GENERATING FUNCTIONS OF q-CHROMATIC POLYNOMIALS

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Abstract. Given a graph $G=(V,E)$ and a linear form $\lambda \in \mathbb{Z}_{>0}^V$, Bajo et al. (2025) introduced the q-chromatic polynomial $\chi_G^\lambda(q,n) := \sum q^{\sum_{v\in V} \lambda_v c(v)}$ where the sum is over all proper colorings $c: V \to [n] := \{1,2,\dots,n\}$; they showed that $\chi_G^\lambda(q,n)$ is a polynomial in $[n]_q := 1+q+\cdots+q^{n-1}$ with coefficients in $\mathbb{Z}(q)$. For $d \in \mathbb{Z}_{>0}$ and the linear form given by $(d,d^2,\dots,d^d)$, we show that the q-chromatic polynomial distinguishes labeled graphs with vertex set $[d]$. Using permutation statistics introduced by Chung–Graham (1995), called G-statistics, and polyhedral geometry, we give the multivariate integer point transform for the region of proper colorings of a given graph G. This integer point transform allows us to find the generating function for the q-chromatic polynomial with respect to any linear form. We further specialize these results to the linear form $\mathbb{1} := (1,1,\dots,1)$, which allows us to write the q-chromatic polynomial in the q-binomial basis, clarifying expressions found by Bajo et al. Moreover, we show that G-statistics are compatible with the theory of order polytopes used by Bajo et al. and Chow (1999). This yields further properties for the generating function of q-chromatic polynomial with linear form $\mathbb{1}$, where certain coefficients of the numerator polynomial are palindromic polynomials in q.

1. Introduction

For a graph $G=(V,E)$ and a linear form $\lambda \in \mathbb{Z}_{>0}^V$, Bajo et al. [1] introduced a q-analogue of the chromatic polynomial,

$$\chi_G^\lambda(q,n) := \sum_{\substack{\text{proper colorings}\\ c:V\to[n]}} q^{\sum_{v\in V} \lambda_v c(v)},$$

which we call the q**-chromatic polynomial** of G, due to the fact that $\chi_G^\lambda(q,n)$ is a polynomial in $[n]_q := 1+q+\cdots+q^{n-1}$ with coefficients in $\mathbb{Z}(q)$ [1]. For $V=\{v_1,\dots,v_d\}$ and linear form $\mathbb{1} := (1,1,\dots,1)$, the q-chromatic polynomial is known to be the **principal specialization** $X_G(q,q^2,\dots,q^n,0,0,\dots)$ of Stanley's **chromatic symmetric function** [10]

$$X_G(x_1,x_2,\dots) := \sum_{\substack{\text{proper colorings}\\ c:V\to\mathbb{Z}_{>0}}} x_{c(v_1)} x_{c(v_2)} \cdots x_{c(v_d)}.$$

Stanley conjectured that the chromatic symmetric function distinguish trees; Loehr and Warrington [9] conjectured, more strongly, that the principal specialization $X_G(q,q^2,\dots,q^n,0,0,\dots)$ distinguishes trees. Since the principal specialization is the q-chromatic polynomial with the particular linear form $\mathbb{1}$, Bajo et al. asked if there exists some linear form $\lambda \in \mathbb{Z}_{>0}^d$ such that the q-chromatic polynomial distinguishes trees. In Theorem 2.1, we answer this question in the affirmative (for all graphs, not just for trees), for $\lambda = (d,d^2,\dots,d^d)$ where $d=|V|$.

In Section 3, we express the generating function of the q-chromatic polynomial of G for any linear form $\lambda \in \mathbb{Z}_{>0}^V$ using polyhedral geometry and certain statistics on permutations and graphs. G-descents were introduced by Chung–Graham [6] in order to give a combinatorial interpretation of the coefficients of the chromatic polynomial $\chi_G(n)$ when expressed in the basis $\left\{\binom{n+k}{d} : 0 \le k \le d\right\}$. Their proof was

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not combinatorial, however, such a proof was provided by Steingrímsson [12]. Using these ideas in conjunction with polyhedral geometry, we express a multivariate generating function of graph colorings (Theorem 3.2 below). Consequently, we prove an expression for the generating function for the q-chromatic polynomial for any linear form (Theorem 3.3).

In Section 4, we apply our general expression of the q-chromatic generating function to the linear form $\mathbb{1} \in \mathbb{Z}^d$. In particular, we give a rational expression of the generating function of $\chi_G^{\mathbb{1}}(q,n)$ for any graph G in Theorem 4.1. From this theorem, we give a method to compute the coefficients of $\chi_G^{\mathbb{1}}(q,n)$ when written in the basis $\left\{ \begin{bmatrix} n+k \\ d \end{bmatrix}_q : 0 \le k \le d \right\}$ using G-statistics, yielding a q-analogue of Chung–Graham's expression of the chromatic polynomial in the binomial basis. Moreover, Chow [5] writes the chromatic symmetric function in terms of fundamental quasisymmetric functions with G-statistics, and the generating function for the principal specialization of Chow's expression coincides with our generating function for the q-chromatic polynomial with linear form $\mathbb{1}$, which we prove in Section 4.2.

In Section 5, emulating work by Chow, we give a bijection between permutations and pairs of acyclic orientations and linear extensions of the induced poset (Theorem 5.3). In addition, Bajo et al. show the q-chromatic polynomial can be expressed as a sum of q-Ehrhart polynomials of order polytopes. Using their result, in Theorem 5.4 we express the generating function of the q-chromatic polynomial with linear form $\mathbb{1}$ using pairs of acyclic orientations and linear extensions of induced posets which agrees with our previous method using G-statistics. We use our bijection and different expressions of the generating function of the q-chromatic polynomial to prove the following:

(1) In Section 5.2, we write the leading term of $\widetilde{\chi}_G^{\mathbb{1}}(q,x)$ as a single sum relating to the G-major index polynomial (Corollary 5.6). Moreover, we show that the degree of the G-major index polynomial can be computed by minimizing $\sum_{v \in V} c(v)$ over all proper colorings c in Corollary 5.8. In particular, we give bounds for the degree of the G-major index polynomial for trees (Corollary 5.10).
(2) In Corollary 6.3 we prove that the coefficients of $\chi_G^{\mathbb{1}}(q,n)$ when written in the basis $\left\{ \begin{bmatrix} n+k \\ d \end{bmatrix}_q : 0 \le k \le d \right\}$ are palindromic polynomials, up to a shift.

2. The linear form $\lambda = (k, k^2, \dots, k^d)$

Let $G = ([d], E)$ be a graph, let $k \in \mathbb{Z}_{>0}$ such that $k \ge d$, and consider the linear form $\lambda = (k, k^2, \dots, k^d)$. We will now show that $\chi_G^\lambda(q,n)$ is unique for every graph G. The key tool will be the unique base-k representation of a natural number.

Theorem 2.1. *For $d \in \mathbb{Z}_{>0}$, let $k \in \mathbb{Z}_{>0}$ such that $k \ge d$ and let $\lambda_i := k^i$. If G and H are graphs on $[d]$ such that $\chi_G^\lambda(q, d-1) = \chi_H^\lambda(q, d-1)$, then $G = H$.*

Proof. Suppose $\chi_G^\lambda(q, d-1) = \chi_H^\lambda(q, d-1)$, that is,

$$\sum_{\substack{\text{proper } G-\text{coloring} \\ c:[d]\to[d-1]}} q^{\sum_{i\in[d]} \lambda_i c(i)} = \sum_{\substack{\text{proper } H-\text{coloring} \\ f:[d]\to[d-1]}} q^{\sum_{i\in[d]} \lambda_i f(i)}.$$

By the uniqueness of the base-k representation of a natural number, each proper coloring of G gives a unique power of q on the left-hand side and, similarly, each proper coloring of H gives unique powers of q on the right-hand side. In order for equality to hold, the powers appearing must be the same and hence every proper coloring of G using at most $d-1$ colors is also a proper coloring of H and vice versa. In other words, the two graphs share the same $(d-1)$-colorings.

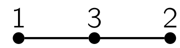


Figure 1. Path graph on three vertices

In particular, some of the $(d-1)$-colorings encode the edges of a graph since we can use them to test if we have an edge between any two vertices, as follows. Suppose $v, w \in [d]$ are vertices of some graph on $[d]$ and consider the coloring $f : [d] \to [d-1]$ defined by $f(v) = f(w) = 1$ and each other vertex being assigned a unique color between 2 and $d-1$. If v, w form an edge, f will not be a proper coloring, otherwise it will be. Hence if the power of q corresponding to f appears we do not have the edge v, w and it does not appear we do have an edge. Since these $(d-1)$-colorings of G and H must coincide, G and H have the same edge sets, showing $G = H$. $\square$

Theorem 2.1 shows that it can be useful to consider various linear forms aside from the all-ones vector $\mathbb{1}$.

3. G-statistics and the regions of proper colorings

3.1. G**-statistics.** Let $G = (V, E)$ be a graph with $V = [d]$. The following definition was introduced by Chung–Graham [6]. Given a permutation $\pi = [\pi(1) \cdots \pi(d)] \in S_d$ written in one-line notation, define the **rank** of $\pi(i)$ with respect to π, denoted $p_\pi(\pi(i))$, to be the largest $r \in \mathbb{Z}_{>0}$ for which there exist positions $i_1 < i_2 < \cdots < i_r = i$ such that $\{\pi(i_j), \pi(i_{j+1})\} \in E$ for $1 \le j < r$. This gives a function $p_\pi : [d] \to [d]$ which we call the **rank function of** π. Steingrímsson used a variation of rank to instead respect an ordered set partition of the vertex set [12]. Given an ordered set partition $A := (A_1, A_2, \dots, A_m)$ of V, define the **rank** of an element $v \in V$ with respect A, denoted $p_A(v)$, as follows. For $v \in V$, let $v_{i_1}, v_{i_2}, \dots, v_{i_k} = v$ be a longest path in G ending in v such that $v_{i_j} \in A_{i_j}$ for each j and $i_1 < i_2 < \cdots < i_k$, then we define $p_A(v) = k$. This gives a function $p_A : [d] \to [d]$, which we call the **rank function of** A.

We use the same terminology since the rank function of a permutation can be seen as a special case of the rank function of an ordered set partition. For a permutation $\pi = [\pi(1) \cdots \pi(d)]$, the ordered set partition consisting of singletons $A := (\pi(1), \dots, \pi(d))$, $p_\pi = p_A$ yields the same rank function; other ordered set partitions can also yield the same rank functions, see Example 1. The rank with respect to an ordered set partition will be helpful in the proof of Proposition 3.1 since for one containment we do not start with a permutation.

Example 1. Let G be the path graph depicted in Figure 1, and let $\pi = [321] \in S_3$. To find $p_\pi(\pi(3))$, we have positions $1 < 3$ so that $\{\pi(1), \pi(3)\} = \{3, 1\} \in E$. However, for the positions $2 < 3$ we have $\{\pi(2), \pi(3)\} = \{2, 1\} \notin E$. This means $1 < 3$ is the longest increasing sequence of positions ending at 3 satisfying our edge condition, showing $p_\pi(\pi(3)) = p_\pi(1) = 2$.

For the same graph, now consider the ordered set partition $A := (\{3\}, \{1, 2\})$ of $[3]$. To find $p_A(1)$ we need to find a longest path in G using elements in increasing blocks. Here we can pick the path $3-1$, and so $p_A(1) = 2$. In fact, both rank functions above give the same values $p_\pi(3) = 1 = p_A(3)$, $p_\pi(2) = 2 = p_A(2)$, $p_\pi(1) = 2 = p_A(1)$.

The rank of a permutation gives rise to the following central G-statistic [6]. We say $\pi \in S_d$ has a G**-descent** at $i \in [d-1]$ if either

(1) $p_\pi(\pi(i)) > p_\pi(\pi(i+1))$, or
(2) $p_\pi(\pi(i)) = p_\pi(\pi(i+1))$ and $\pi(i) > \pi(i+1)$.

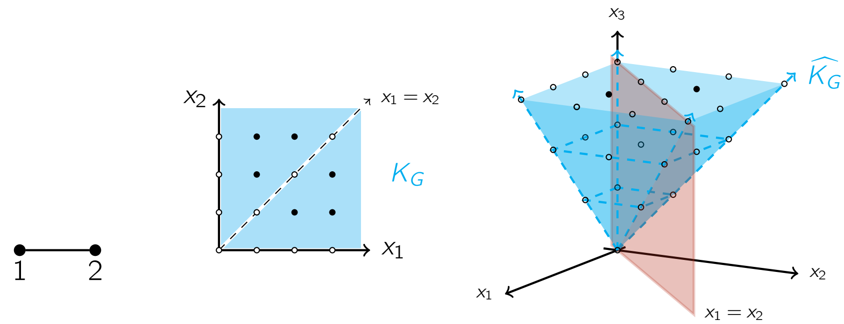


Figure 2. For G the path graph on two vertices, the center figure shows K_G, and the right figure shows $\widehat{K_G}$.

