\hat{\eta}}$ is isomorphic to either the domain or the codomain of the degree 2 homomorphism $\rho : \mathrm{Spin}(V_{\hat{\eta}}) \to SO_+(V_{\hat{\eta}})$. The lemma also compute $\mathrm{Spin}(V_{\mathbb{Q}})_{\hat{\eta}}$ explicitly.

Let $\{e_{\hat{\sigma},1},\dots,e_{\hat{\sigma},d}\}$ be an orthogonal basis of $V_{\hat{\sigma},\mathbb{R}}$ satisfying $(e_{\hat{\sigma},i}, e_{\hat{\sigma},i}) \in \{2,-2\}$. Then $e_{\hat{\sigma},i}\in C(V_{\mathbb{R}})$ belongs to the Clifford group and $\rho_{e_{\hat{\sigma},i}}$ acts on $V_{\mathbb{R}}$ as minus the reflection in the hyperplane orthogonal to $e_{\hat{\sigma},i}$. Note that $e_{\hat{\sigma},i}e_{\hat{\sigma},j} = -e_{\hat{\sigma},i}e_{\hat{\sigma},j}$, if $i\neq j$, but the images of $e_{\hat{\sigma},i}$ and $e_{\hat{\sigma},j}$ in $O(V_{\mathbb{R}})$ commute. Set

$$g_{\hat{\sigma}} := \prod_{i=1}^{d} e_{\hat{\sigma},i}. \tag{4.2.2}$$

[2]Note that even tensor factors commute in the tensor product $\otimes_{\hat{\sigma}\in\hat{\Sigma}} C(V_{\hat{\sigma}})$ in the category of $\mathbb{Z}_2$-graded algebras.

Choose an embedding $\hat{\sigma}_0 : F \to \mathbb{R}$ and let $\tilde{F} \subset \mathbb{R}$ be the Galois closure of $\hat{\sigma}_0(F)$ over $\mathbb{Q}$. The subfield $\tilde{F}$ of $\mathbb{R}$ is independent of $\hat{\sigma}_0$ and any embedding $\hat{\sigma} \in \hat{\Sigma}$ factors through an embedding of F in $\tilde{F}$, which we denote by $\hat{\sigma}$ as well. Diagram (4.1.2) holds, with $V_{\mathbb{R}}$ replaced by $V_{\tilde{F}}$ and $V_{\hat{\sigma}}$ now denotes the corresponding direct summand of $V_{\tilde{F}}$. The element $g_{\hat{\sigma}}$ belongs to the Clifford subgroup of $C(V_{\tilde{F}})^{even,\times}$, which belongs to $\mathrm{Spin}(V_{\tilde{F}})$ if and only if $d/2$ is even, since the signature of $V_{\hat{\sigma}}$ is $(d/2, d/2)$. Note that $g_{\hat{\sigma}}^2 = (-1)^{d/2}$. The element $g_{\hat{\sigma}}$ maps to $SO(V_{\tilde{F}})$ acting on $V_{\hat{\sigma}}$ via multiplication by -1 and it acts as the identity on $V_{\hat{\sigma}'}$, for $\hat{\sigma}' \neq \hat{\sigma}$. Under the natural embedding of $C(V_{\hat{\sigma}_1})$ in $C(V_{\tilde{F}})$, the element $g_{\hat{\sigma}_2}$ commutes with $C(V_{\hat{\sigma}_1})^{even}$ and anti-commutes with $C(V_{\hat{\sigma}_1})^{odd}$, for all $\hat{\sigma}_1$, $\hat{\sigma}_2$ in $\hat{\Sigma}$, by [Ch, II.2.4].

Let $SO_+(V_{\hat{\sigma}})$ be the image of $\mathrm{Spin}(V_{\hat{\sigma}})$ in $SO(V_{\hat{\sigma}})$. We have the short exact sequences

$$\begin{array}{rcl} 0 \to SO_+(V_{\mathbb{Q}}) \to & SO(V_{\mathbb{Q}}) & \to \mathbb{Q}^\times/\mathbb{Q}^{\times,2} \to 0, \\ 0 \to SO_+(V_{\hat{\eta}}) \to & SO(V_{\hat{\eta}}) & \to F^\times/F^{\times,2} \to 0, \\ 0 \to SO_+(V_{\hat{\sigma}}) \to & SO(V_{\hat{\sigma}}) & \to \tilde{F}^\times/\tilde{F}^{\times,2} \to 0, \end{array}$$

where $\mathbb{Q}^{\times,2}$, $F^{\times,2}$, and $\tilde{F}^{\times,2}$ are the groups of squares of non-zero elements (see [Ch, II.3.7]). Furthermore, $SO_+(\bullet)$ is the commutator subgroup of $SO(\bullet)$, for $\bullet = V_{\mathbb{Q}}$, $V_{\hat{\eta}}$, and $V_{\hat{\sigma}}$ [Ch, II.3.9]. Hence, for every $g \in \mathrm{Spin}(V_{\mathbb{Q}})_{\hat{\eta}}$, ρ_g restricts to the subspace $V_{\hat{\sigma}}$ of $V_{\tilde{F}}$ as an element of $SO_+(V_{\hat{\sigma}})$.

Let $\mathrm{Spin}(V_{\mathbb{Q}})_{\{g_{\hat{\sigma}}\}}$ be the subgroup of $\mathrm{Spin}(V_{\mathbb{Q}})$ consisting of elements, which commute in $C(V_{\tilde{F}})^{even,\times}$ with $g_{\hat{\sigma}}$, for all $\hat{\sigma} \in \hat{\Sigma}$. Let $\mathrm{Spin}(V_{\mathbb{Q}})''_{\{g_{\hat{\sigma}}\}}$ be the commutator subgroup of the commutator subgroup of $\mathrm{Spin}(V_{\mathbb{Q}})_{\{g_{\hat{\sigma}}\}}$ (the second derived subgroup). The group $\mathrm{Spin}(V_{\mathbb{Q}})_{\{g_{\hat{\sigma}}\}}$ is independent of the choice of the bases $\{e_{\hat{\sigma},i}\}$, since $g_{\hat{\sigma}}$ is determined, up to sign, by its image $\rho_{g_{\hat{\sigma}}}$ in $SO(V_{\tilde{F}})$ and $\rho_{g_{\hat{\sigma}}}$ is independent of the choice of the bases.

Lemma 4.2.1. *The homomorphism (4.1.3) maps* $\mathrm{Spin}(V_{\hat{\eta}})$ *onto* $\mathrm{Spin}(V_{\mathbb{Q}})''_{\{g_{\hat{\sigma}}\}}$, *so the latter is equal to* $\mathrm{Spin}(V_{\mathbb{Q}})_{\hat{\eta}}$. *The resulting homomorphism*

$$\zeta : \mathrm{Spin}(V_{\hat{\eta}}) \to \mathrm{Spin}(V_{\mathbb{Q}})_{\hat{\eta}} \tag{4.2.3}$$

is an isomorphism, if $\dim_{\mathbb{Q}}(F)$ *is odd, and its kernel is* -1, *if* $\dim_{\mathbb{Q}}(F)$ *is even. The outer square of the diagram*

$$\begin{tikzcd} \mathrm{Spin}(V_{\hat{\eta}}) \arrow[r, "\zeta"] \arrow[d, "\rho"'] & \mathrm{Spin}(V_{\mathbb{Q}})_{\hat{\eta}} \arrow[r, "\subset"] \arrow[dl, "\bar{\zeta}"] & \mathrm{Spin}(V_{\mathbb{Q}}) \arrow[d, "\rho"'] \\ SO_+(V_{\hat{\eta}}) \arrow[rr, "\subset"'] & & SO_+(V_{\mathbb{Q}}) \end{tikzcd}$$

is commutative. Consequently, ζ *factors through a homomorphism* $\bar{\zeta} : \mathrm{Spin}(V_{\mathbb{Q}})_{\hat{\eta}} \to SO_+(V_{\hat{\eta}})$, *which is an isomorphism, if* $\dim_{\mathbb{Q}}(F)$ *is even, and is equal to* $\rho \circ \zeta^{-1}$, *if* $\dim_{\mathbb{Q}}(F)$ *is odd.*

Proof. The homomorphism (4.1.3) factors through $C(V_{\tilde{F}})^{even,\times}$. The group $Gal(\tilde{F}/\mathbb{Q})$ acts on $C(V_{\tilde{F}}) = C(V_{\mathbb{Q}}) \otimes_{\mathbb{Q}} \tilde{F}$ via its action on the second tensor factor and the subset $C(V_{\tilde{F}})^{even,\times}$ is $Gal(\tilde{F}/\mathbb{Q})$-invariant. The homomorphism (4.1.3) is $Gal(\tilde{F}/\mathbb{Q})$-eqvariant, as the top left homomorphism in diagram (4.1.2) is the diagonal homomorphism. Hence, the image of $C(V_{\hat{\eta}})^{even,\times}$ in $C(V_{\tilde{F}})^{even,\times}$, via the homomorphism (4.1.3), is contained in $C(V_{\mathbb{Q}})^{even,\times}$. The image of $Spin(V_{\hat{\eta}})$ is thus contained in $\mathrm{Spin}(V_{\mathbb{Q}})$, by the commutativity of the diagram (4.1.2). The last statement about the kernel follows from the equality $Nm_{F/\mathbb{Q}}(-1) = (-1)^{\dim_{\mathbb{Q}}(F)}$.

The group $\mathrm{Spin}(V_{\hat{\eta}})$ is equal to its commutator subgroup, being a simple Lie group [Mi2, Def. 12.45 and Cor. 21.50]. The element $g_{\hat{\sigma}}$ belongs to the center of

$$\left[\otimes_{\hat{\sigma}\in\hat{\Sigma}}C(V_{\hat{\sigma}})\right]^{even,\times} \cap \left[\otimes_{\hat{\sigma}\in\hat{\Sigma}}C(V_{\hat{\sigma}})^{even}\right],$$

by [Ch, II.2.4]. Hence, the group $\mathrm{Spin}(V_{\hat{\eta}})$ is mapped via (4.1.3) into $\mathrm{Spin}(V_{\mathbb{Q}})''_{\{g_{\hat{\sigma}}\}}$.

It remains to prove that the subgroup $\mathrm{Spin}(V_{\mathbb{Q}})_{\hat{\eta}}$ is the whole of $\mathrm{Spin}(V_{\mathbb{Q}})''_{\{g_{\hat{\sigma}}\}}$. The pairing $(\bullet,\bullet)_{\hat{\eta}}$ is the unique F-valued symmetric F-bilinear pairing on $V_{\hat{\eta}}$, such that $tr_{F/\mathbb{Q}} \circ (\bullet,\bullet)_{\hat{\eta}} = (\bullet,\bullet)_V$. Indeed, $f \mapsto tr_{F/\mathbb{Q}} \circ f : \mathrm{Hom}_F(V_{\hat{\eta}}, F) \to \mathrm{Hom}_{\mathbb{Q}}(V_{\mathbb{Q}}, \mathbb{Q})$ is an isomorphism of $\mathbb{Q}$-vector spaces and the pairing $(\bullet,\bullet)_{\hat{\eta}}$ corresponds to an element of the subspace $\mathrm{Hom}_F(V_{\hat{\eta}}, \mathrm{Hom}_F(V_{\hat{\eta}}, F))$ of $\mathrm{Hom}_{\mathbb{Q}}(V_{\hat{\eta}}, \mathrm{Hom}_F(V_{\hat{\eta}}, F))$. Hence, if g is an isometry of $(V_{\mathbb{Q}}, (\bullet,\bullet)_V)$, which commutes with $\hat{\eta}(F)$, then g is an F-linear isometry of $(V_{\hat{\eta}}, (\bullet,\bullet)_{V_{\hat{\eta}}})$. Indeed, in that case $tr_{F/\mathbb{Q}}(g(x), g(y))_{V_{\hat{\eta}}} = (g(x), g(y))_V = (x,y)_V$, and $(g(\bullet), g(\bullet))_{V_{\hat{\eta}}}$ is an F-valued symmetric F-bilinear pairing on $V_{\hat{\eta}}$. Hence, for $g \in \mathrm{Spin}(V_{\mathbb{Q}})_{\{g_{\hat{\sigma}}\}}$, the isometry ρ_g of $(V_{\mathbb{Q}}, (\bullet,\bullet)_V)$ is also an isometry of $(\bullet,\bullet)_{V_{\hat{\eta}}}$. Thus, if g belongs to $\mathrm{Spin}(V_{\mathbb{Q}})''_{\{g_{\hat{\sigma}}\}}$, then ρ_g belongs to the commutator subgroup $SO_+(V_{\hat{\eta}})$ of $O(V_{\hat{\eta}})$, where $O(V_{\hat{\eta}})$ is the subgroup of $O(V_{\mathbb{Q}})$ of F-linear automorphisms preserving the F-valued pairing $(\bullet,\bullet)_{V_{\hat{\eta}}}$. Consequently, $\rho_g = \rho_{\tilde{g}}$, for some $\tilde{g} \in \mathrm{Spin}(V_{\hat{\eta}})$, by [Ch, II.3.8]. So $g = \pm\zeta(\tilde{g})$. If $\dim_{\mathbb{Q}}(F)$ is odd, then $\zeta(-1) = -1$, and so g belongs to the image of ζ. If $\dim_{\mathbb{Q}}(F)$ is even, then the above argument shows that the image of ζ has index at most 2 in the first derived subgroup of $\mathrm{Spin}(V_{\mathbb{Q}})_{\{g_{\hat{\sigma}}\}}$. If the index is 2, then the image of ζ is contained in the second derived subgroup and is a normal subgroup of the first derived subgroup with an abelian quotient, and so the image of ζ is equal to the second derived subgroup $\mathrm{Spin}(V_{\mathbb{Q}})''_{\{g_{\hat{\sigma}}\}}$. □

5. Complex multiplication by K from a maximal isotropic subspace of $H^1(X \times \hat{X}, \mathbb{Q}) \otimes_F K$

5.1. **Complex multiplication by the CM-field K.** Set $V_{\hat{\eta},K} := H^1(X\times\hat{X},\mathbb{Q})\otimes_F K$. We have the isomorphism

$$\mathrm{Hom}_K(H^1(X,\mathbb{Q}) \otimes_F K, K) \stackrel{\cong}{\to} \mathrm{Hom}_{\mathbb{Q}}(H^1(X,\mathbb{Q}) \otimes_F K, \mathbb{Q}),$$

sending $h \in \mathrm{Hom}_K(H^1(X,\mathbb{Q}) \otimes_F K, K)$ to $tr_{K/\mathbb{Q}} \circ h$. We get the K bilinear pairing

$$(\bullet,\bullet)_{V_{\hat{\eta},K}} : V_{\hat{\eta},K} \otimes_K V_{\hat{\eta},K} \to K \tag{5.1.1}$$

analogous to $(\bullet,\bullet)_{V_{\hat{\eta}}}$. We define

$$\mathrm{Spin}(V_{\hat{\eta},K}) \tag{5.1.2}$$

as the associated spin group. Let

$$\mathrm{Spin}(V_{\hat{\eta},K})_{\mathbb{Q}} \tag{5.1.3}$$

be the subgroup of $\mathrm{Spin}(V_{\hat{\eta},K})$ leaving the subset $V_{\mathbb{Q}}$ of $V_{\hat{\eta},K}$ invariant.

Remark 5.1.1. The group $\mathrm{Spin}(V_{\hat{\eta}})$ is a subgroup of $\mathrm{Spin}(V_{\hat{\eta},K})_{\mathbb{Q}}$. The difference is explained by the following diagram. The Galois involution ι acts on $C(V_{\hat{\eta}} \otimes_F K) \cong C(V_{\hat{\eta}}) \otimes_F K$ via its action on the second tensor factor K. The subset $\mathrm{Spin}(V_{\hat{\eta},K})$ of $C(V_{\hat{\eta}} \otimes_F K)$ is ι-invariant and is thus endowed with a $Gal(K/F)$-action.

$$\begin{tikzcd}
0 \arrow[r] & \{\pm 1\} \arrow[r] \arrow[d, "\cong"] & \mathrm{Spin}(V_{\hat{\eta}}) \arrow[r] \arrow[d, "\cong"] & SO(V_{\hat{\eta}}) \arrow[r] \arrow[d, "\cong"] & F^{\times}/F^{\times,2} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \{\pm 1\} \arrow[r] \arrow[d, "\cong"] & \mathrm{Spin}(V_{\hat{\eta},K})^{\iota} \arrow[r] \arrow[d, "\cap"] & SO(V_{\hat{\eta}} \otimes_F K)^{\iota} \arrow[r] \arrow[d, "="] & [K^{\times}/K^{\times,2}]^{\iota} \arrow[d, "="] & \\
0 \arrow[r] & \{\pm 1\} \arrow[r] \arrow[d, "\cong"] & \mathrm{Spin}(V_{\hat{\eta},K})_{\mathbb{Q}} \arrow[r] \arrow[d, "\cap"] & SO(V_{\hat{\eta}} \otimes_F K)^{\iota} \arrow[r] \arrow[d, "\cap"] & [K^{\times}/K^{\times,2}]^{\iota} \arrow[d, "\cap"] & \\
0 \arrow[r] & \{\pm 1\} \arrow[r] & \mathrm{Spin}(V_{\hat{\eta},K}) \arrow[r] & SO(V_{\hat{\eta}} \otimes_F K) \arrow[r] & K^{\times}/K^{\times,2} \arrow[r] & 0
\end{tikzcd}$$

The top and bottom rows are exact, by [Ch, II.3.7]. The third row is exact, by definition of $\mathrm{Spin}(V_{\hat{\eta},K})_{\mathbb{Q}}$. The second row is not exact at $SO(V_{\hat{\eta}} \otimes_F K)^{\iota}$, since the homomorphism $F^{\times}/F^{\times,2} \to K^{\times}/K^{\times,2}$ is not injective. Hence, $\mathrm{Spin}(V_{\hat{\eta}})$ is a proper subgroup of $\mathrm{Spin}(V_{\hat{\eta},K})_{\mathbb{Q}}$ of index equal to the cardinality of the kernel of $F^{\times}/F^{\times,2} \to K^{\times}/K^{\times,2}$.

Choose an element g_0 of $\mathrm{Spin}(V_{\hat{\eta},K})$. Set

$$W := \rho_{g_0}(H^1(\hat{X}, \mathbb{Q}) \otimes_F K). \tag{5.1.4}$$

Its pure spinor in $\wedge^*_K[H^1(X,\mathbb{Q}) \otimes_F K] \cong [\wedge^*_F[H^1(X,\mathbb{Q})] \otimes_F K]$ is $m_{g_0}(1)$. Define $\eta : K \to \mathrm{End}(V_{\hat{\eta},K})$, so that $\eta(t)$ acts on W via its K-subspace structure and $\eta(t)$ acts on $\bar{W}$ via scalar multiplication by $\iota(t)$, where ι is the generator of $Gal(K/F)$. The vector space $V_{\hat{\eta}} = H^1(X,\mathbb{Q}) \oplus H^1(X,\mathbb{Q})^*$ is an F-subspace of $V_{\hat{\eta},K}$ of half the dimension. It is equal to[3]

$$V_{\hat{\eta}} = \{w + \bar{w} \ : \ w \in W\}, \tag{5.1.5}$$

provided

$$W \cap \iota(W) = (0), \tag{5.1.6}$$

which we assume. Hence, the subspace $V_{\hat{\eta}}$ is invariant under $\eta(t)$ and we get an embedding

$$\eta : K \to \mathrm{End}_{\mathbb{Q}}\left[H^1(X,\mathbb{Q}) \oplus H^1(X,\mathbb{Q})^*\right]. \tag{5.1.7}$$

Extending coefficient linearly from $\mathbb{Q}$ to $\mathbb{C}$ we get the embedding

$$\eta : K \to \mathrm{End}_{\mathbb{C}}\left[H^1(X,\mathbb{C}) \oplus H^1(\hat{X},\mathbb{C})\right]. \tag{5.1.8}$$

Lemma 5.1.2. *The endomorphism $\eta(\iota(t))$ of $V_{\mathbb{Q}}$ is the adjoint of $\eta(t)$ with respect to both $(\bullet,\bullet)_{V_{\hat{\eta},K}}$ and $(\bullet,\bullet)_{V_{\mathbb{Q}}}$, for all $t \in K$. Consequently, $\eta(t)$ belongs to the image of $\mathrm{Spin}(V_{\hat{\eta},K})_{\mathbb{Q}}$ in $SO(V_{\mathbb{Q}})$, if and only if $t\iota(t) = 1$.*

[3] Here we abuse notation and write $\bar{w}$ for $(id_{V_{\mathbb{Q}}} \otimes \iota)(w)$, where ι is the generator of $Gal(K/F)$.

Proof. It suffices to prove the statement for $\eta : K \to V_{\hat{\eta},K}$ and the K valued bilinear pairing $(\bullet,\bullet)_{V_{\hat{\eta},K}}$, since $(x,y)_{V_{\mathbb{Q}}} = tr_{K/\mathbb{Q}}((x,y)_{V_{\hat{\eta},K}})$, for all $x,y \in V_{\mathbb{Q}}$. Let $x \in W$ and $y \in \bar{W}$. We have

$$\begin{aligned}(\eta(t)x,y)_{V_{\hat{\eta},K}} &= (tx,y)_{V_{\hat{\eta},K}} = t(x,y)_{V_{\hat{\eta},K}} = (x,ty)_{V_{\hat{\eta},K}} = (x,\iota^2(t)y)_{V_{\hat{\eta},K}} \\ &= (x,\eta(\iota(t))y)_{V_{\hat{\eta},K}},\end{aligned}$$

where the first and last equality are by definition of η. The statement follows for all $x,y \in V_{\hat{\eta},K}$, since W and $\bar{W}$ are complementary and isotropic.

If $\eta(t)$ belongs to the image of $\mathrm{Spin}(V_{\hat{\eta},K})$, then it is an isometry and so $\eta(t)\eta(\iota(t)) = id$. Hence, $t\iota(t) = 1$. Conversely, if $t\iota(t) = 1$, then $\eta(t)$ act on the complementary isotropic subspaces W and $\bar{W}$ via scalar multiplication by t and t^{-1} and so belongs to the image of $\mathrm{Spin}(V_{\hat{\eta},K})$. As $\eta(t)$ leaves $V_{\mathbb{Q}}$ invariant, it belongs to the image of $\mathrm{Spin}(V_{\hat{\eta},K})_{\mathbb{Q}}$. □

Remark 5.1.3. Note that there exists an element $t \in K$ satisfying $K = \mathbb{Q}(t)$ and $t\iota(t) = 1$, by [H2, Theorem 3.7]. The statement of [H2, Theorem 3.7] assumes that the CM-field is associated to the transcendental sublattice of a lattice of $K3$-type, but the proof of the existence of such a primitive element is valid for any CM-field.

5.2. **F-invariant 2-forms on $V_{\mathbb{Q}}$ from totally imaginary elements of K.** Let K_- be the -1-eigenspace of $\iota : K \to K$. Given $t \in K_-$, set

$$\begin{aligned}\tilde{\Xi}_t(x,y) &:= (\eta(t)x,y)_{V_{\hat{\eta}}}, \\ \Xi_t(x,y) &:= (\eta(t)x,y)_V.\end{aligned} \tag{5.2.1}$$

We have $\Xi_t = tr_{F/\mathbb{Q}} \circ \tilde{\Xi}_t$.

Corollary 5.2.1. *$\tilde{\Xi}_t$ is anti-symmetric and for $s \in K$ we have*

$$\tilde{\Xi}_t(\eta(s)x,\eta(s)y) = \tilde{\Xi}_t(\hat{\eta}(s\iota(s))x,y).$$

We get the injective homomorphism $\Xi : K_- \to \wedge^2_F V^*_{\mathbb{Q}} \subset \wedge^2_{\mathbb{Q}} V^*_{\mathbb{Q}}$, given by $t \mapsto \Xi_t$.

