WULFF ISOPERIMETRY ON CAYLEY GRAPHS: SUBMODULAR BV, TEMPERED FØLNER, AND PROFILE RATIO BOUNDS

MAYUKH MUKHERJEE

Abstract. We develop a discrete BV framework on Cayley graphs that is tailored to anisotropic geometric analysis. The core analytic outputs are: *(i)* a sharp discrete Wulff isoperimetric inequality with the best constant tied to the generating set/stencil S and asymptotic attainment by Wulff samplers; *(ii)* a quantitative Γ-convergence of discrete total variation to the continuum anisotropic perimeter (with algebraic rates in crystalline regimes); and *(iii)* a gauge construction in the Heisenberg group that neutralizes shear at the level of the discrete energy, yielding an *exact per-edge* BV + shear identity, scale-sharp compactness, and matched Γ-limsup/liminf bounds.

As an application we revisit a question of Gromov (2008) on the ratio between the exact isoperimetric profile and its greatest nondecreasing minorant. We show that this ratio is uniformly bounded on non-amenable groups and on two-ended groups, and using our sharp Wulff inequality and Γ-convergence, we give a self-contained, constant-tracked proof of profile ratio boundedness for all virtually nilpotent groups. We isolate a *Tempered Følner* criterion (TF) (an exhaustion principle with controlled increment size), which forces bounded profile ratio in general. We verify *full* (TF) with explicit constants in two large families: finite-lamp wreath products over (TF) bases and lamplighters over amenable bases, where we build nested exhaustions with global (TF)(ii) and a cofinal subsequence of near-minimizers (TF)(i_e) (hence full (TF)). For semidirect products $\mathbb{Z}^d \rtimes_A \mathbb{Z}$ we construct layer-nested sets of logarithmic height that are (A-covariantly) nested, Følner, and satisfy (TF)(ii) with a constant independent of $A \in \mathrm{GL}(d, \mathbb{Z})$; within the class of layer-nested sets they are near-optimal up to constants, and when A is hyperbolic (no eigenvalue on the unit circle) known L^1-isoperimetric lower bounds give (TF)(i_e) along these sets, hence *full* (TF) in this case. Thus Gromov's problem is settled in broad regimes, and elsewhere reduced to a clean structural hypothesis; in this context, we also formulate *gap conjectures* that would settle the question for all amenable Cayley graphs. We also present an amenable "cursor-lamplighter" *graph model* that exhibits an unbounded ratio when (TF) fails, clarifying the sharpness of our assumptions.

The BV/variational viewpoint has direct spectral and analytic consequences. From the same inputs we derive Cheeger-type, Faber-Krahn, and Nash inequalities and obtain quantitative heat-kernel/mixing bounds on finite quotients, with constants depending explicitly on S and the Wulff constant and tracked throughout [Che70, Bus82, Dod84, Moh89, VSCC92, SC02]. The (TF) control further yields a tempered form of Property A with scale calibration, leading to explicit coarse embeddings of Cayley graphs (and box spaces) into Hilbert space with compression $\rho(t) \gtrsim t^{1/2}/\log t$ and tracked constants. Finally, we show that (TF) is a robust reflection of bi-equivariant geometry, and encodes a two-sided regularity principle: on the $(\Gamma \times \Gamma)$-homogeneous-space graph the boundary energy splits *exactly* as the sum of the left and right cuts (Dirichlet forms of the two regular representations), and in virtually nilpotent classes this *doubles* the sharp Wulff constant asymptotically–refining and answering another question of Gromov.

Our techniques-calibration-style proofs for the discrete Wulff bound (cf. [Tay78]), a combinatorial Hodge/curl-fitting lemma with explicit constants, a gauge-based shear split on Heisenberg (in the sub-Riemannian spirit of [Pan82, LM05, LR03, MR09]), and a padding/gluing scheme behind (TF) (cf. quasi-tilings [OW87], [DH17], [CJK+18])–might be of independent interest.

Contents

1. Introduction

1.1. Motivation.

Isoperimetry on finitely generated groups sits at the meeting point of geometric analysis, probability, and combinatorics. In the continuum, anisotropic surface energies are minimized by Wulff shapes; sharp constants follow from calibrations and BV methods [Tay78]. On Cayley graphs one seeks a discrete counterpart that

(1) is *sharp in the optimal constant determined by S*;
(2) is stable under sampling; and
(3) interfaces cleanly with spectral and heat-kernel inequalities.