Denote the set of G-descents of π by $\mathrm{Des}_G(\pi) \subseteq [d-1]$. We say π has a G-**ascent** at $i \in [d-1]$ if either

(1) $p_\pi(\pi(i)) < p_\pi(\pi(i+1))$, or
(2) $p_\pi(\pi(i)) = p_\pi(\pi(i+1))$ and $\pi(i) < \pi(i+1)$.

Denote the set of G-ascents of π by $\mathrm{Asc}_G(\pi) \subseteq [d-1]$. Given a permutation $\pi \in S_d$, let the G-**major index** be $\mathrm{maj}_G(\pi) := \sum_{i \in \mathrm{Des}_G(\pi)} i$. Let $\mathrm{asc}_G(\pi) := |\mathrm{Asc}_G(\pi)|$ and $\mathrm{des}_G(\pi) := |\mathrm{Des}_G(\pi)|$. When $E = \varnothing$, these descent statistics stem from the usual descents of a permutation.

Example 2. For the same path graph G as in Example 1, consider again the permutation $\pi = [321] \in S_3$. We computed the ranks $p_\pi(3) = 1$, $p_\pi(2) = 2$, $p_\pi(1) = 2$. Since $p_\pi(\pi(1)) < p_\pi(\pi(2))$, we have $1 \in \mathrm{Asc}_G(\pi)$. Furthermore, $p_\pi(\pi(2)) = p_\pi(\pi(3))$ and $\pi(2) = 2 > 1 = \pi(3)$, so $2 \in \mathrm{Des}_G(\pi)$. Note that the G-ascent set in this example is not the same as the usual ascent set on [321] (which is the empty set).

3.2. **Chromatic generating functions.** We will now describe the polyhedral structure encoding proper colorings of G and then exhibit the connection to G-ascents.

Given an edge $e = ij \in E(G)$, define the hyperplane $H_e := \left\{x \in \mathbb{R}^d : x_i = x_j\right\}$. We can express any proper coloring of G as an integer point $a \in \mathbb{Z}^d_{>0}$ such that $a_i \neq a_j$ whenever $ij \in E$. Moreover, we can also see such colorings using at most n colors as lying in the $(n+1)$-dilate of the open cube $(0,1)^d$ with the graphical hyperplane arrangement $\{H_e\}_{e \in E}$ removed, following the ideas behind inside-out polytopes [3]. We define

$$K_G := \left\{x \in \mathbb{R}^d_{>0} : x_i \neq x_j \text{ for } ij \in E\right\}$$

and so the integral points of K_G correspond to all proper colorings of G. It will be helpful to record which dilate of the cube $(0,1)^d$ our lattice points are in, as this encodes how many colors are available for a coloring. To capture this information, define

$$\widehat{K_G} := \left\{x \in \mathbb{R}^{d+1} : 0 < x_1, \ldots, x_d < x_{d+1},\, x_i \neq x_j \text{ for } ij \in E\right\},$$

which we may think of as the homogenization of K_G. See Figure 2 for an example.

For these polyhedral structures, we now create a disjoint polyhedral decomposition using G-ascents. We will turn our focus to $\widehat{K_G}$ for the rest of this section, as this will connect to the q-chromatic polynomial, but the arguments and proofs are analogous for K_G, which gives the G-partition generating function defined by Bajo et al. [1]. For $\pi \in S_d$ define the half-open simplicial cone

$$\widehat{\Delta^G_\pi} := \left\{ x \in \mathbb{R}^{d+1} : \begin{array}{l} 0 < x_{\pi(1)} \le x_{\pi(2)} \le \cdots \le x_{\pi(d)} < x_{d+1} \\ x_{\pi(i)} < x_{\pi(i+1)} \text{ whenever } i \in \mathrm{Asc}_G(\pi) \end{array} \right\}.$$

We denote $\widehat{\Delta^G_\pi}$ by $\widehat{\Delta_\pi}$ when G is clear from the context.

Proposition 3.1. *Let $G = ([d], E)$ be a graph. Then*

$$\widehat{K_G} = \biguplus_{\pi \in S_d} \widehat{\Delta^G_\pi}.$$

Proof. We remark that the general idea of the following argument is essentially given by Steingrímsson [12, Theorem 6]. We first show $\widehat{K_G} \supseteq \biguplus_{\pi\in S_d} \widehat{\Delta^G_\pi}$. Let $\pi \in S_d$; it suffices prove $\widehat{\Delta_\pi} \cap H_e = \varnothing$ for all $e \in E$. Let $e = \{v_1, v_2\} \in E$, so there exist $i_1, i_2 \in [d]$ such that $\pi(i_1) = v_1$ and $\pi(i_2) = v_2$; without loss of generality suppose $i_1 < i_2$. This means $p_\pi(v_2) \ge p_\pi(v_1) + 1$ since when computing the rank for $\pi(i_2)$, a longest path in the definition of rank of $\pi(i_2)$ must be at least as long as the longest path defining the rank of $\pi(i_1)$ with the edge $v_1v_2 \in E$. Towards a contradiction, if all positions j where $i_1 \le j < i_2$ are G-descents, then ranks must be weakly decreasing, $p_\pi(v_1) \ge p_\pi(\pi(i_1+1)) \ge \cdots \ge p_\pi(v_2)$, which gives a contradiction. So there must exist a G-ascent at some position j where $i_1 \le j < i_2$, yielding the strict inequality $x_{v_1} < x_{v_2}$ in the definition of $\widehat{\Delta_\pi}$. As the edge was arbitrary, $\widehat{\Delta_\pi} \cap H_e = \varnothing$ for all $e \in E$.

We now show that $\widehat{K_G} \subseteq \biguplus_{\pi\in S_d} \widehat{\Delta^G_\pi}$, that is, a given $a := (a_1, \ldots, a_{d+1}) \in \widehat{K_G}$ satisfies $a \in \widehat{\Delta_\pi}$ for some unique $\pi \in S_d$. Let $b_1 < b_2 < \cdots < b_k < b_{k+1}$ be all the distinct values among the entries of a. By definition of $\widehat{K_G}$, we know a_{d+1} is the only term achieving the highest value b_{k+1}. For each $j \in [k]$ define $A_j := \{i \in [d] : a_i = b_j\}$. This gives an ordered set partition $(A_j, \ldots, A_k)$ of the vertex set $[d]$. To form our permutation, we first place the blocks in order $A_1, \ldots, A_k$. Then, within each block we order the vertices decreasingly by their rank with respect to the ordered set partition $(A_1, \ldots, A_k)$. Lastly, we order consecutive vertices that share the same rank decreasingly by their label. Since each block of A are vertices which share the same value from the point a, each block is an independent set in G. Thus, ordering decreasingly by rank in each block is well-defined because the rank function depends only on the previous blocks, and the lack of edges within each block implies that reordering vertices within a block does not change the rank function. See Example 3 for an example of this process. Ordering the vertices in this manner gives a well-defined permutation $\pi \in S_d$.

We next claim $p_A = p_\pi$. Let $\pi(i) \in [d]$ and set $r := p_\pi(\pi(i))$ and $p := p_A(\pi(i))$. By definition of r, there exist positions $i_1 < i_2 < \cdots < i_r = i$ such that $\{\pi(i_j), \pi(i_{j+1})\} \in E$ for $1 \le j < r$. So the path formed by $\pi(i_1)\pi(i_2)\ldots\pi(i)$ must use vertices in different blocks of A of increasing index. Since p is defined from the longest such path satisfying this condition, we have $r \le p$. Similarly, by the definition of p, there exists a longest path $v_{j_1}, v_{j_2}, \ldots, v_{j_p} = \pi(i)$ ending in $\pi(i)$ such that $v_{j_k} \in A_{j_k}$ for each k and $j_1 < j_2 < \cdots < j_p$. These vertices must appear in increasing positions of π by construction since they are in blocks of A of increasing index. Since r is defined from the longest such path satisfying this condition, we have $p \le r$. This shows $r = p$. Since this was for an arbitrary element $\pi(i)$, the rank functions are equal.

Observe next that by construction of π and since the rank functions coincide, G-ascents are only possible between blocks in our ordering. Note that it might be the case that only some of these positions are G-ascents, but it is certain that no G-ascents occur within a block A_j of elements in the ordering. Since a must satisfy equality between elements in the order within a fixed block A_j, and a is able to satisfy strict inequalities between blocks in our ordering, where a_{d+1} is strictly greater than any other entry, we have $a \in \widehat{\Delta_\pi}$.

Finally, we show that a cannot be an element of any $\widehat{\Delta_\tau}$ for $\tau \neq \pi$. Specifically, since the elements of a within a block A_j are all equal, those elements must occur in G-descending order. Since the values of a are distinct among blocks, for any τ with $a \in \widehat{\Delta_\tau}$, it must be that the blocks of elements A_j appear in strictly increasing order with respect to τ. Thus, we are forced to have the blocks in order, as well as to avoid any G-ascents within a block, which leads that any such τ must equal π, and our proof is complete. □

Example 3. We will give an example of how a point in $\widehat{K_G}$ lies in some $\widehat{\Delta_\pi^G}$ for some $\pi \in S_d$, as suggested by Proposition 3.1. Let G be the path graph from Example 1. We have $a = (a_1, a_2, a_3, a_4) := (1.1, 1.1, 2, 3) \in \widehat{K_G}$. To construct a permutation π such that $a \in \widehat{\Delta_\pi}$, we have distinct values $a_1 = 1.1 < 2 = a_3 < 3 = a_4$. This gives an ordered set partition $A = (\{1,2\},\{3\})$ of [3]. To construct our permutation, we place our blocks in order of the ordered set partition and then order within each block by rank with respect to A or labels if ranks are equal. For instance, in the first block note that 1 and 2 both have rank 1 with respect to A, and since $1 < 2$ then we have 21 first appearing. The block $\{3\}$ comes after and so we have the permtuation $[213] \in S_3$. We have $\mathrm{Asc}_G([213]) = \{2\}$, and so $a \in \widehat{\Delta_{[213]}} = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : 0 < x_2 \leq x_1 < x_3 < x_4\}$.

We give another example of this process. Let H be the bowtie graph given in Figure 3. Now consider $b = (b_1, b_2, b_3, b_4, b_5, b_6) := (2, 3, 2, 1, 4, 5) \in \widehat{K_H}$. To construct a permutation τ such that $b \in \widehat{\Delta_\tau}$ we first create the ordered set partition of [5] from our distinct values. This ordered set partition is $B = (\{4\},\{1,3\},\{2\},\{5\})$. To construct τ we place blocks in order and in order to decide the order within block $\{1,3\}$ we must decrease by rank with respect to this ordered set partition. We have $p_B(1) = 2 > 1 = p_B(3)$ since $14 \in E(H)$. This means $\tau = [41325]$. Since $\mathrm{Asc}_H(\tau) = \{1,3,4\}$, then $b \in \widehat{\Delta_{[41325]}} = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : 0 < x_4 < x_1 \leq x_3 < x_2 < x_5 < x_6\}$.

We now study the main multivariate generating function behind proper colorings. The **integer point transform** of a set $S \subseteq \mathbb{R}^n$ is

$$\sigma_S(z_1, \ldots, z_n) := \sum_{m \in S \cap \mathbb{Z}^n} z^m$$

where $z^m = z_1^{m_1} z_2^{m_2} \cdots z_n^{m_n}$ for $m = (m_1, \ldots, m_n)$. We can compute $\sigma_{\widehat{K_G}}$ from $\sigma_{\widehat{\Delta_\pi}}$ via our decomposition in Proposition 3.1, and each $\sigma_{\widehat{\Delta_\pi}}$ can be computed by standard methods (see, e.g., [2]).