Proof. We have

$$\tilde{\Xi}_t(x,y) = (\eta(t)x,y)_{V_{\hat{\eta}}} \overset{\text{Lemma 5.1.2}}{=} (x,\eta(\iota(t))y)_{V_{\hat{\eta}}} = -(\eta(t)y,x)_{V_{\hat{\eta}}} = -\tilde{\Xi}_t(y,x).$$

Hence $\tilde{\Xi}_t$ is anti-symmetric. We have $\Xi_t(\eta(f)x,y) = \Xi_t(x,\eta(f)y)$, for all $f \in F$, by Lemma 5.1.2 again. Hence Ξ_t belongs to $\wedge^2_F V^*$. □

Composing the homomorphism in Corollary 5.2.1 with the isomorphism $V \cong V^*$ we get an injective homomorphism from K_- to $\wedge^2_F H^1(X \times \hat{X}, \mathbb{Q})$. Note that the Neron-Severi group of a simple abelian variety with complex multiplication by K has rank $\dim_{\mathbb{Q}}(F) = \dim_{\mathbb{Q}}(K_-)$, by [BL, Prop. 5.5.7].

5.3. **A maximal isotropic subspace W_T of $V_{\mathbb{C}}$ associated to a CM-type T.** We have two isomorphisms $V_{\hat{\sigma},\mathbb{R}} \otimes_F K \to V_{\hat{\sigma},\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$, one via $id_{V_{\hat{\sigma},\mathbb{R}}} \otimes \sigma$, the other via $id_{V_{\hat{\sigma},\mathbb{R}}} \otimes \bar{\sigma}$, where σ is an embedding restricting to F as $\hat{\sigma}$. A CM-type T thus provides an embedding of $V_{\mathbb{R}} \otimes_F K = \oplus_{\{\hat{\sigma}\in\hat{\Sigma}\}} V_{\hat{\sigma},\mathbb{R}} \otimes_F K$ into $V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$. Precomposing with the obvious embedding of $V_{\mathbb{Q}}\otimes_F K$ into $V_{\mathbb{R}}\otimes_F K$ and post composing with the isomorphism $V_{\mathbb{R}}\otimes_{\mathbb{R}}\mathbb{C}\cong V_{\mathbb{Q}}\otimes_{\mathbb{Q}}\mathbb{C}$ we get the embedding

$$e_T : V_{\mathbb{Q}}\otimes_F K \to V_{\mathbb{Q}}\otimes_{\mathbb{Q}}\mathbb{C}. \tag{5.3.1}$$

Set

$$W_{T,\mathbb{C}} := \mathrm{span}_{\mathbb{R}}(e_T(W)). \tag{5.3.2}$$

Note that $W_{T,\mathbb{C}}$ is a complex subspace, as $\mathrm{span}_{\mathbb{R}}(e_T(W))=\mathrm{span}_{\mathbb{C}}(e_T(W))$.

The subspace $V_{\hat{\sigma},\mathbb{R}}\otimes_F K$ of $(V_{\hat{\eta}}\otimes_{\mathbb{Q}}\mathbb{R})\otimes_F K$ decomposes as the direct sum of its intersections $W_{\hat{\sigma}} := [V_{\hat{\sigma},\mathbb{R}}\otimes_F K]\cap[W\otimes_{\mathbb{Q}}\mathbb{R}]$ and $\bar{W}_{\hat{\sigma}} := [V_{\hat{\sigma},\mathbb{R}}\otimes_F K]\cap[\bar{W}\otimes_{\mathbb{Q}}\mathbb{R}]$, since W and $\bar{W}$ are complementary $\hat{\eta}(F)$-subspaces of $V_{\hat{\eta}}\otimes_F K$. Set

$$\begin{aligned} W_{T,\hat{\sigma}} &:= [id_{V_{\hat{\sigma}},\mathbb{R}}\otimes T(\hat{\sigma})](W_{\hat{\sigma}}),\\ \bar{W}_{T,\hat{\sigma}} &:= [id_{V_{\hat{\sigma}},\mathbb{R}}\otimes T(\hat{\sigma})](\bar{W}_{\hat{\sigma}}). \end{aligned} \tag{5.3.3}$$

If $\sigma := T(\hat{\sigma})$, $v\in V_{\hat{\sigma},\mathbb{R}}$, and $t\in K$, then

$$v\otimes_F \sigma(\iota(t)) = v\otimes_F \overline{\sigma(t)} = \overline{v\otimes_F \sigma(t)}.$$

Hence, the subspace $\bar{W}_{T,\hat{\sigma}}$ of $V_{\mathbb{C}}$ is the complex conjugate of $W_{T,\hat{\sigma}}$. We have $W_{T,\mathbb{C}} = \oplus_{\hat{\sigma}\in\hat{\Sigma}}W_{T,\hat{\sigma}}$ and $\bar{W}_{T,\mathbb{C}} = \oplus_{\hat{\sigma}\in\hat{\Sigma}}\bar{W}_{T,\hat{\sigma}} = \oplus_{\hat{\sigma}\in\hat{\Sigma}}\overline{W_{T,\hat{\sigma}}}$. By the definition of the complex multiplication $\eta : K\to \mathrm{End}(V_{\mathbb{C}})$ in (5.1.8), $\eta(t)$ acts on the complex subspace $W_{T,\hat{\sigma}}$ of $V_{\mathbb{C}}$ via multiplication by $\sigma(t):=T(\hat{\sigma})(t)$ and on $\bar{W}_{T,\hat{\sigma}}$ via multiplication by $\bar{\sigma}(t)=\overline{\sigma(t)}$. If $T(\hat{\sigma})=\sigma$, then under the decomposition $V_{\mathbb{C}}=\oplus_{\sigma\in\Sigma}V_\sigma$ associated to η, we have $V_\sigma = W_{T,\hat{\sigma}}$, and $V_{\bar{\sigma}}=\bar{W}_{T,\hat{\sigma}}$. So $W_T=\oplus_{\hat{\sigma}\in\hat{\Sigma}}V_{T(\hat{\sigma})}$.

6. The Galois group action on the set $\mathcal{T}_K$ of CM-types

Let $\sigma_0 : K\to\mathbb{C}$ be an embedding. Let $\tilde{K}$ be the Galois closure of $\sigma_0(K)$ in $\mathbb{C}$. The subfield $\tilde{K}$ of $\mathbb{C}$ is independent of σ_0. The set of embedding of K in $\tilde{K}$ are in bijection to the set Σ of its embeddings in $\mathbb{C}$, as any embedding of K in $\mathbb{C}$ factors through $\tilde{K}$. Define $\tilde{F}\subset\mathbb{R}$ similarly. Let $\mathcal{T}'$ be the set of unital homomorphisms of F-algebras

$$T : K\to F\otimes_{\mathbb{Q}}\tilde{K}.$$

Above, inclusion of F in $F\otimes_{\mathbb{Q}}\tilde{K}$ is via $f\mapsto f\otimes 1$. We have the decomposition $F\otimes_{\mathbb{Q}}\tilde{F}\cong\oplus_{\hat{\sigma}\in\hat{\Sigma}}\hat{\sigma}$, where we denote by $\hat{\sigma}$ both a field embedding $\hat{\sigma}:F\to\tilde{F}$ and a 1-dimensional vector space over $\tilde{F}$ endowed with an F vector space structure, such that for all $f\in F$ and $v\in\hat{\sigma}$ the equality $fv=\hat{\sigma}(f)v$ holds. We have the isomorphisms

$$F\otimes_{\mathbb{Q}}\tilde{K}\cong(F\otimes_{\mathbb{Q}}\tilde{F})\otimes_{\tilde{F}}\tilde{K}\cong\oplus_{\hat{\sigma}\in\hat{\Sigma}}\hat{\sigma}\otimes_{\tilde{F}}\tilde{K}.$$

Note that the direct summand $\hat{\sigma}\otimes_{\tilde{F}}\tilde{K}$ is naturally isomorphic to $\tilde{K}$. Composing an element $T:K\to F\otimes_{\mathbb{Q}}\tilde{K}$ of $\mathcal{T}'$ with the projection onto the direct summand $\hat{\sigma}\otimes_{\tilde{F}}\tilde{K}$ we get an embedding $\sigma:K\to\tilde{K}$, which restricts to F as $\hat{\sigma}$. Hence, the set $\mathcal{T}'$ is in natural bijection with the set $\mathcal{T}_K$ of CM-types.

The group $Gal(\tilde{K}/\mathbb{Q})$ acts on $F\otimes_{\mathbb{Q}}\tilde{K}$ via its action on the right tensor factor. Hence $Gal(\tilde{K}/\mathbb{Q})$ acts on $\mathcal{T}'$ via $g(T):=(id_F\otimes g)\circ T$, for all $g\in Gal(\tilde{K}/\mathbb{Q})$ and $T\in\mathcal{T}'$. Let us describe the corresponding action on $\mathcal{T}_K$. Let $r:\Sigma\to\hat{\Sigma}$ send $\sigma:K\to\tilde{K}$ to its restriction to F. Then $\mathcal{T}_K = \{f : \hat{\Sigma} \to \Sigma \ : \ r \circ f = id_{\hat{\Sigma}}\}$. Given $g \in Gal(\tilde{K}/\mathbb{Q})$, define $m_g : \Sigma \to \Sigma$ by $m_g(\sigma) := g \circ \sigma$. Denote by $\hat{g} \in Gal(\tilde{F}/\mathbb{Q})$ the restriction of g to $\tilde{F}$. Define $m_{\hat{g}} : \hat{\Sigma} \to \hat{\Sigma}$ by $m_{\hat{g}}(\hat{\sigma}) = \hat{g} \circ \hat{\sigma}$. Then $Gal(\tilde{K}/\mathbb{Q})$ acts on $\mathcal{T}_K$ sending $f \in \mathcal{T}_K$ to

$$g(f) := m_g \circ f \circ m_{\hat{g}^{-1}}.$$

Given $T \in \mathcal{T}'$, we get the embedding

$$e_T : V_{\mathbb{Q}} \otimes_F K \stackrel{id_V \otimes T}{\to} V_{\mathbb{Q}} \otimes_F (F \otimes_{\mathbb{Q}} \tilde{K}) \cong V_{\mathbb{Q}} \otimes_{\mathbb{Q}} \tilde{K}. \tag{6.0.1}$$

The embedding in (5.3.1) naturally factors through e_T above, hence the use of the same notation. Given an d-dimensional K-subspace W of $V_{\mathbb{Q}} \otimes_F K$, let

$$W_T$$

be the subspace of $V_{\mathbb{Q}} \otimes_{\mathbb{Q}} \tilde{K}$ spanned over $\tilde{K}$ by $e_T(W)$. The set

$$\{(id_V \otimes g)(W_T) \ : \ g \in Gal(\tilde{K}/\mathbb{Q})\}$$

determines a reduced subscheme of $Gr(2n, V_{\mathbb{Q}} \otimes_{\mathbb{Q}} \tilde{K})$ defined over $\mathbb{Q}$. The set

$$\{W_T \ : \ T \in \mathcal{T}'\}$$

is the union of $Gal(\tilde{K}/\mathbb{Q})$-orbits, hence it corresponds to a reduced subscheme of $Gr(2n, V_{\mathbb{Q}} \otimes_{\mathbb{Q}} \tilde{K})$ defined over $\mathbb{Q}$. Assume that W_T is an even maximal isotropic subspace of $V_{\mathbb{Q}} \otimes_{\mathbb{Q}} \tilde{K}$, for all $T \in \mathcal{T}'$. Let $\ell_T \in \mathbb{P}(S^+_{\tilde{K}})$ be the pure spinor of W_T. The isomorphism between the spinorial variety in $\mathbb{P}(S^+_{\tilde{K}})$ and the even component of the grassmannian of maximal isotropic subspaces of $V_{\mathbb{Q}} \otimes_{\mathbb{Q}} \tilde{K}$ is defined over $\mathbb{Q}$. Hence, the set $\{\ell_T \ : \ T \in \mathcal{T}'\}$ corresponds to a reduced subscheme defined over $\mathbb{Q}$ and its span in $S^+_{\tilde{K}}$ is of the form $B \otimes_{\mathbb{Q}} \tilde{K}$, for some subspace B of $S^+_{\mathbb{Q}}$.

Remark 6.0.1. The embedding $e_T : V_{\hat{\eta}} \otimes_F K \to V_{\mathbb{Q}} \otimes_{\mathbb{Q}} \tilde{K}$ extends to an injective algebra homomorphism $\tilde{e}_T : C(V_{\hat{\eta}} \otimes_F K) \to C(V_{\mathbb{Q}} \otimes_{\mathbb{Q}} \tilde{K})$ and yields an analogue of Diagram (4.1.2) with $\mathrm{Spin}(V_{\hat{\eta}} \otimes_F K)$ replacing $\mathrm{Spin}(V_{\hat{\eta}})$ and $\mathrm{Spin}(V_{\mathbb{Q}} \otimes_{\mathbb{Q}} \tilde{K})$ replacing $\mathrm{Spin}(V_{\mathbb{R}})$. Note, however, that the algebras embedding $\tilde{e}_T$ depends on the choice of a CM-type T.

7. A linear space B secant to the spinorial variety

Let $W \subset V \otimes_F K$ be the maximal isotropic subspace (5.1.4). We have two K-actions on $V_{\mathbb{Q}} \otimes_F K$, one via η on the left factor $V_{\mathbb{Q}}$, and the obvious one on the right factor. Each of W and $\bar{W}$ is invariant with respect to both. The two actions coincide on W and are ι-conjugate on $\bar{W}$. The 2-dimensional vector K-subspace $\wedge^d_K W \oplus \wedge^d_K \bar{W}$ of $\wedge^d_K(V_{\mathbb{Q}} \otimes_F K) \cong (\wedge^d_F V_{\mathbb{Q}}) \otimes_F K$ is ι-invariant and is hence of the form $HW \otimes_F K$ for a 2-dimensional F-subspace HW of $\wedge^d_F V_{\mathbb{Q}} \subset H^d(X \times \hat{X}, \mathbb{Q})$. Now $HW \otimes_F K$ is also a subspace of $\wedge^d_{\eta(K)}[V_{\mathbb{Q}} \otimes_F K]$, and so HW is a 1-dimensional $\eta(K)$-subspace of $\wedge^d_{\eta(K)} V_{\mathbb{Q}}$. The latter is one-dimensional over K and so we get the equality

$$HW = \wedge^d_{\eta(K)} H^1(X \times \hat{X}, \mathbb{Q}). \tag{7.0.1}$$

Given an embedding $\sigma \in \Sigma$ of K, denote by $V_{\sigma,\mathbb{C}}$ the subspace of $V_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C}$ on which the η action of K acts via the character $\sigma : K \to \mathbb{C}$. We have the equality

$$HW \otimes_{\mathbb{Q}} \mathbb{C} \ = \ \oplus_{\sigma\in\Sigma} \wedge^d_{\mathbb{C}} V_{\sigma,\mathbb{C}}. \tag{7.0.2}$$

Our goal in this section is to show that HW is spanned by Hodge classes, if B is spanned by Hodge classes (Lemma 7.4.1). In Section 7.1 we show that W_T is a maximal isotropic subspace of $V_{\mathbb{C}}$, for each CM-type T. In Section 7.2 we introduce the groups $\mathrm{Spin}(V_{\mathbb{Q}})_\eta$ and $\mathrm{Spin}(V_{\mathbb{Q}})_{\eta,B}$, the latter being the special Mumford-Tate group for a generic deformation of $(X \times \hat{X}, \eta)$. We relate that characters ℓ_T and $\wedge^d V_\sigma$ of $\mathrm{Spin}(V_{\mathbb{Q}})_\eta$ and use it to conclude that the lines $\{\ell_T \ : T \in \mathcal{T}_K\}$ are linearly independent. In Section 7.3 we construct a $\mathrm{Spin}(V_{\mathbb{Q}})_\eta$-invariant hermitian form. In Section 7.4 we prove Lemma 7.4.1.

7.1. **Pure spinors.**

Let

$$V_\sigma \tag{7.1.1}$$

be the direct summand of $V_{\mathbb{Q}} \otimes_{\mathbb{Q}} \tilde{K}$ on which $\eta(K)$ acts via the character σ, where the algebraic closure $\tilde{K}$ of K in $\mathbb{C}$ is introduced in section 6. If σ belongs to a CM-type T, then

$$V_{\sigma,\mathbb{C}} = W_{T,\hat{\sigma}} \ \text{ and } \ V_{\bar{\sigma},\mathbb{C}} = \bar{W}_{T,\hat{\sigma}},$$

where $W_{T,\hat{\sigma}}$ is given in (5.3.3). We have the direct sum decomposition $V_{\mathbb{C}} = \oplus_{\sigma\in\Sigma} V_{\sigma,\mathbb{C}}$ and

$$W_{T,\mathbb{C}} = \oplus_{\{\hat{\sigma}\in\hat{\Sigma}\}} V_{T(\hat{\sigma}),\mathbb{C}}. \tag{7.1.2}$$

Remark 7.1.1. If K is a Galois extension of $\mathbb{Q}$, then $Gal(K/\mathbb{Q})$ acts transitively on the set of characters via $\sigma \mapsto \sigma \circ g^{-1}$, for all $g \in Gal(K/\mathbb{Q})$. In that case the subfield $\sigma(K) \subset \mathbb{C}$ is independent of σ and we may identify K with this subfield by choosing an embedding σ_0. The subspace V_σ is then the subspace of $V_{\mathbb{Q}} \otimes_{\mathbb{Q}} \sigma_0(K)$, where $\eta(a)$ acts via $id_{V_{\mathbb{Q}}} \otimes \sigma(a)$, for all $a \in K$. Similarly, the subspaces $V_{\hat{\sigma}}$ are the analogous subspaces of $V_{\mathbb{Q}} \otimes_{\mathbb{Q}} \hat{\sigma}_0(F)$.

For each CM-type T consider the subspace $W_{T,\mathbb{C}}$ of $V_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C}$.

Lemma 7.1.2. *The subspace $W_{T,\mathbb{C}}$ of $V_{\mathbb{C}}$ is maximal isotropic, for every CM-type T.*

Proof. If $\hat{\sigma}_1 \neq \hat{\sigma}_2$, then $V_{\sigma_1,\mathbb{C}}$ and $V_{\sigma_2,\mathbb{C}}$ are orthogonal, since $V_{\hat{\sigma}_1}$ and $V_{\hat{\sigma}_2}$ are, by Lemma 3.1.1. Ditto for $V_{\bar{\sigma}_1,\mathbb{C}}$ and $V_{\sigma_2,\mathbb{C}}$ and for $V_{\bar{\sigma}_1,\mathbb{C}}$ and $V_{\bar{\sigma}_2,\mathbb{C}}$. The subspace $V_{\sigma,\mathbb{C}}$ is isotropic, since $W_{T,\mathbb{C}}$ is, and $V_{\bar{\sigma},\mathbb{C}}$ is isotropic, since $\bar{W}_{T,\mathbb{C}}$ is. Hence, $W_{T,\mathbb{C}}$ is a maximal isotropic subspace of $V_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C}$ for every CM-type T. □

Denote by $\mathcal{T}_K$ the set of $2^{e/2}$ complex multiplication types. The element ι of $Gal(K/\mathbb{Q})$ induces an involution of $\mathcal{T}_K$. Let $\ell_T \subset H^{ev}(X, \mathbb{C})$ be the one-dimensional subspace spanned by a pure spinor corresponding to the subspace $W_{T,\mathbb{C}}$, $T \in \mathcal{T}_K$.

Lemma 7.1.3. *The linear subspace of $H^{ev}(X, \mathbb{C})$ spanned by $\{\ell_T \ : \mathcal{T} \in \mathcal{T}_K\}$, is of the form $B \otimes_{\mathbb{Q}} \mathbb{C}$, where B is a rational subspace*

$$B \subset H^{ev}(X, \mathbb{Q}).$$

Proof. The statement follows from Lemma 7.1.2 and the discussion in Section 6. □

Let $\ell_\sigma \in S^+_{\hat{\sigma}} \otimes_{\mathbb{R}} \mathbb{C}$ be the pure spinor of the maximal isotropic subspace V_σ of $V_{\hat{\sigma}}$. Under the tensor decomposition (3.2.4) of pure spinors of F-invariant maximal isotropic subspaces of $V_{\mathbb{C}}$, we have

$$\ell_T = \otimes_{\hat{\sigma}\in\hat{\Sigma}} \ell_{T(\hat{\sigma})}, \tag{7.1.3}$$

by the equality (7.1.2). Denote by $P_{\hat{\sigma}} \subset S^+_{\hat{\sigma}}$ the secant plane spanned by ℓ_σ and $\ell_{\bar{\sigma}}$. Then

$$B_{\mathbb{R}} = \otimes_{\hat{\sigma}\in\hat{\Sigma}} P_{\hat{\sigma}}. \tag{7.1.4}$$

7.2. **An isotypic decomposition of the** $\mathrm{Spin}(V_{\mathbb{Q}})_\eta$**-invariant space** $B \otimes B$**.** Let

$$\mathrm{Spin}(V_{\mathbb{Q}})_\eta \tag{7.2.1}$$

be the subgroup of the group $\mathrm{Spin}(V_{\mathbb{Q}})_{\hat{\eta}}$, given in (4.2.1), leaving invariant the subspace V_σ, for all $\sigma \in \Sigma$. Let $\mathrm{Spin}(V)_\eta$ be the analogous subgroup of $\mathrm{Spin}(V)_{\hat{\eta}}$. We will see that $\mathrm{Spin}(V)_\eta/\{\pm 1\}$ is an arithmetic subgroup of a product of $e/2$ copies of the unitary group $U(d,d)$ (see Equation (7.2.6) and Lemma 7.3.1 below).

Set $V_{\tilde{K}} := V_{\mathbb{Q}} \otimes_{\mathbb{Q}} \tilde{K}$. Let $\mathrm{Spin}(V_{\tilde{K}})_{\{g_{\hat{\sigma}}\}}$ be the subgroup of $\mathrm{Spin}(V_{\tilde{K}})$ consisting of elements g commuting with $g_{\hat{\sigma}}$, given in (4.2.2), for all $\hat{\sigma} \in \hat{\Sigma}$. Let $\mathrm{Spin}(V_{\tilde{K}})_{\hat{\eta}}$ be the commutator subgroup of $\mathrm{Spin}(V_{\tilde{K}})_{\{g_{\hat{\sigma}}\}}$. We have the short exact sequence

$$0 \to \{\pm 1\} \to \mathrm{Spin}(V_{\tilde{K}})_{\hat{\eta}} \to \prod_{\hat{\sigma}\in\hat{\Sigma}} SO_+(V_{\hat{\sigma}}) \to 0.$$

The group $\mathrm{Spin}(V_{\tilde{K}})_{\hat{\eta}}$ is equal to its commutator subgroup, since that property holds for each $\mathrm{Spin}(V_{\hat{\sigma}} \otimes_{\tilde{F}} \tilde{K})$ and the images of the latter via the natural inclusion $C(V_{\hat{\sigma}} \otimes_{\tilde{F}} \tilde{K}) \subset C(V_{\tilde{K}})$ generate $\mathrm{Spin}(V_{\tilde{K}})_{\hat{\eta}}$.

Let

$$\mathrm{Spin}(V_{\tilde{K}})_\eta \tag{7.2.2}$$

be the subgroup of $\mathrm{Spin}(V_{\tilde{K}})_{\hat{\eta}}$ consisting of elements g such that V_σ is ρ_g-invariant, for all $\sigma \in \Sigma$. Note that $\mathrm{Spin}(V_{\mathbb{Q}})_\eta$ is contained in the $Gal(\tilde{K}/\mathbb{Q})$-invariant subgroup of $\mathrm{Spin}(V_{\tilde{K}})_\eta$, where the $Gal(\tilde{K}/\mathbb{Q})$-action is induced by that on $C(V_{\tilde{K}}) \cong C(V_{\mathbb{Q}}) \otimes_{\mathbb{Q}} \tilde{K}$ via the action on the second tensor factor.