The tensions are most pronounced in sub-Riemannian geometries (e.g. Heisenberg), where the relevant exponent is the homogeneous dimension Q and non-Abelian features such as shear frustrate naive discretizations of continuum heuristics [Pan82, LM05, MR09, AKLD09]. Our approach replaces smooth BV by a *submodular BV calculus* on Cayley graphs and introduces a discrete calibration adapted to the anisotropy τ_S induced by S *(the support function of the horizontal zonotope $Z_S := \sum_{s\in S_+}[-X_s, X_s]$).*

A second, more group-theoretic thread concerns the *exact isoperimetric profile I°* and its greatest nondecreasing minorant I°_{incr}. Gromov asked whether the ratio $I^\circ/I^\circ_{\mathrm{incr}}$ is uniformly bounded on every infinite finitely generated group [Gro08b, p. 4]. We prove boundedness in several large and natural families and identify a structural criterion that implies it in general.

1.2. Main contributions.

We develop a submodular BV calculus on Cayley graphs and use it to obtain the following results.

(1) **Asymptotically sharp discrete Wulff isoperimetry.** Let (Γ, S) be either $\mathbb{Z}^d$ or a lattice in a Carnot group of homogeneous dimension Q. There exists a constant $h_S > 0$, determined explicitly by the anisotropy τ_S (see Definition 3.2), such that for all finite $Y \subset \Gamma$,
$$|\partial^{\mathrm{e}}_S Y| \;\ge\; h_S |Y|^{(Q-1)/Q},$$
with exponent $(d-1)/d$ in the Abelian case ($Q = d$). Moreover, there are *Wulff samplers* Y_k for which
$$|\partial^{\mathrm{e}}_S Y_k| \le (1 + o(1))h_S|Y_k|^{(Q-1)/Q} \qquad \text{as } |Y_k| \to \infty.$$
This yields a discrete counterpart of the continuum Wulff theory with explicit constant and asymptotic attainment. *(See Theorems 3.25 and 4.1.)*

(2) **Quantitative Γ-convergence with an algebraic rate.** For discrete total variation $\mathrm{Per}_{\tau_S,h}$ on $\mathbb{Z}^d$ sampled at scale $h \downarrow 0$, we prove
$$\mathrm{Per}_{\tau_S,h} \xrightarrow[\Gamma]{L^1} \mathrm{Per}_{\tau_S} \qquad \text{with error } \mathcal{O}(h^\alpha),$$

for crystalline anisotropies (including orthotropic), with explicit $\alpha > 0$ and rate-tracked recovery sequences. A new *oblique-counting lemma* isolates the sampling error arising from facet orientations relative to the lattice. *(See Theorems 5.7 and 5.16.)*

(3) **Heisenberg shear: constructive gauge and exact BV + shear split.** On the discrete Heisenberg group we introduce a quadratic gauge g (defined in Definition 7.2) that neutralizes horizontal shear and prove the exact per-edge identity

$$\mathrm{Per}_{\mathrm{Heis}}(Y) \;=\; \mathrm{Per}_{\mathrm{BV}}(Y) \;+\; \mathrm{Shear}_g(Y),$$

valid for all finite Y, with a quantified cap-loss correction at truncation scales. This yields compactness at the correct scaling and matching Γ-lim sup / lim inf bounds. *(See Theorems 7.11 and 7.15 and Corollary 7.13.)*

(4) **Gromov's profile question: bounded ratios in large classes, a Tempered Følner criterion, and permanence.** Let $I^\circ(n) := \min_{|Y|=n} |\partial_S^{\mathrm{e}} Y|$ and $I^\circ_{\mathrm{incr}}(n) := \inf_{m \geq n} I^\circ(m)$ (the right monotone minorant; it is nondecreasing in n). We prove the bounded ratio property (BRP), $\sup_n I^\circ(n)/I^\circ_{\mathrm{incr}}(n) < \infty$, for all virtually nilpotent groups via our discrete Wulff two-sided asymptotics. We introduce the *Tempered Følner* (TF) property (Definition 9.11) and show it *quantitatively* forces BRP:

$$\text{(TF)} \implies \sup_n \frac{I^\circ(n)}{I^\circ_{\mathrm{incr}}(n)} \;\leq\; A + \Delta AB \qquad (\Delta = |S|),$$

(see Theorem 9.31). Beyond a criterion, we give a constructive route: assuming only *nested near-minimizers* (NNM) in edge normalization, our interleaving scheme produces a single nested chain with full (TF) (under amenability), hence BRP, with explicit constants (Theorems 9.20, 9.100 and 9.101). We verify this program in three canonical amenable families with tracked constants: $\mathbb{Z}^d$ and Carnot lattices (nested Wulff samplers; Theorem 9.30), finite-lamp lamplighters $H^{(\Gamma)} \rtimes \Gamma$ (explicit tempered expansion + Erschler checkpoints; Theorems 9.39, 9.42 and 9.48), and hyperbolic semidirects $\mathbb{Z}^d \rtimes_A \mathbb{Z}$ (logarithmic layer-nested sets + L^1-profile lower bounds; Theorems 9.75 and 9.81). Permanence is established under finite index and finite extensions, and we give a product tensoring principle that preserves NNM and yields (TF) on direct products with controlled constants (Theorems 9.58, 9.59, 9.61 and 9.63). At the level of minorants, we prove a clean quasi-isometry comparison (no additive loss) and a scaled transfer of BRP; under a mild doubling of I°_{incr} this recovers full QI-invariance (Theorem 9.90).

