# STABILITY OF STANDING WAVES FOR ALL FREQUENCIES TO NONLINEAR SCHRÖDINGER EQUATIONS WITH POTENTIALS IN ONE DIMENSION

NORIYOSHI FUKAYA, MASAHIRO IKEDA, AND HIROAKI KIKUCHI

*Dedicated to Professor Yoshio Tsutsumi on the occasion of his 70th birthday.*

Abstract. In this paper, we study the orbital stability of standing waves for one-dimensional nonlinear Schrödinger equations with potentials. We show that the standing waves are orbitally stable for all frequencies in the $L^2$- subcritical and critical cases. Since the presence of potentials breaks the scale invariance of the equations, it is a delicate problem to apply the abstract theory of Grillakis, Shatah, and Strauss (1987) directly without a perturbative argument. For this reason, little is known about the orbital stability of standing waves for *all* frequencies in the non-scale-invariant setting. We overcome this difficulty by employing the approach of Noris, Tavares, and Verzini (2014).

## 1. Introduction

In this paper, we consider the following one-dimensional nonlinear Schrödinger equations with a potential:

$$i\psi_t = -\psi_{xx} + V(x)\psi - |\psi|^{p-1}\psi, \quad (t,x) \in \mathbb{R} \times \mathbb{R}, \tag{1.1}$$

where $1 < p < \infty$, $V$ is a real-valued function, and $\psi = \psi(t,x)$ is the complex-valued unknown function.

Equation (1.1) is known as a model proposed to describe the local dynamics at a nucleation site. Here, the potential $V$ simulates a local depression in the ion density (see Rose and Weinstein [24]). In addition, (1.1) also appears in the Bose-Einstein condensate. The following three-dimensional nonlinear Schrönger equation

$$i\psi_t = -\Delta\psi + |x|^2\psi - |\psi|^2\psi \quad \text{in } \mathbb{R} \times \mathbb{R}^3 \tag{1.2}$$

describes the Bose-Einstein condensate with attractive inter-particle interactions under a magnetic trap. Equation (1.1) is derived from (1.2) and seems appropriate when its longitudinal dimension is much longer than its transverse ones (see [4] for the derivation of (1.1)).

Now, we state our assumption on the potential $V$:

(V1) $V \in L^1_{\mathrm{loc}}(\mathbb{R};\mathbb{R}) \cap C^1(\mathbb{R} \setminus \{0\})$ is even, non-decreasing function of $r = |x|$, and $\partial_x V \not\equiv 0$.

---

*Date*: September 30, 2025.

2020 *Mathematics Subject Classification.* 35Q55, 35B35.

*Key words and phrases.* Nonlinear Schrödinger equation, standing wave, stability.

Put

$$\omega_1 := -\inf\Big\{\int_{\mathbb{R}}(|u_x|^2+V(x)|u|^2)dx\colon u\in H^1(\mathbb{R}) \text{ with } V|u|^2\in L^1(\mathbb{R}),\ \|u\|_{L^2}=1\Big\}. \tag{1.3}$$

Under the assumption (V1), we see that $\omega_1$ is well-defined, that is, $\omega_1<\infty$ (see Lemma A.1 (i) below).

**Definition 1.1.** Let $\omega>\omega_1$. We define a real Hilbert space $X$ by

$$X := \{u\in H^1(\mathbb{R})\colon V(x)|u|^2\in L^1(\mathbb{R})\}$$

equipped with the inner product

$$(u,v)_{X_\omega} := \operatorname{Re}\int_{\mathbb{R}}\big(u_x(x)\overline{v_x(x)}+V(x)u(x)\overline{v(x)}+\omega u(x)\overline{v(x)}\big)dx.$$

The norm of $X$ is denoted by $\|\cdot\|_{X_\omega}$. In particular, we use $\|\cdot\|_X := \|\cdot\|_{X_{\omega_1+1}}$. In addition, we denote by $X^*$ the dual of the function space $X$.

We also find from the assumption (V1) that

$$X\hookrightarrow H^1(\mathbb{R}). \tag{1.4}$$

See Lemma A.1 (ii) below.

Throughout this paper, we always assume the following:

**Assumption 1.1.** For any $\psi_0\in X$, there exist $T_{\max}=T_{\max}(\|\psi_0\|_X)>0$ and a solution $\psi\in C([0,T_{\max}),X)$ to (1.7) with $\psi|_{t=0}=\psi_0$ satisfying

$$E(\psi(t))=E(\psi_0),\quad \|\psi(t)\|_{L^2}=\|\psi_0\|_{L^2}\quad \text{for all } t\in[0,T_{\max}),$$

where $E$ is the energy defined by

$$E(u)=\frac{1}{2}\int_{\mathbb{R}}(|u_x(x)|^2+V(x)|u(x)|^2)dx-\frac{1}{p+1}\|u\|_{L^{p+1}}^{p+1}.$$

*Remark* 1.1. The assumption 1.1 holds if there exist real valued functions $V_1$ and $V_2$ such that $V(x)=V_1(x)+V_2(x)$, where $V_1$ and $V_2$ satisfy the following:

(i) $V_1\in C^\infty(\mathbb{R})$, $V_1(x)\geq 0$ for $x\in\mathbb{R}$, $\partial_x^m V_1\in L^\infty(\mathbb{R})$ for $m\geq 2$
(ii) $V_2\in L^1(\mathbb{R})+L^\infty(\mathbb{R})$.

(see [6, Theorem 3.3.1, 3.3.5, 3.3.9, Proposition 4.2.3 and Theorem 9.2.6] and [19]).

Here, we are interested in a standing wave solution to (1.7). By a *standing wave*, we mean a solution to (1.1) of the form

$$\psi(t,x)=e^{i\omega t}\phi_\omega(x)\qquad(\omega\in\mathbb{R}).$$

Then we see that the function $\phi_\omega$ satisfies the following elliptic equations:

$$-\phi_{xx}+V(x)\phi+\omega\phi-|\phi|^{p-1}\phi=0,\quad x\in\mathbb{R}. \tag{1.5}$$

We define the action functional $S_\omega \in C^1(X,\mathbb{R})$ by

$$S_\omega(u) := E(u) + \frac{\omega}{2}\|u\|_{L^2}^2.$$

Then we see that $u \in X$ is a weak solution to (1.5) if and only if $S'_\omega(u) = 0$ in $X^*$. We call by *ground state* a solution to (1.5) which minimizes the action functional $S_\omega$ among all non-trivial solutions to (1.5) in $X$. We denote by $\mathcal{A}_\omega$ the set of all non-trivial solutions to (1.5) in $X$ and by $\mathcal{G}_\omega$ that of all ground states to (1.5). Namely,

$$\mathcal{A}_\omega := \{u \in X \setminus \{0\} \colon S'_\omega(u) = 0 \text{ in } X^*\}$$

and

$$\mathcal{G}_\omega := \{u \in \mathcal{A}_\omega \colon S_\omega(u) \le S_\omega(v) \text{ for all } v \in \mathcal{A}_\omega\}. \tag{1.6}$$

The following properties of ground states are well-known. For the readers' convenience, we give these proofs later.

**Theorem 1.1.** *Let $\omega > \omega_1$ and $1 < p < \infty$. Assume that (V1). Then there exists $(\phi_\omega)_{\omega>\omega_1}$ such that the following is true.*

(i) $\phi_\omega \in \mathcal{A}_\omega$ *is positive, even, and decreasing in* $r = |x|$.
(ii) $\mathcal{G}_\omega = \{e^{i\theta}\phi_\omega \colon \theta \in \mathbb{R}\}$.
(iii) $S''_\omega(\phi_\omega)$ *has a unique negative eigenvalue* $\nu_\omega$, *which is simple.*
(iv) $\ker S''_\omega(\phi_\omega) = \operatorname{span}\{i\phi_\omega\}$.
(v) $\sigma(S''_\omega(\phi_\omega)) \setminus \{\nu_\omega, 0\} \subset [\delta,\infty)$ *for some* $\delta > 0$, *where* $\sigma(S''_\omega(\phi_\omega))$ *is the set of spectrum of the operator* $S''_\omega(\phi_\omega)$.

Next, we study the orbital stability of standing waves $e^{i\omega t}\phi_\omega$. The orbital stability/instability of standing waves is defined by the following:

**Definition 1.2.** We say that the standing wave $e^{i\omega t}\phi_\omega$ is orbitally stable if for any $\varepsilon > 0$, there exists $\delta > 0$ such that if

$$\|\psi_0 - \phi_\omega\|_X < \delta,$$

then the corresponding solution $\psi$ to (1.1) with $\psi|_{t=0} = \psi_0$ exists globally and satisfies

$$\sup_{t>0}\inf_{\theta\in\mathbb{R}} \|\psi(t) - e^{i\theta}\phi_\omega\|_X < \varepsilon.$$

Otherwise, we say that the standing wave $e^{i\omega t}\phi_\omega$ is orbitally unstable

The orbital stability and instability of the standing waves to nonlinear Schrödinger equations with potentials has been studied so far (see [1, 2, 5, 9, 11, 12, 13, 17, 21, 22, 24, 28, 29] and references therein). Let us recall some of them. To this end, we consider the following nonlinear Schrödinger equations in general dimensions:

$$i\psi_t = -\Delta\psi + V(x)\psi - |\psi|^{p-1}\psi \quad \text{in } \mathbb{R}\times\mathbb{R}^d, \tag{1.7}$$

where $d \geq 1$, $1 < p < 2^* - 1$, and $V$ is a real valued function. Here

$$2^* = \begin{cases} \infty & \text{when } d = 1, 2, \\ \dfrac{2d}{d-2} & \text{when } d \geq 3. \end{cases}$$

When we consider the standing waves $\psi(t, x) = e^{i\omega t}\phi_\omega(x)$, then $\phi_\omega$ satisfies

$$-\Delta\phi + V(x)\phi + \omega\phi - |\phi|^{p-1}\phi = 0 \quad \text{in } \mathbb{R}^d. \tag{1.8}$$

We first consider the case where $V \equiv 0$. In this case, the equation (1.7) is invariant under the scaling

$$\psi(t, x) \mapsto \psi_\lambda(t, x) = \lambda^{\frac{2}{p-1}}\psi(\lambda^2 t, \lambda x). \tag{1.9}$$

Note that

$$\|\psi_\lambda(0, \cdot)\|_{L^2} = \lambda^{\frac{2}{p-1}-\frac{d}{2}}\|\psi(0, \cdot)\|_{L^2}. \tag{1.10}$$

Thus, we see that the scaling (1.9) preserves the $L^2$-norm $\|\cdot\|_{L^2}$ when $p = 1 + 4/d$. For this reason, the exponent $p = 1+4/d$ is referred to as "$L^2$-critical". Let $\phi_\omega$ be the ground state of (1.8). Cazenave and Lions [5] proved that standing waves $e^{i\omega t}\phi_\omega$ are stable when $L^2$-subcritical case $1 < p < 1 + 4/d$. On the other hand, Berestycki and Cazenave [2] showed that standing waves $e^{i\omega t}\phi_\omega$ are unstable when $L^2$-supercritical case $1 + 4/d < p < 2^* - 1$. Concerning with the $L^2$-critical case $p = 1 + 4/d$, Weinstein [29] proved that the standing waves $e^{i\omega t}\phi_\omega$ are unstable for any $\omega > 0$, where $\phi_\omega \in H^1(\mathbb{R}^d)$ is any non-trivial solution to (1.8) (not necessary the ground state). When we consider the ground state $\phi_\omega$ to (1.8), we can also study the stability of standing waves $e^{i\omega t}\phi_\omega$ by applying the abstract theory of Grillakis, Shatah and Strauss [14], in which they showed that the standing wave $e^{i\omega t}\phi_\omega$ is stable if $\frac{d}{d\omega}\|\phi_\omega\|_{L^2}^2 > 0$ and unstable if $\frac{d}{d\omega}\|\phi_\omega\|_{L^2}^2 < 0$ under several assumptions. See Comech and Pelinovsky [7], Ohta [20], and Maeda [16] for the case of $\frac{d}{d\omega}\|\phi_\omega\|_{L^2}^2 = 0$. When $V \equiv 0$, it follows from (1.10) that

$$\|\phi_\omega\|_{L^2}^2 = \omega^{\frac{2}{p-1}-\frac{d}{2}}\|\phi_1\|_{L^2}^2. \tag{1.11}$$