Let

$$\mathrm{Spin}(V)_{\eta,B} \tag{7.2.3}$$

be the subgroup of $\mathrm{Spin}(V)_\eta$ fixing every point of the subspace B of $H^{ev}(X,\mathbb{Q})$. Denote by $\mathrm{Spin}(V_{\tilde{K}})_{\eta,B}$ the analogous subgroup of $\mathrm{Spin}(V_{\tilde{K}})_\eta$. When $F = \mathbb{Q}$, the group $\mathrm{Spin}(V_{\tilde{K}})_\eta$ specializes to the group $\mathrm{Spin}(V_K)_{\ell_1,\ell_2}$ in [M2, Eq. (2.2.1)] and the group $\mathrm{Spin}(V)_{\eta,B}$ specializes to $\mathrm{Spin}(V)_P$, given in [M2, Eq. (2.2.2)].

The group $\mathrm{Spin}(V_{\tilde{K}})_\eta$ leaves invariant each of the subspaces W_T, $T \in \mathcal{T}_K$, by Equation (7.1.2). Hence, each of the lines ℓ_T in $H^{ev}(X,\tilde{K})$ is $\mathrm{Spin}(V_{\tilde{K}})_\eta$-invariant. Thus, the commutator subgroup of $\mathrm{Spin}(V_{\tilde{K}})_\eta$ is contained in $\mathrm{Spin}(V_{\tilde{K}})_{\eta,B}$.

$$\begin{tikzcd}
\mathrm{Spin}(V)_{\hat{\eta}} \arrow[r, "\subset"] & \mathrm{Spin}(V_{\mathbb{Q}})_{\hat{\eta}} & \mathrm{Spin}(V_{\hat{\eta}}) \arrow[l, "(4.2.3)"'] \arrow[r, "\subset"] & \mathrm{Spin}(V_{\hat{\eta},K})_{\mathbb{Q}} \arrow[r, "\subset"] & \mathrm{Spin}(V_{\hat{\eta}} \otimes_F K) \\
\mathrm{Spin}(V)_{\eta} \arrow[u, "\cup"] \arrow[r, "\subset"] & \mathrm{Spin}(V_{\mathbb{Q}})_{\eta} \arrow[u, "\cup"] \arrow[r, "\subset"] & \mathrm{Spin}(V_{\tilde{K}})_{\eta} & & \\
\mathrm{Spin}(V)_{\eta,B} \arrow[u, "\cup"] \arrow[r, "\subset"'] & \mathrm{Spin}(V_{\mathbb{Q}})_{\eta,B} \arrow[u, "\cup"] \arrow[r, "\subset"'] & \mathrm{Spin}(V_{\tilde{K}})_{\eta,B}. \arrow[u, "\cup"] & &
\end{tikzcd}$$

Note that $W_T \cap W_{T'} = V_\sigma$, if σ is the only common value of T and T'. Hence, $\mathrm{Spin}(V)_{\eta,B}$ is equal to the subgroup of $\mathrm{Spin}(V)_{\hat{\eta}}$ fixing every point of B.

For each $\sigma \in \Sigma$ let $\mathrm{Spin}(V_\sigma \oplus V_{\bar{\sigma}})_\eta$ be the subgroup of $\mathrm{Spin}(V_\sigma \oplus V_{\bar{\sigma}})$ leaving each of V_σ and $V_{\bar{\sigma}}$ invariant. We have a natural homomorphism

$$\prod_{\hat{\sigma}\in\hat{\Sigma}} \mathrm{Spin}(V_\sigma \oplus V_{\bar{\sigma}})_{\eta,P_{\hat{\sigma}}} \to \mathrm{Spin}(V_{\tilde{K}})_{\eta,B} \tag{7.2.4}$$

induced by the isomorphism

$$C(V_{\tilde{K}}) \cong \otimes_{\hat{\sigma}\in\hat{\Sigma}} C(V_\sigma \oplus V_{\bar{\sigma}}),$$

where the tensor product is in the category of $\mathbb{Z}_2$-graded algebras, as in (4.1.1). Above σ restricts to F as $\hat{\sigma}$. Now, $\mathrm{Spin}(V_\sigma \oplus V_{\bar{\sigma}})_{\eta,P_{\hat{\sigma}}}$ is isomorphic to the group $SL(V_\sigma)$ of linear transformations of V_σ, as a d-dimensional vector space over $\tilde{K}$, of determinant 1. An elements g of $SL(V_\sigma)$ acts on $V_{\bar{\sigma}}$ by $(g^*)^{-1}$, under the identification of $V_{\bar{\sigma}}$ with V_σ^* via the bilinear pairing $(\bullet,\bullet)_V$. The proof is identical to that of [M2, Lemma 2.2.2].

Similarly, we have the homomorphism

$$\prod_{\hat{\sigma}\in\hat{\Sigma}} \mathrm{Spin}(V_\sigma \oplus V_{\bar{\sigma}})_{\eta} \to \mathrm{Spin}(V_{\tilde{K}})_{\eta},$$

and $\mathrm{Spin}(V_\sigma \oplus V_{\bar{\sigma}})_\eta/\{\pm 1\}$ is isomorphic to $GL(V_\sigma)$, by the proof of [I, Lemma 1]. An elements g of $GL(V_\sigma)$ acts on $V_{\bar{\sigma}}$ by $(g^*)^{-1}$, under the identification of $V_{\bar{\sigma}}$ with V_σ^* via the bilinear pairing $(\bullet,\bullet)_V$.

$$\begin{tikzcd}
\prod_{\hat{\sigma}\in\hat{\Sigma}} C(V_\sigma \oplus V_{\bar{\sigma}})^{even,\times} \arrow[r] & \left[\otimes_{\hat{\sigma}\in\hat{\Sigma}}(V_\sigma \oplus V_{\bar{\sigma}})\right]^{even,\times} \arrow[r, "\cong"] & C(V_{\tilde{K}})^{even,\times} \\
\prod_{\hat{\sigma}\in\hat{\Sigma}} \mathrm{Spin}(V_\sigma \oplus V_{\bar{\sigma}})_\eta \arrow[u, "\cup"] \arrow[rr] & & \mathrm{Spin}(V_{\tilde{K}})_\eta. \arrow[u, "\cup"]
\end{tikzcd} \tag{7.2.5}$$

The bottom horizontal homomorphism in (7.2.5) induces the isomorphism

$$\prod_{\hat{\sigma}\in\hat{\Sigma}} [\mathrm{Spin}(V_\sigma \oplus V_{\bar{\sigma}})_\eta/\{\pm 1\}] \to \mathrm{Spin}(V_{\tilde{K}})_\eta/\{\pm 1\},$$

by an argument similar to the proof of Lemma 4.2.1. We get the isomorphism

$$\mathrm{Spin}(V_{\tilde{K}})_\eta/\{\pm 1\} \cong \prod_{\hat{\sigma}\in\hat{\Sigma}} GL(V_\sigma), \tag{7.2.6}$$

where each σ is chosen to restrict to F as $\hat{\sigma}$.

Given $\sigma_0 \in \Sigma$, let the character $\det_{\sigma_0}$ be the composition

$$\mathrm{Spin}(V_{\tilde{K}})_\eta \to \mathrm{Spin}(V_{\tilde{K}})_\eta/\{\pm 1\} \stackrel{\cong}{\to} \prod_{\hat{\sigma}\in\hat{\Sigma}} GL(V_\sigma) \to GL(V_{\sigma_0}) \stackrel{\det}{\to} \tilde{K}^\times, \tag{7.2.7}$$

where the third arrow is the projection onto the factor associated to σ_0. The character ℓ_T of $\mathrm{Spin}(V_{\tilde{K}})_\eta$ satisfies

$$\ell_T \otimes \ell_T \cong \bigotimes_{\hat{\sigma}\in\hat{\Sigma}} \det{}_{T(\hat{\sigma})}$$

More generally, we have

Lemma 7.2.1. $\ell_{T_1} \otimes \ell_{T_2} \cong \bigotimes_{\{\hat{\sigma} \ : \ T_1(\hat{\sigma})=T_2(\hat{\sigma})\}} det_{T_1(\hat{\sigma})}$.

Proof. The isomorphism $\ell_\sigma \otimes \ell_\sigma \cong \det_\sigma$ of the $\mathrm{Spin}(V_\sigma \oplus V_{\bar{\sigma}})_\eta/\pm 1 \cong GL(V_\sigma)$ characters follows from [Ch, III.3.2]. Hence, $\ell_\sigma \otimes \ell_{\bar{\sigma}}$ is the trivial characxter, as its square is $\det_{\bar{\sigma}} \otimes \det_\sigma$, which is the trivial character. The statement now follows from the factorization (7.1.3). ☐

Corollary 7.2.2. *The lines ℓ_T, $T \in \mathcal{T}_K$, in $S^+_{\tilde{K}}$ are linearly independent over $\tilde{K}$.*

Proof. The group $\mathrm{Spin}(V_{\tilde{K}})_\eta$ leaves $B\otimes_{\mathbb{Q}}\tilde{K}$ invariant and acts on the lines $\{\ell_T \ : \ T \in \mathcal{T}_K\}$ as characters. These characters are distinct, by Lemma 7.2.1. ☐

The image m_g, of $g \in \mathrm{Spin}(V)_{\eta,B}$ via the even half spin representation, leaves invariant each pure spinor ℓ_T. Given an element g of $\mathrm{Spin}(V)_{\eta,B}$, its image ρ_g via the vector representation leaves each of W_T, $T \in \mathcal{T}_K$, invariant and acts on each with trivial determinant. Furthermore, ρ_g commutes with $\hat{\eta}(F)$ and thus leaves invariant each of the subspaces $V_{\hat{\sigma},\mathbb{R}}$ of $V_{\mathbb{R}}$, for all $\hat{\sigma} \in \hat{\Sigma}$. Hence, ρ_g leaves invariant each of the subspaces $V_{\sigma,\mathbb{C}}$, for all $\sigma \in \Sigma$. Denote by $\rho_{g,\sigma}$ the restriction of ρ_g to $V_{\sigma,\mathbb{C}}$. Note that $\det_\sigma(g) = \det(\rho_{g,\sigma})$.

Lemma 7.2.3. *For $g \in \mathrm{Spin}(V_{\mathbb{Q}})_{\eta,B}$ we have $\det(\rho_{g,\sigma}) = 1$, for all $\sigma \in \Sigma$.*

Proof. Fix $\sigma_0 \in \Sigma$ restricting to F as $\hat{\sigma}_0 \in \hat{\Sigma}$. Choose CM-types T_1 and T_2, such that $T_1(\hat{\sigma}_0) = T_2(\hat{\sigma}_0)$ and $T_1(\hat{\sigma}) \neq T_2(\hat{\sigma})$, for $\hat{\sigma} \neq \hat{\sigma}_0$. Then $\ell_{T_1} \otimes \ell_{T_2}$ is the character $\det_{\sigma_0}$, by Lemma 7.2.1. Thus, $\det_{\sigma_0}(g) = 1$, for $g \in \mathrm{Spin}(V_{\mathbb{Q}})_{\eta,B}$, since m_g fixes every point in B and so $m_g \otimes m_g^\dagger$ fixes every point in $B \otimes B$. ☐

Lemma 7.2.4. *The composition* $\mathrm{Spin}(V_{\tilde{K}})_{\eta,B} \to \mathrm{Spin}(V_{\tilde{K}})_\eta/\{\pm 1\} \stackrel{(7.2.6)}{\to} \prod_{\hat{\sigma}\in\hat{\Sigma}} GL(V_\sigma)$ *maps* $\mathrm{Spin}(V_{\tilde{K}})_{\eta,B}$ *isomorphically onto* $\prod_{\hat{\sigma}\in\hat{\Sigma}} SL(V_\sigma)$*, where each σ is chosen to restrict to F as $\hat{\sigma}$.*

Proof. The composition is injective, since (7.2.6) is an isomorphism and -1 does not belong to $\mathrm{Spin}(V_{\tilde{K}})_{\eta,B}$. The image of the composition is contained in $\prod_{\hat{\sigma}\in\hat{\Sigma}} SL(V_\sigma)$, by Lemma 7.2.3. The image of the composition is the whole of $\prod_{\hat{\sigma}\in\hat{\Sigma}} SL(V_\sigma)$, since the subgroup of $\mathrm{Spin}(V_{\tilde{K}})_{\eta,B}$, generated by the images of $\mathrm{Spin}(V_\sigma \oplus V_{\bar{\sigma}})_{\eta,P_{\hat{\sigma}}}$, surjects onto $\prod_{\hat{\sigma}\in\hat{\Sigma}} SL(V_\sigma)$. ☐

Corollary 7.2.5. *The subgroup* $\mathrm{Spin}(V_{\tilde{K}})_B$ *fixing all points of* B *is equal to the subgroup* $\mathrm{Spin}(V_{\tilde{K}})_{\eta,B}$.

Proof. The homomorphism $\prod_{\hat{\sigma}\in\hat{\Sigma}} \mathrm{Spin}(V_\sigma \oplus V_{\bar{\sigma}})_{\eta,P_{\hat{\sigma}}} \to \mathrm{Spin}(V_{\tilde{K}})_{\eta,B}$, given in (7.2.4), is an isomorphism, by Lemma 7.2.4. The subgroup $\mathrm{Spin}(V_\sigma \oplus V_{\bar{\sigma}})_{\eta,P_{\hat{\sigma}}}$ is equal to the subgroup $\mathrm{Spin}(V_\sigma \oplus V_{\bar{\sigma}})_{P_{\hat{\sigma}}}$ fixing every point of $P_{\hat{\sigma}}$, by [M2, Lemma 2.2.2]. Finally, $\mathrm{Spin}(V_{\tilde{K}})_B$ is generated by the images of $\mathrm{Spin}(V_\sigma \oplus V_{\bar{\sigma}})_{P_{\hat{\sigma}}}$, $\hat{\sigma} \in \hat{\Sigma}$. □

7.3. $\mathrm{Spin}(V)_{\eta,B}$**-invariant hermitian forms.** Let t be a non-zero element of the (-1)-eigenspace K_- of $\iota : K \to K$.

Let $H_t : V_{\mathbb{Q}} \times V_{\mathbb{Q}} \to K$ be given by

$$H_t(x,y) := (-t^2)(x,y)_{V_{\hat{\eta}}} + t(\eta(t)x,y)_{V_{\hat{\eta}}}. \tag{7.3.1}$$

Lemma 7.3.1. *The form* H_t *is hermitian, i.e., it satisfies* $H_t(x,y) = \iota(H_t(y,x))$ *and* $H_t(x,\eta(\lambda)y) = \lambda H_t(x,y)$. *Furthermore,* H_t *is* $\mathrm{Spin}(V_{\mathbb{Q}})_\eta$*-invariant.*

Proof. The equality $H_t(x,y) = \iota(H_t(y,x))$ follows from Lemma 5.1.2. The form H_t is F-bilinear. Hence, the equality $H_t(x,\eta(\lambda)y) = \lambda H_t(x,y)$, for all $\lambda \in K$, would follow from the case $\lambda = t$. We have

$$\begin{aligned}
H_t(x,\eta(t)y) &= -t^2(x,\eta(t)y)_{V_{\hat{\eta}}} + t(\eta(t)x,\eta(t)y)_{V_{\hat{\eta}}} \\
&\overset{\text{Lemma 5.1.2}}{=} -t^2(x,\eta(t)y)_{V_{\hat{\eta}}} + t(\hat{\eta}(-t^2)x,y)_{V_{\hat{\eta}}} \\
&= -t^2(x,\eta(t)y)_{V_{\hat{\eta}}} + -t^3(x,y)_{V_{\hat{\eta}}} \\
&\overset{\text{Lemma 5.1.2}}{=} t\left[-t^2(x,y)_{V_{\hat{\eta}}} + t(\eta(t)x,y)_{V_{\hat{\eta}}}\right] = tH_t(x,y).
\end{aligned}$$

The pairing $(\bullet,\bullet)_{V_{\hat{\eta}}}$ is $\mathrm{Spin}(V_{\hat{\eta}})$-invariant. It follows that it is also $\mathrm{Spin}(V_{\mathbb{Q}})_{\hat{\eta}}$-invariant, by Lemma 4.2.1, which implies that the two groups have the same image in $GL(V_{\mathbb{Q}})$. Now $\mathrm{Spin}(V_{\mathbb{Q}})_\eta$ is a subgroup of $\mathrm{Spin}(V_{\mathbb{Q}})_{\hat{\eta}}$, by definition. The form H_t is $\mathrm{Spin}(V_{\mathbb{Q}})_\eta$-invariant, since $(\bullet,\bullet)_{V_{\hat{\eta}}}$ is and $\eta(t)\rho_g(x) = \rho_g(\eta(t)x)$, by definition of $\mathrm{Spin}(V_{\mathbb{Q}})_\eta$. □

Let $H : V_\eta \times V_\eta \to K$ be a non-degenerate hermitian form. Choose an orthogonal basis $\mathcal{B} = \{v_1,\dots,v_d\}$ of V_η as a vector space over K and let $[H]_{\mathcal{B}}$ be the hermitian matrix of H in the basis $\mathcal{B}$. Given an embedding $\hat{\sigma} : F \to \mathbb{R}$, let $a_{\hat{\sigma}}$ be the number of basis elements v_i, such that $\hat{\sigma}(H(v_i,v_i)) > 0$ and let $b_{\hat{\sigma}}$ be the number of basis elements v_i, such that $\hat{\sigma}(H(v_i,v_i)) < 0$. The non-degeneracy of H implies that $a_{\hat{\sigma}} + b_{\hat{\sigma}} = d$.

Definition 7.3.2. ([DM, Sec. 4]) The *discriminant* $\mathrm{disc}(H)$ is the coset of $\det([H]_{\mathcal{B}})$ in $F^\times/Nm_{K/F}(K^\times)$ and $\mathrm{disc}(H)$ is independent of the choice of $\mathcal{B}$.

Lemma 7.3.3. (1) *(*[Lan]*) The signatures* $(a_{\hat{\sigma}}, d-a_{\hat{\sigma}})$, $\hat{\sigma} \in \hat{\Sigma}$, *and the discriminant* $\mathrm{disc}(H_t)$ *determine the isomorphism class of* (V_η, H). *Given integers* $0 \le a_{\hat{\sigma}} \le d$ *and an element* $\delta \in F^\times/Nm_{K/F}(K^\times)$, *there exists a* K*-valued hermitian form* H *on* V_η *with these invariants.*

(2) *(*[DM, Cor. 4.2]*) The following are equivalent:*
- $a_{\hat{\sigma}} = b_{\hat{\sigma}}$, *for all* $\hat{\sigma} \in \hat{\Sigma}$, *and* $\mathrm{disc}(H) = (-1)^{d/2}$;
- *there exists an isotropic subspace of* (V_η, H) *of dimension* $d/2$.

Definition 7.3.4. The pair (V_η, H) is said to be of *split type*, if it satisfies the conditions of Lemma 7.3.3 (2).

7.4. **If B is spanned by Hodge classes then HW is as well.** In this section we show that if the secant space B is spanned by Hodge classes, then condition (1.1.1) is satisfied for the K-action η on $H^1(X\times\hat{X},\mathbb{Q})$, and so the subspace HW of $H^d(X\times\hat{X},\mathbb{Q})$ consists of Hodge classes.

Lemma 7.4.1. *Assume that B is spanned by Hodge classes. Then the complex structure $I_{X\times\hat{X}}: H^1(X\times\hat{X},\mathbb{R})\rightarrow H^1(X\times\hat{X},\mathbb{R})$ of $X\times\hat{X}$ commutes with the action of $\eta(K)$. Furthermore,*

$$\dim(V^{0,1}_{\sigma,\mathbb{C}})=\dim(V^{1,0}_{\sigma,\mathbb{C}}), \tag{7.4.1}$$

for all $\sigma\in\Sigma$, and consequently HW is spanned by Hodge classes.

Proof. Set $I:=I_{X\times\hat{X}}$. Let $T\in\mathcal{T}_K$ be a CM-type. We get the two complex conjugate lines ℓ_T and $\ell_{\iota(T)}$ in B, each spanned by a Hodge class. The proof of [M2, Lemma 2.2.6] establishes that each of $W_{T,\mathbb{C}}$ and $\bar{W}_{T,\mathbb{C}}=W_{\iota(T),\mathbb{C}}$ is I invariant and the dimensions of the I-eigenspaces in each are equal,

$$\begin{aligned}\dim(W^{1,0}_{T,\mathbb{C}}) &= \dim(W^{0,1}_{T,\mathbb{C}})=2n\\ \dim(\bar{W}^{1,0}_{T,\mathbb{C}}) &= \dim(\bar{W}^{0,1}_{T,\mathbb{C}})=2n.\end{aligned} \tag{7.4.2}$$

The endomorphisms in $\eta(F)$ commute with I, by assumption. Hence, the subspace $V_{\hat{\sigma},\mathbb{R}}:=H^1(X\times\hat{X},\mathbb{R})_{\hat{\sigma}}$ is I-invariant, for all $\hat{\sigma}$ in $\hat{\Sigma}$. We get the equality

$$\dim(V^{1,0}_{\hat{\sigma},\mathbb{C}})=\dim(V^{0,1}_{\hat{\sigma},\mathbb{C}})=d, \tag{7.4.3}$$

for all $\hat{\sigma}$ in $\hat{\Sigma}$.

The following equalities hold for all $\sigma\in T$

$$\begin{aligned}V_{\sigma,\mathbb{C}} &= [V_{\hat{\sigma},\mathbb{R}}\otimes_{\mathbb{R}}\mathbb{C}]\cap W_{T,\mathbb{C}},\\ V_{\bar{\sigma},\mathbb{C}} &= [V_{\hat{\sigma},\mathbb{R}}\otimes_{\mathbb{R}}\mathbb{C}]\cap \bar{W}_{T,\mathbb{C}},\end{aligned}$$

by construction. Hence, the $\eta(K)$ eigenspaces $V_{\sigma,\mathbb{C}}$ are I-invariant, for all $\sigma\in\Sigma$.

We know the equality

$$\sum_{\sigma\in T}\dim(V^{0,1}_{\sigma,\mathbb{C}})=\sum_{\sigma\in T}\dim(V^{1,0}_{\sigma,\mathbb{C}})=2n, \tag{7.4.4}$$

by equality (7.4.2). We prove Equation (7.4.1) by contradiction. Assume that $\dim(V^{1,0}_{\sigma_0,\mathbb{C}})>\dim(V^{0,1}_{\sigma_0,\mathbb{C}})$, for some $\sigma_0\in\Sigma$. Then $\dim(V^{1,0}_{\bar{\sigma}_0,\mathbb{C}})<\dim(V^{0,1}_{\bar{\sigma}_0,\mathbb{C}})$, and so $\dim(V^{1,0}_{\sigma_0,\mathbb{C}})>d/2>\dim(V^{1,0}_{\bar{\sigma}_0,\mathbb{C}})$, by (7.4.3). Thus, changing the value of a CM-type T at σ_0 changes the sum in (7.4.4). This contradicts the fact that the sum is $2n$, for all CM-types T. Equation (7.4.1) thus hold for all $\sigma\in\Sigma$. The equalities (7.4.1) hold for all $\sigma\in\Sigma$, if and only if HW consists of rational Hodge classes, by [DM, Prop. 4.4]. $\square$

8. The $\mathrm{Spin}(V)_{\eta,B}$-invariant subalgebra of $H^*(X\times \hat{X},\mathbb{Q})$

Let W be the maximal isotropic subspace (5.1.4). The associated complex multiplication $\eta: K\rightarrow \mathrm{End}_{\mathbb{Q}}[H^1(X,\mathbb{Q})\oplus H^1(\hat{X},\mathbb{Q})]=\mathrm{End}_{\mathbb{Q}}(V_{\mathbb{Q}})$ is given in (5.1.7). Let $B\subset H^{ev}(X,\mathbb{Q})$ be the associated secant linear subspace in Lemma 7.1.3. Let $\mathcal{A}:=(\wedge^*V_{\mathbb{Q}})^{\mathrm{Spin}(V)_{\eta,B}}$ be the subalgebra of $\wedge^*V_{\mathbb{Q}}$ of $\mathrm{Spin}(V)_{\eta,B}$-invariant classes with respect to the natural extension of the representation $\rho:\mathrm{Spin}(V_{\mathbb{Q}})\rightarrow SO_+(V_{\mathbb{Q}})$ to the exterior algebra.