Limits and an open problem. We build an amenable (non-vertex-transitive) "balloon-chain" graph with $\sup_r I^\circ(r)/I^\circ_{\mathrm{incr}}(r) = \infty$, and we explain why this single-edge-plateau mechanism cannot occur verbatim in Cayley graphs (a left-Lipschitz trimming obstruction; Theorems 9.52 and 9.55). This leaves a question: does there exist an amenable *Cayley* graph with unbounded ratio? (Question 9.1.)

We now state informal versions of the main results.

Throughout Theorem A, φ denotes the directed-edge integrand from the CI/zonoid model (Definition 3.2), specialized to the split-pair normalization $a_s = a_{s^{-1}} = \frac{1}{2}$, i.e. $\varphi = \tau_S$ in Remark 3.4.

Theorem A (Discrete Wulff isoperimetry: sharp constants in the cases of interest)**.** *Let* $\mathrm{Per}_{S,1}$ *be the* directed *edge perimeter (Definition 3.22) and set*

$$h_S \;:=\; \lim_{n\to\infty} n^{-\frac{D-1}{D}} \,\inf\{\mathrm{Per}_{S,1}(Y) :\; Y \subset \Gamma,\; |Y| = n\},$$

where D is the relevant dimension ($D = d$ for $\Gamma \cong \mathbb{Z}^d$, $D = Q$ for a lattice in a Carnot group). With φ as in Definition 3.2 and Remark 3.4, the limit exists and

$$h_S \;=\; c_{\text{Wulff}}(\varphi).$$

In particular:

(i) Unimodular axis stencil on $\mathbb{Z}^d$. *If $S = \{\pm b_1, \dots, \pm b_d\}$ with $B = [b_1 \cdots b_d] \in GL(d, \mathbb{Z})$, then $\varphi(\xi) = \sum_{i=1}^d |\langle b_i, \xi\rangle|$ and $h_S = 2d$.*

(ii) Uniform lattices in Carnot groups (horizontal anisotropy). *If Γ is a uniform lattice in a Carnot group of homogeneous dimension Q with symmetric S, then*

$$\varphi(\xi) \;=\; \tau_S(\xi) \;:=\; \sum_{s\in S_+} \big|\langle \xi, X_s\rangle\big|, \qquad h_S \;=\; c_{\text{Wulff}}(\tau_S).$$

(iii) Virtually Abelian with basis fiber. *If there is a surjection $\pi : \Gamma \to \mathbb{Z}^d$ with finite kernel K of size m and S contains a subset mapping bijectively to a unimodular basis of $\mathbb{Z}^d$, then*

$$h_S \;=\; m^{1/d} c_{\text{Wulff}}(\tau_S) \;=\; 2dm^{1/d}.$$

Moreover, there exist Wulff samplers Y_k such that $\mathrm{Per}_{S,1}(Y_k) \le (1 + o(1))h_S|Y_k|^{(D-1)/D}$.

Proof pointers. Theorem A is proved in Section 3 via a submodular BV compactness/liminf argument together with a sampler/phase-averaging limsup (a calibration-style step), yielding the sharp Wulff constant. Section 5 then *quantifies* the attainment: for orthotropic and finite-stencil crystalline anisotropies one obtains an algebraic rate $O(h^\beta)$ (with $\beta \in (0, 1]$ tied to the boundary regularity) of approach to the Wulff value.

Theorem B (Quantitative Γ-convergence for canonical/crystalline energies). *On $\mathbb{Z}^d$ we treat two cases.*

(i) Orthotropic/canonical axis stencil. *If $\varphi(\xi) = \sum_{i=1}^d \omega_i|\xi_i|$ and E_h is the axis-face energy from §5, then $E_h \xrightarrow[\Gamma]{L^1_{\text{loc}}} TV_\varphi$, and for E with $C^{1,\beta}$ boundary and half-open samplers A_h,*

$$\big|E_h(A_h) - TV_\varphi(\chi_E)\big| \;\le\; Ch^\beta,$$

with $\beta \in (0, 1]$ the boundary regularity ($\beta = 1$ for $C^{1,1}$) and C depending on $d, \omega, \|\partial E\|_{C^{1,\beta}}, \mathcal{H}^{d-1}(\partial E)$.