From this, we can check the sign of $\frac{d}{d\omega}\|\phi_\omega\|_{L^2}^2$ and find that

$$\frac{d}{d\omega}\|\phi_\omega\|_{L^2}^2 \begin{cases} > 0 & \text{if } 1 < p < 1 + \dfrac{4}{d}, \\ = 0 & \text{if } p = 1 + \dfrac{4}{d}, \\ < 0 & \text{if } 1 + \dfrac{4}{d} < p < 2^* - 1. \end{cases}$$

It is well-known that the ground state $\phi_\omega$ satisfies the assumptions of the abstract theory of Grillakis, Shatah and Strauss [14]. Thus, we can find that the standing wave $e^{i\omega t}\phi_\omega$ is stable for $1 < p < 1+4/d$ and unstable for $1 + 4/d < p < 2^* - 1$.

Now we pay our attention to the case of $V \not\equiv 0$. In this case, the equations (1.7) are not scale invariant, so that (1.11) does not hold. For this reason, we cannot check the sign of $\frac{d}{d\omega}\|\phi_\omega\|_{L^2}^2$ at least directly. Rose and Weinstein [24] employed the bifurcation theory and computed the sign of $\frac{d}{d\omega}\|\phi_\omega\|_{L^2}^2$ for $\omega \in \mathbb{R}$ which is close to $\omega_1$. After that, Fukuizumi and Ohta [12] gave another sufficient condition of stability which is based on the result of Grillakis, Shatah, and Strauss [14] and showed that standing waves $e^{i\omega t}\phi_\omega$ ($\phi_\omega \in \mathcal{G}_\omega$) are stable for sufficiently large $\omega > 0$ when $1 < p < 1 + 4/d$. Fukuizumi [10] also showed the stability for large $\omega$ when $p = 1+4/d$. Since they used a kind of perturbation argument to verify the sufficient condition, some restrictions on the frequency $\omega \in \mathbb{R}$ were required. There are several results in which the stability of standing waves $e^{i\omega t}\phi_\omega$ is obtained under some conditions on the frequency $\omega \in \mathbb{R}$. Compared with these, there are few results in which the stability of standing waves are obtained without any additional conditions on the frequency $\omega \in \mathbb{R}$. McLeod, Stuart, and Troy [17] studied the behavior of $\partial_\omega\phi_\omega$ using the continuity argument and obtained the stability of standing waves in one dimensional case $d = 1$. In particular, the standing waves $e^{i\omega t}\phi_\omega$ are stable for any $\omega > \omega_1$ if $1 < p \leq 5$. However, they assumed the boundedness on the potential $V$. Thus, the harmonic potential $V(x) = |x|^2$ and the inverse potential $V(x) = -|x|^{-\theta}$ ($\theta \in (0,1)$) were not treated in their result.

Here, using the argument of [18], we study the stability of standing waves $e^{i\omega t}\phi_\omega$ ($\phi_\omega \in \mathcal{G}_\omega$) for all $\omega > \omega_1$. Our result includes unbounded potentials such as $V(x) = |x|^2, -|x|^{-\theta}$ ($\theta \in (0,1)$) and so on. To study the orbital stability of the standing waves $e^{i\omega t}\phi_\omega$, we need the following additional condition:

(V2) $2V + xV'$ is a non-decreasing function of $r = |x|$. Furthermore, $2V + xV'$ is not a constant function if $p = 5 = 1 + 4/d|_{d=1}$.

*Remark* 1.2. Concerning the assumption (V2), note first that in the one-dimensional setting the case where $2V + xV'$ is constant is already excluded by (V1). Indeed, in this case we can write $V(r) = c_1 + c_2 r^{-2}$ for some $c_1 \in \mathbb{R}$ and $c_2 \in \mathbb{R}$. If $c_2 \neq 0$, then $V \notin L^1_{\mathrm{loc}}(\mathbb{R})$; whereas if $c_2 = 0$, then $\partial_x V \equiv 0$.

In higher dimensions, however, one may consider the case where $2V + xV'$ is constant, in particular $V(r) = -r^{-2}$. In this situation, equation (1.7) is invariant under the scaling (1.9). Hence, when $p = 1+4/d$, it follows from the same argument as in [29] that all standing waves are strongly unstable. Therefore, in the assumption (V2), it is essential to exclude the case where $2V + xV'$ is constant for $p = 1 + 4/d$.

Then we obtain the following:

**Theorem 1.2.** *Let $d = 1, 1 < p \le 5$, and $\phi_\omega \in \mathcal{G}_\omega$. Assume that Assumption 1.1, (V1), and (V2). Then the standing wave $e^{i\omega t}\phi_\omega$ is stable for any $\omega > \omega_1$.*

*Remark* 1.3.
(i) Our result covers $V(x) = |x|^2$, $V(x) = -|x|^{-\theta}$ with $\theta \in (0, 1)$, and combinations of these. As far as the authors know, in these cases, the stability of standing waves for all frequencies has not been studied so far.
(ii) For the case of $1+4/d < p < 2^* - 1$, it is known that the standing waves $e^{i\omega t}\phi_\omega$ ($\phi_\omega \in \mathcal{G}_\omega$) are unstable for sufficiently large $\omega > 0$. Thus, we cannot obtain the stability for all frequency in this case, so that the condition $1 < p \le 1 + 4/d$ in Theorem 1.2 is essential.

To prove Theorem 1.2, we use the argument of [18], in which they studied the stability for the following nonlinear Schrödinger equation on the unit ball $B_1$ with the Dirichlet boundary condition:

$$\begin{cases} i\psi_t = -\Delta\psi - |\psi|^{p-1}\psi & \text{in } \mathbb{R} \times B_1, \\ \psi = 0 & \text{in } \mathbb{R} \times \partial B_1. \end{cases} \tag{1.12}$$

Actually, since the equations (1.12) are not scale-invariant, we cannot check the sign of $\frac{d}{d\omega}\|\phi_\omega\|_{L^2}^2$ at least directly. To overcome the difficulty, Noris, Tavares, and Verzini [18] considered the following optimization problem with two constraints:

$$M_\alpha := \sup\Big\{\int_{B_1} |u(x)|^{p+1}dx \colon u \in H_0^1(B_1),\ \int_{B_1} |u(x)|^2 dx = 1,\ \int_{B_1} |\nabla u(x)|^2 dx = \alpha\Big\}$$

for $\alpha > 0$. Then they showed that for any $\alpha > \lambda_1(B_1)$, where $\lambda_1(B_1)$ is the first eigenvalue of the Dirichlet Laplacian in the unit ball $B_1$, there exists a maximizer $u_\alpha \in H_0^1(B_1)$ for $M_\alpha$. In addition, $u_\alpha \in H_0^1(B_1)$ is a positive function and there exist $\mu_\alpha > 0$ and $\omega_\alpha > -\lambda_1(B_1)$ such that

$$-\Delta u_\alpha + \omega_\alpha u_\alpha = \mu_\alpha u_\alpha^p, \qquad \int_{B_1} |u_\alpha(x)|^2 dx = 1, \qquad \int_{B_1} |\nabla u_\alpha(x)|^2 dx = \alpha.$$

Then they showed that

$$\frac{d}{d\omega}\|\phi_\omega\|_{L^2}^2 = C_\alpha \frac{\partial \mu_\alpha}{\partial \alpha},$$

where $C_\alpha$ is a positive constant depending on $\alpha > \lambda_1(B_1)$. Thus, in order to check the sign of $\frac{d}{d\omega}\|\phi_\omega\|_{L^2}^2$, it suffices to do that of $\partial_\alpha \mu_\alpha$. Then by exploiting the Dirichlet boundary condition, they showed that $\partial_\alpha \mu_\alpha > 0$ for all $\alpha > \lambda_1(B_1)$ when $1 < p \le 1 + 4/d$.

Here, we simplify the above argument and determine the sign of $\frac{d}{d\omega}\|\phi_\omega\|_{L^2}^2$ directly. We also note that since our spatial domain is the entire space, we cannot apply the argument of [18] directly. To overcome this difficulty, we investigate the asymptotic behavior of $\partial_\omega \phi_\omega$ more precisely and use (V2).

*Remark* 1.4. Here, we emphasize that we essentially need the condition $d = 1$ for obtaining $\partial_\omega\phi_\omega(0) > 0$ (see Lemma 6.5 below for details). Once $\partial_\omega\phi_\omega(0) > 0$ is established, our result can be extended to higher dimensions under certain assumptions.

The rest of this paper is organized as follows. In Section 2, we show the existence of ground states. In Section 3, we prove the positivity and evenness of the ground states. In Section 4, we show the uniqueness of the ground state. In Section 5, we establish its non-degeneracy. In Section 6, we study the orbital stability of the standing waves $e^{i\omega t}\phi_\omega$ and give the proof of Theorem 1.2. In Appendix A we prove the finiteness of $\omega_1$ and (1.4). Finally, in Appendix B, we prove an auxiliary result that is needed for the non-degeneracy of the ground state.

## 2. Existence of ground state

In this section, we show the existence of the ground state of (1.5). To this end, we consider the following minimization problem:

$$d(\omega) = \inf\{S_\omega(u)\colon u \in X \setminus \{0\},\ K_\omega(u) = 0\}, \tag{2.1}$$

where

$$K_\omega(u) := \langle S'_\omega(u), u\rangle = \|u\|_{X_\omega}^2 - \|u\|_{L^{p+1}}^{p+1} \tag{2.2}$$

is the Nehari functional. Concerning the minimization problem, we obtain the following.