By $\wedge^*V_{\tilde{F}}$ we will refer to $\wedge^*_{\tilde{F}}V_{\tilde{F}}$, where $V_{\tilde{F}}=V_{\mathbb{Q}}\otimes_{\mathbb{Q}}\tilde{F}$.

Proposition 8.0.1. (1) *The subspace $(\wedge^2V_{\mathbb{Q}})^{\mathrm{Spin}(V)_{\eta,B}}$ is equal to the image of the injective homomorphism $\Xi: K_-\rightarrow\wedge^2V_{\mathbb{Q}}$ of Corollary (5.2.1). This subspace is thus $(e/2)$-dimensional over $\mathbb{Q}$ and is naturally a 1-dimensional vector space over F, and it consists of $\mathrm{Spin}(V)_\eta$-invariant classes.*

(2) *The subalgebra $\mathcal{A}$ is commutative and is generated by the $(e/2)$-dimensional $\mathbb{Q}$-vector space $(\wedge^2V_{\mathbb{Q}})^{\mathrm{Spin}(V)_{\eta,B}}$ and the e-dimensional $\mathbb{Q}$-vector space $HW=\wedge^d_{\eta(K)}V_{\mathbb{Q}}$.*

(3) *The subspace $HW\otimes_{\mathbb{Q}}\tilde{K}$ decomposes as a direct sum of the characters $\det_\sigma$, $\sigma\in\Sigma$, of $\mathrm{Spin}(V_{\tilde{K}})_\eta$, each with multiplicity 1.*

(4) *The subalgebra $\mathcal{R}:=(\wedge^*V_{\mathbb{Q}})^{\mathrm{Spin}(V)_\eta}$ is generated by $(\wedge^2V_{\mathbb{Q}})^{\mathrm{Spin}(V)_{\eta,B}}$.*

Proof. (2) Step 1: We prove first that the subalgebra $(\wedge^*V_{\tilde{F}})^{\mathrm{Spin}(V)_{\eta,B}}$ is commutative and is generated by the $(e/2)$-dimensional $\tilde{F}$-vector space $(\wedge^2V_{\mathbb{Q}})^{\mathrm{Spin}(V)_{\eta,B}}\otimes_{\mathbb{Q}}\tilde{F}$ and the e-dimensional $\tilde{F}$-vector space $HW\otimes_{\mathbb{Q}}\tilde{F}$.

We have the isomorphism
$$\wedge^*V_{\tilde{F}}\cong\bigotimes_{\hat{\sigma}\in\hat{\Sigma}}\wedge^*V_{\hat{\sigma}}.$$
The $\mathrm{Spin}(V)_{\eta,B}$-invariant subalgebra is equal to the $\mathrm{Spin}(V_{\tilde{F}})_{\eta,B}$-invariant subalgebra. The factorization (7.2.4) is defined over $\tilde{F}$ and yields an analogous factorization of $\mathrm{Spin}(V_{\tilde{F}})_{\eta,B}$. Using the latter factorization we get the isomorphism
$$(\wedge^*V_{\tilde{F}})^{\mathrm{Spin}(V)_{\eta,B}}\cong\bigotimes_{\hat{\sigma}\in\hat{\Sigma}}(\wedge^*V_{\hat{\sigma}})^{\mathrm{Spin}(V_{\hat{\sigma}})_{P_{\hat{\sigma}}}}.\tag{8.0.1}$$
The argument of [M2, Lemma 2.2.7] applies, replacing $\mathbb{Q}$ by $\tilde{F}$ and K by $\tilde{K}$, and yields that
$$\dim_{\tilde{F}}\left((\wedge^kV_{\hat{\sigma}})^{\mathrm{Spin}(V_{\hat{\sigma}})_{P_{\hat{\sigma}}}}\right)=\left\{\begin{array}{lll}0 & \text{if} & k \text{ is odd}\\ 1 & \text{if} & k \text{ is even and } k\neq d\\ 3 & \text{if} & k=d.\end{array}\right.$$
Furthermore, $(\wedge^dV_{\hat{\sigma}})^{\mathrm{Spin}(V_{\hat{\sigma}})_{P_{\hat{\sigma}}}}$ decomposes as a direct sum of three one-dimensional $\mathrm{Spin}(V_{\hat{\sigma}}\otimes_{\tilde{F}}\tilde{K})_\eta$-invariant subspaces corresponding to the trivial character and the characters $\det_\sigma$ and $\det_{\bar{\sigma}}$, where σ restricts to $\hat{\sigma}$. The direct sum of the latter two is of the form $HW_{\hat{\sigma}}\otimes_{\tilde{F}}\tilde{K}$ of a 2-dimensional $\tilde{F}$ subspace $HW_{\hat{\sigma}}$. Then $HW\otimes_{\mathbb{Q}}\tilde{K}=\oplus_{\hat{\sigma}\in\hat{\Sigma}}HW_{\hat{\sigma}}$. Choose an element $\Xi_{\hat{\sigma}}$ spanning $(\wedge^2V_{\hat{\sigma}})^{\mathrm{Spin}(V_{\hat{\sigma}})_{P_{\hat{\sigma}}}}$. Then $\Xi_{\hat{\sigma}}^{d/2}$ spans that trivial character in $(\wedge^dV_{\hat{\sigma}})^{\mathrm{Spin}(V_{\hat{\sigma}})_{P_{\hat{\sigma}}}}$ and the symmetric square of the subspace $HW_{\hat{\sigma}}$ is mapped onto

$\wedge^{2d}V_{\hat{\sigma}}$, which is spanned by $\Xi^d_{\hat{\sigma}}$. The tensor factors on the righthand side of (8.0.1) are all even, hence they commute with respect to wedge-product. We conclude that $\mathcal{A}_{\tilde{F}} := (\wedge^* V_{\tilde{F}})^{\mathrm{Spin}(V)_{\eta,B}}$ is an commutative graded subalgebra of $\wedge^* V_{\tilde{F}}$ and it is generated by $HW \otimes_{\mathbb{Q}} \tilde{F}$ and its graded summand $\mathcal{A}^2_{\tilde{F}}$ in degree 2.

We conclude that $(\wedge^2 V_{\tilde{F}})^{\mathrm{Spin}(V)_{\eta,B}}$ is an $(e/2)$-dimensional $\tilde{F}$-vector space with basis $\{\Xi_{\hat{\sigma}} \ : \ \hat{\sigma} \in \hat{\Sigma}\}$. Taking $Gal(\tilde{F}/\mathbb{Q})$ invariants we get that $(\wedge^2 V_{\mathbb{Q}})^{\mathrm{Spin}(V)_{\eta,B}}$ is an $(e/2)$-dimensional $\mathbb{Q}$-vector space. Furthermore,

$$(\wedge^2 V_{\mathbb{Q}})^{\mathrm{Spin}(V)_{\eta,B}} = \{\Xi_t \ : \ t \in K_-\},$$

where Ξ_t is given in (5.2.1).

Step 2: Choose a basis $\{\beta_1, \dots, \beta_{e/2}\}$ of $\mathcal{A}^2_{\mathbb{Q}} := (\wedge^2 V_{\mathbb{Q}})^{\mathrm{Spin}(V)_{\eta,B}}$ and a basis $\{\delta_1, \dots, \delta_e\}$ of HW. Then every element of $(\wedge^* V_{\tilde{F}})^{\mathrm{Spin}(V)_{\eta,B}}$ is the sum $\gamma := \sum_i c_i f_i$, where $c_i \in \tilde{F}$ and f_i is a polynomial in the basis elements β_j's and δ_k's. These two bases consist of rational classes. We may further assume that the f_i's are linearly independent over $\mathbb{Q}$. Then they are also linearly independent over $\tilde{F}$, since the homomorphism $\wedge^*_{\mathbb{Q}} V_{\mathbb{Q}} \to (\wedge^*_{\mathbb{Q}} V_{\mathbb{Q}}) \otimes_{\mathbb{Q}} \tilde{F}$, given by $f \mapsto f \otimes 1$, sends linearly independent subsets over $\mathbb{Q}$ to linearly independent subsets over $\tilde{F}$. Hence, if γ is $Gal(\tilde{F}/\mathbb{Q})$-invariant, then so is each of the coefficients c_i. Consequently, the algebra $(\wedge^* V_{\mathbb{Q}})^{\mathrm{Spin}(V)_{\eta,B}}$ is generated by $\mathcal{A}^2_{\mathbb{Q}}$ and HW.

(1) Let us verify the $\mathrm{Spin}(V)_\eta$-invariance of $(\wedge^2 V_{\mathbb{Q}})^{\mathrm{Spin}(V)_{\eta,B}}$. For $g \in \mathrm{Spin}(V)_\eta$ and $x, y \in V_{\mathbb{Q}}$ we have

$$\Xi_t(\rho_g(x), \rho_g(y)) := (\eta(t)\rho_g(x), \rho_g(y))_V = (\rho_g(\eta(t)x), \rho_g(y))_V = (\eta(t)x, y)_V = \Xi_t(x, y).$$

(3) Follows from the direct sum decomposition (7.0.2).

(4) Assume that $x \in \wedge^*_{\mathbb{Q}} V_{\mathbb{Q}}$ is $\mathrm{Spin}(V)_\eta$-invariant. Then each of its factors $x_{\hat{\sigma}}$ in $(\wedge^* V_{\hat{\sigma}})^{\mathrm{Spin}(V)_{\hat{\sigma}}}$, with respect to the factorization (8.0.1), is $\mathrm{Spin}(V)_\eta$-invariant, as different factors contribute linearly independent characters over $\mathbb{Z}$. Hence, $x_{\hat{\sigma}}$ belongs to the subalgebra generated by the one-dimensional subspace $\Xi_{\hat{\sigma}}$ of $\wedge^2 V_{\hat{\sigma}}$. Thus, x is a rational class of the subalgebra generated by $(\wedge^2 V_{\tilde{F}})^{\mathrm{Spin}(V)_{\eta,B}}$. Arguing as in Step 2 of the proof of part (2) we conclude that x belongs to $\mathcal{R}$. □

Let $\Sigma' \subset \Sigma$ be a subset, such that $\Sigma' \cap \iota(\Sigma') = \emptyset$. Set $\det_{\Sigma'} := \otimes_{\{\sigma \in \Sigma'\}} \det_\sigma$. Let k be the cardinality of Σ'. Denote by $(\wedge^{dj} V_{\tilde{K}})_{\Sigma'}$ the isotypic subspace of $\wedge^{dj} V_{\tilde{K}}$ corresponding to the character $\det_{\Sigma'}$ of $\mathrm{Spin}(V_{\tilde{K}})_\eta$. Note that $(\wedge^{dj} V_{\tilde{K}})_{\Sigma'}$ vanishes, if $j < k$ or $j > e-k$, and it is one-dimensional if $j = k$ or if $j = e-k$. Let $(\wedge^{dj} V_{\tilde{K}})_k$ be the direct sum of all $(\wedge^{dj} V_{\tilde{K}})_{\Sigma'}$, for subsets $\Sigma' \subset \Sigma$ as above of cardinality k. The dimension of $(\wedge^{dk} V_{\tilde{K}})_k$ is $\binom{e/2}{k} 2^k$. Note that $(\wedge^{dk} V_{\tilde{K}})_k$ is contained in the subalgebra generated by HW and $(\wedge^d V_{\tilde{K}})_d = HW \otimes_{\mathbb{Q}} \tilde{K}$.

Lemma 8.0.2. *If $d > 2$, then HW is contained in the primitive cohomology with respect to every* $\mathrm{Spin}(V)_{\eta,B}$*-invariant ample class. More generally, the same is true for* $(\wedge^{dk} V_{\tilde{K}})_k$.

Proof. Let h be a class in $(\wedge^2 V_{\mathbb{Q}})^{\mathrm{Spin}(V)_{\eta,B}}$. Then h is $\mathrm{Spin}(V_{\mathbb{Q}})_\eta$-invariant, by Proposition 8.0.1(1). Thus $h^{1+d(e-2k)/2} (\wedge^{dk} V_{\tilde{K}})_k$ is contained in $(\wedge^{d(e-k)+2} V_{\tilde{K}})_k$, which vanishes. □

9. An adjoint orbit in $\mathrm{Spin}(V_{\mathbb{R}})_B$ as a period domain of abelian varieties of Weil type

9.1. **The group** $\mathrm{Spin}(V_{\mathbb{R}})_B$**.** Let $\mathrm{Spin}(V_{\mathbb{R}})_B$ be the subgroup of $\mathrm{Spin}(V_{\mathbb{R}})$ consisting of elements g, such that $m(g)$ leaves every point of B (and hence also of $B_{\mathbb{R}} := B \otimes_{\mathbb{Q}} \mathbb{R}$) invariant. Choose a non-zero element t of K_-. Given $\sigma \in \Sigma$ restricting to F as $\hat{\sigma}$, let

$$SO_+(V_{\hat{\sigma},\mathbb{R}})_{\eta_t} \tag{9.1.1}$$

be the subgroup of $SO_+(V_{\hat{\sigma},\mathbb{R}})$ of elements g, which commute with the restriction of η_t to $V_{\hat{\sigma},\mathbb{R}}$ and which restriction to each of $V_{\sigma,\mathbb{C}}$ and $V_{\bar{\sigma},\mathbb{C}}$ has determinant 1.

Lemma 9.1.1. *The representation* $\rho : \mathrm{Spin}(V_{\mathbb{R}}) \to SO_+(V_{\mathbb{R}})$ *maps* $\mathrm{Spin}(V_{\mathbb{R}})_B$ *isomorphically onto* $\prod_{\hat{\sigma}\in\hat{\Sigma}} SO_+(V_{\hat{\sigma},\mathbb{R}})_{\eta_t}$.

Proof. The representation ρ maps $\mathrm{Spin}(V_{\mathbb{R}})_B$ injectively into $SO_+(V_{\mathbb{R}})$, since $-1 \in \mathrm{Spin}(V_{\mathbb{R}})$ does not belong to $\mathrm{Spin}(V_{\mathbb{R}})_B$. We prove first the inclusion of $\rho(\mathrm{Spin}(V_{\mathbb{R}})_B) \subset \prod_{\hat{\sigma}\in\hat{\Sigma}} SO_+(V_{\hat{\sigma},\mathbb{R}})_{\eta_t}$. We have the inclusion

$$\rho(\mathrm{Spin}(V_{\mathbb{R}})_B) \subset \{g \in SO_+(V_{\mathbb{R}}) \ : \ g(V_\sigma) = V_\sigma \text{ and } \det(g_{|V_\sigma}) = 1, \forall\ \sigma \in \Sigma\}. \tag{9.1.2}$$

Indeed, fix $\sigma_0 \in \Sigma$ restricting to $\hat{\sigma}_0 \in \hat{\Sigma}$ and let T_1 and T_2 be two CM-types, such that $T_1(\hat{\sigma}_0) = T_2(\hat{\sigma}_0) = \sigma_0$ and $T_1(\hat{\sigma}) \neq T_2(\hat{\sigma})$, for all $\hat{\sigma} \neq \hat{\sigma}_0$. Then $g \in \rho(\mathrm{Spin}(V_{\mathbb{R}})_B)$ maps ℓ_{T_i} to itself, for $i = 1, 2$, hence it maps W_{T_i} to itself, for $i = 1, 2$, and thus it maps $V_\sigma = W_{T_1} \cap W_{T_2}$ to itself. The equality $\det(g_{|V_\sigma}) = 1$ is proved by the same argument proving Lemma 7.2.3. We conclude that $\rho(\mathrm{Spin}(V_{\mathbb{R}})_B)$ commutes with $\eta(K)$. The right hand side of (9.1.2) is equal to $\prod_{\hat{\sigma}\in\hat{\Sigma}} SO_+(V_{\hat{\sigma},\mathbb{R}})_{\eta_t}$.

Conversely, let $g = \prod_{\hat{\sigma}\in\hat{\Sigma}} g_{\hat{\sigma}}$ be an element of $\prod_{\hat{\sigma}\in\hat{\Sigma}} SO_+(V_{\hat{\sigma},\mathbb{R}})_{\eta_t}$. Then each $g_{\hat{\sigma}}$ lifts to an element of $\mathrm{Spin}(V_{\hat{\sigma},\mathbb{R}})_{P_{\hat{\sigma}}}$, by [M2, Lemma 3.1.1]. Hence, g lifts to $\prod_{\hat{\sigma}\in\hat{\Sigma}} \mathrm{Spin}(V_{\hat{\sigma},\mathbb{R}})_{P_{\hat{\sigma}}}$, which maps to $\mathrm{Spin}(V_{\mathbb{R}})_B$ via the right bottom arrow in (4.1.2). □

Remark 9.1.2. An element of $SO_+(V_{\hat{\sigma},\mathbb{R}})$ leaves invariant the restriction of the $\mathbb{C}$-values hermitian form $\sigma \circ H_t$ to $V_{\hat{\sigma},\mathbb{R}}$ if and only if it commutes with η_t. Hence, $SO_+(V_{\hat{\sigma},\mathbb{R}})_{\eta_t}$ is equal to the subgroup of $SO_+(V_{\hat{\sigma},\mathbb{R}})$ of elements leaving $\sigma \circ H_t$ invariant and whose restrictions to each of $V_{\sigma,\mathbb{C}}$ and $V_{\bar{\sigma},\mathbb{C}}$ have determinant 1. Let $U_+(V_{\hat{\sigma},\mathbb{R}})$ be the subgroup of $SO_+(V_{\hat{\sigma},\mathbb{R}})$ leaving the restriction of $\sigma \circ H_t$ invariant. The determinant of the restriction of elements of $U_+(V_{\hat{\sigma},\mathbb{R}})$ to $V_{\sigma,\mathbb{R}}$ is a non-trivial character of $U_+(V_{\hat{\sigma},\mathbb{R}})$. The subgroup $SO_+(V_{\hat{\sigma},\mathbb{R}})_{\eta_t}$ thus deserves to be denoted $SU(V_{\hat{\sigma},\mathbb{R}})$, since the imaginary parts of $\sigma \circ H_t$ and of $\bar{\sigma} \circ H_t$ defer only by a sign and their restrictions to $V_{\hat{\sigma},\mathbb{R}}$ depend on t only up to a real factor, and so $SO_+(V_{\hat{\sigma},\mathbb{R}})_{\eta_t}$ is independent of σ and of t.

9.2. **A period domain of abelian varieties of Weil type.** Fix a non-zero $t \in K_-$. Set $T_t := \{\sigma \in \Sigma \ : \ Im(\sigma(t)) > 0\}$. T_t is a CM-type and all CM-types arise[4] this way. Given $I \in \rho(\mathrm{Spin}(V_{\mathbb{R}})_B)$, satisfying $I^2 = -id_V$, set

$$g_I(x, y) := \Xi_t(x, I(y)) \stackrel{(5.2.1)}{=} (\eta_t(x), I(y))_V. \tag{9.2.1}$$

[4] Choose $t_0 \in K_- \setminus \{0\}$. It suffices to show that $K_- \ni t \mapsto (\mathrm{sign}(\hat{\sigma}(t/t_0))_{\hat{\sigma}\in\hat{\Sigma}} \in \{\pm 1\}^{e/2}$ is surjective. This follows from the fact that the map $F \to \mathbb{R}^{\hat{\Sigma}}$, given by $x \mapsto (\hat{\sigma}(x))_{\hat{\sigma}\in\hat{\Sigma}}$, contains a full lattice, for example the image of the ring of algebraic integers in F.

The automorphisms I and η_t of $V_\mathbb{R}$ commute and both are anti-self-dual with respect to the symmetric bilinear pairing $(\bullet,\bullet)_V$ on $V_\mathbb{R}$. Hence, g_I is symmetric. Let

$$\Omega_{B,t}\subset\rho(\mathrm{Spin}(V_\mathbb{R})_B) \tag{9.2.2}$$

be the subset of elements I, such that I is a complex structure on $V_\mathbb{R}$, the eigenspaces $V^{1,0}_{\sigma,\mathbb{C}}$ and $V^{1,0}_{\bar{\sigma},\mathbb{C}}$ of the restriction of I to $V_{\sigma,\mathbb{C}}$ are both $d/2$-dimensional, for all $\sigma\in\Sigma$, and the symmetric bilinear form (9.2.1) is positive definite. The subset $\Omega_{B,t}$ depends only on the CM-type T_t associated to $t\in K_-\setminus\{0\}$, but the polarization Ξ_t depends on $t\in K_-$. Denote $\Omega_{B,t}$ by Ω_{B,T_t} as well and set

$$\Omega_B:=\cup_{T\in\mathcal{T}_K}\Omega_{B,T}. \tag{9.2.3}$$

Given two CM-types T_1 and T_2, define $\delta_{T_1,T_2}:V_\mathbb{R}\to V_\mathbb{R}$ as the isometry acting on $V_{\hat{\sigma},\mathbb{R}}$ by multiplication by -1, if $T_1(\hat{\sigma})\neq T_2(\hat{\sigma})$ and by 1 if $T_1(\hat{\sigma})=T_2(\hat{\sigma})$. The isometry δ_{T_1,T_2} belongs to $\mathrm{Spin}(V_\mathbb{R})_B$, by Lemma 9.1.1, since $-id_{V_{\hat{\sigma},\mathbb{R}}}$ belongs to $SO_+(V_{\hat{\sigma},\mathbb{R}})_{\eta_t}$, as d is even. Hence, given $I_1\in\Omega_{B,T_1}$, the complex structure $I_2:=\delta_{T_1,T_2}\circ I_1$ belongs to Ω_{B,T_2}.

Remark 9.2.1. The secant subspace B does not determine the embedding $\eta:K\to\mathrm{End}_{Hdg}(H^1(X\times\hat{X},\mathbb{Q}))$. Given $g\in Gal(K/\mathbb{Q})$, the embedding $\eta\circ g$ has the same secant subspace B. If we incorporate the embedding η in the notation and denote $\Omega_{B,T}$ by $\Omega_{B,\eta,T}$, then $\Omega_{B,\eta\circ g,T}=\Omega_{B,\eta,g^{-1}(T)}$. Consider for example the involution $\iota\in Gal(K/F)\subset Gal(K/\mathbb{Q})$. If $I_{X\times\hat{X}}$ belongs to $\Omega_{B,\eta,T}$, then it belongs also to $\Omega_{B,\eta\circ\iota,\bar{T}}$ and $-I_{X\times\hat{X}}$ belongs to $\Omega_{B,\eta,\bar{T}}$.

Lemma 9.2.2. *The subset $\Omega_{B,t}$ is the image via ρ of an adjoint orbit of* $\mathrm{Spin}(V_\mathbb{R})_B$.

Proof. Step 1: (Reduction to $V_{\hat{\sigma},\mathbb{R}}$). Set $T:=T_t$. On each direct summand $V_{\hat{\sigma},\mathbb{R}}$ we have the equalities $\rho(\mathrm{Spin}(V_{\hat{\sigma},\mathbb{R}})_{P_{\hat{\sigma}}})=SO_+(V_{\hat{\sigma},\mathbb{R}})_{\eta_t}$ and

$$\rho(\mathrm{Spin}(V_\mathbb{R})_B)=\prod_{\hat{\sigma}\in\hat{\Sigma}}\rho(\mathrm{Spin}(V_{\hat{\sigma},\mathbb{R}})_{P_{\hat{\sigma}}}),$$

established in the proof of Lemma 9.1.1. Hence, $\Omega_{B,t}=\prod_{\hat{\sigma}\in\hat{\Sigma}}\Omega_{B,t,\hat{\sigma}}$, where $\Omega_{B,t,\hat{\sigma}}$ is the subset of $\rho(\mathrm{Spin}(V_{\hat{\sigma},\mathbb{R}})_{P_{\hat{\sigma}}})$ consisting of I, such that $I^2=-id$, the eigenspaces of its restrictions to $V^{1,0}_{T(\hat{\sigma}),\mathbb{C}}$ and $V^{1,0}_{\bar{T}(\hat{\sigma}),\mathbb{C}}$ are $d/2$-dimensional, and $(\eta_t(\bullet),I(\bullet))_{V_{\hat{\sigma}}}$ is positive definite. It suffices to show that $\Omega_{B,t,\hat{\sigma}}$ is an adjoint orbit in $\rho(\mathrm{Spin}(V_{\hat{\sigma},\mathbb{R}})_{P_{\hat{\sigma}}})$.