Theorem 3.2. *Given a graph $G = ([d], E)$,*

$$\sigma_{\widehat{K_G}}(z_1, \ldots, z_d, z_{d+1}) = \sum_{\pi \in S_d} \frac{z_{d+1}^2 z_1 \cdots z_d \prod_{j \in \mathrm{Asc}_G(\pi)} z_{\pi(j+1)} \cdots z_{\pi(d)} z_{d+1}}{(1 - z_{d+1}) \prod_{i=1}^{d} (1 - z_{\pi(i)} \cdots z_{\pi(d)} z_{d+1})} .$$

Proof. Given $\pi \in S_d$, let v_i^π be the ray of $\widehat{\Delta_\pi}$ given by the vector $v_i^\pi = x \in \mathbb{R}^{d+1}$ with entries

$$x_{\pi(i+1)} = \cdots = x_{\pi(d)} = x_{d+1} = 1$$

and remaining entries equal to 0, where v_d^π sets only x_{d+1} to 1. So,

$$\widehat{\Delta_\pi} = \mathbb{R}_{>0} v_0^\pi + \mathbb{R}_{>0} v_d^\pi + \sum_{j \in \mathrm{Asc}_G(\pi)} \mathbb{R}_{>0} v_j^\pi + \sum_{j \in \mathrm{Des}_G(\pi)} \mathbb{R}_{\geq 0} v_j^\pi .$$

This half-open simplicial cone comes with the fundamental parallelepiped

$$\Box_{\widehat{\Delta_\pi}} = (0,1] v_0^\pi + (0,1] v_d^\pi + \sum_{j \in \mathrm{Asc}_G(\pi)} (0,1] v_j^\pi + \sum_{j \in \mathrm{Des}_G(\pi)} [0,1) v_j^\pi .$$

By Proposition 3.1,

$$\sigma_{\widehat{K_G}}(z) = \sum_{\pi \in S_d} \sigma_{\widehat{\Delta_\pi}}(z) = \sum_{\pi \in S_d} \frac{\sigma_{\square_{\widehat{\Delta_\pi}}}(z)}{\prod_{i=0}^{d}(1-z^{v_i^\pi})} = \sum_{\pi \in S_d} \frac{z^{v_0^\pi} z^{v_d^\pi} \prod_{j \in \mathrm{Asc}_G(\pi)} z^{v_j^\pi}}{\prod_{i=0}^{d}(1-z^{v_i^\pi})}.$$

Since $v_0^\pi = (1,\dots,1)$ and $v_d^\pi = (0,\dots,0,1)$ in $\mathbb{R}^{d+1}$ for any $\pi \in S_d$,

$$\sigma_{\widehat{K_G}}(z) = \sum_{\pi \in S_d} \frac{(z_1 \cdots z_{d+1})(z_{d+1}) \prod_{j \in \mathrm{Asc}_G(\pi)} z_{\pi(j+1)} \cdots z_{\pi(d)} z_{d+1}}{(1-z_{d+1}) \prod_{i=1}^{d} 1 - z_{\pi(i)} \cdots z_{\pi(d)} z_{d+1}}$$

as desired. $\square$

For any linear form $\lambda \in \mathbb{Z}^d$, we now specialize this integer-point transform to find the generating function for the q-chromatic polynomial.

Theorem 3.3. *Given a graph $G = ([d], E)$ and a linear form $\lambda = (\lambda_1, \dots, \lambda_d) \in \mathbb{Z}^d$, let $\Lambda := \sum_{i \in [d]} \lambda_i$. Then*

$$\sum_{n \geq 0} \chi_G^\lambda(q,n) z^n = z^{-1} \sigma_{\widehat{K_G}}(q^{\lambda_1}, \dots, q^{\lambda_d}, z)$$
$$= \sum_{\pi \in S_d} \frac{q^{\Lambda + \sum_{j \in \mathrm{Asc}_G(\pi)} \lambda_{\pi(j+1)} + \cdots + \lambda_{\pi(d)}} z^{\mathrm{asc}_G(\pi)+1}}{(1-z)(1-q^{\lambda_{\pi(d)}} z)(1-q^{\lambda_{\pi(d)} + \lambda_{\pi(d-1)}} z) \cdots (1-q^\Lambda z)}.$$

Proof. We set $z_i = q^{\lambda_i}$ for $i \in [d]$ and $z_{d+1} = z$ and consider what this does for the integer point transform by definition. Recall that each lattice point in K_G can be seen as a lattice point in the $(n+1)$-dilate of the cube $(0,1)^d$, removing points on H_e for all $e \in E$, and any such a point corresponds to a proper n-coloring. With this perspective,

$$\sigma_{\widehat{K_G}}(q^{\lambda_1}, \dots, q^{\lambda_d}, z) = \sum_{m \in \widehat{K_G} \cap \mathbb{Z}^{d+1}} q^{\lambda_1 m_1 + \cdots + \lambda_d m_d} z^{m_{d+1}} = \sum_{n \geq 0} \sum_{\substack{\text{proper} \\ n-\text{colorings } c}} q^{\lambda_1 c(1) + \cdots + \lambda_d c(d)} z^{n+1}$$
$$= z \cdot \sum_{n \geq 0} \chi_G^\lambda(q,n) z^n.$$

Next, we specialize the expression given by Theorem 3.2:

$$\sigma_{\widehat{K_G}}(q^{\lambda_1}, \dots, q^{\lambda_d}, z) = \sum_{\pi \in S_d} \frac{z^2 q^{\lambda_1} \cdots q^{\lambda_d} \prod_{j \in \mathrm{Asc}_G(\pi)} q^{\lambda_{\pi(j+1)}} \cdots q^{\lambda_{\pi(d)}} z}{(1-z) \prod_{i=1}^{d} (1 - q^{\lambda_{\pi(i)}} \cdots q^{\lambda_{\pi(d)}} z)}$$
$$= z \sum_{\pi \in S_d} \frac{q^{\Lambda + \sum_{j \in \mathrm{Asc}_G(\pi)} \lambda_{\pi(j+1)} + \cdots + \lambda_{\pi(d)}} z^{\mathrm{asc}_G(\pi)+1}}{(1-z) \prod_{i=1}^{d} (1 - q^{\lambda_{\pi(i)}} \cdots q^{\lambda_{\pi(d)}} z)}.$$

Comparing the two expressions for $\sigma_{\widehat{K_G}}(q^{\lambda_1}, \dots, q^{\lambda_d}, z)$ yields

$$\sum_{n \geq 0} \chi_G^\lambda(q,n) z^n = \sum_{\pi \in S_d} \frac{q^{\Lambda + \sum_{j \in \mathrm{Asc}_G(\pi)} \lambda_{\pi(j+1)} + \cdots + \lambda_{\pi(d)}} z^{\mathrm{asc}_G(\pi)+1}}{(1-z) \prod_{i=1}^{d} (1 - q^{\lambda_{\pi(i)}} \cdots q^{\lambda_{\pi(d)}} z)}.$$ $\square$

Remarks. Further setting $q = 1$ gives a generating-function formula for the chromatic polynomial due to Chung–Graham [6]; see also Corollary 4.3 below.

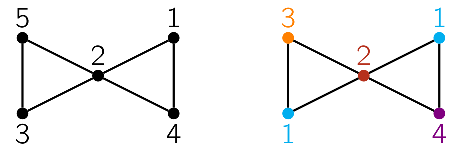


Figure 3. The bowtie graph and its G-sequence coloring for $\pi = [31254]$.

3.3. **Alternative characterizations.** We will now reformulate some expressions coming from the generating function of the q-chromatic polynomial using G-sequences, which were introduced by Steingrímsson [12]. They give an alternate perspective to our chromatic generating functions, as well as insight for future results.

Let $G = ([d], E)$ be a graph and $\pi = [\pi(1) \cdots \pi(d)] \in S_d$. The G-**sequence** of π is an ordered set partition of $[d]$ defined as follows:

- If $\mathrm{Asc}_G(\pi) = \varnothing$, then our ordered set partition is the singular block $A_1 := [d]$.
- If $\mathrm{Asc}_G(\pi) = \{i_1 < \cdots < i_{\mathrm{asc}_G(\pi)}\}$, then our ordered set partition is $(A_1, \ldots, A_{\mathrm{asc}_G(\pi)+1})$ where

$$\begin{aligned} A_1 &:= \{\pi(1), \pi(2), \ldots, \pi(i_1)\} \\ A_m &:= \{\pi(i_{m-1}+1), \ldots, \pi(i_m)\} \quad \text{for} \quad 2 \le m \le \mathrm{asc}_G(\pi) \\ A_{\mathrm{asc}_G(\pi)+1} &:= \{\pi(i_{\mathrm{asc}_G(\pi)}+1), \ldots, \pi(d)\}\,. \end{aligned}$$

Example 4. Consider the bowtie graph G in Figure 3 and the permutation $\pi = [31254] \in S_5$. Then $p_\pi(3) = 1$, $p_\pi(1) = 1$, $p_\pi(2) = 2$, $p_\pi(5) = 3$, and $p_\pi(4) = 4$. With these rank values we can find that $\mathrm{Asc}_G(\pi) = \{2, 3, 4\}$, and so our G-sequence is the ordered set partition $31/2/5/4$.

Moreover, Steingrímsson proved the following [12, Lemma 4].

Lemma 3.4 (Steingrímsson)**.** *Let $(A_1, \ldots, A_{\mathrm{asc}_G(\pi)+1})$ be the G-sequence of a permutation $\pi \in S_d$. Each A_j is an independent set in G, i.e., no two vertices in A_j are adjacent, for any j.*

It is straightforward to see why Lemma 3.4 holds, as an edge within a block would force the existence of a new ascent. Due to Lemma 3.4, we can use the G-sequence to define a proper coloring, which we will show relates to the numerators in the expression of the q-chromatic generating function. For a graph G on $[d]$ and $\pi \in S_d$, suppose π has G-sequence $(A_1, \ldots, A_{\mathrm{asc}_G(\pi)+1})$. We define the G-**sequence coloring of** π as

$$w_\pi : [d] \to [\mathrm{asc}_G(\pi) + 1] \quad \text{where } w_\pi(i) := j \text{ if } i \in A_j.$$

The G-sequence coloring of Example 4 is shown in Figure 3.

We now prove that the numerators found in Theorem 3.3 can be computed via w_π.

Proposition 3.5. *Let G be a graph on $[d]$, let $\pi \in S_d$ be a permutation and $\lambda \in \mathbb{Z}^d_{>0}$ a linear form. Then*

$$\Lambda + \sum_{j \in \mathrm{Asc}_G(\pi)} \lambda_{\pi(j+1)} + \cdots + \lambda_{\pi(d)} = \sum_{i \in [d]} \lambda_i w_\pi(i)\,.$$

Proof. For $\pi = [\pi(1)\pi(2) \cdots \pi(d)] \in S_d$, let $i_2 < \cdots < i_{\mathrm{asc}_G(\pi)+1}$ be the elements of $\mathrm{Asc}_G(\pi)$, and set $i_1 := 0$, $i_{\mathrm{asc}_G(\pi)+2} := d$. Then by definition $A_j = \{\pi(i_j+1), \pi(i_j+2), \ldots, \pi(i_{j+1})\}$, and so position i_2 is given by $|A_1|$, i_3 is given by $|A_1| + |A_2|$, and so on. Thus

$$\Lambda + \sum_{j \in A} \lambda_{\pi(j+1)} + \cdots + \lambda_{\pi(d)} = \Lambda + \lambda_{\pi(i_2+1)} + \cdots + \lambda_{\pi(i_3)} + \lambda_{\pi(i_3+1)} + \cdots + \lambda_{\pi(i_{\mathrm{asc}_G(\pi)+1}+1)} + \cdots + \lambda_{\pi(d)}$$

$$+\lambda_{\pi(i_3+1)}+\cdots+\lambda_{\pi(i_{\mathrm{asc}_G(\pi)+1}+1)}+\cdots+\lambda_{\pi(d)}$$
$$\vdots$$
$$+\lambda_{\pi(i_{\mathrm{asc}_G(\pi)+1}+1)}+\cdots+\lambda_{\pi(d)}$$
$$\begin{aligned}
&=\Lambda+\sum_{k=1}^{\mathrm{asc}_G(\pi)+1}(k-1)(\lambda_{\pi(i_k+1)}+\cdots+\lambda_{\pi(i_{k+1})})\\
&=\Lambda+\sum_{k=1}^{\mathrm{asc}_G(\pi)+1}(k-1)\left(\sum_{a\in A_k}\lambda_a\right)\\
&=\Lambda-\sum_{k=1}^{\mathrm{asc}_G(\pi)+1}\sum_{a\in A_k}\lambda_a+\sum_{k=1}^{\mathrm{asc}_G(\pi)+1}\sum_{a\in A_k}k\lambda_a\\
&=\Lambda-\Lambda+\sum_{k=1}^{\mathrm{asc}_G(\pi)+1}\sum_{a\in A_k}w_\pi(a)\lambda_a\\
&=\sum_{i\in[d]}w_\pi(\pi(i))\lambda_{\pi(i)}\,.
\end{aligned}$$

The third equality holds as $A_j=\{\pi(i_j+1),\pi(i_j+2),\ldots,\pi(i_{j+1})\}$. The fifth equality holds since $A_1,A_2,\ldots$ partition $[d]$; hence summing over all sets in this partition gives Λ. Moreover, each element $a\in A_k$ is given the weight $k=w_{\pi,A}(a)$ and λ_a. We can further simplify

$$\sum_{i\in[d]}w_\pi(\pi(i))\lambda_{\pi(i)}=\sum_{\pi^{-1}(j)\in[d]}w_\pi(j)\lambda_j=\sum_{i\in[d]}w_\pi(i)\lambda_i\,,$$

proving the desired equation. $\square$

Corollary 3.6. *For G a graph on $[d]$ and a linear form $\lambda\in\mathbb{Z}^d$,*

$$\sum_{n\geq 0}\chi_G^\lambda(q,n)z^n=\sum_{\pi\in S_d}\frac{q^{\sum_{i\in[d]}\lambda_i w_\pi(i)}z^{\mathrm{asc}_G(\pi)+1}}{(1-z)(1-q^{\lambda_{\pi(d)}}z)(1-q^{\lambda_{\pi(d)}+\lambda_{\pi(d-1)}}z)\cdots(1-q^\Lambda z)}\,.$$

4. The linear form $\lambda=\mathbb{1}$

In this section, we focus our attention on the special case $\lambda=\mathbb{1}$. We investigate consequences of our multivariate generating function identities and connections with quasisymmetric functions.