**Theorem 2.1.** *Assume that (V1). There exists a minimizer $\phi_\omega \in X \setminus \{0\}$ for $d(\omega)$.*

We put

$$Q_\omega(u) = S_\omega(u) - \frac{1}{p+1}K_\omega(u) = \frac{p-1}{2(p+1)}\|u\|_{X_\omega}^2, \tag{2.3}$$

$$N(u) = S_\omega(u) - \frac{1}{2}K_\omega(u) = \frac{p-1}{2(p+1)}\|u\|_{L^{p+1}}^{p+1}. \tag{2.4}$$

Then we have

$$\begin{aligned} d(\omega) &= \inf\{Q_\omega(u)\colon u \in X \setminus \{0\},\ K_\omega(u) = 0\} \\ &= \inf\{N(u)\colon u \in X \setminus \{0\},\ K_\omega(u) = 0\}. \end{aligned} \tag{2.5}$$

**Lemma 2.1.** $d(\omega) > 0$ *for* $\omega > \omega_1$.

*Proof.* Let $u \in X \setminus \{0\}$ satisfy $K_\omega(u) = 0$. By the Sobolev embedding (1.4), we have

$$\|u\|_{X_\omega}^2 = \|u\|_{L^{p+1}}^{p+1} \le C\|u\|_{X_\omega}^{p+1}$$

for some $C > 0$, which is independent of $u \in X$. This implies that $\|u\|_{X_\omega} \ge (1/C)^{\frac{1}{p-1}}$. From this and (2.5), we have obtained the desired result. □

**Lemma 2.2.** *If $u \in X$ satisfies $Q_\omega(u) \le d(\omega) \le N(u)$, then $u$ is a minimizer for $d(\omega)$.*

*Proof.* Note that $u \neq 0$ because $N(u) \ge d(\omega) > 0$. We put $\lambda_0 := (Q_\omega(u)N(u)^{-1})^{\frac{1}{p-1}} \in (0,1]$. Then we have $K_\omega(\lambda_0 u) = 0$. From (2.5) and our assumption, we obtain

$$Q_\omega(u) \le d(\omega) \le Q_\omega(\lambda_0 u) = \lambda_0^2 Q_\omega(u) \le Q_\omega(u).$$

This implies $Q_\omega(u) = d(\omega)$, $\lambda_0 = 1$, and $K_\omega(u) = 0$. □

*Proof of Theorem 2.1.* Let $\{u_n\}$ be a minimizing sequence of $d(\omega)$, that is, $K_\omega(u_n) = 0$ and

$$Q_\omega(u_n) = N(u_n) \to d(\omega) \quad \text{as } n \to \infty.$$

Note that $\{u_n\}$ is bounded in $X$. We put $v_n := |u_n|^*$, where $f^*$ is the Schwarz rearrangement of the function $f$. Then we see that $\|v_n\|_{L^q} = \|u_n\|_{L^q}$ for any $q \ge 1$ and $\|\nabla v_n\|_{L^2} \le \|\nabla u_n\|_{L^2}$ (see [15, Section 3.3 and Lemma 7.17]). Moreover, for any $f \in X$, the inequality

$$\int_{\mathbb{R}} V(x)|f^*|^2 dx \le \int_{\mathbb{R}} V(x)|f|^2 dx \tag{2.6}$$

holds. Indeed, by the increasing monotonicity of $V$, we have $\int_{\mathbb{R}} V(x)\chi_{\{x\colon |f(x)|^2>s\}}\, ds \ge \int_{\mathbb{R}} V(x)\chi_{\{x\colon |f^*(x)|^2>s\}}\, ds$ for all $s > 0$. From this and the layer cake representation $|f(x)|^2 = \int_0^\infty \chi_{\{x\colon |f(x)|^2>s\}}\, ds$ (see [15, Theorem 1.13 (4)]), we obtain

$$\begin{aligned}\int_{\mathbb{R}} V(x)|f^*|^2\, dx &= \int_{\mathbb{R}}\int_0^\infty V(x)\chi_{\{x\colon |f^*(x)|^2>s\}}\, ds\, dx \\ &\le \int_0^\infty \int_{\mathbb{R}} V(x)\chi_{\{x\colon |f(x)|^2>s\}}\, ds\, dx = \int_{\mathbb{R}} V(x)|f|^2\, dx.\end{aligned}$$

Therefore, we have $\|v_n\|_X \le \|u_n\|_X$. This implies $\{v_n\}$ is also bounded in $X$. Then up to subsequence (we still denote it by the same symbol), there exists $u_\infty \in X$ such that $v_n$ converges weakly to $u_\infty$ in $X$. By the weak lower semi-continuity of norms, we have

$$Q_\omega(u_\infty) \le \lim_{n\to\infty} Q_\omega(v_n) \le \lim_{n\to\infty} Q_\omega(u_n) = d(\omega).$$

Moreover, by the radial monotonicity compactness lemma [6, Proposition 1.7.1], we have

$$N(v_\infty) = \lim_{n\to\infty} N(v_n) = \lim_{n\to\infty} N(u_n) = d(\omega). \tag{2.7}$$

By Lemma 2.2, $u_\infty \in X$ is a minimizer for $d(\omega)$. This completes the proof. □

We set

$$\mathcal{M}_\omega := \{u \in X : K_\omega(u) = 0,\ S_\omega(u) = d(\omega)\},$$

which is the set of all minimizers for $d(\omega)$.

**Theorem 2.2.** *For any $\omega > 0$, we have $\mathcal{M}_\omega = \mathcal{G}_\omega$.*

*Proof.* Let $\phi_\omega \in \mathcal{G}_\omega$. Then we have $K_\omega(\phi_\omega) = \langle S'_\omega(\phi_\omega), \phi_\omega\rangle = 0$. This together with (2.1) yields that

$$d(\omega) \le S_\omega(\phi_\omega). \tag{2.8}$$

Next, let $v_\omega \in \mathcal{M}_\omega$. Then there exists a Lagrange multiplier $\lambda_\omega \in \mathbb{R}$ such that $S'_\omega(v_\omega) = \lambda_\omega K'_\omega(v_\omega)$. Since $K_\omega(v_\omega) = 0$, we obtain $\langle S'_\omega(v_\omega), v_\omega\rangle = K_\omega(v_\omega) = 0$ and $\langle K'_\omega(v_\omega), v_\omega\rangle = -(p-1)\|v_\omega\|_{L^{p+1}}^{p+1} \neq 0$. Hence, we find that $\lambda_\omega = 0$, that is, $S'_\omega(v_\omega) = 0$. Thus, by (1.6), we obtain $S_\omega(\phi_\omega) \le S_\omega(v_\omega) = d(\omega)$. This together with (2.8) yields that $S_\omega(\phi_\omega) = S_\omega(v_\omega) = d(\omega)$. This implies that $\phi_\omega \in \mathcal{M}_\omega$ and $v_\omega \in \mathcal{G}_\omega$. Therefore, we conclude that $\mathcal{M}_\omega = \mathcal{G}_\omega$. □

Now, we give the proof of Theorem 1.1 (i).

*Proof of Theorem 1.1 (i).* We see from the proof of Theorem 2.1 that there exists a minimizer $\phi_\omega \in X$ of $d(\omega)$, which is positive, even and decreasing in $r = |x|$. By Theorem 2.2, we see that $\phi_\omega \in \mathcal{G}_\omega$. This completes the proof. □

## 3. Positivity and evenness of ground state

In this section, we show that the ground states are positive up to phase and even functions.

**Lemma 3.1.** *Assume that (V1). Then $\mathcal{A}_\omega \subset C^1(\mathbb{R}) \cap C^3(\mathbb{R}\setminus\{0\})$.*

*Proof.* The assertion follows from [15, Theorem 11.7 (i) and (vi)]. □

**Theorem 3.1.** *Assume that (V1). Let $u \in \mathcal{M}_\omega$. There exists $\theta \in \mathbb{R}$ such that $e^{i\theta}u$ is a positive function.*

*Proof.* Take $\theta \in \mathbb{R}$ such that $e^{i\theta}u(0) \ge 0$, and put $v := e^{i\theta}u$.

First, we show the positivity of $v$. We put $v_1 := |\operatorname{Re} v|$ and $v_2 := |\operatorname{Im} v|$. Then we have $|v_1 + iv_2| = |u|$ and $|\nabla(v_1 + iv_2)| = |\nabla u|$, and so $v_1 + iv_2 \in \mathcal{M}_\omega$. Since $\mathcal{M}_\omega \subset \mathcal{A}_\omega$, $v_1 + iv_2$ solves (1.5). In, particular, $v_j$ solves

$$-v_j'' + (\omega + V + |u|^{p-1})v_j = 0, \quad j = 1, 2.$$

Moreover, Lemma 3.1 implies $v_j \in C^1(\mathbb{R}) \cap C^3(\mathbb{R}\setminus\{0\})$. Since $v_2(0) = v_2'(0) = 0$, where $v_2'(0) = 0$ follows from $v_2 \ge 0$ and $v_2 \in C^1(\mathbb{R})$, the uniqueness of the solution of the initial value problem for the ODE implies $v_2 \equiv 0$. Moreover, if $v_1(x_0) = 0$ for some $x_0 \in \mathbb{R}$, since $v_1 \ge 0$ and $v_1 \in C^1(\mathbb{R})$, we have $v_1'(x_0) = 0$, which implies $v_1 \equiv 0$. This contradicts to $u \not\equiv 0$. Thus $v_1 > 0$ on $\mathbb{R}$. Moreover, since $|v| = v_1 > 0$, $v(0) > 0$, and $v$ is continuous, we see that $v > 0$ on $\mathbb{R}$. □

**Lemma 3.2.** *Assume (V1). If a positive function $u \in X$ satisfies*

$$(u^*)'(r) < 0 \quad \text{for all } r > 0, \qquad \|u_x\|_{L^2} = \|(u^*)_x\|_{L^2}, \qquad \int_{\mathbb{R}} V|u|^2 dx = \int_{\mathbb{R}^d} V|u^*|^2 dx,$$

*where $u^*$ is the Schwarz rearrangement of the function $u$, then $u = u^*$.*

*Proof.* Since $(u^*)' < 0$, we have $\{x \in \mathbb{R}\colon 0 < u^*(x) < \|u^*\|_{L^\infty},\ |\nabla u^*(x)| = 0\} = \emptyset$. Therefore, by applying [3, Theorem 1.1] together with $\|u_x\|_{L^2} = \|(u^*)_x\|_{L^2}$, we obtain $u = u^*(\cdot - y)$ for some $y \in \mathbb{R}$.

Suppose $y \neq 0$ and derive a contradiction. By the strict monotonicity of $u^*$, we see that $\{x \in \mathbb{R}\colon u^*(x)^2 > s\} = B_{R(s)}(0)$ for some continuous and strictly decreasing function $R(s)$ with $R(s) \to \infty$ as $s \downarrow 0$, where $B_R(y)$ denotes the open ball centered at $y$ with radius $R$. By the monotonicity of $V$, we have $\int_{B_{R(s)}(0)} V(x)\,dx \le \int_{B_{R(s)}(y)} V(x)\,dx$ for all $s > 0$. Moreover, since $V'(r_0) > 0$ for some $r_0 > 0$, we can take $s_0 > 0$ sufficiently small so that $r_0 \in B_{R(s_0)}(0)$. Then it is clear that $\int_{B_{R(s_0)}(0)} V(x)\,dx < \int_{B_{R(s_0)}(y)} V(x)\,dx$. Thus, noting that $s \mapsto \int_{B_{R(s)}(0)} V(x)\,dx$ and $s \mapsto \int_{B_{R(s)}(y)} V(x)\,dx$ are continuous because $V \in L^1_{\mathrm{loc}}(\mathbb{R})$, we obtain

$$\begin{aligned}\int_{\mathbb{R}} V(x)|u^*|^2 dx &= \int_0^\infty \int_{B_{R(s)}(0)} V(x)\,dx\,ds < \int_0^\infty \int_{B_{R(s)}(y)} V(x)\,dx\,ds \\ &= \int_{\mathbb{R}} V(x) u^*(x-y)^2\,dx = \int_{\mathbb{R}} V(x)|u|^2\,dx,\end{aligned}$$

which contradicts $\int_{\mathbb{R}} V(x)|u^*|^2\,dx = \int_{\mathbb{R}} V(x)|u|^2\,dx$. Therefore, $y = 0$. This completes the proof. $\square$

**Theorem 3.2.** *Assume that (V1). Let $u \in \mathcal{M}_\omega$ be a positive ground state. Then $u$ is even and strictly decreasing.*

*Proof.* First, by $\|(u^*)_x\|_{L^2} \le \|u_x\|_{L^2}$ and (2.6), we have $Q_\omega(u^*) \le Q_\omega(u) = d(\omega) = N(u) = N(u^*)$, which, by Lemma 2.2, implies $u^* \in \mathcal{M}_\omega$. In particular, $u^* \in \mathcal{A}_\omega$, and moreover $\|u_x\|_{L^2} = \|(u^*)_x\|_{L^2}$ and $\int_{\mathbb{R}} V|u|^2 dx = \int_{\mathbb{R}} V|u^*|^2 dx$.