Step 2: Let I be an element in $\rho(\mathrm{Spin}(V_{\hat{\sigma},\mathbb{R}})_{P_{\hat{\sigma}}})$ with $I^2=-id$, such that the eigenspaces $V^{1,0}_{T(\hat{\sigma}),\mathbb{C}}$ and $V^{1,0}_{\bar{T}(\hat{\sigma}),\mathbb{C}}$ of its restrictions to $V_{T(\hat{\sigma})}$ are $d/2$-dimensional. We claim that $(\eta_t(\bullet),I(\bullet))_{V_{\hat{\sigma}}}$ is positive definite, if and only if the subspace $U^+_\mathbb{C}:=V^{1,0}_{T(\hat{\sigma})}\oplus V^{0,1}_{\bar{T}(\hat{\sigma})}$ is positive definite with respect to the restriction of $(\bullet,\bullet)_{V_{\hat{\sigma}}}$ and $U^-_\mathbb{C}:=V^{1,0}_{\bar{T}(\hat{\sigma})}\oplus V^{0,1}_{T(\hat{\sigma})}$ is negative definite. Indeed, both subspaces are self conjugate, corresponding to subspaces U^+ and U^- of $V_{\hat{\sigma},\mathbb{R}}$, and given $x,y\in V^{1,0}_{T(\hat{\sigma})}\oplus V^{0,1}_{\bar{T}(\hat{\sigma})}$, we have $\eta_tI(y)=T(\hat{\sigma})(t)iy$ and

$$\Xi_t(x,I(y))=(\eta_t(x),I(y))_V=(x,-\eta_tI(y))_V=-T(\hat{\sigma})(t)i(x,y)=|T(\hat{\sigma})(t)|(x,y).$$

Given $x,y\in V^{1,0}_{\bar{T}(\hat{\sigma})}\oplus V^{0,1}_{T(\hat{\sigma})}$, we have $\eta_tI(y)=-T(\hat{\sigma})(t)iy$ and

$$\Xi_t(x,I(y))=(\eta_t(x),I(y))_V=(x,-\eta_tI(y))_V=T(\hat{\sigma})(t)i(x,y)=-|T(\hat{\sigma})(t)|(x,y).$$

Step 3: Let I_1, I_2 be elements of $\Omega_{B,t,\hat{\sigma}}$. Let $V_{T(\hat{\sigma})}^{(1,0),I_j}$ and $V_{T(\hat{\sigma})}^{(0,1),I_j}$ be the eigenspaces of the restriction of I_j to $V_{T(\hat{\sigma})}$, $j = 1, 2$. Set $U_{\mathbb{C}}^{+,I_j} := V_{T(\hat{\sigma})}^{(1,0),I_j} \oplus V_{\bar{T}(\hat{\sigma})}^{(0,1),I_j}$ and define $U_{\mathbb{C}}^{-,I_j}$ analogously and let U^{+,I_j} and U^{-,I_j} be the corresponding real subspaces of $V_{\hat{\sigma},\mathbb{R}}$, $j = 1, 2$. We claim that there exists an element $g \in \rho(\mathrm{Spin}(V_{\hat{\sigma},\mathbb{R}})_{P_{\hat{\sigma}}})$ satisfying

$$\begin{aligned} g(U^{+,I_1}) &= U^{+,I_2}, \\ g(U^{-,I_1}) &= U^{-,I_2} \end{aligned} \tag{9.2.4}$$

H_t restricts to $V_{\hat{\sigma},\mathbb{R}}$ as a $K \otimes_{F,\hat{\sigma}} \mathbb{R}$-valued hermitian form and its signature is the same whether we identify $K \otimes_{F,\hat{\sigma}} \mathbb{R}$ with $\mathbb{C}$ via $T(\hat{\sigma})$ or via $\bar{T}(\hat{\sigma})$. The four subspaces U^{+,I_j}, U^{-,I_j}, $j = 1, 2$, are $K \otimes_{F,\hat{\sigma}} \mathbb{R}$ subspaces of $V_{\hat{\sigma},\mathbb{R}}$ and there exist isomorphisms of hermitian spaces $g_+ : U^{+,I_1} \to U^{+,I_2}$ and $g_- : U^{-,I_1} \to U^{-,I_2}$, as the signatures of the restriction of H_t are the same, by Step 2, hence we get a unitary automorphism g of $(V_{\hat{\sigma},\mathbb{R}}, H_t)$ satisfying (9.2.4). The embedding $\eta : K \to \mathrm{End}(V_{\hat{\sigma},\mathbb{R}})$ extends to an embedding of $K \otimes_{F,\hat{\sigma}} \mathbb{R}$ and, possibly after composing g with η_s for a suitable $s \in K \otimes_{F,\hat{\sigma}} \mathbb{R}$ of absolute value 1, we may assume that g belongs to $SU(V_{\hat{\sigma},\mathbb{R}}) = \mathrm{Spin}(V_{\hat{\sigma},\mathbb{R}})_{P_{\hat{\sigma}}}$ (see Remark 9.1.2).

The definition of H_t implies that g is an isometry of $V_{\hat{\sigma},\mathbb{R}}$ with respect to $(\bullet, \bullet)_{V_{\hat{\sigma}}}$ and it commutes with the restriction of η_t. By construction, we have $g(\eta_t I_1)g^{-1} = \eta_t I_2$. Combining with $g\eta_t = \eta_t g$ we get that $gI_1g^{-1} = I_2$.

Step 4: We have seen that $\Omega_{B,t,\hat{\sigma}}$ is contained in a single adjoint orbit in $\mathrm{Spin}(V_{\hat{\sigma},\mathbb{R}})_{P_{\hat{\sigma}}}$. It remains to show the reverse inclusion, i.e., that $\Omega_{B,t,\hat{\sigma}}$ is invariant under the conjugation action of $\mathrm{Spin}(V_{\hat{\sigma},\mathbb{R}})_{P_{\hat{\sigma}}}$. The check is straightforward. □

The connected component $\Omega_{B,t}$ may be regarded as a $\mathrm{Spin}(V_{\mathbb{R}})_B$-adjoint orbit, since ρ maps $\mathrm{Spin}(V_{\mathbb{R}})_B$ injectively into $SO_+(V_{\mathbb{R}})$. Let A be the differentiable manifold underlying $X \times \hat{X}$. Given $I \in \Omega_{B,t}$ denote by (A, I) the abelian variety with complex structure I. The period domain $\Omega_{B,t}$ parametrizes polarized abelian varieties of Weil type $((A, I), \eta, \Xi_t)$ with complex multiplication by K. In Section 11 we will consider examples in which the complex structure $I_{X\times\hat{X}}$ belongs to $\Omega_{B,t}$ and $\Omega_{B,t}$ parametrizes abelian varieties of split Weil type, i.e., for which (V_η, H_t) is of split type (see Definition 7.3.4 and Lemma 11.1.2).

Lemma 9.2.3. *Assume that $(X \times \hat{X}, \eta, \Xi_t)$ is of split Weil type. Every $2n$-dimensional polarized abelian variety of split Weil type (A', η', h), $\eta' : K \to \mathrm{End}_{Hdg}(H^1(A', \mathbb{Q}))$, is isomorphic to $((A, I), \eta, \Xi_{t'})$, for some $I \in \Omega_{B,t'}$ and some non-zero $t' \in K_-$.*

Proof. There exists a unique hermitian form $H : H_1(A', \mathbb{Q}) \times H_1(A', \mathbb{Q}) \to K$ and a non-zero element $s \in K_-$, such that the polarization h of (A', η', h) satisfies $h(x, y) = tr_{K/\mathbb{Q}}(sH(x, y))$, for all $x, y \in H_1(A', \mathbb{Q})$, by [DM, Lem. 4.6]. There exists an isomorphism $f : (H_1(A', \mathbb{Q}), H) \to (V_{\mathbb{Q}}, H_t)$, by Lemma 7.3.3, as both hermitian spaces have the same dimension over K and both are of split type. Hence, f conjugates $h(x, y)$ to

$$tr_{K/\mathbb{Q}}(sH_t(x, y)) = tr_{K/\mathbb{Q}}(st(\eta_t(x), y)_{V_{\hat{\eta}}}) = tr_{K/\mathbb{Q}}((\eta_{st^2}(x), y)_{V_{\hat{\eta}}}) = 2\Xi_{st^2}(x, y).$$

Let $I' : H_1(A', \mathbb{R}) \to H_1(A', \mathbb{R})$ be the complex structure of A'. Then the conjugate $I := f \circ I' \circ f^{-1} : V_{\mathbb{R}} \to V_{\mathbb{R}}$ is such that $tr_{K/\mathbb{Q}}((\eta_{st^2}(x), I(y))_{V_{\hat{\eta}}})$ is positive definite. Hence, so is $tr_{K/\mathbb{Q}}((\eta_{-st^2}(x), -I(y))_{V_{\hat{\eta}}})$. Now, the complex structure on $H_1(X \times \hat{X}, \mathbb{Q})$

is related to minus the complex structure of $H^1(X \times \hat{X}, \mathbb{Q})$ via the isomorphism $V_{\mathbb{R}} \ni x \mapsto (x, \bullet)_V \in V^*_{\mathbb{R}}$. So (A', η', h) is isomorphic to $((A, -I), \eta, \Xi_{-st^2})$ and $-I$ belongs to $\Omega_{B,t'}$, where $t' = -st^2$. $\square$

Remark 9.2.4. The statement and proof of Lemma 9.2.3 hold after replacing the assumption that $(X \times \hat{X}, \eta, \Xi_t)$ and (A', η', h) are of split Weil type by the assumption that their hermition forms have the same signatures and discriminant invariants (see Lemma 7.3.3). In particular, $\Omega_{B,t}$ parametrizes a complete family of abelian varieties of Weil type, even if those are not of split type.

The connected component $\Omega_{B,t}$ is isomorphic to an open subset of $\prod_{\hat{\sigma}\in\hat{\Sigma}} Gr(d/2, V_{T_t(\hat{\sigma}),\mathbb{C}})$ in the analytic topology, by Lemma [M2, 4.0.1] applied to each factor separately.

Remark 9.2.5. The period domains of abelian varieties of split Weil type are described in [DM] as symmetric hermitian doimains (see also [CS, Lemma 11.5.25]). The construction in [DM] of the period domain of abelian varieties of split Weil type starts with the polarization Ξ_t (called there the Riemann form) rather than the pairing $(\bullet, \bullet)_{V_{\hat{\eta}}}$. Note that we have the equalities $\Xi_t = tr_{F/\mathbb{Q}} \circ \tilde{\Xi}_t$,

$$\begin{aligned} (x, y)_{V_{\hat{\eta}}} &= \tilde{\Xi}_t(\eta_{t^{-1}}(x), y), \\ H_t(x, y) &= (-t^2)\tilde{\Xi}_t(\eta_{t^{-1}}(x), y) + t\tilde{\Xi}_t(x, y). \end{aligned}$$

Strictly speaking, $\tilde{\Xi}_t$ is an element of $\wedge^2_F \mathrm{Hom}_F(V_{\hat{\eta}}, F)$ which is dual to $\wedge^2_F H^1(X \times \hat{X}, \mathbb{Q})$. We regard $\Xi_t := tr_{F/\mathbb{Q}} \circ \tilde{\Xi}_t$ as a class in $\wedge^2_F H^1(X \times \hat{X}, \mathbb{Q}) \subset H^2(X \times \hat{X}, \mathbb{Q})$ by identifying $V_{\hat{\eta}}$ with its dual via $(\bullet, \bullet)_{V_{\hat{\eta}}}$. The diffeomorphism induced by $(\bullet, \bullet)_V$, of the compact complex tori $X \times \hat{X}$ and its dual $(V_{\mathbb{R}}/V, I)$, is an isomorphism of abelian varieties, for all complex structures I in $\Omega_{B,t}$, since ρ_I is an isometry of $V_{\mathbb{R}}$, so the induced isomorphism $V_{\mathbb{R}} \to V^*_{\mathbb{R}}$ satisfies $I(v) \mapsto (Iv, \bullet) = (v, -I(\bullet)) = I_{X\times\hat{X}}(v, \bullet)$.

10. Hodge-Weil classes, pure spinors, and Orlov's equivalence

In section 10.1 we recall Orlov's derived equivalence $\Phi : D^b(X \times X) \to D^b(X \times \hat{X})$ and its Spin(V)-equivariance properties. Given two coherent sheaves F_i, $i = 1, 2$, on X with $ch(F_i) \in B$ and satisfying rank$(\Phi(F_1 \boxtimes F_2^\vee)) \neq 0$, the class $\kappa(\Phi(F_1 \boxtimes F_2^\vee))$ is Spin$(V)_{\eta,B}$-invariant. Hence, $\kappa_{d/2}(\Phi(F_1 \boxtimes F_2^\vee))$ belongs to $\mathcal{A}^d = \mathcal{R}^d \oplus HW(X \times \hat{X}, \eta)$. We introduce a grading on $B \otimes B$ in section 10.1 and use it in section 10.2 to give a sufficient condition for the class $\kappa_{d/2}(\Phi(F_1 \boxtimes F_2^\vee))$ to project to a nonzero class in the direct summand $HW(X \times \hat{X}, \eta)$ of $\mathcal{A}^d$.

10.1. **A grading of $B \otimes B$.** We have the decomposition $B \otimes_{\mathbb{Q}} \tilde{K} = \oplus_{T\in\mathcal{T}_K} \ell_T$, by Corollary 7.2.2. Given $T, T' \in \mathcal{T}_K$, set $T \cap T' := \{\hat{\sigma} \in \hat{\Sigma} \ : \ T(\hat{\sigma}) = T'(\hat{\sigma})\}$. Let $|T \cap T'|$ denote the cardinality of $T \cap T'$. Then $\dim_{\tilde{K}}(W_T \cap W_{T'}) = \sum_{\hat{\sigma}\in T\cap T'} \dim_{\tilde{K}}(V_{T(\hat{\sigma})}) = d|T \cap T'|$.

The subspace

$$BB_{i,\tilde{K}} := \bigoplus_{\{(T,T')\in\mathcal{T}_K\times\mathcal{T}_K \ : \ |T\cap T'|=i\}} \ell_T \otimes \ell_{T'}$$

of $S^+_{\mathbb{Q}} \otimes_{\mathbb{Q}} S^+_{\mathbb{Q}} \otimes_{\mathbb{Q}} \tilde{K}$ is $Gal(\tilde{K}/\mathbb{Q})$-invariant, hence of the form $BB_i \otimes_{\mathbb{Q}} \tilde{K}$, for a subspace BB_i of $S^+_{\mathbb{Q}} \otimes_{\mathbb{Q}} S^+_{\mathbb{Q}}$. The vector space $B \otimes_{\mathbb{Q}} B$ decomposes as the direct sum

$$B \otimes_{\mathbb{Q}} B = \oplus_{i=0}^{e/2} BB_i.$$

Note that the characters $\ell_{T_1} \otimes \ell_{T_1'}$ and $\ell_{T_2} \otimes \ell_{T_2'}$ of $\mathrm{Spin}(V_{\mathbb{Q}})_\eta$ are isomorphic, if and only if $T_1 \cap T_1' = T_2 \cap T_2'$, by Lemma 7.2.1. Consequently, the characters $\ell_{T_1} \otimes \ell_{T_1'}$ and $\ell_{T_2} \otimes \ell_{T_2'}$ are different, if $|T_1 \cap T_1'| \neq |T_2 \cap T_2'|$.

The subspace HW of $\wedge^d V_{\mathbb{Q}}$ is the intersection $\wedge^d V_{\mathbb{Q}} \cap [\oplus_{\sigma \in \Sigma} \wedge^d V_\sigma]$ in $\wedge^d (V \otimes_{\mathbb{Z}} \tilde{K})$. Denote by $\langle HW \rangle$ the intersection of $\wedge^* V_{\mathbb{Q}}$ with the subalgebra of $\wedge^*(V \otimes_{\mathbb{Z}} \tilde{K})$ generated by $HW \otimes_{\mathbb{Q}} \tilde{K} = \oplus_{\sigma\in\Sigma} \wedge^d V_\sigma$. Then $\langle HW \rangle$ is a graded subalgebra and $\langle HW \rangle = \oplus_{k=0}^{e} \langle HW \rangle^{kd}$, where $\langle HW \rangle^{kd}$ consists of the rational points of the direct sum of all wedge products $\wedge_{i=1}^{k} \wedge^d V_{\sigma_i}$. Let $\tilde{\varphi} : S \otimes_{\mathbb{Z}} S \to \wedge^* V$ be the isomorphism given in (3.2.5) (and over $\mathbb{Z}$ in [M2, Eq. (2.3.2]).

Consider the increasing filtration $F^k(\wedge^* V) := \oplus_{i \leq k} \wedge^i V$ and the decreasing filtration $F_k(\wedge^* V) := \oplus_{i \geq k} \wedge^i V$. Define $F^k(\wedge^* V_{\mathbb{Q}})$ and $F_k(\wedge^* V_{\mathbb{Q}})$ analogously.

Lemma 10.1.1. *If $|T \cap T'| = k$, then the line $\tilde{\varphi}(\ell_T \otimes \ell_{T'})$ belongs to $F^{d(e-k)}(\wedge^* V_{\mathbb{C}})$ and it projects onto the line $\wedge^{d(e-k)}[W_T + W_{T'}]$ in $\wedge^{d(e-k)} V_{\mathbb{C}}$. The image of the composition*

$$BB_1 \xrightarrow{\tilde{\varphi}} F^{d(e-1)}(\wedge^* V_{\mathbb{Q}}) \to \wedge^{d(e-1)} V_{\mathbb{Q}} \tag{10.1.1}$$

is the e-dimensional subspace $\langle HW \rangle^{d(e-1)}$.

Note that $\dim_{\mathbb{Q}}(B) = 2^{\dim_{\mathbb{Q}}(F)} = 2^{e/2}$ and so $\dim_{\mathbb{Q}}(B \otimes_{\mathbb{Q}} B) = 2^e$. The character $\det_\sigma$ of $\mathrm{Spin}(V_{\tilde{K}})_\eta$ appears in $BB_k \otimes_{\mathbb{Q}} \tilde{K}$, if and only if $k = 1$ and its multiplicity in $BB_1 \otimes_{\mathbb{Q}} \tilde{K}$ is $2^{(\frac{e}{2}-1)}$. We see that $\dim_{\mathbb{Q}}(BB_1) = e2^{(\frac{e}{2}-1)}$ and so the composition (10.1.1) is injective, if and only if $e = 2$.

Proof. If $d/2$ is even, then the line $\ell_\sigma \ell_{\bar{\sigma}}$ in the subspace $\mathrm{Sym}^2 P_{\hat{\sigma}}$ of $S^+_{\hat{\sigma}} \otimes S^+_{\hat{\sigma}}$ is mapped via $\tilde{\varphi}_{\hat{\sigma}}$, given in (3.2.6), to a line in $\wedge^* V_{\hat{\sigma}}$ which projects onto the top graded summand $\wedge^{2d} V_{\hat{\sigma}}$, by [M2, Lemma 2.3.2]. In that case the image $\tilde{\varphi}_{\hat{\sigma}}(\ell_\sigma \wedge \ell_{\bar{\sigma}})$, of the line $\ell_\sigma \wedge \ell_{\bar{\sigma}}$ in the subspace $\wedge^2 P_{\hat{\sigma}}$ of $S^+_{\hat{\sigma}} \otimes S^+_{\hat{\sigma}}$, is contained in $F^{2d-2}(\wedge^* V_{\hat{\sigma}})$, by [M2, Lemma 2.3.2].

If $d/2$ is odd, then the line $\ell_\sigma \wedge \ell_{\bar{\sigma}}$ in the subspace $\wedge^2 P_{\hat{\sigma}}$ of $S^+_{\hat{\sigma}} \otimes S^+_{\hat{\sigma}}$ is mapped via $\tilde{\varphi}_{\hat{\sigma}}$ to a line in $\wedge^* V_{\hat{\sigma}}$ which projects onto the top graded summand $\wedge^{2d} V_{\hat{\sigma}}$, by [M2, Lemma 2.3.2]. In that case the line $\tilde{\varphi}_{\hat{\sigma}}(\ell_\sigma \ell_{\bar{\sigma}})$ is contained in $F^{2d-2}(\wedge^* V_{\hat{\sigma}})$.

The line $\ell_\sigma \otimes \ell_\sigma$ is mapped via $\tilde{\varphi}_{\hat{\sigma}}$ to a line in $\wedge^* V_{\hat{\sigma}}$ which belongs to $F^d(\wedge^* V_{\hat{\sigma}})$, but not to $F^{d-1}(\wedge^* V_{\hat{\sigma}})$, and projects onto the line $\wedge^d V_\sigma$ in $\wedge^d V_{\hat{\sigma}}$, by [Ch, III.3.2]. Assume that $T, T' \in \mathcal{T}_K$ satisfy $|T \cap T'| = 1$. Say $T(\hat{\sigma}_0) = T'(\hat{\sigma}_0) = \sigma_0$. Then

$$\ell_T \otimes \ell_{T'} = (\ell_{\sigma_0} \otimes \ell_{\sigma_0}) \otimes \bigotimes_{\{\hat{\sigma} \in \hat{\Sigma} \ : \ \hat{\sigma} \neq \hat{\sigma}_0\}} (\ell_{T(\hat{\sigma})} \otimes \ell_{T'(\hat{\sigma})}),$$

by (7.1.3). The line $\tilde{\varphi}(\ell_T \otimes \ell_{T'})$ thus belongs to $F^{d(e-1)}(\wedge^* V_{\mathbb{C}})$ and projects onto the line[5] $\wedge^{d(e-1)}[W_T + W_{T'}]$ in $\wedge^{d(e-1)} V_{\mathbb{C}}$. The last statement follows also from the proof of [Ch, III.3.3]. We conclude that $\tilde{\varphi}(BB_1) \otimes_{\mathbb{Q}} \mathbb{C}$ is contained in $F^{d(e-1)}(\wedge^* V_{\mathbb{C}})$ and it

[5]It follows that the isomorphism $\phi : S^+_{\mathbb{C}} \otimes S^+_{\mathbb{C}} \to \wedge^* V_{\mathbb{C}}$, given in (10.1.4), maps the line $\ell_T \otimes \ell_{T'}$ into a line in $F_d(\wedge^* V_{\mathbb{C}})$, which projects onto the line $\wedge^d (W_T \cap W_{T'})$ in $HW \otimes_{\mathbb{Q}} \mathbb{C}$.

projects onto $\oplus_{\sigma\in\Sigma}\wedge^{d(e-1)}[(V_\sigma)^\perp]$. In particular, the image of the composition (10.1.1) is e-dimensional.