4.1. **A combinatorial interprentation of q-binomial basis representations.** We will now apply the results from Section 3 to the special linear form $\lambda=\mathbb{1}\in\mathbb{Z}^d$, giving an expression of the coefficients of $\chi_G^\lambda(q,n)$ in the basis $\left\{\begin{bmatrix}n+k\\ d\end{bmatrix}_q:0\leq k\leq d\right\}$. In addition, via G-sequences we can omit some summands in this basis expression.

Theorem 4.1. *For a graph $G=([d],E)$,*

$$\sum_{n\geq 0}\chi_G^{\mathbb{1}}(q,n)z^n=\frac{\sum_{\pi\in S_d}q^{d+\sum_{j\in\mathrm{Asc}_G(\pi)}d-j}z^{\mathrm{asc}_G(\pi)+1}}{(1-z)(1-qz)(1-q^2z)\cdots(1-q^dz)}\,.$$

Proof. Upon setting $\lambda=\mathbb{1}$ in Theorem 3.3, all permutations have the same denominator. Moreover, for each permutation $\pi\in S_d$,

$$q^{\Lambda+\sum_{j\in\mathrm{Asc}_G(\pi)}\lambda_{\pi(j+1)}+\cdots+\lambda_{\pi(d)}}z^{\mathrm{asc}_G(\pi)+1}=q^{d+\sum_{j\in\mathrm{Asc}_G(\pi)}d-j}z^{\mathrm{asc}_G(\pi)+1}.$$ $\square$

Corollary 4.2. *Let ξ be the chromatic number of G. Then*

$$\chi_G^{\mathbb{1}}(q,n) = \sum_{j=0}^{d-\xi} \left(\begin{bmatrix} n+j \\ d \end{bmatrix}_q \sum_{\substack{\pi \in S_d \\ \mathrm{des}_G(\pi)=j}} q^{d+\sum_{j\in \mathrm{Asc}_G(\pi)} d-j} \right).$$

Proof. For $\pi \in S_d$, set $\alpha_\pi = d + \sum_{j\in \mathrm{Asc}_G(\pi)} d-j$, and let

$$\sum_{\pi\in S_d} q^{\alpha_\pi} z^{\mathrm{asc}_G(\pi)+1} = \sum_{i=0}^{m} a_i(q) z^i \in \mathbb{Z}[q,z],$$

so $a_i(q) = \sum q^{\alpha_\pi}$ where we sum over all permutations π such that $\mathrm{asc}_G(\pi)+1 = i$, i.e., $\mathrm{asc}_G(\pi) = i-1$. Hence

$$\begin{aligned}
\sum_{t\geq 0} \chi_G^{\mathbb{1}}(q,t) z^t &= \frac{a_0(q)+\cdots+a_m(q)z^m}{(1-z)(1-qz)\cdots(1-q^d z)} \\
&= (a_0(q)+\cdots+a_m(q)z^m) \left(\sum_{i\geq 0} \begin{bmatrix} d+i \\ d \end{bmatrix}_q z^i \right) \\
&= \sum_{j\geq 0} \left(\sum_{i=0}^{j} \begin{bmatrix} d+j-i \\ d \end{bmatrix}_q a_i(q) \right) z^j,
\end{aligned}$$

and so $\chi_G^{\mathbb{1}}(q,n) = \sum_{i=0}^{n} \begin{bmatrix} d+n-i \\ d \end{bmatrix}_q a_i(q)$.

We will now show that certain summands in this q-binomial expression are zero. If $i > d$ then $\mathrm{asc}_G(\pi)+1 > d$ and so $\mathrm{asc}_G(\pi) > d-1$. However, $\mathrm{Asc}_G(\pi) \subseteq [d-1]$ so the terms with $i > d$ are zero. If $i < \xi$ then $\mathrm{asc}_G(\pi)+1 < \xi$ and so the G-sequence would give a proper coloring of less than ξ colors, which contradicts Lemma 3.4. So these terms are also zero. Thus we may change indices to deduce

$$\begin{aligned}
\chi_G^{\mathbb{1}}(q,n) &= \sum_{i=0}^{n} \begin{bmatrix} d+n-i \\ d \end{bmatrix}_q a_i(q) = \sum_{i=\xi}^{d} \begin{bmatrix} d+n-i \\ d \end{bmatrix}_q a_i(q) = \sum_{k=0}^{d-\xi} \begin{bmatrix} n+d-\xi-k \\ d \end{bmatrix}_q a_{\xi+k}(q) \\
&= \sum_{j=0}^{d-\xi} \begin{bmatrix} n+j \\ d \end{bmatrix}_q a_{d-j}(q).
\end{aligned}$$

Since permutations indexed by $a_{d-j}(q)$ have $\mathrm{asc}_G(\pi)+1 = d-j$ which implies $\mathrm{asc}_G(\pi) = d-1-j$, we must have $\mathrm{des}_G(\pi) = j$. This proves

$$\chi_G^{\mathbb{1}}(q,n) = \sum_{j=0}^{d-\xi} \left(\begin{bmatrix} n+j \\ d \end{bmatrix}_q \sum_{\substack{\pi \in S_d \\ \mathrm{des}_G(\pi)=j}} q^{d+\sum_{j\in \mathrm{Asc}_G(\pi)} d-j} \right).$$

□

Remarks. For a graph G on $[d]$ with chromatic number ξ, we can express

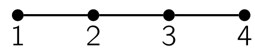
$$\sum_{\pi\in S_d} q^{\alpha_\pi} z^{\mathrm{asc}_G(\pi)+1} = a_\xi(q) z^\xi + \cdots + a_d(q) z^d \in \mathbb{Z}[q,z]$$

where $\pi \in S_d$, $\alpha_\pi := d + \sum_{j\in \mathrm{Asc}_G(\pi)} d-j$, and $a_i(q) := \sum q^{\alpha_\pi}$ where we sum over all permutations π such that $\mathrm{asc}_G(\pi)+1 = i$, by the argument given in the proof of Corollary 4.2.

G_1

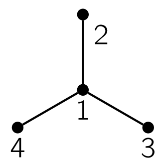
$$\frac{q^{10}z^4+(3q^9+5q^8+3q^7)z^3+(3q^7+5q^6+3q^5)z^2+q^4z}{(1-z)(1-qz)(1-q^2z)(1-q^3z)(1-q^4z)}$$

G_2



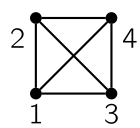
$$\frac{8q^{10}z^4+(4q^9+6q^8+4q^7)z^3+2q^6z^2}{(1-z)(1-qz)(1-q^2z)(1-q^3z)(1-q^4z)}$$

G_3



$$\frac{8q^{10}z^4+(5q^9+4q^8+5q^7)z^3+(q^7+q^5)z^2}{(1-z)(1-qz)(1-q^2z)(1-q^3z)(1-q^4z)}$$

G_4



$$\frac{24q^{10}z^4}{(1-z)(1-qz)(1-q^2z)(1-q^3z)(1-q^4z)}$$

Figure 4. Graphs and their corresponding generating function of the q-chromatic polynomial with linear form $\mathbb{1}$.

Example 5. We give the generating functions of the q-chromatic polynomials with linear form $\mathbb{1}$ for the graphs in Figure 4. By Corollary 4.2, we can express the q-chromatic polynomials as

$$\chi^{\mathbb{1}}_{G_1}(q,n) = \begin{bmatrix} n+3 \\ 4 \end{bmatrix}_q q^4 + \begin{bmatrix} n+2 \\ 4 \end{bmatrix}_q (3q^7+5q^6+3q^5) + \begin{bmatrix} n+1 \\ 4 \end{bmatrix}_q (3q^9+5q^8+3q^7) + \begin{bmatrix} n \\ 4 \end{bmatrix}_q q^{10},$$

$$\chi^{\mathbb{1}}_{G_2}(q,n) = \begin{bmatrix} n+2 \\ 4 \end{bmatrix}_q (2q^6) + \begin{bmatrix} n+1 \\ 4 \end{bmatrix}_q (4q^9+6q^8+4q^7) + \begin{bmatrix} n \\ 4 \end{bmatrix}_q (8q^{10}),$$

$$\chi^{\mathbb{1}}_{G_3}(q,n) = \begin{bmatrix} n+2 \\ 4 \end{bmatrix}_q (q^7+q^5) + \begin{bmatrix} n+1 \\ 4 \end{bmatrix}_q (5q^9+4q^8+5q^7) + \begin{bmatrix} n \\ 4 \end{bmatrix}_q (8q^{10}),$$

$$\chi^{\mathbb{1}}_{G_4}(q,n) = \begin{bmatrix} n \\ 4 \end{bmatrix}_q (24q^{10}).$$

Corollary 4.2 gives a q-analogue of the above-mentioned result of Chung–Graham [6] via setting $q=1$.

Corollary 4.3 (Chung–Graham)**.** *For a graph G on d vertices,*

$$\chi_G(n) = \chi^{\mathbb{1}}_G(1,n) = \sum_{j=0}^{d-\xi} \binom{n+j}{d} \left|\{\pi \in S_d : \mathrm{des}_G(\pi) = j\}\right|.$$

4.2. **Connections to symmetric functions.** The integer point transform encodes integer points as exponents of monomials. We could have instead encoded integer points as the indices of monomials; doing so for the integer points of K_G defines the chromatic symmetric function. By partitioning K_G, we expressed the integer point transform σ_{K_G} as a sum of rational functions in Theorem 3.2. Chow [5] provided an analogous result for the chromatic symmetric function using Stanley's theory of P-partitions and quasisymmetric functions [10]. For this section, we assume readers have some background on quasisymmetric functions but we review some important terms [11, Section 7.19]. Let $\mathcal{Q}$ denote the algebra of quasisymmetric functions. For any subset $S \subseteq [d-1]$ the **fundamental**

quasisymmetric function is

$$F_{S,d}(x) := \sum_{\substack{i_1 \le \cdots \le i_d \\ i_j < i_{j+1} \text{ if } j \in S}} x_{i_1} x_{i_2} \cdots x_{i_d} \,.$$

These fundamental quasisymmetric functions form a basis for quasisymmetric functions and Chow wrote the chromatic symmetric function in this basis [5, Corollary 1].

Theorem 4.4 (Chow)**.** *Let G be a graph on $[d]$, then*

$$X_G = \sum_{S \subseteq [d-1]} N_S F_{S,d}$$

where N_S is the number of permutations with G-ascent set S.

Since the q-chromatic polynomial with the linear form $\mathbb{1}$ can be recovered from the principal specialization of the chromatic symmetric function, we can apply the principal specialization to Theorem 4.4 to deduce formulas for the q-chromatic polynomial. In particular, we will see that the generating function for the principal specialization of order m applied to Theorem 4.4 gives the same expression of the generating function we find for the q-chromatic polynomial with linear form $\mathbb{1}$. The **principal specialization of order** m is the ring homomorphism $\mathbf{ps}_m : \mathcal{Q} \to \mathbb{Q}[q]$ given by

$$\mathbf{ps}_m(x_i) = \begin{cases} q^{i-1} & \text{if } 1 \le i \le m, \\ 0 & \text{otherwise.} \end{cases}$$

Hence for a graph G on $[d]$ and $n \in \mathbb{Z}_{>0}$,

$$q^d \mathbf{ps}_n(X_G) = q^d X_G(1, q, \ldots, q^{n-1}, 0, \ldots) = q^d \sum_{\substack{\text{proper colorings} \\ c:[d]\to[n]}} q^{\sum_{i\in[d]} c(i)-1} = \sum_{\substack{\text{proper colorings} \\ c:[d]\to[n]}} q^{\sum_{i\in[d]} c(i)}$$
$$= \chi_G^{\mathbb{1}}(q,n)\,.$$

Gessel gave the following expression for the generating function of the principal specialization of fundamental quasisymmetric functions [7, Lemma 5.2].