(ii) Finite-stencil crystalline integrands. *Let $\mathcal{P} \subset \mathbb{Z}^d \setminus \{0\}$ be finite with weights $\alpha_p > 0$. Fix a set of representatives $\mathcal{P}_+$ containing exactly one vector from each antipodal pair in $\mathcal{P} \cup (-\mathcal{P})$ and set $\tilde\alpha_p := \alpha_p + \alpha_{-p}$ (with $\alpha_{-p} := 0$ if $-p \notin \mathcal{P}$). Define*

$$\phi_{\mathcal{P}}(\nu) \;:=\; \sum_{p\in\mathcal{P}_+} \tilde\alpha_p|\nu\cdot p|, \qquad E_h^{\mathcal{P}}(A_h) \;:=\; h^{d-1}\sum_{p\in\mathcal{P}_+}\tilde\alpha_p \sum_{\substack{x\in\mathbb{Z}_h^d\\ x,x+hp\in\mathbb{Z}_h^d}} \big|\chi_{A_h}(x+hp) - \chi_{A_h}(x)\big|.$$

*(*Primitivity of p is not required; any length factor is absorbed into $\tilde\alpha_p$.*) Then $E_h^{\mathcal{P}} \xrightarrow[\Gamma]{L^1_{\text{loc}}} TV_{\phi_{\mathcal{P}}}$. Moreover, if $\partial E \in C^{1,\beta}$ and A_h are the half-open samplers, then*

$$\big|E_h^{\mathcal{P}}(A_h) - TV_{\phi_{\mathcal{P}}}(\chi_E)\big| \;\le\; C_{\mathcal{P},E}h^\beta,$$

with $C_{\mathcal{P},E}$ depending only on $d, \{\tilde\alpha_p\}_{p\in\mathcal{P}_+}, \max_{p\in\mathcal{P}_+}|p|, \|\partial E\|_{C^{1,\beta}}$, and $\mathcal{H}^{d-1}(\partial E)$. In particular, $\beta = 1$ for $C^{1,1}$ boundaries.

Theorem C (Heisenberg: exact BV+shear identity and quantified bounds). *In the discrete Heisenberg group with generators $S = \{a^{\pm1}, b^{\pm1}\}$, right Cayley action, and the gauge $T(x, y, z) = (x, y, h)$ with*

$h = z - xy$, every finite gauge-column-convex Y *with footprint E admits the exact* undirected-edge *identity*

$$B_S(Y) \;=\; \underbrace{\sum_{e\in\mathcal{E}_{\mathrm{bd}}(E)} \alpha(e) \;+\; \sum_{e\in\mathcal{E}_{\mathrm{int}}(E)} \delta(e)}_{:=\,\mathrm{Per}_{\mathrm{BV}}(Y)} \;+\; \underbrace{2 \sum_{e\in\mathcal{E}_{\mathrm{int}}(E)} \min\{d(t(e)), m(e)\}}_{:=\,\mathrm{S}(Y)}\,.$$

Here $\alpha(e) = \ell(u)$ on boundary edges $e = (u \to u+v)$, while on internal edges $\delta(e) = |\ell(u) - \ell(u+v)|$, $\quad m(e) = \min\{\ell(u), \ell(u+v)\}$, $\quad t(e) = h(u+v) - h(u) + \sigma_v(u)$, with the alignment shifts σ_v from Lemma 7.4 and half-open columns $I_u = [h(u), h(u)+\ell(u))$. The function $d(\cdot)$ is as in Lemma 7.7. Moreover, along gauge-column-convex stacks Y_ρ with $|Y_\rho| \asymp \rho^4$ one has compactness and two-sided bounds at the scale ρ^3 as in Theorem 7.15, with the cap-loss error vanishing by Proposition 7.20.

Remark 1.1 (Normalization for Theorem C)**.** Throughout Theorem C and its proof we use the *undirected* horizontal edge boundary $B_S(\cdot)$ (each broken undirected edge counted once). If one prefers the directed normalization, multiply all B_S identities by 2; cf. Remark 7.25.

Theorem D (Bounded profile ratio in large classes; (TF) as a sharp criterion)**.** *Let $I^\circ(r)$ be the exact vertex profile and $I^{\mathrm{incr}\circ}(r)$ its greatest nondecreasing minorant. Then*

$$\sup_{r\ge 1} \frac{I^\circ(r)}{I^{\mathrm{incr}\circ}(r)} < \infty$$

for each of the following classes:

(1) *non-amenable groups (with bound $\le \Delta/h$; here $h := \inf_{\emptyset\ne Y\subset_{\mathrm{fin}}\Gamma} |\partial_{\mathcal{S}} Y|/|Y|$ is the vertex Cheeger constant);*
(2) *two-ended groups (virtually $\mathbb{Z}$). In this case there exist constants $0 < c_-(\mathcal{S}) \le c_+(\mathcal{S}) < \infty$ such that $c_-(\mathcal{S}) \le I^\circ(r) \le c_+(\mathcal{S})$ for all r; see Remark 9.7 for explicit choices of $c_\pm(\mathcal{S})$;*
(3) *virtually nilpotent groups (polynomial growth of homogeneous dimension Q; $I^\circ(r) \asymp r^{(Q-1)/Q}$ and the ratio is bounded by a constant depending only on $(\Gamma, \mathcal{S})$).*