Next, we show that $(u^*)' < 0$ on $(0,\infty)$. Clearly $(u^*)' \le 0$. Suppose $(u^*)'(r_0) = 0$ for some $r_0 > 0$. Then $(u^*)''(r_0) \ge 0$ and $u^*(r_0) > 0$. We put $e(r) := \frac{1}{2}(u^*)'(r)^2 - \frac{1}{2}(\omega + V(r))u^*(r)^2 + \frac{1}{p+1}u^*(r)^{p+1}$. Since $u \in X$, we have $\int_0^\infty e(r)\,dr < \infty$. Moreover, by (1.5), $e'(r) = -\frac{1}{2}V'(r)(u^*)^2 \le 0$. Thus, $\lim_{r\to\infty} e(r) = 0$ and $e(r) \ge 0$ for all $r \ge 0$. In particular, $\frac{1}{2}(\omega + V(r_0))u^*(r_0)^2 \le \frac{1}{p+1}u^*(r_0)^{p+1}$. Therefore,

$$(u^*)''(r_0) = (\omega + V(r_0))u^*(r_0) - u^*(r_0)^p \le -\frac{p-1}{p+1}u^*(r_0)^p < 0,$$

which contradicts $(u^*)''(r_0) \ge 0$. Hence $(u^*)' < 0$ on $(0,\infty)$.

Therefore, by Lemma 3.2, we obtain $u = u^*$ and $u'(r) < 0$ on $(0,\infty)$. This completes the proof. $\square$

## 4. Uniqueness of ground state

This section is devoted to the following:

**Theorem 4.1.** *Assume that (V1). Then the positive solution $\phi_\omega \in X$ to* (1.5) *is unique.*

Let $\phi_\omega \in X$ be a positive solution to (1.5). Then we see from Theorem 3.2 that $\phi_\omega$ becomes even and $\phi_\omega'(x) < 0$ for all $x > 0$. Thus, it suffices to consider even positive solutions to (1.5). Let $r = |x|$. This together with Lemma 3.1 implies that (1.5) is reduced to the following:

$$\begin{cases} \phi'' - (V(r) + \omega)\phi + \phi^p = 0 & \text{on } (0, \infty), \\ \phi'(0) = 0 \end{cases} \tag{4.1}$$

where the prime mark denotes the differentiation with respect to $r$. We will adapt the argument of [8, Theorem 3.3] who proved the uniqueness of positive solution to semilinear elliptic equations in one dimension by using the argument of [26, 27]. To this end, we consider the general ordinary differential equation:

$$\phi'' + \frac{f'(r)}{f(r)}\phi' - g(r)\phi + h(r)\phi^p = 0. \tag{4.2}$$

We put

$$\begin{aligned} a(r) &:= f(r)^{2(p+1)/(p+3)} h(r)^{-2/(p+3)}, \\ b(r) &:= -\frac{1}{2}a'(r) + \frac{f'(r)}{f(r)}a(r), \\ c(r) &:= -b'(r) + \frac{f'(r)}{f(r)}b(r), \\ G(r) &:= b(r)g(r) + \frac{1}{2}c'(r) - \frac{1}{2}(ag)'(r). \end{aligned}$$

Let $\phi_\omega$ be a positive solution to (4.2). We put

$$J(r;\phi) := \frac{1}{2}a(r)(\phi'(r))^2 + b(r)\phi'(r)\phi(r) + \frac{1}{2}c(r)\phi^2(r) - \frac{1}{2}a(r)g(r)\phi^2(r) + \frac{1}{p+1}a(r)h(r)\phi^{p+1}(r).$$

Then it is known that the following Pohozaev identity holds:

**Proposition 4.1.** *Let $\phi$ be a positive solution to* (4.2)*. Then we have*

$$\frac{d}{dr}J(r;\phi) = G(r)\phi^2(r) \tag{4.3}$$

*for all $r > 0$.*

We now modify the result of [8, Theorem 3.3]. To state the result, we prepare the following:

$$U_1(r) := \frac{1}{f(r)} \int_0^r f(\tau)(|g(\tau)| + h(\tau))d\tau,$$
$$U_2(r) := \frac{1}{f(r)} \int_0^r f(\tau)h(\tau)d\tau,$$
$$D(r) := b(r)^2 - a(r)(c(r) - a(r)g(r)).$$

Then we impose the following conditions:

(I) $f, g \in C^3(0,\infty)$ are positive functions and $g \in C^1(0,\infty)$.

(II) There exists $R > 0$ such that

  (a) $f(|g| + h) \in L^1(0,R)$,

  (b) $\tau \to f(\tau)(|g(\tau)| + h(\tau)) \int_\tau^R f^{-1}(\sigma)d\sigma \in L^1(0,R)$.

(III) $\lim_{r\to 0} a(r)U_1(r)U_2(r) = \lim_{r\to 0} b(r)U_2(r) = 0$.

(IV) $G \le 0$ and $G \not\equiv 0$.

Then the following theorem holds:

**Theorem 4.2.** *Let $p > 1$ and assume that conditions (I)–(IV). Then the problem (4.2) has at most one positive solution $\phi$ which satisfies $J(r;\phi) \to 0$ as $r \to \infty$ and $\phi'(0) = 0$.*

*Proof.* By Proposition 4.1 and (IV), for any positive solution $\phi$ to (4.2) satisfying $J(r;\phi) \to 0$ as $r \to \infty$, we have $J(r;\phi_\omega) \ge 0$ for all $r > 0$. Hence, the assertion follows from the same argument as in the proof of [8, Theorem 3.3]. □

*Remark* 4.1. In our setting, [8, Theorem 3.3] cannot be applied directly, since the assumption [8, (IV)] may fail. These assumptions were used there to prove the positivity of $J(r;\phi)$. However, under our stronger assumption (IV), the positivity of $J(r;\phi)$ follows immediately, so [8, (IV)] is not required.

*Proof of Theorem 4.1.* First, we can easily find that (4.1) coincides (4.2) with

$$f \equiv 1, \qquad g(r) = V(r) + \omega, \qquad h \equiv 1. \tag{4.4}$$

Thus, we find that (4.1) corresponds to

$$a \equiv 1, \qquad b \equiv 0, \qquad c \equiv 0 \tag{4.5}$$

and

$$G(r) = -\frac{1}{2}V'(r). \tag{4.6}$$

Furthermore, we see that (4.1) corresponds to

$$\begin{aligned} U_1(r) &= \frac{1}{f(r)} \int_0^r f(\tau)(|g(\tau)| + h(\tau))d\tau = \int_0^r (|V(\tau) + \omega| + 1)d\tau, \\ U_2(r) &= \frac{1}{f(r)} \int_0^r f(\tau)h(\tau)d\tau = r. \end{aligned} \tag{4.7}$$

It follows from (V1) and (4.4) that $f, h \in C^3(0,\infty)$ are positive functions and $g \in C^1(0,\infty)$. Thus, condition (I) holds.

Next, we will check that condition (II) is satisfied. It follows from (V1) and (4.4) that

$$f(|g|+h) = (|V(r)+\omega|+1) \in L^1(0,R).$$

In addition, by (V1) and (4.4), one has

$$\tau \mapsto f(\tau)(|g(\tau)|+h(\tau)) \int_\tau^R f(\sigma)^{-1} d\sigma = (|V(\tau)+\omega|+1)(R-\tau) \in L^1(0,R).$$

Thus, we can verify condition (II).

By (4.4), (4.5), (4.7), and (V1), we obtain

$$\lim_{r\to 0} a(r)U_1(r)U_2(r) = \lim_{r\to 0} r \int_0^r (|V(\tau)+\omega|+1)d\tau = 0, \qquad \lim_{r\to 0} b(r)U_2(r) = \lim_{r\to 0} 0 = 0.$$

Therefore, we see that condition (III) is satisfied.

By (4.6) and (V1), we have (IV).

Finally, we show that any positive solution $\phi \in X$ of (4.1) satisfy $J(r;\phi) \to 0$ as $r \to \infty$. we see that

$$J(r;\phi) = \frac{1}{2}\phi'(r)^2 - \frac{g(r)}{2}\phi(r)^2 + \frac{1}{p+1}\phi(r)^{p+1}. \tag{4.8}$$

Since $\phi \in X$, we have $J(r;\phi_\omega) \in L^1(0,\infty)$. This means that there exists $(r_n)_n$ such that $r_n \to \infty$ and $J(r_n;\phi) \to 0$ as $n \to \infty$. Since $J$ is decreasing from Proposition 4.1 and (IV) we obtain $J(r;\phi) \to 0$ as $r \to \infty$. Therefore, the assertion follows from Theorem 4.2. □

We conclude this section with the proof of Theorem 1.1 (ii).

*Proof of Theorem 1.1 (ii).* Let $u \in \mathcal{G}_\omega$. Then by Theorems 2.2 and 3.1, there exists $\theta \in \mathbb{R}$ such that $e^{i\theta}u$ is a positive solution to (1.5). Since we see from Theorem 4.1 that the positive solution to (1.5) is unique, we have $e^{i\theta}u = \phi_\omega$, where $\phi_\omega$ is the one given in Theorem 1.1 (i). This completes the proof. □

## 5. Non-degeneracy of the ground state

5.1. **Non-degenracy of the ground state in $X_{\mathbf{rad}}$.** Throughout this subsection, we consider only the real-valued function.