Lemma 4.5 (Gessel)**.**

$$\sum_{m\ge 0} \mathbf{ps}_m(F_{S,d}) z^m = \frac{z^{|S|+1} q^{\sum_{i\in S} d-i}}{(1-z)(1-qz)\cdots(1-q^d z)} \,.$$

With Theorem 4.4, we can find q-chromatic generating functions by applying the principal specialization:

$$\begin{aligned}
\sum_{n\ge 0} \chi_G^{\mathbb{1}}(q,n) z^n &= \sum_{n\ge 0} q^d \mathbf{ps}_n(X_G) z^n \\
&= q^d \sum_{n\ge 0} \mathbf{ps}_n \left(\sum_{S\subseteq[d-1]} N_S F_{S,d} \right) z^n \\
&= q^d \sum_{S\subseteq[d-1]} N_S \sum_{n\ge 0} \mathbf{ps}_n(F_{S,d}) z^n \\
&= q^d \sum_{S\subseteq[d-1]} N_S \frac{z^{|S|+1} q^{\sum_{i\in S} d-i}}{(1-z)(1-qz)\cdots(1-q^d z)}
\end{aligned}$$

$$= \frac{\sum_{\pi\in S_d} q^{d+\sum_{i\in \mathrm{Asc}_G(\pi)} d-i} z^{\mathrm{asc}_G(\pi)+1}}{(1-z)(1-qz)\cdots(1-q^d z)}$$

where the last equality holds by definition of N_S. This is exactly the expression in Theorem 4.1.

5. G-statistics and acyclic orientations

The q-chromatic polynomial was explored by Bajo et al. via the q-Ehrhart theory of order polytopes, analogous to the non-q setting. This relates to Stanley's theory of P-partitions used by Chow to prove Theorem 4.4. We will see that these approaches can be unified via G-statistics.

Given a poset $Q = ([d], \preceq)$, its **order polytope** is

$$\mathcal{O}(Q) = \left\{(x_1,\dots,x_d) \in [0,1]^d : x_i \le x_j \text{ if } i \preceq j\right\}.$$

Let $\lambda \in \mathbb{Z}^d$ be some linear form and let $P \subseteq \mathbb{R}^d$ be a lattice polytope, Chapoton [4] defines

$$\mathrm{ehr}_P^\lambda(q,n) := \sum_{m\in nP\cap\mathbb{Z}^d} q^{\lambda\cdot m}.$$

Bajo et al. use the following key lemma relating these q-Ehrhart polynomials of (open) order polytopes to the q-chromatic polynomial [1, Lemma 3]; all notation used in the sum is defined in the next subsection.

Lemma 5.1 (Bajo et al)**.** *Let G be a graph on $[d]$ and let $\lambda \in \mathbb{Z}^d$ be a linear form. Then*

$$\chi_G^\lambda(q,n) = \sum_{\rho\in\mathcal{A}(G)} \mathrm{ehr}^\lambda_{\mathcal{O}(G_\rho)^\circ}(q,n+1).$$

To compute the Ehrhart polynomials in the above sum, one considers acyclic orientations of G and linear extensions of induced posets related to those acyclic orientations. Thus, to connect G-statistics to the work of Bajo et al., we need to establish a direct connection between permutations and pairs of acyclic orientations and linear extensions, which we do next.

5.1. The main bijection.

Let P be a poset on $[d]$. A **linear extension** $\tau : P \to [d]$ is an order preserving bijection. A **natural labeling** of a poset P is some fixed linear extension $\omega : P \to [d]$ and we denote the labeled poset as (P,ω). When given a natural labeling, we can identify a linear extension τ as a permutation given by the word $[\omega(\tau^{-1}(1))\omega(\tau^{-1}(2))\cdots\omega(\tau^{-1}(d))]$ of the labels of (P,ω). The collection of all such permutations is the **Jordan–Hölder set**, denoted $\mathcal{L}(P,\omega)$ or $\mathcal{L}(P)$ if the labeling is clear from the context. For a poset P with a natural labeling ω and linear extension τ, we say that the linear extension has a **descent** at $i \in [d-1]$ if for the associated word $\omega\circ\tau^{-1}$ we have $\omega\circ\tau^{-1}(i) > \omega\circ\tau^{-1}(i+1)$. Denote the set of all descents by $\mathrm{Des}(\omega\circ\tau^{-1})$.

Example 6. Let P be the poset given in Figure 5, then $\omega : P \to [4]$ where $\omega(1) = 1$, $\omega(2) = 3$, $\omega(3) = 4$, and $\omega(4) = 2$ is a natural labeling. We can also see this in Figure 5 since the labeled poset respects the usual order $1 \le 2 \le 3 \le 4$. Moreover $\tau : P \to [4]$ defined by

$$\tau(1) = 2,\ \tau(2) = 4,\ \tau(3) = 3,\ \tau(4) = 1$$

is a linear extension because $i \le_P j$ for $i,j \in P$ implies $\tau(i) \le \tau(j)$. We can think of the image of τ as defining the positions of elements of P for a linear order that respects the relations of P.

To see the corresponding word in $\mathcal{L}(P,\omega)$ for τ we have

$$[\omega(\tau^{-1}(1))\omega(\tau^{-1}(2))\omega(\tau^{-1}(3))\omega(\tau^{-1}(4))] = [\omega(4)\omega(1)\omega(3)\omega(2)] = [2143].$$

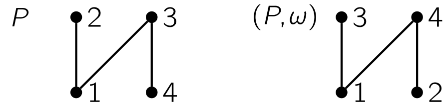


Figure 5. On the left, a poset P with elements $[4]$. On the right, a natural labeling ω of P given in Example 6.

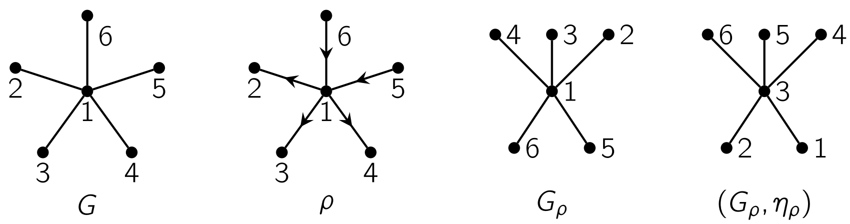


Figure 6. A graph, an acyclic orientation, the corresponding induced poset, and the induced poset with the rank labeling.

Note that τ^{-1} can be seen as a permutation of elements of P, and applying ω ensures we have a word of the labels of P. We also have $\mathrm{Des}([2143]) = \{1,3\}$.

Let $\mathcal{A}(G)$ denote the set of acyclic orientations of a given graph G. For $\rho \in \mathcal{A}(G)$, let G_ρ denote the **induced poset** on elements $[d]$, were $i \to j$ in ρ gives the relation $i \le j$ in G_ρ, followed by taking the transitive closure to produce a poset. For a graph G and $\rho \in \mathcal{A}(G)$, let $\{v_1, \ldots, v_{i_1}\}$ be all minimal elements of G_ρ, with $v_1 < \cdots < v_{i_1}$; next let $\{v_{i_1+1}, \ldots, v_{i_2}\}$ be all the minimal elements after removing $\{v_1, \ldots, v_{i_1}\}$ from G_ρ, with $v_{i_1+1} < \cdots < v_{i_2}$, and repeat until we have listed all elements of G_ρ. We define the **rank labeling** $\eta_\rho : G_\rho \to [d]$ by $\eta_\rho(v_j) = j$.

Example 7. Figure 6 gives an example of an induced poset for the star graph on 5 vertices for some acyclic orientation. For the induced poset, we have the rank labeling η_ρ where

$$\eta_\rho(1) = 3,\ \eta_\rho(2) = 4,\ \eta_\rho(3) = 5,\ \eta_\rho(4) = 6,\ \eta_\rho(5) = 1,\ \eta_\rho(6) = 2.$$

The following lemma will play a key role in our arguments in this section.

Lemma 5.2. *Let G be a graph on $[d]$ and $\pi \in S_d$. If $\pi(i)\pi(j) \in E(G)$ and $i < j$, then $p_\pi(\pi(i)) < p_\pi(\pi(j))$.*

Proof. When computing the rank $p_\pi(\pi(j))$, consider a path defining $p_\pi(\pi(i))$ along with the edge $\pi(i)\pi(j)$. In other words, we know for $r := p_\pi(\pi(i))$ there exist positions $i_1 < i_2 < \cdots < i_r = i$ such that $\{\pi(i_j), \pi(i_{j+1})\} \in E$ for $1 \le j < r$. Thus $\pi(i_1)\pi(i_2)\ldots\pi(i)\pi(j)$ is a possible path satisfying the definition in rank for $\pi(j)$. Since $p_\pi(\pi(j))$ is defined from the longest such path satisfying this condition,

$$p_\pi(\pi(j)) \ge p_\pi(\pi(i)) + 1 > p_\pi(\pi(i)). \qquad \square$$

Let G be a graph on $[d]$. For $\pi \in S_d$ we define the **π-rank acyclic orientation**, denoted ρ_π, to be the orientation where

$$\rho_\pi \text{ orients the edge } ij \in E(G) \text{ by } i \to j \text{ whenever } p_\pi(i) < p_\pi(j).$$

To show ρ_π is well defined, consider two adjacent vertices v_1, v_2. Then there exist i_1, i_2 such that $\pi(i_1) = v_1, \pi(i_2) = v_2$; without loss of generality say $i_1 < i_2$. Then $p_\pi(v_1) < p_\pi(v_2)$ by Lemma 5.2.

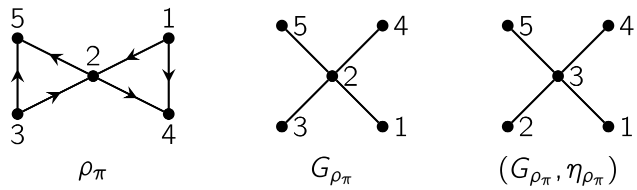


Figure 7. The π-rank acyclic orientation ρ_π of the bowtie graph in Figure 3 for $\pi = [31254]$, the induced poset G_{ρ_π}, and the induced poset with the rank labeling $(G_{\rho_\pi}, \eta_{\rho_\pi})$.

To show that this orientation is acyclic, note that on a directed path we must have ranks be strictly increasing and so having a cycle would contradict these strict inequalities of ranks. Given a natural labeling ω of G_{ρ_π}, we also define

$$\sigma_\pi := \omega \circ \pi = \omega \circ (\pi^{-1})^{-1}$$

which we can identify as the word $[\omega(\pi(1))\omega(\pi(2))\cdots\omega(\pi(d))]$.

Example 8. Consider the bowtie graph G from Figure 3 and $\pi = [31254]$. Recall from Example 4 that the rank function p_π gives $p_\pi(3) = 1$, $p_\pi(1) = 1$, $p_\pi(2) = 2$, $p_\pi(5) = 3$, and $p_\pi(4) = 4$. The π-rank acyclic orientation is shown in Figure 7. Moreover note that the rank labeling is a natural labeling for G_{ρ_π} and so for this labeling $\sigma_\pi = \eta_\rho \circ \pi$ as a word is [21354].

Theorem 5.3 below exhibits that the rank functions for G-statistics encode acyclic orientations, i.e., the induced posets of G, by showing that each permutation in S_d corresponds uniquely to a linear extension of an induced poset. Our proof below closely mirrors a proof due to Chow [5] arising in the work discussed previously regarding quasisymmetric functions and the chromatic symmetric function. Specifically, Chow used Stanley's [10] theory of P-partition functions to establish a bijection between permutations and pairs of acyclic orientations and linear extensions, where each acyclic orientation is given an order-reversing labeling of the induced poset. He also gave a labeling of these induced posets to show that the usual descent set of a permutation is equal to the descent set of the corresponding linear extension under this bijection and labeling. Since he used P-partitions, i.e., order-reversing maps from a poset P to $\mathbb{R}_{\geq 0}$, his results have direct analogues to order-preserving maps of posets, which we use for order polytopes in Theorem 5.3.

Further, Chow assigned the acyclic orientation in a manner that does not involve rank functions, though the resulting orientations are in fact equivalent. Translating Chow's work to our context of order-preserving maps, he considers the following: for G a graph on $[d]$, a permutation $\pi \in S_d$, and an edge $\pi(i)\pi(j) \in E(G)$, orient $\pi(i) \to \pi(j)$ for ρ_π if $i < j$. The resulting acyclic orientation is identical to our π-rank acyclic orientation defined above since for G a graph on $[d]$, $\pi \in S_d$, and $\pi(i)\pi(j) \in E(G)$, we have $p_\pi(\pi(i)) < p_\pi(\pi(j))$ if and only if $i < j$. The backwards direction of this equivalence is Lemma 5.2, while the forward direction can also be proven with Lemma 5.2 by contrapositive.

Thus, while the structure of the proof of Theorem 5.3 closely mirrors Chow's work, to set this in the context considered in this paper, we provide a complete proof of the bijection and apply it to our setting of order-preserving maps.