More generally, if $(\Gamma, \mathcal{S})$ satisfies Tempered Følner *(TF), then*

$$\sup_{r\ge 1} \frac{I^\circ(r)}{I^{\mathrm{incr}\circ}(r)} \;\le\; A + \Delta AB,$$

with (A, B) the (TF) *constants.*

Remark 1.2 (Normalization for Theorem D)**.** Throughout Theorem D we work with the *vertex* boundary $\partial_{\mathcal{S}} Y = \mathcal{S}Y \setminus Y$ and the corresponding exact/increasing profiles $I^\circ, I^{\mathrm{incr}\circ}$. The directed-edge profiles satisfy $I^\circ \le I^\circ_{\mathrm{edge}} \le \Delta I^\circ$ and $I^{\mathrm{incr}\circ} \le I^{\mathrm{incr}\circ}_{\mathrm{edge}} \le \Delta I^{\mathrm{incr}\circ}$ (Lemma 9.15); hence all ratio bounds are invariant up to the factor Δ when switching normalization.

Theorem E (Stability of (TF))**.** *Work in the* directed-edge *normalization; the vertex version follows from Lemma A.5.*

(i) Finite index and finite extensions. (TF) *is preserved under finite index passage and finite extensions. Quantitatively: if $H \le G$ has finite index or fits in $1 \to F \to G \to Q \to 1$ with $|F| < \infty$, then* (TF) *holds on G iff it holds on H (resp. Q), with constants changing by a factor that depends only on the indices and on a bounded change of generators (Lemma A.6).*

(ii) Direct products (product generating set). *Let $(G_i, \mathcal{S}_i)$, $i = 1, 2$, satisfy* (TF) *with edge constants $(A_e^{(i)}, B^{(i)})$. Equip $G_1 \times G_2$ with $\mathcal{S} := (\mathcal{S}_1 \times \{e\}) \cup (\{e\} \times \mathcal{S}_2)$. Then:*

- (TF)(ii) *holds on $G_1 \times G_2$* unconditionally *with*

$$B \;\le\; B^{(1)} + B^{(2)} \qquad \textit{(up to a factor depending only on } \Delta_1, \Delta_2).$$

- *If, in addition, both factors admit* nested near-minimizers (NNM) *at large volumes (Definition 9.20), then* (TF)(i$_e$) *holds along a cofinal subsequence for* $G_1 \times G_2$*; hence full* (TF) *holds on* $G_1 \times G_2$ *with*

$$A_e \;\le\; C\big(A_e^{(1)} + A_e^{(2)}\big), \qquad B \;\le\; C\big(B^{(1)} + B^{(2)}\big),$$

where C *depends only on* (Δ_1, Δ_2).

(iii) Finite-lamp wreath products over a (TF) base. *Let* $G = H^{(\Gamma)} \rtimes \Gamma$ *with* H *finite, nontrivial, and equip* G *with the standard "toggle+base" generating set. If* $(\Gamma, \mathcal{S}_\Gamma)$ *satisfies* (TF) *with constants* (A_e^Γ, B_Γ)*, then there exists a nested chain in* G *with global* (TF)(ii) *and checkpoints obeying* (TF)(i$_e$) *with constants depending only on* H *and* (A_e^Γ, B_Γ)*. If, in addition,* G *admits nested near-minimizers at large volumes, then full* (TF) *holds on* G.

Corollary 1.3 (Finite-lamp wreaths over a (TF) base: NNM (cofinal) and full (TF))**.** Under the hypotheses of Theorem E (iii) (with base $(\Gamma, \mathcal{S}_\Gamma)$ satisfying (TF)), the constructed nested chain in $G = H^{(\Gamma)} \rtimes \Gamma$ admits infinitely many checkpoints obeying (i$_e$) with constants depending only on H and the base (TF) data. Hence, by Lemma 9.21, G has *NNM (cofinal)*. In particular, full (TF) follows with constants depending only on H and the base (TF) constants.