**Theorem 5.1.** *Assume that (V1). Let $\phi_\omega \in X$ be the positive ground state of* (1.5)*. Then $\phi_\omega \in X$ is non-degenerate in $X_{rad}$.*

*Proof of Theorem 5.1.* We still employ the notations (4.4), (4.5), and (4.6) in Section 4. It follows from condition (V1) that $G = -\frac{1}{2}V' \not\equiv 0$ in $(0,\infty)$. This implies that we can find a closed interval $[r_1, r_2] \subset (0,\infty)$ such that

$$\min_{r\in[r_1,r_2]} |G(r)| > 0.$$

We choose $\gamma \in C_0^\infty(0,\infty) \setminus \{0\}$ such that $\gamma \geq 0$ and $\operatorname{supp}\gamma = [r_1, r_2]$. We take $\delta > 0$ and fix it later. We define

$$\begin{aligned}
g_\delta(r) &:= g(r) + \delta\gamma(r)h(r)\phi_\omega^{p-1}(r) = V(r) + \omega + \delta\gamma(r) \times 1 \times \phi_\omega^{p-1}(r) = V(r) + \omega + \delta\gamma(r)\phi_\omega^{p-1}(r),\\
h_\delta(r) &:= (1 + \delta\gamma(r))h(r) = 1 + \delta\gamma(r)
\end{aligned}$$

in $(0,\infty)$. We define $a_\delta, b_\delta, c_\delta, G_\delta$ and $D_\delta$ in $(0,\infty)$ as follows:

$$\begin{aligned}
a_\delta &:= f^{\frac{2(p+1)}{p+3}} h_\delta^{-\frac{2}{p+3}} = h_\delta^{-\frac{2}{p+3}}, \qquad b_\delta := -\frac{1}{2}a_\delta' + \frac{f'}{f}a_\delta = -\frac{1}{2}a_\delta',\\
c_\delta &:= -b_\delta' + \frac{f'}{f}b_\delta = -b_\delta', \qquad G_\delta := b_\delta g_\delta + \frac{1}{2}c_{\delta,r} - \frac{1}{2}(a_\delta g_\delta)',\\
D_\delta &:= b_\delta^2 - a_\delta(c_\delta - a_\delta g_\delta).
\end{aligned}$$

Since $g_\delta = g = V + \omega$ and $h_\delta = h = 1$ in $(0,\infty) \setminus [r_1, r_2]$, we can easily see that

$$a_\delta = a = 1, \qquad b_\delta = b = 0, \qquad c_\delta = c = 0, \qquad G_\delta = -\frac{1}{2}ag = G, \qquad D_\delta = g = D$$

in $(0,\infty) \setminus [r_1, r_2]$. We fix $\delta > 0$ small enough such that

$$\min_{r\in[r_1,r_2]} |G_\delta(r)| > 0.$$

We see that

$$G_\delta(r) \leq 0 \qquad \text{in } (0,\infty). \tag{5.1}$$

Indeed, if $r \notin [r_1, r_2]$, we have by condition (V1) that $G_\delta(r) = G(r) = -\frac{1}{2}V'(r) \leq 0$. In addition, since $\min_{r\in[r_1,r_2]} |G_\delta(r)| > 0$ and $G(r) \leq 0$ in $(0,\infty)$, we find that (5.1) holds. Note that $\phi_\omega$ satisfies

$$\begin{aligned}
\phi_\omega'' - g_\delta(r)\phi_\omega + h_\delta(r)\phi_\omega^p &= \phi_\omega'' - (V(r) + \omega)\phi_\omega - \delta\gamma(r)\phi_\omega^p + \phi_\omega^p + \delta\gamma(r)\phi_\omega^p\\
&= \phi_\omega'' - (V(r) + \omega)\phi_\omega + \phi_\omega^p = 0.
\end{aligned}$$

$\phi_\omega(0) \in (0,\infty)$ and $\lim_{r\to\infty} \phi_\omega(r) = 0$. Namely, $\phi_\omega$ is the positive solution to

$$\begin{cases}
\phi_\omega'' - g_\delta(r)\phi_\omega + h_\delta(r)\phi_\omega^p = 0, \qquad 0 < r < \infty,\\
\lim_{r\to\infty} \phi_\omega(r) = 0.
\end{cases} \tag{5.2}$$

We can find that positive solution to (5.2) is unique for sufficiently small $\delta > 0$ from Section 4 and (5.1).

We shall show that $\phi_\omega$ is non-degenerate in $X_{\rm rad}$. Suppose to the contrary that $\phi_\omega$ is degenerate in $X_{\rm rad}$. Then there exists $\psi_\omega \in X_{\rm rad} \setminus \{0\}$ such that

$$\psi_\omega'' - g(r)\psi_\omega + p\phi_\omega^{p-1}\psi_\omega = 0 \qquad \text{for } 0 < r < \infty. \tag{5.3}$$

We define $C^2$-functional $S_{\omega,\delta}$ by

$$S_{\omega,\delta}(u) = 2\int_0^\infty \Big(\frac{1}{2}(|u'|^2 + g_\delta(r)|u|^2) - \frac{1}{p+1}h_\delta(r)|u|^{p+1}\Big)dr \quad \text{for } u \in X.$$

We can easily see that

$$\langle S_{\omega,\delta}''(\phi_\omega)v, v\rangle = \langle S_\omega''(\phi_\omega)v, v\rangle - 2\delta(p-1)\int_0^\infty \gamma(r)\phi_\omega^{p-1}(r)v^2(r)dr$$

for each $v \in X_{\rm rad}$ and

$$\langle S_\omega''(\phi_\omega)\phi_\omega, \phi_\omega\rangle = -2(p-1)\int_0^\infty \phi_\omega^{p+1}(r)dr < 0.$$

Moreover, we have

$$\int_0^\infty \gamma(r)\phi_\omega^{p-1}(r)\psi_\omega^2(r)dr > 0. \tag{5.4}$$

Indeed, if not, we have $\int_0^\infty \gamma(r)\phi_\omega^{p-1}(r)\psi_\omega^2(r)dr = 0$. Since $\phi_\omega(r) > 0$ for any $r > 0$, we see that $\psi_\omega \equiv 0$ on $\operatorname{supp}\gamma$. From the uniqueness of the initial value problem of (5.3), we see that $\psi_\omega \equiv 0$ in $(0,\infty)$, which contradicts $\psi_\omega \in X_{\rm rad} \setminus \{0\}$. Thus, by (5.4), we obtain

$$\begin{aligned}\langle S_{\omega,\delta}''(\phi_\omega)\alpha\phi_\omega + \beta\psi_\omega, \alpha\phi_\omega + \beta\psi_\omega\rangle &= -2(p-1)\int_0^\infty \phi_\omega^{p+1}(r)dr\alpha^2 \\ &\quad - 2\delta(p-1)\int_0^\infty \gamma(r)\phi_\omega^{p+1}(r)dr\alpha^2 \\ &\quad - 2\delta(p-1)\int_0^\infty \gamma(r)\phi_\omega^{p-1}(r)\psi_\omega^2(r)dr\beta^2 \\ &\quad - 4\delta(p-1)\int_0^\infty \gamma(r)\phi_\omega^{p-1}(r)\phi_\omega(r)\psi_\omega(r)dr\alpha\beta < 0\end{aligned} \tag{5.5}$$

for each $(\alpha,\beta) \in \mathbb{R}^2 \setminus \{(0,0)\}$. In addition, we can easily verify that $\phi_\omega$ does not satisfy (5.3), so that $\phi_\omega$ and $\psi_\omega$ are linearly independent. Thus, we see from (5.5) that the Morse index of $S_{\omega,\delta}''(\phi_\omega)$ is at least two. However, since the Morse index of $S_{\omega,\delta}''(\phi_\omega)$ is one (see Section B below), we obtain a contradiction. Hence, we have shown that $\phi_\omega$ is non-degenerate in $X_{\rm rad}$. □

5.2. **Proof of Theorem 1.1 (iv).** Here, we consider not the real-valued functions but the complex-valued ones. We define the following operators $L_{\omega,+}$ and $L_{\omega,-}$ by

$$L_{\omega,+} = -\partial_{xx} + V(x) + \omega - p\phi_\omega^{p-1}, \qquad L_{\omega,-} = -\partial_{xx} + V(x) + \omega - p\phi_\omega^{p-1}.$$

For each $\varphi \in X$, we put $\varphi_1 = \operatorname{Re}\varphi$ and $\varphi_2 = \operatorname{Im}\varphi$. Then one has

$$S_\omega''(\phi_\omega)\varphi = L_{\omega,+}\varphi_1 + iL_{\omega,-}\varphi_2.$$

In order to prove $\ker S''_\omega(\phi_\omega) = \operatorname{span}\{i\phi_\omega\}$, it suffices to show that $\ker L_{\omega,+} = \{0\}$ and $\ker L_{\omega,-} = \operatorname{span}\{\phi_\omega\}$. To this end, we prepare the following lemma:

**Lemma 5.1.** *All eigenvalues of the operators $L_{\omega,+}$ and $L_{\omega,-}$ are simple.*

*Proof.* Let $\varphi_\omega \in X$ and $\psi_\omega \in X$ be the eigenfunctions of $L_{\omega,+}$ associated with an eigenvalue $\lambda$. Then we have

$$\partial_x(\varphi_\omega \partial_x \psi_\omega - \partial_x \varphi_\omega \psi_\omega) = \varphi_\omega \partial_{xx}\psi_\omega - \partial_{xx}\varphi_\omega \psi_\omega = 0.$$

This implies that there exists a constant $C$ such that $\varphi_\omega \partial_x \psi_\omega - \partial_x \varphi_\omega \psi_\omega = C$ on $\mathbb{R}$. Since $\lim_{|x|\to\infty}(\varphi_\omega \partial_x \psi_\omega - \partial_x \varphi_\omega \psi_\omega) = 0$, one has $C = 0$. Thus, $\varphi_\omega$ and $\psi_\omega$ are linearly dependent. We can prove the simpleness of the eigenvalues of $L_{\omega,-}$ similarly. Thus, we obtained the desired result. $\square$

*Proof of Theorem 1.1 (iv).* We first consider the operator $L_{\omega,+}$. Suppose that $\varphi_\omega \in X$ satisfies $L_{\omega,+}\varphi_\omega = 0$. Then we decompose $\varphi_\omega$ into the even part and the odd part as

$$\varphi_\omega = \varphi_{1,\omega} + \varphi_{2,\omega}, \qquad \varphi_{1,\omega} = \frac{\varphi_\omega(x) + \varphi_\omega(-x)}{2}, \qquad \varphi_{2,\omega} = \frac{\varphi_\omega(x) - \varphi_\omega(-x)}{2}.$$

Then we see that $\varphi_{1,\omega} \in X_{\mathrm{rad}}$ and also satisfies $L_{\omega,+}\varphi_{1,\omega} = 0$. By Theorem 5.1, we have $\varphi_{1,\omega} = 0$. Observe that $\varphi_{2,\omega}$ satisfies $\varphi_{2,\omega}(0) = 0$ and

$$-\varphi''_{2,\omega} + V(r)\varphi_{2,\omega} + \omega\varphi_{2,\omega} - p\phi_\omega^{p-1}(r)\varphi_{2,\omega} = 0, \qquad r > 0.$$

Then by the similar argument as in [8, Proof of Theorem 4.4], we obtain $\varphi_{2,\omega} = 0$. Thus, we see that $\ker L_{\omega,+} = \{0\}$.

Next, we study the operator $L_{\omega,-}$. We can easily find that $L_{\omega,-}\phi_\omega = 0$. Namely, $\phi_\omega$ is the zero eigenfunction of $L_{\omega,-}$. It follows from Lemma 5.1 that $\ker L_{\omega,-} = \operatorname{span}\{\phi_\omega\}$. This completes the proof. $\square$

## 6. Proof of stability of the ground state

### 6.1. **Spectral conditions.**

*Proof of Theorem 1.1 (iii) and (v).* Let $\omega > \omega_1$. We consider the following minimization problem:

$$\nu_\omega := \inf\{\langle L_{\omega,+}u, u\rangle \colon u \in X,\ \|u\|_{L^2} = 1\}.$$

Since $\langle L_{\omega,+}\phi_\omega, \phi_\omega\rangle = -(p-1)\|\phi_\omega\|_{L^{p+1}}^{p+1} < 0$, we have $\nu_\omega < 0$. Then using a similar argument to [6, Lemma 8.1.11], there exists a minimizer $\chi_\omega \in X$ that satisfies $L_{\omega,+}\chi_\omega = \nu_\omega\chi_\omega$. In addition, it follows from the min-max theorem (see e.g. Reed and Simon [23, Theorem XIII.1]), $\nu_\omega$ is the first eigenvalue of $L_{\omega,+}$. By Lemma 5.1, we see that $\nu_\omega$ is simple. In addition, since $L_{\omega,-}\phi_\omega = 0$ and $\phi_\omega$ is positive, we find that $L_{\omega,-}$ is a non-negative operator. Thus, Theorem 1.1 (iii) holds.