If $|T\cap T'|=k$, then

$$\ell_T\otimes\ell_{T'}=\bigotimes_{\{\hat{\sigma}\in\hat{\Sigma}\ :\ T(\hat{\sigma})=T'(\hat{\sigma})\}}(\ell_{T(\hat{\sigma})}\otimes\ell_{T(\hat{\sigma})})\otimes\bigotimes_{\{\hat{\sigma}\in\hat{\Sigma}\ :\ T(\hat{\sigma})\neq T'(\hat{\sigma})\}}(\ell_{T(\hat{\sigma})}\otimes\ell_{T'(\hat{\sigma})}),$$

and the above argument applies verbatim to show that $\tilde{\varphi}(\ell_T\otimes\ell_{T'})$ belongs to $F^{d(e-k)}(\wedge^*V_{\mathbb{C}})$ and in projects onto the line $\wedge^{d(e-k)}[W_T+W_{T'}]$ in $\wedge^{d(e-k)}V_{\mathbb{C}}$. □

Remark 10.1.2. Let KB_1 be the kernel of the composition (10.1.1). Let us describe KB_1 explicitly. It suffices to describe $KB_1\otimes_{\mathbb{Q}}\tilde{K}$. Given $\sigma\in\Sigma$, let $(\mathcal{T}_K\times\mathcal{T}_K)_\sigma$ be the set of ordered pairs $(T,T')\in\mathcal{T}_K\times\mathcal{T}_K$, such that $T\cap T'=\{\sigma\}$. The equality $W_T+W_{T'}=\wedge^{d(e-1)}[\oplus_{\{\sigma'\in\Sigma\ :\sigma'\neq\bar{\sigma}\}}V_{\sigma'}]=(V_\sigma)^\perp$ holds, for each pair $(T,T')\in(\mathcal{T}_K\times\mathcal{T}_K)_\sigma$. The line $\ell_T\otimes\ell_{T'}$ is canonically isomorphic to the line $\wedge^{d(e-1)}(V_\sigma^\perp)$, for each pair $(T,T')\in(\mathcal{T}_K\times\mathcal{T}_K)_\sigma$, by Lemma 10.1.1. In particular, the lines $\ell_{T_1}\otimes\ell_{T_1'}$ and $\ell_{T_2}\otimes\ell_{T_2'}$ are canonically isomorphic, for two pairs (T_1,T_1') and (T_2,T_2') in $(\mathcal{T}_K\times\mathcal{T}_K)_\sigma$. Choosing a non-zero vector t_σ in $\wedge^{d(e-1)}[(V_\sigma)^\perp]$, for each $\sigma\in\Sigma$, we get a non-zero vector $\lambda_{(T,T')}$ in $\ell_T\otimes\ell_{T'}$ mapping to t_σ via (10.1.1), for each (T,T') with $|T\cap T'|=1$. The subspace $KB_1\otimes_{\mathbb{Q}}\tilde{K}$ is explicitly given by

(10.1.2)
$$KB_1\otimes_{\mathbb{Q}}\tilde{K}\ =\ \left\{\sum_{\{(T,T')\ :\ |T\cap T'|=1\}}c_{(T,T')}\lambda_{(T,T')}\ :\ \sum_{(T,T')\in(\mathcal{T}_K\times\mathcal{T}_K)_\sigma}c_{(T,T')}=0,\ \ \forall\sigma\in\Sigma\right\},$$

where $c_{(T,T')}\in\tilde{K}$. □

Let $\mathcal{P}$ be the normalized Poincaré line bundle over $\hat{X}\times X$. Let $\mu:X\times X\to X\times X$ be the automorphism $\mu(x,y)=(x+y,y)$. Let $\Phi_{\mathcal{P}}:D^b(\hat{X})\to D^b(X)$ be the equivalence with Fourier-Mukai kernel $\mathcal{P}$. We get the equivalence $id\times\Phi_{\mathcal{P}}:D^b(X\times\hat{X})\to D^b(X\times X)$ with Fourier-Mukai kernel $\pi_{1,3}^*(\mathcal{O}_\Delta)\otimes\pi_{2,4}^*\mathcal{P}$, where $\pi_{i,j}$ is the projection from $X\times\hat{X}\times X\times X$ onto the product of the i-th and j-th factors and Δ is the diagonal in $X\times X$. Orlov's equivalence

$$\Phi:D^b(X\times X)\to D^b(X\times\hat{X}) \tag{10.1.3}$$

is the inverse of $\mu_*\circ(id\times\Phi_{\mathcal{P}}):D^b(X\times\hat{X})\to D^b(X\times X)$. Let

$$\phi:H^*(X\times X,\mathbb{Z})\to H^*(X\times\hat{X},\mathbb{Z}) \tag{10.1.4}$$

be the isomorphism induced by the cohomological action of Φ. Set

$$\tilde{\phi}:=\exp(-c_1(\mathcal{P})/2)\cup\phi\circ(id\otimes\tau):H^*(X\times X,\mathbb{Q})\to H^*(X\times\hat{X},\mathbb{Q}),$$

where

$$\tau:H^*(X,\mathbb{Z})\to H^*(X,\mathbb{Z}) \tag{10.1.5}$$

acts on $H^i(X,\mathbb{Z})$ by multiplication by $(-1)^{i(i-1)/2}$. The linear transformation $\tilde{\phi}$ is an isomorphism of two $\mathrm{Spin}(V_{\mathbb{Q}})$-representations, the tensor square $S_{\mathbb{Q}}\otimes S_{\mathbb{Q}}$ of the spin representation and $\wedge^* V_{\mathbb{Q}}$, by [M2, Prop. 1.3.1]. Note the inclusion $\tilde{\phi}(B \otimes B) \subset \mathcal{A}$, where $\mathcal{A} := (\wedge^* V_{\mathbb{Q}})^{\mathrm{Spin}(V)_{\eta,B}}$.

Lemma 10.1.3. *If $|T \cap T'| = k$, then the line $\phi(\ell_T \otimes \tau(\ell_{T'}))$ belongs to $F_{dk}(\wedge^* V_{\mathbb{C}})$ and it projects onto the line $\wedge^{dk}[W_T \cap W_{T'}]$ in $\wedge^{dk} V_{\mathbb{C}}$. The image of the composition*

$$BB_1 \overset{\phi\circ(id\otimes\tau)}{\rightarrow} F_d(\wedge^* V_{\mathbb{Q}}) \rightarrow \wedge^d V_{\mathbb{Q}} \tag{10.1.6}$$

is HW.

Proof. Let $\phi_{\mathcal{P}} : H^*(X, \mathbb{Z}) \rightarrow H^*(\hat{X}, \mathbb{Z})$ be the isomorphism induced by $ch(\mathcal{P})$. The statement follows from Lemma 10.1.1, by the equality $\phi \circ (id \otimes \tau) = (\phi_{\mathcal{P}} \otimes \phi_{\mathcal{P}}^{-1}) \circ \tilde{\varphi}$ [M2, Lemma 6.3.1] and the fact that $\phi_{\mathcal{P}} \otimes \phi_{\mathcal{P}}^{-1}$ restricts to each graded summand of $\wedge^* V_{\mathbb{Q}}$ as Poincaré duality, up to sign [M2, Lemma 6.3.2]. We claim that Poincaré duality followed by the cohomological action of the automorphism interchanging the factors X and $\hat{X}$ results in an automorphism of $\wedge^* V$ which maps $\langle HW \rangle^{d(e-1)}$ to $\langle HW \rangle^d$. It suffices to show that it maps $\wedge^{d(e-1)}[W_T + W_{T'}]$ to $\wedge^d[W_T \cap W_{T'}]$. Indeed, Poincaré duality maps the top exterior power $\wedge^t_{\tilde{K}} Z$ of a t-dimensional subspace Z of $V_{\tilde{K}}$ to the top exterior power $\wedge^{de-t}_{\tilde{K}}(\mathrm{ann}(Z))$, where $\mathrm{ann}(Z)$ is the annihilator of Z in $V^*_{\tilde{K}} = \mathrm{Hom}(V_{\tilde{K}}, \tilde{K})$ with respect to the evaluation pairing. Interchanging the factors of X and $\hat{X}$ induces on the level of first cohomologies the isomorphism $V \cong V^*$ induced by the pairing $(\bullet, \bullet)_V$, which maps $\mathrm{ann}(Z)$ to the orthogonal complement $Z^\perp$ with respect to $(\bullet, \bullet)_V$. Now note that $W_T \cap W_{T'}$ is the orthogonal complement of $W_T + W_{T'}$ with respect to $(\bullet, \bullet)_V$ and given $\sigma \in \Sigma$ restricting to $\hat{\sigma} \in \hat{\Sigma}$, the subspace V_σ is the orthogonal complement of $V_\sigma \oplus [\oplus_{\{\hat{\sigma}' \in \hat{\Sigma} \,:\, \hat{\sigma}' \neq \hat{\sigma}\}} V_{\hat{\sigma}'}]$. The subspace $\langle HW \rangle^d$ is the direct sum of the top exterior powers of the former subspaces and $\langle HW \rangle^{d(e-1}$ is the direct sum of the top exterior powers of the latter subspaces. □

The character $\det_{\sigma_1} \otimes \cdots \otimes \det_{\sigma_k}$, with $\hat{\sigma}_1, \ldots, \hat{\sigma}_k$ distinct in $\hat{\Sigma}$, appears only in $BB_k \otimes_{\mathbb{Q}} \tilde{K}$ and it appears in $BB_k \otimes_{\mathbb{Q}} \tilde{K}$ with multiplicity $2^{(\frac{e}{2}-k)}$. Denote by

$$BB_{T\cap T'}$$

the subspace of $BB_k \otimes_{\mathbb{Q}} \tilde{K}$, $k = |T \cap T'|$, such that $BB_{T\cap T'}$ is the direct sum of all characters of $\mathrm{Spin}(V_{\tilde{K}})_\eta$ isomorphic to $\otimes_{\{\hat{\sigma} \in T \cap T'\}} \det_{T(\hat{\sigma})}$. There are $2^k \binom{e/2}{k}$ pairwise non-isomorphic such characters, so the dimension of BB_k is $\binom{e/2}{k} 2^{e/2}$.

10.2. **Hodge-Weil classes from pure spinors.** The $\mathbb{Q}$-subalgebra $\langle HW \rangle$ of $\wedge^* V_{\mathbb{Q}}$ in Lemma 10.1.1 is a graded subalgebra and we set $\langle HW \rangle^k := \langle HW \rangle \cap \wedge^k V_{\mathbb{Q}}$. The graded summand $\langle HW \rangle^k$ vanishes, if $d \not| k$, and

$$\dim_{\mathbb{Q}} \langle HW \rangle^{dk} = \binom{e}{k},$$

for $0 \leq k \leq e$. The multiplicity of $\det_\sigma$ in $\langle HW\rangle^{dk} \otimes_{\mathbb{Q}} \tilde{K}$ is 0, if k is even, and it is $\binom{(e/2)-1}{i}$, if $k = 2i+1$, for $0 \leq i \leq \frac{e}{2}-1$. Hence, the multiplicity of $\det_\sigma$ in $\langle HW\rangle \otimes_{\mathbb{Q}} \tilde{K}$ is equal to its multiplicity in $B \otimes_{\mathbb{Q}} B \otimes_{\mathbb{Q}} \tilde{K}$.

Let c be a class in $B \otimes_{\mathbb{Q}} B$. Write $c = \sum_i c_i$, with c_i in BB_i. Set $\check{\phi} := \phi \circ (id \otimes \tau)$. Let $\check{\phi}(c)_i$ be the graded summand of $\check{\phi}(c)$ in $\wedge^{2i} V_{\mathbb{Q}}$. Set $r := \check{\phi}(c)_0$, considered as a rational number via the isomorphism $\wedge^0 V_{\mathbb{Q}} \cong \mathbb{Q}$. Set $\kappa(\check{\phi}(c)) := \check{\phi}(c) \exp\left(-\check{\phi}(c)_1/r\right)$ whenever $r \neq 0$.

Proposition 10.2.1. *Assume that c_1 does not belong to KB_1, $r \neq 0$, and $d > 2$.*

1. *The class $\kappa(\check{\phi}(c))$ is $\mathrm{Spin}(V)_{\eta,B}$-invariant.*
2. *The class $\kappa(\check{\phi}(c_0))$ belongs to $\mathcal{R}$.*
3. *The equality $\check{\phi}(c)_0 = \check{\phi}(c_0)_0$ holds. The difference $\kappa(\check{\phi}(c)) - \kappa(\check{\phi}(c_0))$ belongs to $F_d(\wedge^* V_{\mathbb{Q}})$ and its projection to $\wedge^d V_{\mathbb{Q}}$ is a non-zero element of HW.*

Proof. (1) The class $\kappa(\check{\phi}(c))$ is $\mathrm{Spin}(V)_{\eta,B}$-invariant, by the same argument proving [M2, Corollary 1.3.2].

(2) The class $\check{\phi}(c_k)$ belongs to $F_{dk}(\wedge^* V_{\mathbb{Q}})$, by Lemma 10.1.3, and $d > 2$, by assumption. Hence, $\check{\phi}(c)_0 = \check{\phi}(c_0)_0$ and $\check{\phi}(c)_1 = \check{\phi}(c_0)_1$. We thus have

$$\kappa(\check{\phi}(c)) = \kappa(\check{\phi}(c_0)) + \check{\phi}(c - c_0) \exp\left(-\check{\phi}(c)_1/r\right).$$

The class $\kappa(\check{\phi}(c_0))$ is $\mathrm{Spin}(V)_\eta$-invariant, by the same argument proving [M2, Corollary 1.3.2], and is hence in $\mathcal{R}$, by Proposition 8.0.1(4).

(3) The class $\check{\phi}(c - c_0) \exp\left(-\check{\phi}(c)_1/r\right)$ belongs to $F_d(\wedge^* V_{\mathbb{Q}})$ and its projection to $\wedge^d V_{\mathbb{Q}}$ is equal to that of $\check{\phi}(c - c_0)$, which in turn is equal to the projection of $\check{\phi}(c_1)$. Now, the projection of $\check{\phi}(c_1)$ to $\wedge^d V_{\mathbb{Q}}$ is a non-zero element of HW, by Lemma 10.1.3. $\square$

Remark 10.2.2. There exist classes c of the form $c = \alpha \otimes \beta$, with $\alpha, \beta \in B$, satisfying the assumptions of Proposition 10.2.1. Indeed, any subspace containing classes of the form $\alpha \otimes \beta$, $\alpha, \beta \in B$, is the whole of $B \otimes B$ and so for generic choices of α and β we have that $c_1 \notin KB_1$ and $\check{\phi}(c)_0 \neq 0$. One would like to choose α and β to be Chern classes of objects in $D^b(X)$.

The following Corollary is an immediate consequence of Proposition 10.2.1. It is the reduction of the algebraicity of the Weil classes on abelian varieties with complex multiplication to the variational Hodge conjecture. Let (A, η', h') be a polarized abelian variety of Weil type parametrized by the same connected component of the period domain Ω_B corresponding to $(X \times \hat{X}, \eta, h)$, where $h = \Xi_t$ for some $t \in K_-$, given in (9.2.2).

Corollary 10.2.3. *Assume that the class c is algebraic. If the flat deformation of the algebraic class $\kappa_{d/2}(\check{\phi}(c))$ in $H^d(X \times \hat{X}, \mathbb{Q})$ remains algebraic in $H^d(A, \mathbb{Q})$, then every class in $HW(A, \eta')$ is algebraic.*

11. Examples

Let K be a CM-field and F its maximal totally real subfield. In section 11.1 we construct a K-secant linear subspace $B \subset S^+_{\mathbb{Q}} := H^{ev}(X, \mathbb{Q})$, spanned by Hodge classes, for every simple abelian variety X with an embedding $\hat{\eta} : F \to \mathrm{End}_{\mathbb{Q}}(X)$. Let $\tilde{K}$ be the Galois closure of K over $\mathbb{Q}$ and $\tilde{F}$ that of F. We have the short exact sequence

$$0 \to (\mathbb{Z}/2\mathbb{Z})^{\nu} \to \mathrm{Gal}(\tilde{K}/\mathbb{Q}) \to \mathrm{Gal}(\tilde{F}/\mathbb{Q}) \to 0,$$

where $1 \leq \nu \leq e/2 := [F : \mathbb{Q}]$, by [D, Sec. 1.1]. In section 11.2 we specialize to the case where $\nu = 1$ and the above short exact sequence splits. This is the case when $K = F(\sqrt{-q})$, where q is a positive rational number. In this case we provide an example of a class $c = \alpha_0 \otimes \beta$ in $B \otimes B$, which satisfies the assumptions of Proposition 10.2.1 and is hence mapped to a non-zero class in $HW(X \times \hat{X}, \eta)$ (see Lemma 11.2.9). Here α_0 is a class in the $\mathbb{Q}(\sqrt{-q})$-secant line associated to the pure spinors $\exp(\sqrt{-q}\Theta)$ and its complex conjugate as in [M2, Equation (1.2.4)]. If X is a jacobian and Θ is its principal polarization, a choice of $\alpha_0 = ch(E)$, where E is a coherent sheaf, is discussed in [M2, Example 8.2.3]. The class $\beta \in B$ satisfies an explicit genericity assumption (see Lemma 11.2.9). In case $[F : \mathbb{Q}] = 2$, we provide an example in which the class β is the Chern character of a coherent sheaf (Lemma 11.2.8).

11.1. **An example of a K-secant B spanned by Hodge classes.** The subspace $\wedge^k_F H^1(X, \mathbb{Q})$ consists of elements γ of $\wedge^k_Q H^1(X, \mathbb{Q}) \subset H^1(X, \mathbb{Q})^{\otimes k}$, such that the endomorphism $1 \otimes \cdots \otimes \hat{\eta}_q \otimes \cdots \otimes 1$, with $\hat{\eta}_q$ in the j-th tensor factor, maps γ to the same element $\hat{\eta}_q(\gamma)$, regardless of the choice of j. Wedge products over $\mathbb{Q}$ of such classes need not be such.

Lemma 11.1.1. *If X is simple and $\mathrm{End}_{\mathbb{Q}}(X) = F$, then $H^{1,1}(X, \mathbb{Q})$ is contained in the subspace $\wedge^2_F H^1(X, \mathbb{Q})$ of $H^2(X, \mathbb{Q})$.*

Proof. We have $\dim_{\mathbb{Q}} H^{1,1}(X, \mathbb{Q}) = \dim_{\mathbb{Q}}(F)$, by [BL, Prop. 5.5.7]. In this case $NS(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ is isomorphic to $\mathrm{End}_{\mathbb{Q}}(X)$, by [BL, Prop. 5.2.1]. Let L_0 be a polarization on X and let L be a line bundle. Denote their hermitian forms by H_0 and H. Let $X = V/\Lambda$ and $\hat{X} = \Omega/\hat{\Lambda}$, where $\Omega := \mathrm{Hom}_{\bar{\mathbb{C}}}(V, \mathbb{C})$ and $\hat{\Lambda} = \{\omega \in \Omega \ : \ Im(\omega(\lambda)) \in \mathbb{Z}, \forall \lambda \in \Lambda\}$. Let $\phi_{H_0} : V \to \Omega$ be given by $\phi_{H_0}(v) = H_0(v, \bullet)$ and define ϕ_H similarly. Then ϕ_H is the analytic representation of ϕ_L. Set $\theta := Im(H)$ and $\theta_0 := Im(H_0)$. Let $f := \phi_H^{-1}\phi_{H_0} \in \mathrm{End}_{\mathbb{Q}}(X)$. Then $\phi_{H_0}(x) = \phi_H(f(x))$ and $H_0(x, \bullet) = H(f(x), \bullet)$, for all $x \in V$. Thus,

$$\begin{aligned} \theta(f(x), y) &= ImH(f(x), y) = ImH_0(x, y) = -ImH_0(y, x) = -ImH(f(y), x) \\ &= ImH(x, f(y)) = \theta(x, f(y)). \end{aligned}$$

So θ belongs to $\wedge^2_F \Lambda^*_{\mathbb{Q}} = \wedge^2_F H^1(X, \mathbb{Q})$, since every element of $F = \mathrm{End}_{\mathbb{Q}}(X)$ is represented by such f. □

Assume that X is simple and $\mathrm{End}_{\mathbb{Q}}(X) = F$. Then the subspace $\wedge^2_F H^1(X, \mathbb{Q})$ of $H^2(X, \mathbb{Q})$ contains a Hodge class Θ, by Lemma 11.1.1. Contraction with Θ induces an $\hat{\eta}(F)$-equivariant homomorphism $\Theta\rfloor : H^1(X, \mathbb{Q})^* \to H^1(X, \mathbb{Q})$. The element g of $\mathrm{Spin}(V_{\mathbb{Q}})$ corresponding to cup product with $\exp(\Theta)$ acts on $V_{\mathbb{Q}} := H^1(X, \mathbb{Q}) \oplus H^1(X, \mathbb{Q})^*$ by $\rho_g(w, \xi) = (w - \Theta\rfloor\xi, \xi)$. It follows that ρ_g is $\hat{\eta}(F)$-equivariant

$$f(\rho_g(w, \xi)) = f(w - \Theta\rfloor\xi, \xi) = (f(w) - f(\Theta\rfloor\xi), \xi \circ f) = (f(w) - \Theta\rfloor(\xi \circ f), \xi \circ f) = f(\rho_g(w, \xi)).$$

Hence, the induced action of ρ_g on $\wedge^* V_{\mathbb{Q}}$ commutes with the induced action of $\hat{\eta}(F)$, where $\hat{\eta}_f$ acts via $\wedge^k\hat{\eta}_f$ on $\wedge^k V_{\mathbb{Q}}$, for all $f \in F$. Note that the nilpotent element of square zero $\rho_g - id_{V_{\mathbb{Q}}}$ depends linearly on Θ and $\rho_g = \exp(\rho_g - id_{V_{\mathbb{Q}}})$. We thus define the element $g_0 \in \mathrm{Spin}(V_{\hat{\eta},K})$ to be the unique element, such that m_{g_0} acts on the spin representation $\wedge^*_K[H^1(X,\mathbb{Q})\otimes_F K]$ via cup product with $\exp(\sqrt{-q}\Theta)$, where q is an element of F such that $K = F[\sqrt{-q}]$. The isometry ρ_{g_0} of $V_{\mathbb{Q}}\otimes_F K$ corresponds to $id_{V_{\mathbb{Q}}\otimes_F K} + (\rho_g - id)\otimes\sqrt{-q}$. Explicitly, decompose elements of $V_{\mathbb{Q}}\otimes_F K$ as $(v_1, v_2\otimes\sqrt{-q})$ with $v_i \in V_{\mathbb{Q}}$. Then

$$\begin{aligned}
(\rho_g - id)(v_1, v_2\otimes\sqrt{-q}) &= ((\rho_g - id)(v_1), (\rho_g - id)(v_2)\otimes\sqrt{-q}) \\
\sqrt{-q}(v_1, v_2\otimes\sqrt{-q}) &= (\hat{\eta}_{-q}(v_2), v_1\otimes\sqrt{-q}), \\
(\sqrt{-q}\circ(\rho_g - id))(v_1, v_2\otimes\sqrt{-q}) &= (\hat{\eta}_{-q}((\rho_g - id)(v_2)), (\rho_g - id)(v_1)\otimes\sqrt{-q})
\end{aligned}$$

Set

$$\begin{aligned}
\nu &:= (\sqrt{-q}\circ(\rho_g - id)), \\
\rho_{g_0} &:= \exp(\nu).
\end{aligned}$$

In block matrix form we have:

$$\nu = \begin{pmatrix} 0 & -\hat{\eta}_q(\rho_g - id)) \\ \rho_g - id & 0 \end{pmatrix}, \text{ and } \exp(\nu) = \begin{pmatrix} 1 & -\hat{\eta}_q(\rho_g - id)) \\ \rho_g - id & 1 \end{pmatrix}.$$

The induced action of ν on $\wedge^*_K[V_{\mathbb{Q}}\otimes_F K]$ is nilpotent, but no longer of square zero. The induced action of ρ_{g_0} on $\wedge^*_K[V_{\mathbb{Q}}\otimes_F K]$ is still $\exp(\nu)$. The maximal isotropic subspace $W = \rho_{g_0}(H^1(\hat{X},\mathbb{Q})\otimes_F K)$ is the graph of $\nu_| : H^1(\hat{X},\mathbb{Q})\otimes_F K \to H^1(X,\mathbb{Q})\otimes_F K$ sending $\xi \in H^1(\hat{X},\mathbb{Q})$ to $-(\Theta\rfloor\xi)\otimes\sqrt{-q}$. Clearly, $W\cap\iota(W) = (0)$ verifying Equation (5.1.6).