Theorem 5.3. *For a graph G on $[d]$, for each $\rho \in \mathcal{A}(G)$ let ω_ρ be a natural labeling of G_ρ. Then the map*

$$\varphi : S_d \to \{(\rho, \sigma) \mid \rho \in \mathcal{A}(G), \sigma \in \mathcal{L}(G_\rho, \omega_\rho)\}$$

defined by

$$\pi \mapsto (\rho_\pi, \sigma_\pi)$$

is a bijection. Moreover, if we use the rank labelings for our induced posets with $\sigma_\pi = \eta_{\rho_\pi} \circ \pi$, then $\mathrm{Des}_G(\pi) = \mathrm{Des}(\sigma_\pi)$ *where* Des *is the usual descent statistic.*

There are two key observations to highlight in the theorem above. First, while multiple permutations might induce the same permutation-rank acyclic orientation ρ, hence the same induced poset G_ρ, the choice of ω_ρ depends only on ρ and is independent of which permutation is being considered. As a result of this, the second observation is that each permutation which induces ρ via its rank function will yield a unique linear extension of G_ρ with respect to ω_ρ; it is this pair of an acyclic orientation and a linear extension that yields the bijective correspondence with permutations.

Proof. We will first prove bijectivity and then analyze the descent sets in the case where we use the rank labelings on each induced poset. Suppose $\pi \in S_d$. Recall ρ_π is an acyclic orientation of the edges $i \to j$ whenever $p_\pi(i) < p_\pi(j)$. Also recall $\sigma_\pi := \omega_{\rho_\pi} \circ \pi$, where ω_{ρ_π} is the same for all π and π' such that $\rho_\pi = \rho_{\pi'}$.

We claim that $\sigma_\pi \in \mathcal{L}(G_{\rho_\pi})$. To prove this, we have that $\sigma_\pi = \omega_{\rho_\pi} \circ (\pi^{-1})^{-1}$, where π^{-1} is defined as π is a bijection. We can consider the domain of π^{-1} to be the elements of the induced poset G_{ρ_π} because these elements are given by the set $[d]$. This means we want to show $\pi^{-1} : G_{\rho_\pi} \to [d]$ is a linear extension, i.e., an order preserving bijection from G_{ρ_π} to $[d]$. In other words, for the elements, not the labeling, of G_{ρ_π} we want to show that π^{-1} respects the relations of this poset. We know $\pi^{-1} : G_{\rho_\pi} \to [d]$ is a bijection, so all that is left to show is that it is order preserving. To do so, suppose $i \prec j$ in G_{ρ_π} is a cover relation of elements; hence it must form an edge in G and $i \neq j$. Denote $v_i := \pi^{-1}(i)$ and $v_j := \pi^{-1}(j) \in [d]$. For contradiction suppose $v_j < v_i$. By Lemma 5.2,

$$p_\pi(j) = p_\pi(\pi(v_j)) < p_\pi(\pi(v_i)) = p_\pi(i).$$

This inequality contradicts that $i \preceq j$ in G_{ρ_π} since this relation implies $p_\pi(i) \le p_\pi(j)$. So we must have $v_i \le v_j$, in other words π^{-1} preserves cover relations. This means π^{-1} is order preserving, hence by definition $\sigma_\pi = \omega_\rho \circ \pi \in \mathcal{L}(G_{\rho_\pi})$.

To define the inverse map $\psi : \{(\rho, \sigma) \mid \rho \in \mathcal{A}(G), \sigma \in \mathcal{L}(G_\rho)\} \to S_d$ consider (ρ, σ) such that $\rho \in \mathcal{A}(G)$ and $\sigma \in \mathcal{L}(G_\rho)$. By definition, $\sigma = \omega_\rho \circ \tau^{-1}$ for some order preserving map $\tau : G_\rho \to [d]$. We send $\psi(\rho, \sigma) = \tau^{-1}$. We will now verify that ψ is the inverse of φ. First, $\psi \circ \varphi(\pi) = \psi(\rho_\pi, \sigma_\pi) = \pi$ since $\sigma_\pi = \omega_\rho \circ \pi$. We next want to show $\varphi \circ \psi(\rho, \sigma) = (\rho, \sigma)$ where $\sigma = \omega_\rho \circ \tau^{-1}$. By definition of ψ, we have $\varphi(\psi(\rho, \sigma)) = \varphi(\tau^{-1})$. For $\varphi(\tau^{-1})$, we have $\sigma_{\tau^{-1}} = \omega_\rho \circ \tau^{-1} = \sigma$. We claim $\rho_{\tau^{-1}} = \rho$. By definition of $\rho_{\tau^{-1}}$, we have $i \to j$ in $\rho_{\tau^{-1}}$ whenever $p_{\tau^{-1}}(i) < p_{\tau^{-1}}(j)$ so it suffices to show that for an edge $ij \in E(G)$ we have $i \le j$ in G_ρ if and only if $p_{\tau^{-1}}(i) < p_{\tau^{-1}}(j)$. For the forward direction, suppose $i \preceq j \in G_\rho$, then $\tau(i) \le \tau(j)$ since τ is order preserving. Hence

$$p_{\tau^{-1}}(i) = p_{\tau^{-1}}(\tau^{-1}(\tau(i))) < p_{\tau^{-1}}(\tau^{-1}(\tau(j))) = p_{\tau^{-1}}(j)$$

by Lemma 5.2. For the backward direction, suppose $p_{\tau^{-1}}(i) < p_{\tau^{-1}}(j)$ then we claim $i \preceq j$ in G_ρ, since otherwise $\tau(i) > \tau(j)$ and by the same argument as the forward direction $p_{\tau^{-1}}(i) > p_{\tau^{-1}}(j)$, giving a contradiction. This shows $\rho_{\tau^{-1}} = \rho$ and proves that φ is a bijection.

Next, when for each $\rho \in \mathcal{A}(G)$ we have that η_ρ is the rank labeling, we will show $\mathrm{Des}_G(\pi) = \mathrm{Des}(\sigma_\pi)$ for every permutation $\pi \in S_d$. For a permutation $\pi \in S_d$, we know the rank labeling of G_{ρ_π} does the following. Say $\{v_1, \ldots, v_{i_1}\}$ are the vertices of G minimal in the poset, where $v_1 < \cdots < v_{i_1}$, $\{v_{i_1+1}, \ldots, v_{i_2}\}$ are all vertices of G minimal in $G \setminus \{v_1, \ldots, v_{i_1}\}$, where $v_{i_1+1} < \cdots < v_{i_2}$, and so on, then set $\eta_\rho(v_j) = j$. We claim η_{ρ_π} is a natural labelling since $i \preceq j$ in G_{ρ_π} implies $p_\pi(i) \leq p_\pi(j)$ which must mean $\eta_\rho(i) \leq \eta_\rho(j)$ by construction of our labeling. Additionally, any permutation with the same rank function p_π gives the same acyclic orientation and so η_{ρ_π} depends only on this acyclic orientation. We now show the descent statistics are the same under η_{ρ_π}. By construction of our labeling, $\eta_\rho(i) < \eta_\rho(j)$ if and only if

$$p_\pi(i) < p_\pi(j) \quad \text{or} \quad p_\pi(i) = p_\pi(j) \text{ and } i < j\,.$$

Thus, $i \in \mathrm{Des}_G(\pi)$ if and only if $p_\pi(\pi(i)) > p_\pi(\pi(i+1))$ or $p_\pi(\pi(i)) = p_\pi(\pi(i+1))$ and $\pi(i) > \pi(i+1)$. By our prior observation, this condition is equivalent to $\eta_\rho(\pi(i)) > \eta_\rho(\pi(i+1))$ which holds if and only if $i \in \mathrm{Des}(\eta_\rho \circ \pi) = \mathrm{Des}(\sigma_\pi)$. □

Viewing the rank function as defining an acyclic orientation motivates the definitions for G-statistics. Using the rank function of a permutation to produce a poset makes it also easier to see that the rank labeling is a natural labeling. In addition, Theorem 5.3 shows that for any $\rho \in \mathcal{A}(G)$, the rank labeling of G_ρ ensures that the usual descent set of a linear extension is equal to the G-descent set of the corresponding permutation.

We can apply the same arguments by Bajo et al. for order polytopes to give the generating function of the q-chromatic polynomial using q-Ehrhart theory and linear extensions of induced posets. Bajo et al. give the generating function for G-partitions using linear extensions of induced posets. We can apply the connection they found in Lemma 5.1 to obtain the generating function for the q-chromatic polynomial. We will only sketch the details for the q-chromatic polynomial with linear form $\mathbb{1} \in \mathbb{Z}^d$, as this case has a nice numerator polynomial in the rational expression of the generating function. In Section 6, we will see more properties of this polynomial.

Theorem 5.4. *Let $G = ([d], E)$ be a graph, and for $\rho \in \mathcal{A}(G)$, let ω_ρ be any natural labeling of G_ρ. Then*

$$\frac{\sum_{(\rho,\sigma)} q^{d+\sum_{j\in \mathrm{Asc}(\sigma)} d-j} z^{\mathrm{asc}(\sigma)+1}}{(1-z)(1-qz)\cdots(1-q^d z)} = \frac{\sum_{\pi\in S_d} q^{d+\sum_{j\in \mathrm{Asc}_G(\pi)} d-j} z^{\mathrm{asc}_G(\pi)+1}}{(1-z)(1-qz)\cdots(1-q^d z)}$$

where (ρ,σ) ranges over $\rho \in \mathcal{A}(G)$ and $\sigma \in \mathcal{L}(G_\rho)$ and Asc *is the usual ascent statistic for (G_ρ, ω_ρ).*

Proof. Recall that Bajo et al. proved Lemma 5.1, which states

$$\sum_{n\geq 0} \chi_G^{\mathbb{1}}(q,n) z^n = \sum_{\rho\in\mathcal{A}(G)} \sum_{n\geq 0} \mathrm{ehr}^{\mathbb{1}}_{\mathcal{O}(G_\rho)^\circ}(q,n+1) z^n.$$

We can find this generating function via the integer point transform $\sigma_{K^\circ_{\widehat{G_\rho}}}(z_1,\ldots,z_{d+1})$, where $K^\circ_{\widehat{G_\rho}}$ is the homogenization of the open order polytope of G_ρ with the last coordinate corresponding to height, using descent statistics [2, Exercise 6.23]. For each $\rho \in \mathcal{A}(G)$, the integer point transform is

$$\sigma_{K^\circ_{\widehat{G_\rho}}}(z_1,\ldots,z_{d+1}) = \sum_{\tau\in\mathcal{L}(G_\rho)} \frac{z_1\cdots z_d z_{d+1}^2 \prod_{j\in \mathrm{Asc}(\tau)} z_{\tau(j+1)}\cdots z_{d+1}}{(1-z_{d+1})\prod_{i=1}^d (1-z_{\tau(i)}\cdots z_{d+1})}\,.$$

Specializing this integer point transform to $z_i = q$ for all $i \in [d]$ and $z_{d+1} = z$ yields

$$\sigma_{K^\circ_{\widehat{G_\rho}}}(q,\ldots,q,z) = \sum_{n\geq 1} \mathrm{ehr}^{\mathbb{1}}_{\mathcal{O}(G_\rho)^\circ}(q,n) z^n.$$

Thus

$$\sum_{n\geq 0} \chi_G^{\mathbb{1}}(q,n) z^n = \sum_{\rho\in\mathcal{A}(G)} \sum_{n\geq 0} \mathrm{ehr}^{\mathbb{1}}_{\mathcal{O}(G_\rho)^\circ}(q,n+1) z^n$$

$$
\begin{aligned}
&= \sum_{\rho \in \mathcal{A}(G)} z^{-1} \sum_{n \geq 1} \mathrm{ehr}^{\mathbb{1}}_{\mathcal{O}(G_\rho)^\circ}(q,n) z^n \\
&= \sum_{\rho \in \mathcal{A}(G)} z^{-1} \sigma_{K^\circ_{\widehat{G_\rho}}}(q,\dots,q,z) \\
&= \sum_{\rho \in \mathcal{A}(G)} z^{-1} \sum_{\tau \in \mathcal{L}(G_\rho,\omega_\rho)} \frac{q^{d+\sum_{j\in \mathrm{Asc}(\tau)} d-j} z^{\mathrm{asc}(\tau)+2}}{(1-z)(1-qz)\cdots(1-q^d z)} \\
&= \frac{\sum_{\rho \in \mathcal{A}(G)} \sum_{\tau \in \mathcal{L}(G_\rho)} q^{d+\sum_{j\in \mathrm{Asc}(\tau)} d-j} z^{\mathrm{asc}(\tau)+1}}{(1-z)(1-qz)\cdots(1-q^d z)}.
\end{aligned}
$$

By Theorem 4.1, we know that $\sum_{n\geq 0} \chi^{\mathbb{1}}_G(q,n) z^n = \frac{\sum_{\pi\in S_d} q^{d+\sum_{j\in \mathrm{Asc}_G(\pi)} d-j} z^{\mathrm{asc}_G(\pi)+1}}{(1-z)(1-qz)\cdots(1-q^d z)}$ and so the two rational functions in the statement of the theorem are equal. $\square$

As a consequence of this rational function equality, the numerator polynomial of the q-chromatic polynomial with the linear form $\mathbb{1}$ can be computed using any natural labeling on the induced posets, and so G-statistics unify these expressions.