Next, we will show Theorem 1.1 (v). Since $\omega > \omega_1$, we see that the operator $-\partial_{xx} + V(x) + \omega$ is invertible. In addition, $\lim_{|x|\to\infty} \phi_\omega^{p-1}(x) = 0$. Thus, we see from the Weyl's essential spectrum that $\sigma_{\text{ess}}(L_{\omega,\pm}) = \sigma_{\text{ess}}(-\partial_{xx} + V(x) + \omega) = [\delta, \infty)$. This completes the proof □

6.2. **Slope condition.** In this section, we study the sign of $\frac{d}{d\omega}\|\phi_\omega\|_{L^2}^2$ and give the proof of Theorem 1.2.

In what follows, we investigate the sign of $\frac{d}{d\omega}\|\phi_\omega\|_{L^2}^2$. We note that our argument extends to higher-dimensional cases, except for Lemma 6.5 below. Therefore, in order to clarify why condition $d = 1$ is necessary, we consider the problem in general dimensions.

First, we pay our attention to the regularity of $\phi_\omega$ with respect to $\omega > \omega_1$. Concerning this, we obtain the following:

**Lemma 6.1.** *Let $d = 1$ and $1 < p < 2^* - 1$. Then the mapping $\omega \in (\omega_1, \infty) \mapsto \phi_\omega \in X_{rad}$ is $C^1$.*

We obtain Lemma 6.1 by using a similar argument of Shatah and Strauss [25]. By adapting the abstract result of Grillakis, Shatah and Strauss [14] to our equation (1.1), we obtained the following:

**Theorem 6.1.** *Let $d = 1$ and $\phi_\omega \in \mathcal{G}_\omega$. Assume that Assumption 1.1 and (V1). Then the standing waves $e^{i\omega t}\phi_\omega$ is stable if $\frac{d}{d\omega}\|\phi_\omega\|_{L^2}^2 > 0$ and unstable if $\frac{d}{d\omega}\|\phi_\omega\|_{L^2}^2 < 0$.*

We put

$$u_\omega := \frac{\phi_\omega}{\|\phi_\omega\|_{L^2}}, \qquad\qquad v_\omega := \partial_\omega u_\omega.$$

Then we have

$$\|u_\omega\|_{L^2} = 1, \qquad\qquad \int_{\mathbb{R}^d} u_\omega v_\omega dx = \frac{1}{2}\frac{d}{d\omega}\|u_\omega\|_{L^2}^2 = 0. \tag{6.1}$$

Moreover, $u_\omega$ satisfies the equation

$$-\Delta u_\omega + V(x)u_\omega + \omega u_\omega - \mu(\omega)u_\omega^p = 0 \qquad \text{in } \mathbb{R}^d, \tag{6.2}$$

where $\mu(\omega) := \|\phi_\omega\|_{L^2}^{p-1}$. We note that the sign of $\mu'(\omega)$ coincides with that of $\frac{d}{d\omega}\|\phi_\omega\|_{L^2}^2$.

**Lemma 6.2.** *Let $d \geq 1$ and $1 < p < 2^* - 1$. For $\omega > \omega_1$, the following holds.*

$$\mu(\omega)\int_{\mathbb{R}^d} u_\omega^p v_\omega dx = \frac{1}{p-1}\big(1 - \mu'(\omega)\|\phi_\omega\|_{L^{p+1}}^{p+1}\big), \tag{6.3}$$

$$\int_{\mathbb{R}^d} u_\omega^p v_\omega dx > 0. \tag{6.4}$$

*Proof.* Let $Q_\omega$ be a action functional of (6.2) given by

$$I_\omega(v) := \frac{1}{2}\|\nabla v\|_{L^2}^2 + \frac{1}{2}\int_{\mathbb{R}^d} V(x)|v|^2 + \frac{\omega}{2}\|v\|_{L^2}^2 - \frac{\mu(\omega)}{p+1}\|v\|_{L^{p+1}}^{p+1}$$

We note that $I'_\omega(u_\omega)=0$ and

$$I''_\omega(u_\omega)|_{X(\mathbb{R}^d;\mathbb{R})} = -\Delta + V(x) + \omega - p\mu(\omega)u_\omega^{p-1} = -\Delta + V(x) + \omega - p\phi_\omega^{p-1} = S''_\omega(\phi_\omega)|_{X(\mathbb{R}^d;\mathbb{R})}.$$

In particular,

$$I''_\omega(u_\omega)u_\omega = -(p-1)\mu(\omega)u_\omega^p. \tag{6.5}$$

By differentiating the equation $I'_\omega(u_\omega)=0$ with respect to $\omega$, we obtain

$$I''_\omega(u_\omega)v_\omega = -u_\omega + \mu'(\omega)u_\omega^p. \tag{6.6}$$

By using (6.1), (6.5) and (6.6), we have

$$a := \langle I''_\omega(u_\omega)u_\omega, u_\omega\rangle = -(p-1)\mu(\omega)\int_{\mathbb{R}^d}\phi_\omega^{p+1}dx, \tag{6.7}$$

$$b := \langle I''_\omega(u_\omega)u_\omega, v_\omega\rangle = \begin{cases} -(p-1)\mu(\omega)\displaystyle\int_{\mathbb{R}^d}u_\omega^p v_\omega dx, \\ \mu'(\omega)\displaystyle\int_{\mathbb{R}^d}u_\omega^{p+1}dx - 1, \end{cases} \tag{6.8}$$

$$c := \langle I''_\omega(u_\omega)v_\omega, v_\omega\rangle = \mu'(\omega)\int_{\mathbb{R}^d}u_\omega^p v_\omega dx. \tag{6.9}$$

In particular, (6.3) holds. Let $\chi_\omega$ be the positive and normalized eigenfunction corresponds to the unique negative eigenvalue of $I''_\omega(u_\omega)$. Since $u_\omega$ is also positive function, we have $\int_{\mathbb{R}}\chi_\omega u_\omega dx > 0$. This means that there exists $t_\omega \in \mathbb{R}$ such that $\int_{\mathbb{R}}\chi_\omega(t_\omega u_\omega + v_\omega)dx = 0$. By using this, non-degeneracy of $I''_\omega(u_\omega)|_{X_{\mathrm{rad}}(\mathbb{R}^d;\mathbb{R})} = S''_\omega(\phi_\omega)|_{X_{\mathrm{rad}}(\mathbb{R}^d;\mathbb{R})}$ and Theorem 1.1 (iii)–(v), we have the following positivety:

$$\begin{aligned} 0 &< \langle I''_\omega(u_\omega)(t_\omega u_\omega + v_\omega), t_\omega u_\omega + v_\omega\rangle \\ &= at_\omega^2 + 2bt_\omega + c = a\Big(t_\omega + \frac{b}{a}\Big)^2 - \frac{1}{a}(b^2 - ac). \end{aligned}$$

This together with $a<0$ yields that $0 < b^2 - ac$. We note from the expression (6.7)–(6.9) that $ac = b(b+1)$. Thus, $b = ac - b^2 < 0$. This means (6.4) holds. □

**Lemma 6.3.** *Let $d \ge 1$ and $1 < p < 2^* - 1$. Then there exist positive constants $C_1, C_2 > 0$ such that for $\omega > \omega_1$,*

$$C_1\mu'(\omega)\|u_\omega\|_{L^{p+1}}^{p+1} = C_2\Big(1 + \frac{4}{d} - p\Big) - \int_{\mathbb{R}^d}(2V + x\cdot\nabla V)u_\omega v_\omega dx \tag{6.10}$$

*Proof.* We put $\alpha := d(p-1)/2$. By differentiating the relation

$$I_\omega(t^{d/2}u_\omega(tx)) = \frac{t^2}{2}\|\nabla u_\omega\|_{L^2}^2 + \frac{1}{2}\int_{\mathbb{R}^d}V(x/t)u_\omega^2 dx + \frac{\omega}{2}\|u_\omega\|_{L^2}^2 - \frac{t^\alpha\mu(\omega)}{p+1}\|u_\omega\|_{L^{p+1}}^{p+1}$$

at $t=1$ and using $I'_\omega(u_\omega)=0$, $\|\nabla u_\omega\|_{L^2}^2=\mu(\omega)\|u_\omega\|_{L^{p+1}}^{p+1}-\int_{\mathbb{R}^d}Vu_\omega^2dx-\omega\|u_\omega\|_{L^2}^2$ and $\|u_\omega\|_{L^2}=1$, we have

$$
\begin{aligned}
0&=\|\nabla u_\omega\|_{L^2}^2-\frac{1}{2}\int_{\mathbb{R}^d}(x\cdot\nabla V)u_\omega^2dx-\frac{\alpha\mu(\omega)}{p+1}|u|_{L^{p+1}}^{p+1}\\
&=-\omega-\frac{1}{2}\int_{\mathbb{R}^d}(2V+x\cdot\nabla V)u_\omega^2dx+\Big(1-\frac{\alpha}{p+1}\Big)\mu(\omega)\|u_\omega\|_{L^{p+1}}^{p+1}.
\end{aligned}
$$

Moreover, by differentiating the above identity with respect to $\omega$ and using (6.3), we obtain

$$
\begin{aligned}
&1+\int_{\mathbb{R}^d}(2V+x\cdot\nabla V)u_\omega v_\omega dx\\
&=\Big(1-\frac{\alpha}{p+1}\Big)\mu'(\omega)\|u_\omega\|_{L^{p+1}}^{p+1}+\Big(1-\frac{\alpha}{p+1}\Big)(p+1)\mu(\omega)\int_{\mathbb{R}^d}u_\omega^p v_\omega dx\\
&=\Big(1-\frac{\alpha}{p+1}\Big)\mu'(\omega)\|u_\omega\|_{L^{p+1}}^{p+1}+\Big(1-\frac{\alpha}{p+1}\Big)\frac{p+1}{p-1}(1-\mu'(\omega)\|u_\omega\|_{L^{p+1}}^{p+1})\\
&=-\Big(1-\frac{\alpha}{p+1}\Big)\frac{2}{p-1}\mu'(\omega)\|u_\omega\|_{L^{p+1}}^{p+1}+\Big(1-\frac{\alpha}{p+1}\Big)\frac{p+1}{p-1}.
\end{aligned}
$$

It follows from $\alpha=d(p-1)/2$ that

$$
\begin{aligned}
\Big(1-\frac{\alpha}{p+1}\Big)\frac{2}{p-1}&=\frac{d+2-(d-2)p}{(p-1)(p+1)}>0,\\
\Big(1-\frac{\alpha}{p+1}\Big)\frac{p+1}{p-1}-1&=\frac{d}{2(p-1)}\Big(1+\frac{4}{d}-p\Big).
\end{aligned}
$$

Therefore, we see that (6.10) with $C_1=\frac{d+2-(d-2)p}{(p-1)(p+1)}$ and $C_2=\frac{d}{2(p-1)}$. $\square$

**Lemma 6.4.** *Let $d\geq 1$ and $1<p<2^*-1$. If $\mu'(\omega)\leq 0$, then the set $[v_\omega>0]:=\{r\geq 0\colon v_\omega(r)>0\}$ is a nonempty bounded interval.*

*Proof.* First, it is easy to see that $[v_\omega>0]\neq\emptyset$ due to $\int_{\mathbb{R}^d}u_\omega v_\omega dx=0$ (see (6.1)), $u_\omega(r)>0$ for all $r>0$, and $v_\omega\not\equiv 0$.