Theorem F (Lamplighters: tempered increments, checkpoint control, and sharpness)**.** *Let* $G = H^{(\Gamma)} \rtimes \Gamma$ *with* H *finite, nontrivial, and* Γ *amenable, equipped with the standard "toggle+base" generating set. Then there exists a* nested *chain* (F_n) *with*

$$|F_{n+1} \setminus F_n| \;\le\; \Delta |\partial_{\mathcal{S}} F_n| \qquad \textit{for all } n,$$

i.e. (TF)(ii) *holds globally with* $B = \Delta$*. Moreover, there is an infinite subsequence of* checkpoints (F_{n_j}) *along which* (TF)(i$_e$) *holds with a constant* $A_e = A_0(H, \mathcal{S}_\Gamma, \mathcal{S})$ *(Erschler-type near-minimizers). In particular,* (F_{n_j}) *is a Følner sequence in* G*. Conversely, there exists a finitely generated amenable "cursor lamplighter" with* $\sup_r I^\circ(r)/I^{\mathrm{incr}\circ}(r) = \infty$ *when* (TF) *fails.*

Remark 1.4 (Normalization and scope for Theorem F)**.** Throughout the proof of Theorem F we will:

- work with the *directed* edge boundary $B_{\mathcal{S}}(Y) := \#\{(y,s) \in Y \times \mathcal{S} :\ ys \notin Y\}$;
- pass to the vertex boundary via the general comparison $|\partial_{\mathcal{S}} Y| \le B_{\mathcal{S}}(Y) \le |\mathcal{S}| \cdot |\partial_{\mathcal{S}} Y|$ (see Lemma 9.15);
- use the standard "toggle+base" generating set $\mathcal{S} = \mathcal{S}_\Gamma \cup \mathsf{T}$ for $G = H^{(\Gamma)} \rtimes \Gamma$, with $t := |\mathsf{T}|$ and $q := |H| - 1$.

All constants in (TF) stated in vertex normalization are obtained from the directed version by the factor $|\mathcal{S}|$.

Corollary 1.5 (Lamplighters: NNM (cofinal) and full (TF))**.** In the setting of Theorem F (lamplighter $G = H^{(\Gamma)} \rtimes \Gamma$ with H finite, nontrivial), the nested chain (F_n) produced there admits an infinite subsequence of checkpoints (F_{n_j}) with

$$B_{\mathcal{S}}(F_{n_j}) \;\le\; A_0 I^{\mathrm{incr}\circ}_{\mathrm{edge}}\big(|F_{n_j}|\big) \qquad \text{for all } j,$$

hence G has *NNM (cofinal)* with constant $A_e = A_0$ by Lemma 9.21. So G satisfies full (TF) with constants $(A_e, B) = (A_0, \Delta)$ in vertex normalization, and the profile ratio is uniformly bounded by Theorem D .

Theorem G ($\mathbb{Z}^d \rtimes_A \mathbb{Z}$: logarithmic layer-nested sets are nested, Følner, satisfy (TF)(ii) with an explicit constant, and are near-minimizers within the layer-nested class)**.** *Let* $G = \mathbb{Z}^d \rtimes_A \mathbb{Z}$ *with the left generating set* $\mathcal{S}$ *described in §9.8, and let* $K \subset \mathbb{R}^d$ *be a convex body with piecewise* C^1 *boundary. Fix any* $\Lambda(A) > \rho(\operatorname{cof} A) = \rho(A^{-1})$ *and set*

$$E_R \;:=\; F_{K,R,T(R)}, \qquad T(R) := \lfloor \alpha \log R \rfloor, \quad 0 < \alpha < \frac{1}{\log \Lambda(A)}.$$

Then:

(1) *The family* $(E_R)_{R\geq 2}$ *is* nested *(Lemma 9.66) and* Følner*:*

$$\frac{B_{\mathcal{S}}(E_R)}{|E_R|} \xrightarrow[R\to\infty]{} 0, \qquad B_{\mathcal{S}}(E_R) = B_{\mathrm{hor}}(E_R) + B_{\mathrm{vert}}(E_R), \quad B_{\mathrm{vert}}(E_R) = 2|X_0(R)| \asymp R^d.$$

(2) *There exist constants* $c_1, c_2, c_3 \in (0,\infty)$ *depending only on* $(K, A, \mathcal{S})$ *such that for all* $R \geq 2$*,*

$$c_1 R^d \ \leq\ |X_0(R)| \ \leq\ c_2 R^d, \qquad |X_0(R{+}1) \setminus X_0(R)| \ \leq\ c_3 R^{d-1}.$$

(3) (TF)(ii) with an explicit B. *For all sufficiently large R,*

$$|E_{R+1} \setminus E_R| \ \leq\ B \cdot B_{\mathcal{S}}(E_R),$$

with

$$B \ :=\ \frac{4c_2 + c_3}{2c_1}.$$

Indeed, if $T(R{+}1) = T(R)$ *then* $|E_{R+1} \setminus E_R| \leq (2T(R){+}1)c_3 R^{d-1} \leq \frac{c_3}{2c_1} B_{\mathcal{S}}(E_R)$ *since* $B_{\mathcal{S}}(E_R) \geq B_{\mathrm{vert}}(E_R) = 2|X_0(R)| \geq 2c_1 R^d$ *and, for R large,* $2T(R) + 1 \leq R$ *(recall* $T(R) = \lfloor \alpha \log R \rfloor$*); and if* $T(R{+}1) = T(R){+}1$ *then* $|E_{R+1} \setminus E_R| \leq 2|X_0(R{+}1)| + (2T(R){+}1)c_3 R^{d-1} \leq \frac{4c_2+c_3}{2c_1} B_{\mathcal{S}}(E_R)$.