The pure spinor of the maximal isotropic subspace $H^1(\hat{X},\mathbb{Q})$ is spanned by $1 \in H^{ev}(X,\mathbb{Q})$. The pure spinor of W is spanned by $m_{g_0}(1) \in \wedge^*_K[H^1(X,\mathbb{Q})\otimes_F K] \subset H^*(X,\mathbb{Q})\otimes_F K$. The element $m_{g_0}(1)$ is given by

$$\begin{aligned}
m_{g_0}(1) &= 1 + \Theta\otimes\sqrt{-q} - \hat{\eta}_q(\Theta\wedge_K\Theta) - \hat{\eta}_q(\Theta\wedge_K\Theta\wedge_K\Theta)\otimes\sqrt{-q} + \dots \\
&= (1 - \hat{\eta}_q(\Theta\wedge_K\Theta) + \dots) + (\Theta - \hat{\eta}_q(\Theta\wedge_K\Theta\wedge_K\Theta) + \dots)\otimes\sqrt{-q} \qquad (11.1.1)
\end{aligned}$$

The j-th wedge product $\Theta\wedge_K\cdots\wedge_K\Theta$ is equal to $\Theta\wedge_F\cdots\wedge_F\Theta$, where we consider $\wedge^{2j}_F H^1(X,\mathbb{Q})$ as an F-subspace of $\wedge^{2j}_K[H^1(X,\mathbb{Q})\otimes_F K]$ via the isomorphism

$$[\wedge^{2j}_F H^1(X,\mathbb{Q})]\otimes_F K \cong \wedge^{2j}_K[H^1(X,\mathbb{Q})\otimes_F K].$$

The product $\Theta\wedge_F\cdots\wedge_F\Theta$ is the projection of $\Theta\wedge_{\mathbb{Q}}\cdots\wedge_{\mathbb{Q}}\Theta$ into the direct summand $\wedge^{2j}_F H^1(X,\mathbb{Q})$ of $\wedge^{2j}_{\mathbb{Q}} H^1(X,\mathbb{Q})$. Write $\Theta = \sum_{\hat{\sigma}\in\hat{\Sigma}}\Theta_{\hat{\sigma}}$, where $\Theta_{\hat{\sigma}}$ is in $\wedge^2 H^1_{\hat{\sigma}}(X)$. Under the embedding of $\wedge^{2j}_{\mathbb{Q}} H^1(X,\mathbb{Q})$ in $\wedge^{2j}_{\mathbb{R}} H^1(X,\mathbb{R})$ we get the equality $\Theta\wedge_F\cdots\wedge_F\Theta = \sum_{\hat{\sigma}\in\hat{\Sigma}}\Theta_{\hat{\sigma}}\wedge_{\mathbb{R}}\cdots\wedge_{\mathbb{R}}\Theta_{\hat{\sigma}}$.

The secant space B in this example is computed via the factorization (7.1.4). Write $\Theta = \sum_{\hat{\sigma}\in\hat{\Sigma}}\Theta_{\hat{\sigma}}$, where $\Theta_{\hat{\sigma}}$ is in $\wedge^2 H^1_{\hat{\sigma}}(X)$. $V_{\mathbb{C}} = \oplus_{\hat{\sigma}\in\hat{\Sigma}} V_{\hat{\sigma},\mathbb{R}}\otimes_{\mathbb{R}}\mathbb{C}$, where $V_{\hat{\sigma},\mathbb{R}} = H^1_{\hat{\sigma}}(X)\oplus H^1_{\hat{\sigma}}(\hat{X})$. Choose a CM-type T and let $\sigma = T(\hat{\sigma}) \in \Sigma$ be the embedding restricting to F as $\hat{\sigma}$. The isometry ρ_{g_0} is mapped by the homomorphism $\tilde{e}_T$ of Remark 6.0.1 to an isometry which restricts to $V_{\hat{\sigma},\mathbb{R}}\otimes_{\mathbb{R}}\mathbb{C}$ as $\exp(\sigma(\sqrt{-q})\Theta_{\hat{\sigma}})$. We get the pure spinor in $S^+_{\hat{\sigma}}$ corresponding to the element $\exp(\sigma(\sqrt{-q})\Theta_{\hat{\sigma}})$. We get the complex numbers $[T(\hat{\sigma})(\sqrt{-q})]$ and the product in $S^+_{\mathbb{C}}$

$$\bigwedge_{\hat{\sigma}\in\hat{\Sigma}} \exp\left([T(\hat{\sigma})(\sqrt{-q})]\Theta_{\hat{\sigma}}\right) = \exp\left(\sum_{\hat{\sigma}\in\hat{\Sigma}}[T(\hat{\sigma})(\sqrt{-q})]\Theta_{\hat{\sigma}}\right) \tag{11.1.2}$$

is a pure spinor spanning ℓ_T. The secant space $B\otimes_{\mathbb{Q}}\mathbb{C}$ is the span of all these pure spinors in $S^+_{\mathbb{C}}$. Equivalently, $B\otimes_{\mathbb{Q}}\mathbb{R}$ is the product $\otimes_{\hat{\sigma}\in\hat{\Sigma}}P_{\hat{\sigma}}$ in $S^+_{\mathbb{R}}$, where

$$P_{\hat{\sigma}}\otimes_{\mathbb{R}}\mathbb{C} := \mathrm{span}_{\mathbb{C}}\{\exp(\sigma(\sqrt{-q})\Theta_{\hat{\sigma}}), \exp(\bar{\sigma}(\sqrt{-q})\Theta_{\hat{\sigma}})\}$$

is a plane in $S^+_{\hat{\sigma}}$. The equality $\tau(P_{\hat{\sigma}}) = P_{\hat{\sigma}}$, where τ is the dualization involution given in (10.1.5), implies the equality

$$\tau(B) = B.$$

We claim that B is spanned by Hodge classes. It suffices to prove that the 2-form $\Theta_{\hat{\sigma}}$ is a $(1,1)$-form, for every $\hat{\sigma}\in\hat{\Sigma}$. Now, the F action on $H^1(X,\mathbb{Q})$ preserves the Hodge structure, by assumption. Hence, $H^{1,1}(X,\mathbb{C})$ is $F^\times$-invariant, where the $F^\times$-action is via $\wedge^2\hat{\eta}$. The two form Θ is of type $(1,1)$, by assumption. Hence, $\Theta = \sum_{\{\hat{\sigma}_1,\hat{\sigma}_2\}\subset\hat{\Sigma}}\Theta_{\hat{\sigma}_1\otimes\hat{\sigma}_2}$, where the summand $\Theta_{\hat{\sigma}_1\otimes\hat{\sigma}_2}$ is in the isotypic subspace for the character $\hat{\sigma}_1\otimes\hat{\sigma}_2$ of $F^\times$. The isotypic subspace for $\hat{\sigma}_1\otimes\hat{\sigma}_2$ is invariant with respect to the complex structure I_X of X and so is Θ. Hence, $\Theta_{\hat{\sigma}_1\otimes\hat{\sigma}_2}$ is I_X-invariant and so of type $(1,1)$. Our choice of Θ in $\wedge^2_F H^1(X,\mathbb{C})$ means that $\Theta_{\hat{\sigma}_1\otimes\hat{\sigma}_2}$ is non-zero, if and only if $\hat{\theta}_1 = \hat{\theta}_2$, and the summand $\Theta_{\hat{\sigma}}$ in $\wedge^2 H^1_{\hat{\sigma}}(X)$ is the summand $\Theta_{\hat{\sigma}\otimes\hat{\sigma}}$. Hence, indeed $\Theta_{\hat{\sigma}}$ is of type $(1,1)$.

Lemma 11.1.2. *There exists a non-zero $t\in K_-$, such that the complex structure $I_{X\times\hat{X}}$ of $X\times\hat{X}$ belongs to the period domain $\Omega_{B,t}$ given in (9.2.2). Furthermore, the polarized abelian variety of Weil type $(X\times\hat{X},\eta,\Xi_t)$ is of split type (Def. 7.3.4).*

Proof. Set $I := I_{X\times\hat{X}}$. I belongs to $\rho(Spin(V_{\mathbb{R}})_B)$, since it is the image of the element $\tilde{I}$ of $\mathrm{Spin}(V_{\mathbb{R}})$, which acts in $S_{\mathbb{R}}$ as $m(\tilde{I}) = I_X$, where I_X is the complex structure of X, and $\tilde{I}$ belongs to $\mathrm{Spin}(V_{\mathbb{R}})_B$, since B is spanned by Hodge classes.

We show next that the symmetric bilinear form $g_I(x,y) := (\eta_t(x), I(y))_V$, given in (9.2.1), is positive definite for some $t\in K_-$. It suffices to prove it for the restriction of g_I to $V_{\hat{\sigma},\mathbb{R}}$, for each $\hat{\sigma}\in\hat{\Sigma}$. Now, η_t acts on $V_{\hat{\sigma},\mathbb{C}} = V_{\sigma,\mathbb{C}}\oplus V_{\bar{\sigma},\mathbb{C}}$ by acting on $V_{\sigma,\mathbb{C}}$ via multiplication by $\sigma(t)$ and on $V_{\bar{\sigma},\mathbb{C}}$ via multiplication by $\bar{\sigma}(t)$. The positive-definiteness of the restriction of g_I to $V_{\hat{\sigma},\mathbb{R}}$ now follows by the same argument used to prove [M2, Proposition 2.4.4]. The latter argument shows that the restriction of g_I to $V_{\hat{\sigma},\mathbb{R}}$ is definite, and the choice of t is made so that it is positive definite. Such a choice of t exists, as observed in footnote 4 in Section 9.2.

We show next that each of $V^{1,0}_{\sigma,\mathbb{C}}$ and $V^{0,1}_{\sigma,\mathbb{C}}$ is $d/2$-dimensional. The dimension of $V_{\hat{\sigma},\mathbb{R}}$ is $2d$, for each $\hat{\sigma}\in\hat{\Sigma}$, and the dimension of V_σ is d, for each $\sigma\in\Sigma$. We know that $V_{\sigma,\mathbb{C}}$ is I-invariant, since I commutes with $\eta(K)$. The dimension of $V^{1,0}_{\hat{\sigma},\mathbb{R}}$ is equal to that of $V^{0,1}_{\hat{\sigma},\mathbb{R}}$, as they are complex conjugate. It remains to prove that the dimensions of $V^{0,1}_{\sigma,\mathbb{C}}$ and $V^{1,0}_{\sigma,\mathbb{C}}$ are equal, for all $\sigma\in\Sigma$. The proof is identical to that of [M2, Lemma 3.2.1], with $V_{\hat{\sigma},\mathbb{R}}$ here playing the role of $V_{\mathbb{R}}$ there, $V_{\sigma,\mathbb{C}}$ here playing the role of $W_{1,\mathbb{C}}$ there, and $V_{\bar{\sigma},\mathbb{C}}$ here playing the role of $W_{2,\mathbb{C}}$ there.

Finally, we show that $(X\times \hat{X},\eta,\Xi_t)$ is of split type. Choose a basis $\{y_1,\dots,y_d\}$ of $H^1(\hat{X},\mathbb{Q})\cong \mathrm{Hom}_F(H^1(X,\mathbb{Q}),F)$ as a vector space over F. Recall that Θ is an element of $\wedge^2_F H^1(X,\mathbb{Q})$. Let $\theta: H^1(\hat{X},\mathbb{Q})\rightarrow H^1(X,\mathbb{Q})$ be the F-linear contraction with Θ. We may further assume that $\{y_1,\dots,y_{d/2}\}$ is a basis for an F-subspace isotropic with respect to Θ. Then $y_i(\theta(y_j))=0$, for $1\leq i,j\leq d/2$. The element g_0 of $\mathrm{Spin}(V_K)$ with $m_{g_0}(\bullet)=\exp(\sqrt{-q}\Theta)\cup(\bullet)$ acts on $V_{\hat{\eta}}\otimes_F K$ sending (w,y) to $(w-\theta(y)\otimes\sqrt{-q},y)$ (see [Ch, Ch. III.1.7]). So

$$\begin{array}{rcl}
(0,y) & = & (1/2)[m_{g_0}(0,y)+\overline{m_{g_0}}(0,y)],\\
m_{g_0}(0,y) & = & (-\theta(y)\otimes\sqrt{-q},y),\\
\overline{m_{g_0}}(0,y) & = & (\theta(y)\otimes\sqrt{-q},y),\\
\eta_t(0,y) & = & (1/2)[tm_{g_0}(0,y)-t\overline{m_{g_0}}(0,y)]\\
 & = & (1/2)[(-\theta(y)\otimes t\sqrt{-q},y\otimes t)+(-\theta(y)\otimes t\sqrt{-q},-y\otimes t)\\
 & = & (-\theta(y)\otimes t\sqrt{-q},0)],\\
(\eta_t(0,y_i),(0,y_j))_{V_{\hat{\eta}}} & = & -\Theta(y_i,y_j)t\sqrt{-q}.
\end{array}$$

Note that $\mathcal{B}:=\{(0,y_1),\dots,(0,y_d)\}$ is a K-basis of V_K. It suffices to show that the subspace spanned by $\{(0,y_1),\dots,(0,y_{d/2})\}$ is isotropic with respect to H_t, by Lemma 7.3.3(2). Now,

$$\begin{array}{rcl}
H_t((0,y_i),(0,y_j)) & = & (-t^2)((0,y_i),(0,y_j))_{V_{\hat{\eta}}}+t(\eta_t(0,y_i),(0,y_j))_{V_{\hat{\eta}}}\\
 & = & (-t^2)((0,y_i),(0,y_j))_{V_{\hat{\eta}}}-\Theta(y_i,y_j)t^2\sqrt{-q}.
\end{array}$$

The first summand vanishes, since $H^1(\hat{X},\mathbb{Q})$ is an isotropic subspace of $V_{\hat{\eta}}$ and the second summand vanishes for $i,j\leq d/2$, since $\{y_1,\dots,y_{d/2}\}$ is a basis for an F-subspace isotropic with respect to Θ. $\square$

11.2. **A criterion for a class $\alpha_0\otimes\beta$ in $B\otimes B$ to map to a non-zero class in $HW(X\times\hat{X},\eta)$.** In this subsection we consider the special case where the element q of F actually belongs to $\mathbb{Q}$ (and is positive as K is totally complex), though $e/2:=\dim_{\mathbb{Q}}(F)>1$. Choose $\sqrt{-q}$ to be the square root in the upper half plane. The set (11.1.2) of $2^{e/2}$ pure spinors becomes

$$\exp\left(\sqrt{-q}\sum_{\hat{\sigma}\in\hat{\Sigma}}\pm\Theta_{\hat{\sigma}}\right). \tag{11.2.1}$$

Two of those, namely $\exp(\pm\sqrt{-q}\Theta)$, are defined over the quadratic imaginary number field $K_0:=\mathbb{Q}(\sqrt{-q})$ contained in K and span a secant plane P_{K_0} in $S^+_{\mathbb{C}}$ defined over $\mathbb{Q}$. In Lemma 11.2.9 in section 11.2.2 we provide a criterion for a class $\alpha_0\in P_{K_0}$ and a class $\beta\in B$ so that $\alpha_0\otimes\beta$ satisfies the hypothesis of Proposition 10.2.1 and hence maps to a non-zero class in $HW(X\times\hat{X},\eta)$.

We begin in Section 11.2.1 with the special case where $[F:\mathbb{Q}]=2$. If X is the Jacobian of a genus 4 curve and Θ is the natural principal polarization, then we have seen examples of secant sheave E_0 with Chern character α_0 in P_{K_0} in [M2, Examples 8.2.3 and 8.2.4]. Examples of Jacobians with real multiplication can be found in [E, Sh]. In Lemma 11.2.8 we provide an example of a coherent sheaf E' on X, such that $ch(E_0)\otimes ch(E')$ satisfies the hypothesis of Proposition 10.2.1 and hence maps to a non-zero class in $HW(X\times\hat{X},\eta)$.

11.2.1. *The case* $[K:\mathbb{Q}]=4$ *with* $Gal(K/\mathbb{Q})\cong\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$. Following is an example of a class c in $B\otimes B$ satisfying the hypothesis of Proposition 10.2.1 when $F=\mathbb{Q}(\sqrt{t})$, where t is a positive integer, which is not a perfect square. In this case K is a Galois extension of $\mathbb{Q}$, being the splitting field of $(x^2+q)(x^2-t)$, and $Gal(K/\mathbb{Q})$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$, since it has order 4 and exponent 2. Let $\gamma\in Gal(K/\mathbb{Q})$ send $\sqrt{t}$ to $-\sqrt{t}$ and $\sqrt{-q}$ to itself. Then γ interchanges $\Theta_{\hat{\sigma}_1}$ and $\Theta_{\hat{\sigma}_2}$, since $\Theta=\Theta_{\hat{\sigma}_1}+\Theta_{\hat{\sigma}_2}$ is γ-invariant. Hence, $\sqrt{t}(\Theta_{\hat{\sigma}_1}-\Theta_{\hat{\sigma}_2})$ is rational, $\sqrt{-q}(\Theta_{\hat{\sigma}_1}-\Theta_{\hat{\sigma}_2})$ is $\iota\circ\gamma$ invariant, and the pure spinors $\exp(\pm\sqrt{-q}(\Theta_{\hat{\sigma}_1}-\Theta_{\hat{\sigma}_2}))$ are defined over the imaginary quadratic number subfield $K_1:=\mathbb{Q}(\sqrt{-tq})$ of K. Let P_{K_1} be the plane spanned by these two pure spinors. The $\mathbb{Q}$-linear $Gal(K/\mathbb{Q})$-action on the second factor K of $B\otimes_{\mathbb{Q}}K$ permutes[6] the four pure spinors (11.2.1). This is a lift of the action of $Gal(K/\mathbb{Q})$ on the set $\mathcal{T}_K$ of CM-types in the sense that the map $T\mapsto\ell_T$ is $Gal(K/\mathbb{Q})$-equivariant. The set $\mathcal{T}_K$ consists of two $Gal(K/\mathbb{Q})$-orbits. The two latter pure spinors constitute the orbit of elements fixed by $\iota\circ\gamma$. The secant plane P_{K_0} is spanned by the $Gal(K/\mathbb{Q})$-orbit of pure spinors fixed by γ.

Let $\sigma_i\in\Sigma_i$ be the extension of $\hat{\sigma}_i$ mapping $\sqrt{-q}$ to the upper half plane in $\mathbb{C}$. The four CM-types are

$$\begin{pmatrix}\hat{\sigma}_1 & \hat{\sigma}_2\\ \sigma_1 & \sigma_2\end{pmatrix},\quad \begin{pmatrix}\hat{\sigma}_1 & \hat{\sigma}_2\\ \bar{\sigma}_1 & \bar{\sigma}_2\end{pmatrix},\quad \begin{pmatrix}\hat{\sigma}_1 & \hat{\sigma}_2\\ \bar{\sigma}_1 & \sigma_2\end{pmatrix},\quad \begin{pmatrix}\hat{\sigma}_1 & \hat{\sigma}_2\\ \sigma_1 & \bar{\sigma}_2\end{pmatrix}.$$

Let α_0 be a non-zero rational class in P_{K_0}. Let β be a class in B, which does not belong to neither P_{K_0} nor P_{K_1}. Set $c:=\alpha_0\otimes\beta$. Write $\beta=\beta_0+\beta_1$, with $\beta_i\in P_{K_i}$. This is possible, since B is the direct sum of P_{K_0} and P_{K_1}. Assume that $\check{\phi}(\alpha_0\otimes\beta_0)_0\neq 0$.

Lemma 11.2.1. *The class c satisfies the hypothesis of Proposition 10.2.1, namely $\check{\phi}(c)_0\neq 0$ and c_1 does not belong to KB_1.*

Proof. Let $\mathcal{T}_K^\gamma$ be the set of CM-types fixed by γ and define $\mathcal{T}_K^{\iota\circ\gamma}$ similarly. The set $\mathcal{T}_K^\gamma$ consists of the first two CM-types displayed above and $\mathcal{T}_K^{\iota\circ\gamma}$ consists of the last two. We see that $c_1=\alpha_0\otimes\beta_1$ and $\alpha_0\otimes\beta_0$ belongs to $BB_0\oplus BB_2$. For each $\sigma\in\Sigma$ there exists a unique CM-type $T\in\mathcal{T}_K^\gamma$ of which σ is a value. The same holds for $\mathcal{T}_K^{\iota\circ\gamma}$. Hence, the left hand side of each of the linear homogeneous equations of KB_1 in Equation (10.1.2), for each $\sigma\in\Sigma$, is a sum consisting of precisely one non-zero term $c_{(T,T')}$. Finally, $\check{\phi}(c)_0=\check{\phi}(c_0)_0=\check{\phi}(\alpha_0\otimes\beta_0)_0$, by Proposition 10.2.1, which is assumed not to vanish. □

Question 11.2.2. Let X be the Jacobian of a genus 4 curve. Let α_0 be the Chern character $ch(F_1)$ of one of the K_0-secant sheaves in [M2, Examples 8.2.3 or 8.2.4]. Find

[6] This permutation action on the four K-basis elements of $B\otimes_{\mathbb{Q}}K$ defines a new $Gal(K/\mathbb{Q})$-action on $B\otimes_{\mathbb{Q}}K$ via K-linear transformations. The latter is not natural, as it changes if we rescale the basis elements by a scalar in K.

a coherent sheaf F_2 on X with Chern character β in B, such that α_0 and β satisfy the condition of Lemma 11.2.1, and such that the semiregularity map restricts to the image of $at_{F_2} : HT^2(X) \to \mathrm{Ext}^2(F_2, F_2)$ as an injective map.

We proceed to provide an example of a coherent sheaf F_2 verifying the hypotheses in the above question, except possibly for the condition on at_{F_2}. Note that the 4 dimensional space B of the 2^7-dimensional $S^+_{\mathbb{Q}}$ in Section 11.2.1 depends on W, which depends on the choice of the element of $\mathrm{Spin}(V_{\hat{\eta}} \otimes_F K)$ of cup product with $\exp(\sqrt{-q}\Theta)$. In particular, it depends on the choice of Θ is the 2-dimensional $H^{1,1}(X, \mathbb{Q}) = \wedge^2_F H^1(X, \mathbb{Q})$. Denote it thus by $B_{\sqrt{-q}\Theta}$. The subspace $B_{\sqrt{-q}\Theta}$ is τ-invariant, where τ is the duality involution (10.1.5), and it is contained in the subalgebra $\langle H^{1,1}(X, \mathbb{Q})\rangle$ of $S^+_{\mathbb{Q}} = H^{ev}(X, \mathbb{Q})$ generated by $H^{1,1}(X, \mathbb{Q})$. Write $\Theta = \Theta_{\hat{\sigma}_1} + \Theta_{\hat{\sigma}_2}$ as above. Then $\Theta^3_{\hat{\sigma}_i} = 0$, since $H^1_{\hat{\sigma}_i}(X)$ is 4-dimensional. Thus, the subalgebra $\langle H^{1,1}(X, \mathbb{Q})\rangle$ is 9-dimensional with the basis

$$\{1, \Theta, \sqrt{t}\tilde{\Theta}, \Theta^2, \Theta\tilde{\Theta}, \tilde{\Theta}^2, \Theta^3, \sqrt{t}\tilde{\Theta}^3, [pt]\},$$

where $\tilde{\Theta} := \Theta_{\hat{\sigma}_1} - \Theta_{\hat{\sigma}_2}$ and the multiplication by the positive $\sqrt{t} \in \mathbb{R} \subset \mathbb{C}$ above is via scalar multiplication in the $\mathbb{C}$-vector space structure of $H^2(X, \mathbb{C})$. Note that $\sqrt{t}\tilde{\Theta}$ is the scalar product[7] of the element Θ of the F-vector space $\wedge^2_F H^1(X, \mathbb{Q})$ by the scalar $\sqrt{t}$ in F, if we let the embedding $\hat{\sigma}_1$ of F send $\sqrt{t}$ to the positive square root and $\hat{\sigma}_2$ send $\sqrt{t}$ to the negative square root. The group of units $U := \{a + bt \ : \ a^2 - tb^2 = \pm 1\}$ in the subring of algebraic integers in F is the product of an infinite cyclic group and $\{\pm 1\}$. If we choose X, so that $\hat{\eta}(U) \cap \mathrm{End}(X)$ contains an automorphism $\hat{\eta}(f)$ of X of infinite order, then the rays $\mathbb{R}_{\geq 0}\Theta_{\hat{\sigma}_i}$, $i = 1, 2$, are the two limits $\lim_{n\to\pm\infty} \mathbb{R}_{\geq 0}\hat{\eta}(f^n)^*(\Theta)$. More generally, we obtains these rays as limits of rays in the ample cone, by pulling back Θ via self isogenies in $\hat{\eta}(F) \cap \mathrm{End}(X)$. These two lines form the boundary of the ample cone, since $\Theta^4_{\hat{\sigma}_i} = 0$. We conclude that neither $\sqrt{t}\tilde{\Theta}$ nor $-\sqrt{t}\tilde{\Theta}$ are ample.