Next, suppose that the set $[v_\omega>0]$ consists of more than one connected component. Then there exist $0\leq r_1<r_2\leq r_3<r_4\leq\infty$ such that

$$
r_1=0 \text{ or } v_\omega(r_1)=0,\quad v_\omega(r_2)=v_\omega(r_3)=v_\omega(r_4)=0,\quad v_\omega>0 \text{ on } (r_1,r_2)\cup(r_3,r_4).
$$

We put $I_1:=[r_1,r_2)$, $I_2:=(r_3,r_4)$, and $v_j:=v_\omega 1_{I_j}$, where $1_A$ is the characteristic function of the set $A$. Then we see that $v_j\in X$ and there is $t\in\mathbb{R}$ such that $\int_{\mathbb{R}^d}\chi_\omega(v_1+tv_2)dx=0$, where $\chi_\omega$ is the positive and normalized eigenfunction corresponds to the unique negative eigenvalue of $I''_\omega(u_\omega)$. This implies

$$
\langle I''_\omega(u_\omega)(v_1+tv_2),v_1+tv_2\rangle\geq 0. \tag{6.11}
$$

On the other hand, by using the equation $I''_\omega(u_\omega)v_\omega = -u_\omega + \mu'(\omega)u_\omega^p$ and integration by parts, we obtain

$$\langle I''_\omega(u_\omega)v_j, v_j\rangle = \langle -u_\omega + \mu'(\omega)u_\omega^p, v_j\rangle < 0, \qquad \langle I''_\omega(u_\omega)v_1, v_2\rangle = 0,$$

where we used the assumption $\mu'(\omega) \le 0$, the positivity of $u_\omega, v_j$ and $\operatorname{supp} v_1 \cap \operatorname{supp} v_2 = \emptyset$. Therefore, $\langle I''_\omega(u_\omega)(v_1 + tv_2), v_1 + tv_2\rangle < 0$. This contradicts (6.11). Thus, the set $[v_\omega > 0]$ is connected.

Finally, suppose that $[v_\omega > 0]$ is unbounded. Since $[v_\omega > 0]$ is connected and $\int_{\mathbb{R}^d} u_\omega v_\omega = 0$, it follows that $[v_\omega > 0] = (r_1, \infty)$ for some $r_1 > 0$. Thus, by the positivity and the decreasing monotonicity of $u_\omega$, we have

$$\int_0^{r_1} u_\omega^p v_\omega r^{d-1} dr \le u_\omega(r_1)^{p-1} \int_0^{r_1} u_\omega v_\omega r^{d-1} dr, \qquad \int_{r_1}^\infty u_\omega^p v_\omega r^{d-1} dr \le u_\omega(r_1)^{p-1} \int_{r_1}^\infty u_\omega v_\omega r^{d-1} dr.$$

Therefore, by $\int_{\mathbb{R}^d} u_\omega v_\omega = 0$, we obtain

$$\int_0^\infty u_\omega^p v_\omega r^{d-1} dr \le u_\omega(r_1)^{p-1} \int_0^\infty u_\omega v_\omega r^{d-1} dr = 0.$$

This contradicts (6.4). Hence, $[v_\omega > 0]$ is bounded. $\square$

The following lemma is a key of our proof.

**Lemma 6.5.** *Let $1 < p < 2^* - 1$. If $d = 1$ and $\mu'(\omega) \le 0$, then $v_\omega(0) > 0$.*

*Proof.* Suppose that $v_\omega(0) \le 0$. By Lemma 6.4, there exists $0 \le r_1 < r_2$ such that $v_\omega(r_1) = v_\omega(r_2) = 0$ and $v_\omega > 0$ on $(r_1, r_2)$. Since $u_\omega$ is a positive radial solution of (6.2), we have

$$u''_\omega + \frac{d-1}{r}u'_\omega - (V(r) + \omega)u_\omega + \mu(\omega)u_\omega^p = 0, \quad r > 0. \tag{6.12}$$

Differentiating (6.12) with respect to $r$, we have

$$u'''_\omega + \frac{d-1}{r}u''_\omega - (V(r) + \omega)u'_\omega = \frac{d-1}{r^2}u'_\omega + V'(r)u_\omega - p\mu(\omega)u_\omega^{p-1}u'_\omega. \tag{6.13}$$

Differentiating (6.12) with respect to $\omega$, we have

$$v''_\omega + \frac{d-1}{r}v'_\omega - (V(r) + \omega)v_\omega = u_\omega - p\mu(\omega)u_\omega^{p-1}v_\omega - \mu'(\omega)u_\omega^p. \tag{6.14}$$

By (6.13) and (6.14), we obtain

$$\begin{aligned}
&\big((u''_\omega v_\omega - u'_\omega v'_\omega)r^{d-1}\big)' \\
&= \Big(\Big(u'''_\omega + \frac{d-1}{r}u''_\omega\Big)v_\omega - \Big(v''_\omega + \frac{d-1}{r}v'_\omega\Big)u'_\omega\Big)r^{d-1} \\
&= \Big(\frac{d-1}{r^2}u'_\omega v_\omega + V'(r)u_\omega v_\omega - u_\omega u'_\omega + \mu'(\omega)u_\omega^p u'_\omega\Big)r^{d-1}.
\end{aligned}$$

Integrating the both side over $[r_1, r_2]$ and using $v_\omega(r_1) = v_\omega(r_2) = 0$, we obtain

$$\begin{aligned}&u'_\omega(r_1)v'_\omega(r_1)r_1^{d-1} - u'_\omega(r_2)v'_\omega(r_2)r_2^{d-1} \\ &= \int_{r_1}^{r_2} \Big(\frac{d-1}{r^2}u'_\omega v_\omega + V'(r)u_\omega v_\omega - u_\omega u'_\omega + \mu'(\omega)u_\omega^p u'_\omega\Big) r^{d-1}\, dr.\end{aligned} \tag{6.15}$$

In particular, if $d = 1$,

$$\begin{aligned}&u'_\omega(r_1)v'_\omega(r_1)r_1^{d-1} - u'_\omega(r_2)v'_\omega(r_2)r_2^{d-1} \\ &= \int_{r_1}^{r_2} \Big(V'(r)u_\omega v_\omega - u_\omega u'_\omega + \mu'(\omega)u_\omega^p u'_\omega\Big) r^{d-1}\, dr.\end{aligned} \tag{6.16}$$

Since $u'_\omega < 0$, $v'_\omega(r_1) \geq 0$, and $v'_\omega(r_2) \leq 0$, the left-hand side of (6.16) is non-positive. On the other hand, since $u_\omega > 0$, $v_\omega > 0$, $V' \geq 0$, $u'_\omega < 0$ on $(r_1, r_2)$ and $\mu'(\omega) \leq 0$, the right-hand side is positive. This is a contradiction. Thus, we obtain $v_\omega(0) > 0$. ☐

*Remark* 6.1. Our argument does not work in the case of higher dimensions $d \geq 2$. In this case, we have the relation (6.15) instead of (6.16). However, it is unclear whether the right-hand side of (6.16) is positive, since the term $\frac{d-1}{r^2}u'_\omega v_\omega$ is negative.

*Proof of Theorem 1.2.* From Theorem 6.1 and the fact that the sign of $\mu'(\omega)$ coincides with $\frac{d}{d\omega}\|\phi_\omega\|_{L^2}^2$, it suffces to show that $\mu'(\omega) > 0$ for all $\omega > \omega_1$. Suppose that there exists $\omega \in (\omega_1, \infty)$ such that $\mu'(\omega) \leq 0$. Then (6.10) implies

$$\int_0^\infty (2V + rV')u_\omega v_\omega dx \begin{cases} > 0 & \text{if } 1 < p < 5, \\ \geq 0 & \text{if } p = 5. \end{cases} \tag{6.17}$$

Now by Lemmas 6.4 and 6.5, we see that there exists $r_0 > 0$ such that $v_\omega > 0$ on $[0, r_0)$ and $v_\omega \leq 0$ on $(r_0, \infty)$. We see in fact that $v_\omega < 0$ on $(r_0, \infty)$. Indeed, if $v_\omega(r_1) = 0$ for some $r_1 > r_0$, we see that $v'_\omega(r_1) = 0$ and $v''_\omega(r_1) \leq 0$, which contradict (6.14). Therefore, by the assumption (V2) and (6.1), we obtain

$$\begin{aligned}\int_0^\infty (2V + rV')u_\omega v_\omega dx &= \Big(\int_0^{r_0} + \int_{r_0}^\infty\Big)(2V + rV')u_\omega v_\omega dx \\ &\begin{cases} \leq (2V + rV')(r_0)\int_0^\infty u_\omega v_\omega dx & \text{if } 1 < p < 5, \\ < (2V + rV')(r_0)\int_0^\infty u_\omega v_\omega & \text{if } p = 5 \end{cases} \\ &= 0,\end{aligned}$$

which contradicts (6.17). Thus, $\mu'(\omega) > 0$ for all $\omega > \omega_1$. This completes the proof. ☐

## Appendix A. Bounded from below

**Lemma A.1.** *Let $d = 1$ and assume that* (V1)*. Then the following assertions hold:*

(i) $\omega_1$ *given by* (1.3) *is well-defined, namely,*

$$\inf\Big\{\int_{\mathbb{R}}\big(|u_x|^2+V|u|^2\big)\,dx\colon u\in X,\ \|u\|_{L^2}=1\Big\}>-\infty.$$

(ii) *For any $\omega>\omega_1$, the embedding* (1.4) *holds.*

*Proof.* Let $u \in X$. By the Sobolev inequality, we have

$$\int_{|x|\le\delta} V|u|^2\,dx \le \|u\|_{L^\infty}^2\int_{|x|\le\delta}|V(x)|\,dx \le C\|u\|_{H^1}^2\int_{|x|\le\delta}|V(x)|\,dx, \tag{A.1}$$

where the constant $C>0$ does not depend on $u$. Since $V\in L^1_{\mathrm{loc}}(\mathbb{R})$, there exists $\delta>0$ such that

$$C\int_{|x|\le\delta}|V(x)|\,dx<\frac{1}{2}.$$

Thus, by (A.1), we obtain

$$\begin{aligned}\int_{\mathbb{R}} V|u|^2\,dx &= \int_{|x|\le\delta} V|u|^2\,dx+\int_{|x|\ge\delta}V|u|^2\,dx\\ &\ge -\frac{1}{2}\|u\|_{H^1}^2+V(\delta)\int_{|x|\ge\delta}|u|^2\,dx\\ &\ge -\frac{1}{2}\|u_x\|_{L^2}^2-C\|u\|_{L^2}^2.\end{aligned} \tag{A.2}$$

In particular, (i) holds.