Corollary 5.5. *Let $G = ([d], E)$ be a graph, and for $\rho \in \mathcal{A}(G)$, let ω_ρ be any natural labeling of G_ρ. Then*

$$\sum_{\pi \in S_d} q^{\sum_{j\in \mathrm{Asc}_G(\pi)} d-j} z^{\mathrm{asc}_G(\pi)} = \sum_{(\rho,\sigma)} q^{\sum_{j\in \mathrm{Asc}(\sigma)} d-j} z^{\mathrm{asc}(\sigma)}$$

where (ρ,σ) ranges over $\rho \in \mathcal{A}(G)$ and $\sigma \in \mathcal{L}(G_\rho)$ and Asc *is the usual ascent statistic for (G_ρ, ω_ρ).*

5.2. **The leading coefficient of $\widetilde{\chi}^{\mathbb{1}}_G(q,x)$.** Bajo et al. [1] showed that $\chi^\lambda_G(q,n)$ is a polynomial in $[n]_q := 1+q+\cdots+q^{n-1}$ with coefficients in $\mathbb{Z}(q)$. We denote this polynomial by $\widetilde{\chi}^{\mathbb{1}}_G(q,x) \in \mathbb{Q}(q)[x]$. They show that the leading coefficient with respect to x is the polynomial $\frac{1}{[d]_q!}\sum_{(\rho,\sigma)} q^{d+\mathrm{maj}(\sigma)}$, where we sum over all $\rho \in \mathcal{A}(G)$ and all $\sigma \in \mathcal{L}(G_\rho)$, where G_ρ is given some natural labeling [1].

Example 9. Let G be the path graph on two vertices. Bajo et al. [1] computed $\widetilde{\chi}^{\mathbb{1}}_G(q,x) = \frac{2q^2}{1+q}x^2 + \frac{-2q^2}{1+q}x$. The leading coefficient is $\frac{2q^2}{1+q} = \frac{2q^2}{[2]_q}$. Indeed, $2 = \sum_{(\rho,\sigma)} q^{\mathrm{maj}(\sigma)}$ since we only have two acyclic orientations and each has only one linear extension with no descents.

The following corollary shows how Theorem 5.3 provides an alternative form of the leading term of $\widetilde{\chi}^{\mathbb{1}}_G(q,x)$ involving G-major indices.

Corollary 5.6. *The leading coefficient of $\widetilde{\chi}^{\mathbb{1}}_G(q,x)$ is*

$$\frac{1}{[d]_q!}\sum_{(\rho,\sigma)} q^{d+\mathrm{maj}(\sigma)} = \frac{1}{[d]_q!}\sum_{\pi\in S_d} q^{d+\mathrm{maj}_G(\pi)}.$$

Proof. By our bijection in Theorem 5.3, each (ρ,σ) corresponds uniquely to a permutation where the descents sets are the same under the rank labeling. Hence the G-major index and usual major index must be the same. $\square$

For a graph G on $[d]$, let the G-**major index polynomial** be

$$\sum_{\pi\in S_d} q^{\mathrm{maj}_G(\pi)}.$$

We will now show that the degree of the G-major index polynomial connects the minimum sum coloring problem to the leading coefficient of $\widetilde{\chi}_G^{\mathbb{1}}(q,x)$. The minimum sum coloring problem was first studied using graph theory by Ewa Kubicka [8].

Lemma 5.7. *We have*

$$\min\left\{d+\sum_{j\in\mathrm{Asc}_G(\pi)} d-j : \pi\in S_d\right\} = \min\left\{\sum_{v\in V} f(v) : f : V\to[m] \text{ a coloring}\right\}$$

and the number of colorings achieving the minimum sum of colors is the same as the number of permutations achieving the minimum on the left-hand side.

Proof. Minimizing the sum of colors is equivalent to minimizing $\mathbb{1}\cdot x$ for $x\in K_G\cap\mathbb{Z}^d_{>0}$. In addition, the minimum of $\mathbb{1}\cdot x$ for $x\in\Delta_\pi\cap\mathbb{Z}^d_{>0}$ for $\pi\in S_d$ must occur at the integer point from the G-sequence coloring. In other words, suppose $(S_1,\ldots,S_k)$ is the G-sequence of π, then let $x_i=1$ for $i\in S_1$, $x_j=2$ where $j\in S_2$ and so on. Since any other coloring in Δ_π either splits our G-sequence into more sets or uses the same sets of the G-sequence with higher colors, the G-sequence coloring is a minimum in each cone. Moreover, $d+\sum_{j\in\mathrm{Asc}_G(\pi)} d-j=\sum_{i=1}^k i|S_i|$ by Proposition 3.5. Thus, taking the minimum value over all cones gives $\min\left\{d+\sum_{j\in\mathrm{Asc}_G(\pi)} d-j:\pi\in S_d\right\}$ which must equal $\min\left\{\sum_{v\in V} f(v):f:V\to[m] \text{ a coloring}\right\}$. Moreover, each coloring achieving this minimum must be in some unique cone, by our construction of the subdivision of K_G. So, the number of minimum colorings is also the same as the number of permutations achieving $\min\left\{d+\sum_{j\in\mathrm{Asc}_G(\pi)} d-j:\pi\in S_d\right\}$. ☐

Corollary 5.8. *For a graph $G=([d],E)$, the degree of $\sum_{\pi\in S_d} q^{\mathrm{maj}_G(\pi)}$ equals*

$$\binom{d+1}{2}-\min\left\{\sum_{v\in V} f(v) : f : V\to[m] \text{ a coloring}\right\}$$

and its leading coefficient equals the number of colorings achieving this minimum with respect to the sum of colors.

Proof. We will reformulate the powers of q to apply Lemma 5.7. First, we know for $\pi\in S_d$

$$\mathrm{maj}_G(\pi)+\sum_{j\in\mathrm{Asc}_G(\pi)} j=\sum_{i\in\mathrm{Des}_G(\pi)} i+\sum_{j\in\mathrm{Asc}_G(\pi)} j=\sum_{i\in[d-1]} i=\binom{d}{2},$$

whence $\mathrm{maj}_G(\pi)=\binom{d}{2}-\sum_{j\in\mathrm{Asc}_G(\pi)} j=\binom{d+1}{2}-d-\sum_{j\in\mathrm{Asc}_G(\pi)} j$. This means our G-major index polynomial is

$$\sum_{\pi\in S_d} q^{\mathrm{maj}_G(\pi)}=\sum_{\pi\in S_d} q^{\binom{d+1}{2}-d-\sum_{j\in\mathrm{Asc}_G(\pi)} j}$$

and by Corollary 6.2

$$\sum_{\pi\in S_d} q^{\binom{d+1}{2}-d-\sum_{j\in\mathrm{Asc}_G(\pi)} j}=\sum_{\pi\in S_d} q^{\binom{d}{2}-d-\sum_{j\in\mathrm{Asc}_G(\pi)} d-j}.$$

The highest power of this polynomial occurs when $d+\sum_{j\in\mathrm{Asc}_G(\pi)} d-j$ is minimized over all $\pi\in S_d$ and the corresponding coefficient is given by the number permutations achieving this minimum. By Lemma 5.7 this value is given by $\min\{\sum_{v\in V} f(v):f:V\to[m] \text{ a coloring}\}$. ☐

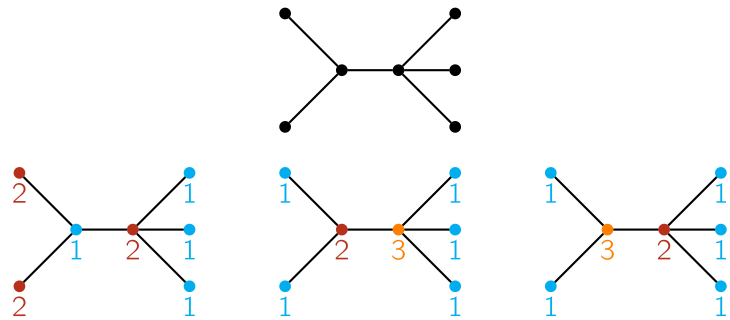


Figure 8. A graph and three colorings achieving the minimum sum of colors.

Example 10. The G-major index polynomial of the graph G in Figure 8 is

$$\sum_{\pi \in S_7} q^{\mathrm{maj}_G(\pi)} = 3q^{18} + 8q^{17} + 20q^{16} + 56q^{15} + 101q^{14} + 167q^{13} + 249q^{12} + 358q^{11} + 448q^{10} + 529q^9 + 555q^8 + 561q^7 + 552q^6 + 470q^5 + 358q^4 + 263q^3 + 170q^2 + 108q + 64.$$

The leading coefficient is 3 and so there are three colorings achieving the chromatic sum. These colorings must sum to

$$\binom{8}{2} - 18 = 28 - 18 = 10.$$

The three colorings are shown in Figure 8. We note that this example also shows that the colorings achieving the minimum sum of colors for trees may require more than two colors.

Bajo et al. [1] made the following conjecture.

Conjecture 5.9 (Bajo et al.)**.** *The leading coefficient of $\widetilde{\chi}_G^{\mathbb{1}}(q,x)$ distinguishes trees.*

This motivates the following bounds on the degrees of the G-major polynomial for trees.

Corollary 5.10. *Let $d \geq 2$. For a tree $T = ([d], E)$, the degree of the G-major polynomial is bounded via*

$$\binom{d+1}{2} - \lfloor 1.5d \rfloor \leq \deg \left(\sum_{\pi \in S_d} q^{\mathrm{maj}_G(\pi)} \right) \leq \binom{d+1}{2} - d - 1 .$$

Moreover, there exists a tree attaining each possible degree between the two bounds.

Proof. We use a result of Kubicka and Schwenk [8, Theorem 3]. Let

$$\Sigma(T) := \min \left\{ \sum_{v \in V} f(v) : f : V \to [m] \text{ a coloring} \right\}$$

and $d := |V|$. We know every coloring of a tree has to use at least 2 colors. Then $\Sigma(T) \geq d+1$ via assigning one vertex the color 2 and the rest the color 1; a star graph attains this lower bound. Since trees are bipartite and thus 2-colorable, we can color the vertices in the larger bipartite set with color 1 and all the vertices in the smaller set as color 2. So $\Sigma(T) \leq \lceil \frac{d}{2} \rceil + 2 \lfloor \frac{d}{2} \rfloor = \lfloor 1.5d \rfloor$, and the path graph achieves this upper bound. Since $d+1 \leq \Sigma(T) \leq \lfloor 1.5d \rfloor$,

$$\binom{d+1}{2} - (d+1) \geq \binom{d+1}{2} - \Sigma(T) \geq \binom{d+1}{2} - \lfloor 1.5d \rfloor ,$$

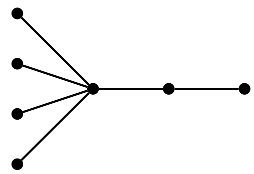


Figure 9. The "broom-like" graph $B(4,3)$ as defined in the proof for Corollary 5.10.

Corollary 5.8 yields our first claim.

To prove the second claim, it suffices to check there exists a tree T with $\Sigma(T) = k$ for every $d+1 \leq k \leq \lfloor 1.5d \rfloor$. Let $B(m,n)$ denote the "broom-like" tree where a path with n vertices has m leaves attached to one end, see Figure 9 for an example. Expressing $k = d+1+b$ for some b, it is not hard to see that $\Sigma(B(d-2b-1, 2b+1)) = k$. □

As a concluding aside for this section, observe that Stanley finds the degree of the polynomial

$$W_P(q) := \sum_{\sigma \in \mathcal{L}(P)} q^{\mathrm{maj}(\sigma)}$$

for a naturally labeled poset P as follows [11, Section 3.15]. For $t \in P$, define $\delta(t)$ to be the number of vertices of the longest chain $t = t_1 < \cdots < t_{\delta(t)}$ of P that starts at t. We further define $\delta(P) = \sum_{t \in P} \delta(t)$. Stanley shows the degree of $W_P(q)$ equals $\binom{|P|+1}{2} - \delta(P)$, and $W_P(q)$ is monic, i.e., only one permutation achieves this maximal major index. We can connect this to our minimum sum coloring since we can compute the degree in two ways. For a graph G on $[d]$ we can use Stanley's result and Theorem 5.3 to compute the degree of $\sum_{\pi \in S_d} q^{\mathrm{maj}_G(\pi)} = \sum_{(\rho,\sigma)} q^{\mathrm{maj}(\sigma)}$ which is

$$\max_{\rho \in \mathcal{A}(G)} \left\{ \binom{d+1}{2} - \delta(G_\rho) \right\} = \binom{d+1}{2} - \min_{\rho \in \mathcal{A}(G)} \{\delta(G_\rho)\}.$$

By Corollary 5.8 this means

$$\min_{\rho \in \mathcal{A}(G)} \{\delta(G_\rho)\} = \min \left\{ \sum_{v \in V} f(v) : f : V \to [m] \text{ a coloring} \right\}.$$

For each acyclic orientation, this gives a candidate for the minimum sum coloring of graphs, namely a coloring $f : V(G) \to [k]$ defined by $f(t) := \delta(t)$ where k is the number of elements of a longest chain in G_ρ.