(4) (Layer-nested near-minimality). *In the* layer-nested *subclass (Definition 9.82), there exists a constant* $A_e(A, \mathcal{S}) \in (0,\infty)$ *such that, for all sufficiently large R,*

$$B_{\mathcal{S}}(E_R) \ \leq\ A_e \ \inf\Big\{ B_{\mathcal{S}}(Y) : \ Y \text{ layer-nested},\ |Y| = |E_R| \Big\}.$$

Equivalently, $\{E_R\}_{R\to\infty}$ *is a nested sequence of near-minimizers within the layer-nested class. (See Proposition 9.84.)*

In particular, clause (TF)(ii) *holds* unconditionally *(no spectral assumptions beyond* $A \in GL(d,\mathbb{Z})$*), and the Følner convergence uses only* $\alpha \log \Lambda(A) < 1$ *so that* $B_{\mathrm{hor}}(E_R) = O(R^{d-1+\alpha\log\Lambda(A)}) = o(|E_R|)$ *while* $B_{\mathrm{vert}}(E_R) \asymp R^d$ *and* $|E_R| \asymp R^d \log R$.

Remark 1.6 (Hyperbolic upgrade)**.** If A is *hyperbolic* (no eigenvalue on the unit circle), then by Corollary 1.7 the logarithmic layer-nested sets (E_R) also satisfy (TF)(i_e) *unconditionally*; together with Theorem G (iii) this yields full (TF) for $G = \mathbb{Z}^d \rtimes_A \mathbb{Z}$ without the NNM hypothesis. The proof is deferred to §9 (profile bounds).

Corollary 1.7 (Hyperbolic semidirects: unconditional full (TF))**.** Let $G = \mathbb{Z}^d \rtimes_A \mathbb{Z}$ with $A \in \mathrm{GL}(d,\mathbb{Z})$ *hyperbolic* (no eigenvalue on the unit circle), and let $\mathcal{S}$ be any finite symmetric generating set. Then there exist constants $c, A_e > 0$ (depending only on $(A, \mathcal{S})$) such that

$$I^{\circ}_{\mathrm{edge}}(v) \ \geq\ c\frac{v}{\log v} \qquad (v \text{ large}),$$

and the logarithmic layer-nested sets $\{E_R\}$ from Theorem satisfy

$$B_{\mathcal{S}}(E_R) \ \leq\ A_e I^{\circ}_{\mathrm{edge}}(|E_R|) \qquad (R \text{ large}).$$

In particular, combining this with Theorem G (iii) yields *full* (TF) for G without assuming nested near-minimizers.

Beyond the hyperbolic case. The following applies to general $A \in \mathrm{GL}(d,\mathbb{Z})$ under the nested near-minimizer (NNM) hypothesis.

Corollary 1.8. If, furthermore, there exist nested near-minimizers at large volumes, then G satisfies full (TF); in particular the profile ratio is uniformly bounded.

As a by-product we obtain tracked-constant analytic consequences for the Cayley graph $(\Gamma, \mathcal{S})$: Cheeger-Buser bounds, Faber-Krahn and Nash inequalities, and mixing estimates on finite quotients, with constants expressed in terms of the relevant isoperimetric data in each regime: the vertex Cheeger constant $h(\Gamma, \mathcal{S})$ in the non-amenable case; the sharp Wulff constant h_S (from Theorem A, directed-edge normalization) in polynomial growth; or, in full generality, the increasing profile $I^{\mathrm{incr}\,\circ}$. The statements below keep vertex normalization and $\Delta := |\mathcal{S}|$.