Moreover, by the definition of $\omega_1$, for $\omega>\omega_1$, we have

$$(\omega-\omega_1)\|u\|_{L^2}^2\le\|u\|_{X_\omega}^2. \tag{A.3}$$

By (A.2),

$$\begin{aligned}\|u_x\|_{L^2}^2 &= \|u\|_{X_\omega}^2-\int_{\mathbb{R}}V|u|^2\,dx-\omega\|u\|_{L^2}^2\\ &\le \|u\|_{X_\omega}^2+\tfrac{1}{2}\|u_x\|_{L^2}^2+C\|u\|_{L^2}^2.\end{aligned}$$

This and (A.3) imply

$$\|u_x\|_{L^2}^2\le C\big(\|u\|_{X_\omega}^2+\|u\|_{L^2}^2\big)\le C\|u\|_{X_\omega}^2.$$

Therefore, combining this with (A.3), we obtain $\|u\|_{H^1}\le C\|u\|_{X_\omega}$. This means (ii). □

## Appendix B. Morse Index of $S''_{\omega,\delta}(\phi_\omega)$

In this section, we show that the Morse index of $S''_{\omega,\delta}(\phi_\omega)$ equals to one. First, we consider the following minimization problem:

$$d(\omega,\delta) := \inf\left\{S_{\omega,\delta}(u)\colon u\in X\setminus\{0\},\ K_{\omega,\delta}(u)=0\right\},$$

where

$$K_{\omega,\delta}(u) := \langle S'_{\omega,\delta}(u),u\rangle = 2\int_0^\infty \left((|u'|^2+g_\delta(r)|u|^2)-h_\delta(r)|u|^{p+1}\right)dr \qquad \text{for } u\in X.$$

By a similar argument of Section 2, we can find that there exists a minimizer, which is positive and even solution to (5.2). We can easily verify that the positive function $\phi_\omega$ satisfies (5.2). Since positive solution to (5.2) is unique (see Section 5), we see that $\phi_\omega$ is also a minimizer of $d(\omega,\delta)$. Using this, we shall show the following:

**Lemma B.1.** *If $u\in H^1_{rad}(\mathbb{R})$ satisfies $\langle K_{\omega,\delta}(\phi_\omega),u\rangle=0$, we have $\langle S''_{\omega,\delta}(\phi_\omega)u,u\rangle\geq 0$.*

*Proof.* We shall show by contradiction. Suppose to the contrary that there exists $u_*\in H^1_{\mathrm{rad}}(\mathbb{R})$ such that $\langle K_{\omega,\delta}(\phi_\omega),u_*\rangle=0$ and $\langle S''_{\omega,\delta}(\phi_\omega)u_*,u_*\rangle=-1$. Define the function $Z:\mathbb{R}^2\to H^1_{\mathrm{rad}}(\mathbb{R})$ by

$$Z(a,b)=\phi_\omega+au_*+b\phi_\omega.$$

We can easily verify that $K_{\omega,\delta}(Z(0,0))=K_{\omega,\delta}(\phi_\omega)=0$. In addition, we have

$$\frac{\partial}{\partial b}K_{\omega,\delta}(Z(a,b))\Big|_{(a,b)=(0,0)} = \langle S''_{\omega,\delta}(\phi_\omega)\phi_\omega,\phi_\omega\rangle = -2(p-1)\int_0^\infty h_\delta(r)|\phi_\omega|^{p+1}dx<0.$$

Hence, it follows from the implicit function theorem that there exists $a_0>0$ and $h\in C^2(-a_0,a_0)$ such that $K_{\omega,\delta}(Z(a,h(a)))=0$ and $h(0)=0$. Since $\phi_\omega$ is a minimizer of $d(\omega,\delta)$, we have $\frac{d^2S_{\omega,\delta}}{da^2}(Z(a,b))\Big|_{a=0}\geq 0$.

On the other hand, differentiating $K_{\omega,\delta}(Z(a,h(a)))=0$ and substituting $a=0$, we obtain

$$\langle K'_{\omega,\delta}(\phi_\omega),u_*\rangle+\frac{dh}{da}(0)\langle K'_{\omega,\delta}(\phi_\omega),\phi_\omega\rangle=0$$

Since $\langle K'_{\omega,\delta}(\phi_\omega),\phi_\omega\rangle=-2(p-1)\int_0^\infty h_\delta(r)|\phi_\omega|^{p+1}dx<0$, we see that $\frac{dh}{da}(0)=0$. It follows that

$$\begin{aligned}\frac{d^2S_{\omega,\delta}}{da^2}(Z(a,h(a))) &= \frac{d}{da}\left(\langle S'_{\omega,\delta}(Z(a,b)),\phi_\omega+\frac{dh}{da}(a)\phi_\omega\rangle\right)\\ &= \langle S''_{\omega,\delta}(Z(a,b))\left(\phi_\omega+\frac{dh}{da}(a)\phi_\omega\right),\phi_\omega+\frac{dh}{da}(a)\phi_\omega\rangle\\ &\quad+\langle S'_{\omega,\delta}(Z(a,b)),\frac{d^2h}{da^2}(a)\phi_\omega\rangle.\end{aligned}$$

This together with $\frac{dh}{da}(0)=0$ implies that

$$\frac{d^2S_{\omega,\delta}}{da^2}(Z(a,h(a)))\Big|_{a=0}=\langle S''_{\omega,\delta}(\phi_\omega)\phi_\omega,\phi_\omega\rangle=-2(p-1)\int_0^\infty h_\delta(r)|\phi_\omega|^{p+1}dx<0,$$

which is a contradiction. This completes the proof. □

From Lemma B.1 and the mini-max theorem, we see that the Morse index of $S''_{\omega,\delta}(\phi_\omega)$ is 1.

**Acknowledgments.** N.F. was supported by JSPS KAKENHI Grant Number JP20K14349 and JP24H00024. M.I was supported by JSPS KAKENHI Grant Number JP25H01453 and JP23K03174. H.K. was supported by JSPS KAKENHI Grant Number JP25H01453 and JP25K07089.

**Data availability.** Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

**Conflict of interest.** The authors declare that they have no conflict of interest.

## References

[1] J. Bellazzini, N. Boussaïd, L. Jeanjean, and N. Visciglia, *Existence and stability of standing waves for supercritical NLS with a partial confinement*, Comm. Math. Phys. **353** (2017), 229–251.

[2] H. Berestycki and T. Cazenave, *Instabilité des états stationnaires dans les équations de Schrödinger et de Klein-Gordon non linéaires*, C. R. Acad. Sci. Paris Sér. I Math. **293** (1981), 489–492.

[3] A. Burchard and A. Ferone, *On the extremals of the Pólya-Szegő inequality*, Indiana Univ. Math. J. **64** (2015), 1447–1463.

[4] L. D. Carr, C. W. Clark, and W. P. Reinhardt, *Stationary solutions of the one-dimensional nonlinear schrödinger equation. i. case of repulsive nonlinearity*, Physical Review A **62** (2000).

[5] T. Cazenave and P.-L. Lions, *Orbital stability of standing waves for some nonlinear Schrödinger equations*, Comm. Math. Phys. **85** (1982), 549–561.

[6] T. Cazenave, *Semilinear Schrödinger equations*, Courant Lecture Notes in Mathematics, vol. 10, New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 2003.

[7] A. Comech and D. Pelinovsky, *Purely nonlinear instability of standing waves with minimal energy*, Comm. Pure Appl. Math. **56** (2003), 1565–1607.

[8] N. Fukaya, *Uniqueness and nondegeneracy of ground states for nonlinear Schrödinger equations with attractive inverse-power potential*, Commun. Pure Appl. Anal. **20** (2021), 121–143.

[9] N. Fukaya and M. Ohta, *Strong instability of standing waves for nonlinear Schrödinger equations with attractive inverse power potential*, Osaka J. Math. **56** (2019), 713–726.

[10] R. Fukuizumi, *Stability of standing waves for nonlinear Schrödinger equations with critical power nonlinearity and potentials*, Adv. Differential Equations **10** (2005), 259–276.

[11] R. Fukuizumi and M. Ohta, *Instability of standing waves for nonlinear Schrödinger equations with potentials*, Differential Integral Equations **16** (2003), 691–706.

[12] R. Fukuizumi and M. Ohta, *Stability of standing waves for nonlinear Schrödinger equations with potentials*, Differential Integral Equations **16** (2003), 111–128.

[13] F. Genoud and C. A. Stuart, *Schrödinger equations with a spatially decaying nonlinearity: existence and stability of standing waves*, Discrete Contin. Dyn. Syst. **21** (2008), 137–186.

[14] M. Grillakis, J. Shatah, and W. Strauss, *Stability theory of solitary waves in the presence of symmetry. I*, J. Funct. Anal. **74** (1987), 160–197.

[15] E. H. Lieb and M. Loss, *Analysis*, vol. 14, American Mathematical Soc., 2001.

[16] M. Maeda, *Stability of bound states of Hamiltonian PDEs in the degenerate cases*, J. Funct. Anal. **263** (2012), 511–528.

[17] J. B. McLeod, C. A. Stuart, and W. C. Troy, *Stability of standing waves for some nonlinear Schrödinger equations*, Differential Integral Equations **16** (2003), 1025–1038.

[18] B. Noris, H. Tavares, and G. Verzini, *Existence and orbital stability of the ground states with prescribed mass for the $L^2$-critical and supercritical NLS on bounded domains*, Anal. PDE **7** (2014), 1807–1838.

[19] Y.-G. Oh, *Cauchy problem and Ehrenfest's law of nonlinear Schrödinger equations with potentials*, J. Differential Equations **81** (1989), 255–274.

[20] M. Ohta, *Instability of bound states for abstract nonlinear Schrödinger equations*, J. Funct. Anal. **261** (2011), 90–110.

[21] M. Ohta, *Strong instability of standing waves for nonlinear Schrödinger equations with a partial confinement*, Commun. Pure Appl. Anal. **17** (2018), 1671–1680.

[22] M. Ohta, *Strong instability of standing waves for nonlinear Schrödinger equations with harmonic potential*, Funkcial. Ekvac. **61** (2018), 135–143.

[23] M. Reed and B. Simon, *Methods of modern mathematical physics. IV. Analysis of operators*, Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1978.

[24] H. A. Rose and M. I. Weinstein, *On the bound states of the nonlinear Schrödinger equation with a linear potential*, Phys. D **30** (1988), 207–218.

[25] J. Shatah and W. Strauss, *Instability of nonlinear bound states*, Comm. Math. Phys. **100** (1985), 173–190.

[26] N. Shioji and K. Watanabe, *A generalized Pohožaev identity and uniqueness of positive radial solutions of* $\Delta u + g(r)u + h(r)u^p = 0$, J. Differential Equations **255** (2013), 4448–4475.

[27] N. Shioji and K. Watanabe, *Uniqueness and nondegeneracy of positive radial solutions of* $\operatorname{div}(\rho\nabla u)+\rho(-gu+hu^p) = 0$, Calc. Var. Partial Differential Equations **55** (2016), Art. 32, 42.

[28] C. A. Stuart, *Uniqueness and stability of ground states for some nonlinear Schrödinger equations*, J. Eur. Math. Soc. (JEMS) **8** (2006), 399–414.

[29] M. I. Weinstein, *Nonlinear Schrödinger equations and sharp interpolation estimates*, Comm. Math. Phys. **87** (1982/83), 567–576.

Noriyoshi Fukaya

Waseda Research Institute for Science and Engineering,

Waseda University,

Tokyo 169-8555, Japan

E-mail:nfukaya@aoni.waseda.jp

Masahiro Ikeda

Graduate School of Information Science and Technology, Osaka University,

1-5, Yamadaoka, Suita-shi, Osaka 565-0871, Japan

E-mail: ikeda@ist.osaka-u.ac.jp

Hiroaki Kikuchi

Department of Mathematics,

Tsuda University,

2-1-1 Tsuda-machi, Kodaira-shi, Tokyo 187-8577, JAPAN

E-mail: hiroaki@tsuda.ac.jp