If we apply an alternative convention to maximize $\sum_{j \in \mathrm{Des}(\sigma)} d - j$ over all (ρ, σ), we can instead use the G-sequence colorings of permutations as described in the proof of Corollary 5.8 and Lemma 5.7. This means that for each acyclic orientation, we have a candidate for the minimum sum coloring of graphs by assigning color based off the rank with respect to G_ρ of an element $t \in G$, the number of vertices in the longest chain in G_ρ ending at t.

6. Symmetry

Theorem 5.3 allows us to simplify some of the expressions arising in Section 3 for the numerator polynomial of the q-chromatic generating function, showing that the coefficients of this numerator polynomial, each of which is a polynomials in q, are each palindromic up to a shift. We begin by introducing several important definitions required to describe the bijection established in Proposition 6.1 below.

Let $s:[d]\to[d]$ be the involution $s(i):=d+1-i$. For a graph G, an acyclic orientation $\rho\in\mathcal{A}(G)$, and a natural labeling ω_ρ of G_ρ, we let ρ^{rev} be the acyclic orientation obtained by reversing all the directions of ρ. We then define

$$\omega'_\rho := s\circ\omega_{\rho^{\mathrm{rev}}}\,.$$

Observe that ω'_ρ is a natural labeling.

For each fixed G and $\rho\in\mathcal{A}(G)$, we next define a map sending a linear extension $\sigma\in\mathcal{L}(G_\rho,\omega_\rho)$ to σ^{op}, a linear extension of $G_{\rho^{\mathrm{rev}}}$. Given σ, we know that there exists an order preserving map $\tau:G_\rho\to[d]$ such that $\sigma=\omega_\rho\circ\tau^{-1}$. We define

$$\sigma^{\mathrm{op}} := \omega'_{\rho^{\mathrm{rev}}}\circ\tau^{-1}\circ s\,.$$

In order to verify that $\sigma^{\mathrm{op}}\in\mathcal{L}(G_{\rho^{\mathrm{rev}}},\omega'_{\rho^{\mathrm{rev}}})$, we must check that $(\tau^{-1}\circ s)^{-1}=s\circ\tau$ is order preserving for $G_{\rho^{\mathrm{rev}}}$. If $i\le j$ in $G_{\rho^{\mathrm{rev}}}$, then $i\ge j$ in G_ρ and so $\tau(i)\ge\tau(j)$, as τ is order preserving. Hence $d+1-\tau(i)\le d+1-\tau(j)$ which shows $s\circ\tau(i)\le s\circ\tau(j)$.

Proposition 6.1. *Let G be a graph on $[d]$ and for $\rho\in\mathcal{A}(G)$, let ω_ρ be a natural labeling of G_ρ. There is a bijection*

$$\psi:\{(\rho,\sigma)\mid\rho\in\mathcal{A}(G),\sigma\in\mathcal{L}(G_\rho,\omega_\rho)\}\to\{(\rho,\sigma)\mid\rho\in\mathcal{A}(G),\sigma\in\mathcal{L}(G_\rho,\omega'_\rho)\}$$

sending

$$(\rho,\sigma)\mapsto(\rho^{\mathrm{rev}},\sigma^{\mathrm{op}})$$

with the property that $\mathrm{Asc}(\sigma)=\{d-i:i\in\mathrm{Asc}(\sigma^{\mathrm{op}})\}$.

Proof. We claim ψ is an involution and therefore also a bijection. Consider $\psi(\psi(\sigma,\rho))=((\rho^{\mathrm{rev}})^{\mathrm{rev}},(\sigma^{\mathrm{op}})^{\mathrm{op}})$. For $\rho\in\mathcal{A}(G)$ we know $(\rho^{\mathrm{rev}})^{\mathrm{rev}}=\rho$, our labels for G_ρ are $(\omega'_\rho)'=s\circ s\circ\omega_\rho=\omega_\rho$, and

$$(\sigma^{\mathrm{op}})^{\mathrm{op}}(i)=(\omega'_\rho)'\circ(\tau^{-1}\circ s)\circ s=\omega_\rho\circ\tau^{-1}=\sigma$$

and so ψ is a bijection.

Moreover, the ascents under this bijection give our desired property. For $\rho\in\mathcal{A}(G)$ and $\sigma\in\mathcal{L}(G_\rho,\omega_\rho)$,

$$\begin{aligned}
j\in\mathrm{Asc}(\sigma)&\Leftrightarrow\omega_\rho\circ\tau^{-1}(j)<\omega_\rho\circ\tau^{-1}(j+1)\\
&\Leftrightarrow d+1-\omega_\rho\circ\tau^{-1}(j)>d+1-\omega_\rho\circ\tau^{-1}(j+1)\\
&\Leftrightarrow \omega'_\rho\circ\tau^{-1}(j)>\omega'_\rho\circ\tau^{-1}(j+1)\\
&\Leftrightarrow \omega'_\rho\circ\tau^{-1}\circ s(d+1-j)>\omega'_\rho\circ\tau^{-1}\circ s(d-j)\\
&\Leftrightarrow d-j\in\mathrm{Asc}(\sigma^{\mathrm{op}})\,,
\end{aligned}$$

hence $\mathrm{Asc}(\sigma)=\{d-i:i\in\mathrm{Asc}(\sigma^{\mathrm{op}})\}$. $\square$

Using Proposition 6.1, we can simplify the expression of the numerator polynomial of the q-chromatic generating function for the linear form $\mathbb{1}\in\mathbb{Z}^d$.

Corollary 6.2. *For G a graph on $[d]$,*

$$\sum_{\pi\in S_d}q^{d+\sum_{j\in\mathrm{Asc}_G(\pi)}d-j}z^{\mathrm{asc}_G(\pi)+1}=\sum_{\pi\in S_d}q^{d+\sum_{j\in\mathrm{Asc}_G(\pi)}j}z^{\mathrm{asc}_G(\pi)+1}=\sum_{\pi\in S_d}q^{\binom{d+1}{2}-\mathrm{maj}_G(\pi)}z^{\mathrm{asc}_G(\pi)+1}.$$

Proof. To prove the first equality of the corollary we have

$$\sum_{\pi\in S_d}q^{\sum_{j\in\mathrm{Asc}_G(\pi)}d-j}z^{\mathrm{asc}_G(\pi)}=\sum_{(\rho,\sigma)}q^{\sum_{j\in\mathrm{Asc}(\sigma)}d-j}z^{\mathrm{asc}(\sigma)}$$

by Corollary 5.5. Further applying Proposition 6.1 results in

$$\sum_{(\rho,\sigma)} q^{\sum_{j\in \mathrm{Asc}(\sigma)} d-j} z^{\mathrm{asc}(\sigma)} = \sum_{(\rho,\sigma)} q^{\sum_{d-j\in \mathrm{Asc}(\sigma^{\mathrm{op}})} d-j} z^{\mathrm{asc}(\sigma^{\mathrm{op}})} = \sum_{(\rho^{\mathrm{rev}},\sigma^{\mathrm{op}})} q^{\sum_{i\in \mathrm{Asc}(\sigma)} i} z^{\mathrm{asc}(\sigma)} = \sum_{(\rho,\sigma)} q^{\sum_{i\in \mathrm{Asc}(\sigma)} i} z^{\mathrm{asc}(\sigma)}$$

where the first equality holds by applying the permutation σ^{op} and the last equality holds, as we have a bijection indexing the sum. Moreover any natural labeling of induced posets produces the same sum. We apply Theorem 5.3, so that

$$\sum_{(\rho,\sigma)} q^{\sum_{i\in \mathrm{Asc}(\sigma)} i} z^{\mathrm{asc}(\sigma)} = \sum_{\pi\in S_d} q^{\sum_{j\in \mathrm{Asc}(\pi)} j} z^{\mathrm{asc}_G(\pi)},$$

giving the first equality of the corollary.

To derive the second equality, we know

$$\binom{d}{2} = \sum_{j\in[d-1]} j = \sum_{j\in \mathrm{Asc}_G(\pi)} j + \sum_{i\in \mathrm{Des}_G(\pi)} i = \sum_{j\in \mathrm{Asc}_G(\pi)} j + \mathrm{maj}_G(\pi)$$

for any $\pi \in S_d$. This means

$$d + \sum_{j\in \mathrm{Asc}_G(\pi)} j = d + \binom{d}{2} - \mathrm{maj}_G(\pi) = \binom{d+1}{2} - \mathrm{maj}_G(\pi),$$

which gives the desired equality. □

Another interesting consequence of Proposition 6.1 is the symmetry of the coefficients of the numerator polynomial of the q-chromatic generating function for the linear form $\mathbb{1} \in \mathbb{Z}^d$.

Corollary 6.3. *For a graph G on $[d]$ with chromatic number ξ, write*

$$\sum_{\pi\in S_d} q^{d+\sum_{j\in \mathrm{Asc}_G(\pi)} d-j} z^{\mathrm{asc}_G(\pi)+1} = a_\xi(q) z^\xi + \cdots + a_d(q) z^d$$

with $a_i(q) := \sum q^{d+\sum_{j\in \mathrm{Asc}_G(\pi)} d-j}$ where we sum over all permutations π such that $\mathrm{asc}_G(\pi)+1 = i$. For each $i \in [\xi, d]$, the polynomial $a_i(q)$ is shifted palindromic, i.e., $q^{d(i+1)} a_i(\frac{1}{q}) = a_i(q)$.

Proof. For $i \in [\xi, d]$ consider $a_i(q)$. By Corollary 6.2, we have $a_i(q) = \sum q^{d+\sum_{j\in \mathrm{Asc}_G(\pi)} j}$ and so

$$\begin{aligned} q^{d(i+1)} a_i\left(\frac{1}{q}\right) &= q^{d(i+1)} \sum_{\substack{\pi\in S_d \\ \mathrm{Asc}_G(\pi)+1=i}} q^{-d-\sum_{j\in \mathrm{Asc}_G} d-j} \\ &= q^{d(\mathrm{asc}_G(\pi)+2)} \sum_{\substack{\pi\in S_d \\ \mathrm{Asc}_G(\pi)+1=i}} q^{-d-d\cdot \mathrm{asc}_G(\pi)+\sum_{j\in \mathrm{asc}_G(\pi)} j} \\ &= \sum_{\substack{\pi\in S_d \\ \mathrm{Asc}_G(\pi)+1=i}} q^{d\cdot \mathrm{asc}_G(\pi)+2d-d-d\cdot \mathrm{asc}_G(\pi)+\sum_{j\in \mathrm{asc}_G(\pi)} j} \\ &= \sum_{\substack{\pi\in S_d \\ \mathrm{Asc}_G(\pi)+1=i}} q^{d+\sum_{j\in \mathrm{Asc}_G} j} \\ &= a_i(q). \end{aligned}$$

This means after factoring out the lowest power of q in $a_i(q)$, we have a palindromic polynomial. □

Example 11. One can see the palindromic polynomials in Example 5. Moreover, the star graph in this example shows these polynomials may not necessarily be unimodal.

We remark further that for a fixed graph G on $[d]$ and a fixed value $n \in \mathbb{Z}_{>0}$, the polynomial $\chi_G^{\mathbb{1}}(q,n)$ is a shifted palindromic polynomial in q. For every n-coloring c, we have an n-coloring $c' := n+1-c$ which is a bijection. Hence

$$q^{d(n+1)}\chi_G^{\mathbb{1}}\left(\frac{1}{q},n\right) = \sum_{\substack{\text{colorings}\\ c:[d]\to[n]}} q^{\sum_{i\in[d]} n+1-\sum_{i\in[d]} c(i)} = \sum_{\substack{\text{colorings}\\ c:[d]\to[n]}} q^{\sum_{i\in[d]} c'(i)} = \sum_{\substack{\text{colorings}\\ c':[d]\to[n]}} q^{\sum_{i\in[d]} c(i)} = \chi_G^{\mathbb{1}}(q,n).$$

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