Set $\alpha := 1 - \frac{q}{2}\Theta^2 + \frac{q^2}{4!}\Theta^4$ and $\beta := \Theta - \frac{q}{3!}\Theta^3$. Then $\exp(\sqrt{-q}\Theta) = \alpha + \sqrt{-q}\beta$. Define $\tilde{\alpha}$ and $\tilde{\beta}$ similarly in terms of $\tilde{\Theta}$. Then $B_{\sqrt{-q}\Theta}$ has the basis

$$\{\alpha, \beta, \tilde{\alpha}, \sqrt{t}\tilde{\beta}\}.$$

Note that $\alpha - \tilde{\alpha} = -2q\Theta_{\hat{\sigma}_1}\Theta_{\hat{\sigma}_2}$ is an element of $B_{\sqrt{-q}\Theta}$ spanning a line which is independent of the choice of Θ.

Given an element $f \in F$ and a class δ in the F-vector space $\wedge^*_F H^1(X, \mathbb{Q})$, denote by $f \cdot \delta$ the scalar multiplication of δ by f, while given $c \in \mathbb{C}$ we denote by $c\delta$ the scalar product in $H^2(X, \mathbb{C})$. We choose $\hat{\sigma}_1$ to map $\sqrt{t}$ to the positive square root and denote $\hat{\sigma}_1(\sqrt{t})$ by $\sqrt{t}$ for short. So $(a + b\sqrt{t})\Theta = (a + b\sqrt{t})\Theta_{\hat{\sigma}_1} + (a + b\sqrt{t})\Theta_{\hat{\sigma}_2}$, while $(a + b\sqrt{t}) \cdot \Theta = (a + b\sqrt{t})\Theta_{\hat{\sigma}_1} + (a - b\sqrt{t})\Theta_{\hat{\sigma}_2}$. The equality $B_{\sqrt{-q}f\cdot\Theta} = B_{\sqrt{-qf^2}\Theta}$ holds for all $f \in F$ satisfying $f^2 \in \mathbb{Q}$, since $\sqrt{-q}f \cdot \Theta = \sqrt{-qf^2}\Theta$ in $\wedge^2_F H^1(X, \mathbb{Q}) \otimes_F K$. In particular, $B_{\sqrt{-q}c\sqrt{t}\cdot\Theta} = B_{\sqrt{-qc^2t}\Theta}$, for all $c \in \mathbb{Q}^\times$.

Given $f \in F$, let $M_f \in \mathrm{End}[\wedge^2_F H^1(X, \mathbb{Q}) \oplus H^6(X, \mathbb{Q})]$ send (x, y) to $(f^2 \cdot x, Nm(f^2)y)$.

[7]In contrast, the rational correspondence $[\hat{\eta}(\sqrt{t})]$ sends Θ to $t\Theta$, as it acts on the second cohomology as the second exterior power of its action on the first cohomology.

Lemma 11.2.3. *The endomorphism M_f maps $B_{\sqrt{-q}\Theta}\cap[\wedge^2_F H^1(X,\mathbb{Q})\oplus H^6(X,\mathbb{Q})]$ to $B_{\sqrt{-q}f\cdot\Theta}\cap[\wedge^2_F H^1(X,\mathbb{Q})\oplus H^6(X,\mathbb{Q})]$.*

Proof. We have the equality

$$-\frac{q}{3!}(a\Theta^3+b\sqrt{t}\tilde{\Theta}^3)=-\frac{q}{2}\Theta_{\hat{\sigma}_1}\Theta_{\hat{\sigma}_2}[(a-b\sqrt{t})\Theta_{\hat{\sigma}_1})+(a+b\sqrt{t})\Theta_{\hat{\sigma}_2})]=-\frac{q}{2}\Theta_{\hat{\sigma}_1}\Theta_{\hat{\sigma}_2}(a-b\sqrt{t})\cdot\Theta.$$

The intersection of $B_{\sqrt{-q}\cdot\Theta}$ with $H^2(X,\mathbb{Q})\oplus H^6(X,\mathbb{Q})$ is

$$\{(a+b\sqrt{t})\cdot\Theta+(-q/2)\Theta_{\hat{\sigma}_1}\Theta_{\hat{\sigma}_2}(a-b\sqrt{t})\cdot\Theta\ :\ a+b\sqrt{t}\in F\}.$$

The intersection of $B_{\sqrt{-q}f\cdot\Theta}$ with $H^2(X,\mathbb{Q})\oplus H^6(X,\mathbb{Q})$ is

$$\{(a+b\sqrt{t})\cdot f\cdot\Theta+(-q/2)Nm(f)\Theta_{\hat{\sigma}_1}\Theta_{\hat{\sigma}_2}(a-b\sqrt{t})\cdot f\cdot\Theta\ :\ a+b\sqrt{t}\in F\}.$$

The intersection is thus the graph of the linear transformation

$$\begin{array}{rcl}\mathcal{B}_{\sqrt{-q}f\cdot\Theta}:\wedge^2_F H^2(X,\mathbb{Q}) & \to & H^6(X,\mathbb{Q})\\ (a+b\sqrt{t})\cdot f\cdot\Theta & \mapsto & (-q/2)Nm(f)\Theta_{\hat{\sigma}_1}\Theta_{\hat{\sigma}_2}(a-b\sqrt{t})\cdot f\cdot\Theta\end{array}$$

Thus,

$$\begin{array}{rcl}\mathcal{B}_{\sqrt{-q}f\cdot\Theta}((a+b\sqrt{t})\cdot f^2\cdot\Theta) & = & (-q/2)Nm(f)\Theta_{\hat{\sigma}_1}\Theta_{\hat{\sigma}_2}(a-b\sqrt{t})\cdot\gamma(f)\cdot f\cdot\Theta\\ & = & (-q/2)Nm(f^2)\Theta_{\hat{\sigma}_1}\Theta_{\hat{\sigma}_2}(a-b\sqrt{t})\cdot\Theta\\ & = & Nm(f^2)\mathcal{B}_{\sqrt{-q}\cdot\Theta}((a+b\sqrt{t})\cdot\Theta)\end{array}$$

We see that $\mathcal{B}_{\sqrt{-q}f\cdot\Theta}\circ(f^2\cdot(\bullet))=Nm(f^2)\mathcal{B}_{\sqrt{-q}\Theta}$. □

Lemma 11.2.4. $B_{\sqrt{-q}\Theta}\cap B_{\sqrt{-q}f\cdot\Theta}\cap[\wedge^2_F H^1(X,\mathbb{Q})\oplus H^6(X,\mathbb{Q})]=(0)$, *for $f\in F^\times\setminus\{\pm 1\}$, and $B_{\sqrt{-q}\Theta}=B_{-\sqrt{-q}\Theta}$.*

Proof. Fix $f\in F^\times$. Let $x\in\wedge^2_F H^1(X,\mathbb{Q})$ be a non-zero element. It can be written in the form $x=(a+b\sqrt{t})f\cdot\Theta$. Then $\mathcal{B}_{\sqrt{-q}\Theta}(x)=-(q/2)\Theta_{\hat{\sigma}_1}\Theta_{\hat{\sigma}_1}(a-b\sqrt{t})\gamma(f)\cdot\Theta$ and $\mathcal{B}_{\sqrt{-q}f\cdot\Theta}(x)=-(q/2)\Theta_{\hat{\sigma}_1}\Theta_{\hat{\sigma}_1}Nm(f)(a-b\sqrt{t})f\cdot\Theta$. The equality $\mathcal{B}_{\sqrt{-q}\Theta}(x)=\mathcal{B}_{\sqrt{-q}f\cdot\Theta}(x)$ is thus equivalent to $\gamma(f)=Nm(f)f$, which is equivalent to $Nm(f)=1$ and $\gamma(f)=f$, which is equivalent to $f=\pm 1$. □

Define an F-vector space structure on $\langle H^{1,1}(X,\mathbb{Q})\rangle^6$ by

$$\begin{array}{rcl}f\cdot\Theta^2_{\hat{\sigma}_1}\Theta_{\hat{\sigma}_2} & = & \hat{\sigma}_1(f)\Theta^2_{\hat{\sigma}_1}\Theta_{\hat{\sigma}_2},\\ f\cdot\Theta_{\hat{\sigma}_1}\Theta^2_{\hat{\sigma}_2} & = & \hat{\sigma}_2(f)\Theta_{\hat{\sigma}_1}\Theta^2_{\hat{\sigma}_2},\end{array}$$

so that $(f\cdot\Theta)^3=Nm(f)f\cdot\Theta^3$.

Lemma 11.2.5. *A non-zero element $f_1\cdot\Theta+f_2\cdot\Theta^3$ of $\langle H^{1,1}(X,\mathbb{Q})\rangle^2\oplus\langle H^{1,1}(X,\mathbb{Q})\rangle^6$ belongs to $\cup_{f\in F\setminus\{0\}}B_{\sqrt{-q}f\cdot\Theta}$ if and only if $f_1\neq 0$ and $\frac{f_2}{(-q/6)\gamma(f_1)}=f^2$, for some non-zero $f\in F$, in which case it belongs to $B_{\sqrt{-q}f\cdot\Theta}$.*

Proof. The statement follows from the equality

$$\begin{array}{rcl}B_{\sqrt{-q}f\cdot\Theta} & = & \{(a+b\sqrt{t})f^{-1}f\cdot\Theta-(q/6)Nm(f)\gamma((a+b\sqrt{t})f^{-1})f\cdot\Theta^3\}\\ & = & \{(a+b\sqrt{t})\Theta-(q/6)(a-b\sqrt{t})f^2\cdot\Theta^3\}.\end{array}$$

□

Corollary 11.2.6. *Let $f \in F$ be a non-zero element satisfying $\hat{\eta}(f) = g^* \in \mathrm{End}_{Hdg}(H^1(X,\mathbb{Q}))$, for some $g \in \mathrm{End}(X)$.*

1. *The class $\Theta - \frac{q}{6}g^*(\Theta^3)$ belongs to $B_{\sqrt{-q}Nm(f)f\cdot\Theta}$.*
2. *The class $g^*\Theta - \frac{q}{6}\Theta^3$ belongs to $B_{\sqrt{-q}\gamma(f^{-1})\cdot\Theta}$.*
3. *If g is an automorphism and $Nm(f) = 1$, then the class $g^*\Theta - \frac{q}{6}(g^{-1})^*(\Theta^3)$ belongs to $B_{\sqrt{-q}\Theta}$.*

Proof. (1) We have the equality $g^*(\Theta^3) = (f^2 \cdot \Theta)^3 = Nm(f)^2 f^2 \cdot \Theta^3$. The statement thus follows from Lemma 11.2.5.

(2) We have the equality $g^*(\Theta) = f^2 \cdot \Theta$. The statement thus follows from Lemma 11.2.5.

(3) $(g^2)^*\Theta - \frac{q}{6}\Theta^3$ belongs to $B_{\sqrt{-q}\gamma(f^{-2})\cdot\Theta}$, by part (2), $\gamma(f^{-2}) = f^2$, and $B_{\sqrt{-q}f^2\cdot\Theta} = B_{\sqrt{-q}g^*(\Theta)}$. The statement follows by applying $(g^{-1})^*$ to both sides of the statement $(g^2)^*\Theta - \frac{q}{6}\Theta^3 \in B_{\sqrt{-q}g^*(\Theta)}$. □

Assume that X is a Jacobian of a genus 4 curve, Θ is its principal polarization, $f \in F$ satisfies $Nm(f) = 1$, $f^2 \neq 1$, and $\hat{\eta}(f) = g^*$, where g is an automorphism of X. We have seen in [M2, Example 8.2.4] that $\Theta - \frac{q}{6}\Theta^3$ is an element of P_{K_0}, which is $ch(E_0)$ for a coherent sheaf E_0.

Example 11.2.7. Following is the construction of a simple coherent sheaf E' on X with Chern class an integer multiple of $\beta' := g^*\Theta - \frac{q}{6}(g^{-1})^*(\Theta^3)$. Choose a positive integer d sufficiently large, so that there exists a positive integer N such that

$$(N/6)\left(dg^*(\Theta^3) - q(g^{-1})^*(\Theta^3)\right)$$

is the class of a curve C' in X. The class $\beta := \Theta - \frac{d}{6}\Theta^3$ is the Chern class of a simple coherent sheaf F', by [M2, Example 8.2.4]. The difference $g^*ch(F') - \beta'$ is $[-dg^*(\Theta^3) + q(g^{-1})^*(\Theta^3)]/6$. Choose N generic translates of $g^*(F')$ so that C' intersects the support of each only at points where the sheaf is a line bundle over its support. Glue the direct sum of these N translates to a line bundle L over C' by choosing isomorphisms of the fibers at each of the points of intersection. For a suitable degree of L the resulting coherent sheaf E' will satisfy $\chi(E') = 0$ and so $ch(E') = N\beta'$.

Let E' be an object in $D^b(X)$ with $ch(E')$ an integer multiple of $g^*\Theta - \frac{q}{6}(g^{-1})^*(\Theta^3)$.

Lemma 11.2.8. *The object $\Phi(E_0 \boxtimes E')$ has non-zero rank and $\kappa(\Phi(E_0 \boxtimes E'))$ remains of Hodge type under any deformation of $(X \times \hat{X}, \eta)$ as an abelian variety with complex multiplication by K. Furthermore, the class $\kappa_2(\Phi(E_0 \boxtimes E'))$ belongs to the direct sum $\langle H^{1,1}(X,\mathbb{Q})\rangle^4 \oplus HW(X \times \hat{X}, \eta)$, but not to $\langle H^{1,1}(X,\mathbb{Q})\rangle^4$.*

Proof. It suffices to verify that the classes $\alpha_0 := \Theta - \frac{q}{6}\Theta^3$ and $\beta' := g^*\Theta - \frac{q}{6}(g^{-1})^*(\Theta^3)$ satisfy the hypotheses of Lemma 11.2.1. The class $ch(E')$ belongs to $B_{\sqrt{-q}\Theta}$, by Corollary 11.2.6, but it belongs to P_{K_0} only if $f^2 = 1$, by Lemma 11.2.4. We assumed that $f^2 \neq 1$. Hence, β' does not belong to P_{K_0}. The class β' does not belong to P_{K_1}, since $P_{K_1} \cap [H^2(X,\mathbb{Q}) \oplus H^6(X,\mathbb{Q})] = \mathrm{span}_{\mathbb{Q}}\{\sqrt{t}(\tilde{\Theta} - (q/3!)\tilde{\Theta}^3)\}$ and so the summand $f^2 \cdot \Theta$ in $H^2(X,\mathbb{Q})$ is linearly independent (over $\mathbb{Q}$) from $\sqrt{t}\tilde{\Theta} = \sqrt{t} \cdot \Theta$. It remains to verify that the rank of $\Phi(E_0 \boxtimes E')$ is non-zero. The rank is equal to $\chi(E_0 \overset{L}{\otimes} E')$. The latter is

$$\begin{array}{rcl} \int_X (\Theta - \frac{q}{2}\Theta_{\hat{\sigma}_1}\Theta_{\hat{\sigma}_2}\Theta)(f^2 \cdot \Theta - \frac{q}{2}\Theta_{\hat{\sigma}_1}\Theta_{\hat{\sigma}_2}f^{-2} \cdot \Theta) & = \\ -\frac{q}{2}(\hat{\sigma}_1(f)^{-2} + \hat{\sigma}_2(f)^{-2} + \hat{\sigma}_1(f)^2 + \hat{\sigma}_2(f)^2) \int_X (\Theta_{\hat{\sigma}_1}\Theta_{\hat{\sigma}_2})^2 & = \\ -\frac{q}{12}(\hat{\sigma}_1(f)^{-2} + \hat{\sigma}_2(f)^{-2} + \hat{\sigma}_1(f)^2 + \hat{\sigma}_2(f)^2) \int_X \Theta^4 \neq 0. & \end{array}$$

□

11.2.2. *The case* $Gal(\tilde{K}/\mathbb{Q}) \cong Gal(\tilde{F}/\mathbb{Q}) \times \mathbb{Z}/2\mathbb{Z}$. Lemma 11.2.1 generalizes to CM-fields K with maximal totally real subfield F, such that $K = F(\sqrt{-q})$, $q \in \mathbb{Q}$ and $q > 0$. In this case the Galois group $\mathrm{Gal}(\tilde{K}/\mathbb{Q})$ of the Galois closure $\tilde{K}$ of K is the cartesian product[8] of $\mathrm{Gal}(\tilde{F}/\mathbb{Q})$ and the involution ι in $\mathrm{Gal}(\tilde{K}/\tilde{F})$, such that $\sigma(\iota(x)) = \overline{\sigma(x)}$, for every embedding $\sigma : \tilde{K} \to \mathbb{C}$

$$\mathrm{Gal}(\tilde{K}/\mathbb{Q}) \cong \mathrm{Gal}(\tilde{F}/\mathbb{Q}) \times \{1, \iota\}.$$

Let T_1 be the CM-type, such that $T_1(\hat{\sigma})$ maps $\sqrt{-q}$ to the upper-half-plane in $\mathbb{C}$, for all $\hat{\sigma} \in \hat{\Sigma}$. Let $G \subset \mathrm{Gal}(\tilde{K}/\mathbb{Q})$ be the subgroup stabilizing $\sqrt{-q}$. The restriction homomorphism from G to $\mathrm{Gal}(\tilde{F}/\mathbb{Q})$ is an isomorphism, by our assumption on K, and both T_1 and $\bar{T}_1$ are G-invariant. Hence, each of ℓ_{T_1} and $\ell_{\bar{T}_1}$ is defined over the subfield $\mathbb{Q}(\sqrt{-q})$ of $\tilde{K}$ fixed by G and the plane P spanned in $S^+_{\mathbb{C}}$ by ℓ_{T_1} and $\ell_{\bar{T}_1}$ is rational.

Let $B_i \subset S^+_{\mathbb{C}}$ be the subspace of B spanned by $\{\ell_T : |T \cap T_1| = i\}$. So $P = B_0 \oplus B_{e/2}$. We get the direct sum decomposition

$$B_{\mathbb{C}} = \oplus_{i=0}^{e/2} B_i.$$

Note that $\bar{B}_i = B_{e/2-i}$ and $B_i \oplus B_{e/2-i}$ is $\mathrm{Gal}(\tilde{K}/\mathbb{Q})$-invariant, hence it is defined over $\mathbb{Q}$. We have the inclusions

$$\begin{array}{rcl} B_0 \otimes B_i & \subset & BB_i, \\ B_{e/2} \otimes B_i & \subset & BB_{e/2-i}, \\ (B_0 \oplus B_{e/2}) \otimes (B_i \oplus B_{e/2-i}) & \subset & BB_i + BB_{e/2-i}. \end{array}$$

Let α_0 be a non-zero rational class in $P = B_0 \oplus B_{e/2}$. Note that P is spanned by $\exp(\sqrt{-q}\Theta)$ and its complex conjugate. Let β be a rational class in B. Then $\beta = \sum_{i=0}^{\lceil e/4 \rceil} \beta_i$, where $\beta_i \in B_i + B_{e/2-i}$.

Lemma 11.2.9. *Assume that* $\beta_0 \neq 0$ *and* $\beta_1 \neq 0$ *and* $\check{\phi}(\alpha_0 \otimes \beta_0)_0 \neq 0$. *The class* $c := \alpha_0 \otimes \beta$ *satisfies the assumptions of Proposition 10.2.1.*

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[8]For more general CM-field, the restriction homomorphism $\mathrm{Gal}(\tilde{K}/\mathbb{Q}) \to \mathrm{Gal}(\tilde{F}/\mathbb{Q})$ is surjective, and its kernel is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^\nu$, where $1 \leq \nu \leq e/2$, by [D, Sec. 1.1]. The assumption that q is in $\mathbb{Q}$ and $q > 0$ implies that $\nu = 1$ and that the short exact sequence $0 \to \{1, \iota\} \to \mathrm{Gal}(\tilde{K}/\mathbb{Q}) \to \mathrm{Gal}(\tilde{F}/\mathbb{Q}) \to 0$ splits. Indeed, if $\tilde{F}$ is the splitting field of a polynomial $f(x) \in \mathbb{Q}[x]$, then $\tilde{F}(\sqrt{-q})$ is the splitting field of $f(x)(x^2 + q)$ and is hence a Galois extension of $\mathbb{Q}$, and so $\nu = 1$. Furthermore, the short exact sequence splits, since (a) the subgroup G of $\mathrm{Gal}(\tilde{K}/\mathbb{Q})$ stabilizing $\sqrt{-q}$ has index 2 in $\mathrm{Gal}(\tilde{K}/\mathbb{Q})$, as the $\mathrm{Gal}(\tilde{K}/\mathbb{Q})$-orbit of $\sqrt{-q}$ is $\pm\sqrt{-q}$, and (b) $\iota \notin G$.

Proof. The class $\alpha_0\otimes\beta_i$ belongs to $BB_i+BB_{e/2-i}$, c_0 is determined by $\alpha_0\otimes\beta_0$ and c_1 by $\alpha_0\otimes\beta_1\in BB_1+BB_{e/2-1}$. The rationality and non-vanishing of each of α_0 and β_1 implies that $c_1\neq 0$. Indeed, if $e=4$, then $c_1=\alpha_0\otimes\beta_1$ and if $e>4$, then $\alpha_0=\alpha_0'+\alpha_0''$, where $\alpha_0'\in B_0$, $\alpha_0''\in B_{e/2}$ and $\bar{\alpha}_0'=\alpha_0''\neq 0$. $\beta_1=\beta_1'+\beta_1''$, where $\beta_1'\in B_1$, $\beta_1''\in B_{e/2-1}$, and $\bar{\beta}_1'=\beta_1''\neq 0$. So $c_1=\alpha_0'\otimes\beta_1'+\alpha_0''\otimes\beta_1''$, where the first summand is in $B_0\otimes B_1$ and the second in $B_{e/2}\otimes B_{e/2-1}$. Hence, $c_1\neq 0$. Again the left hand side of each of the linear homogeneous equations of KB_1 in Equation (10.1.2), for each $\sigma\in\Sigma$, is a sum consisting of precisely one non-zero term $c_{(T,T')}$. Hence, the intersection $BB_1\cap[B_0\oplus B_{e/2}]\otimes[B_1+B_{e/2-1}]$ is e-dimensional and it is complementary of KB_1. We conclude that c_1 does not belong to KB_1. Finally, $\check{\phi}(c)_0=\check{\phi}(c_0)_0=\check{\phi}(\alpha_1\otimes\beta_1)_0$, by Proposition 10.2.1, which is assumed not to vanish. We conclude that c satisfies the assumptions of Proposition 10.2.1. □

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