Theorem H (Tracked analytic consequences)**.** *Let $(\Gamma, \mathcal{S})$ be a finitely generated group of degree $\Delta = |\mathcal{S}|$. Throughout we take P to be the simple (non-lazy) random walk on the Δ-regular Cayley graph and the Dirichlet form $\mathcal{E}(f,f) = \frac{1}{2\Delta}\sum_{x\sim y}(f(x)-f(y))^2$, so the Δ-factors in (H1)-(H4) match the stated normalizations.*

(H1) ***Cheeger-Buser / spectral decay.*** *If Γ is non-amenable with vertex Cheeger constant $h(\Gamma, \mathcal{S}) := \inf_{\emptyset \neq Y \subset_{\mathrm{fin}} \Gamma} \frac{|\partial_S Y|}{|Y|} > 0$, then for the simple random walk on Γ,*

$$\lambda_1(\Gamma) \;\geq\; c\frac{h(\Gamma, \mathcal{S})^2}{\Delta^2}, \qquad \|P_t - \pi\|_{2\to 2} \;\leq\; \exp\Big(-c\frac{h(\Gamma, \mathcal{S})^2}{\Delta^2}t\Big).$$

For any finite quotient X (with the projected generators), the same bounds hold with $h(\Gamma, \mathcal{S})$ replaced by the Cheeger constant of *X, $h(X)$. Uniformity over a family of quotients requires a (τ)-type hypothesis.*

(H2) ***Faber-Krahn at the right scale (general form).*** *For every finite U in a Cayley graph or finite quotient,*

$$\lambda_1^{\mathrm{D}}(U) \;\geq\; c\,\Big(\frac{I^{\mathrm{incr}\,\circ}(|U|)}{\Delta|U|}\Big)^2.$$

In particular, on virtually nilpotent groups of homogeneous dimension Q, where $I^{\mathrm{incr}\circ}(r) \asymp h_S r^{(Q-1)/Q}$ (Theorem A), one has

$$\lambda_1^{\mathrm{D}}(U) \;\geq\; c\Big(\frac{h_S}{\Delta}\Big)^2 |U|^{-2/Q}.$$

(H3) ***Nash inequality and heat-kernel smoothing (polynomial growth).*** *If Γ is virtually nilpotent of homogeneous dimension Q, then for all finitely supported f with $\sum f = 0$,*

$$\|f\|_2^{2+4/Q} \;\leq\; C\,\Big(\frac{\Delta}{h_S}\Big)^2\,\mathcal{E}(f,f)\,\|f\|_1^{4/Q},$$

hence $\|P_t\|_{1\to\infty} \leq C\,t^{-Q/2}$ and $p_t(e,e) \leq C\,t^{-Q/2}$, with constants tracked through h_S and Δ.

(H4) ***Mixing on finite quotients.*** *For every finite quotient X,*

$$\|P_t - \pi\|_{2\to 2} \;\leq\; \exp\Big(-c\,\frac{h(X)^2}{\Delta^2}\,t\Big).$$

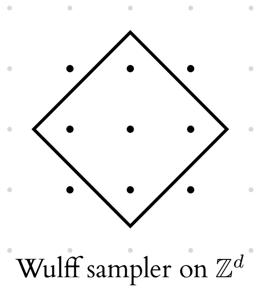
In the virtually nilpotent case, using $I^{\mathrm{incr}\,\circ}(r) \asymp h_S\,r^{(Q-1)/Q}$ and (H2)*, one gets*

$$t_{\mathrm{mix}}^{(2)}(X) \;\leq\; C\,\Big(\frac{\Delta}{h_S}\Big)^2\,|X|^{2/Q}.$$

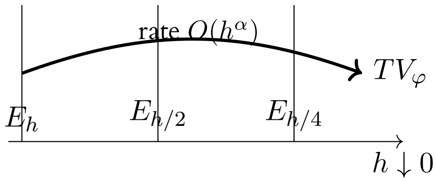
Here $c, C \in (0,\infty)$ are universal constants; all dependencies are explicit and tracked via $h(\Gamma, \mathcal{S})$, h_S, $I^{\mathrm{incr}\,\circ}$ and Δ as indicated.

What is new conceptually. Three methodological ingredients underlie these results.

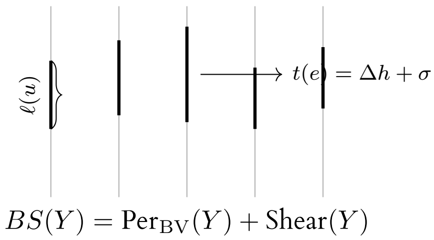
- *Submodular BV calculus with calibration.* Submodularity allows union/gluing without leakage of constants. We pair this with a discrete calibration/coarea scheme adapted to the anisotropy τ_S *(the support function of the horizontal zonotope $Z_S := \sum_{s\in S_+}[-X_s, X_s]$)* to obtain the sharp discrete Wulff bound with asymptotic attainment.



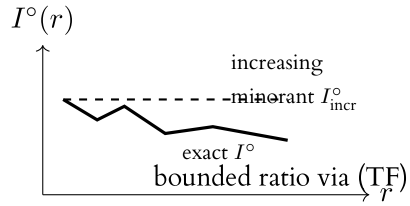
(A) Sharp discrete Wulff with asymptotic attainment (Thm. A).



(B) Quantitative Γ-convergence with algebraic rate (Thm. B).



(C) Heisenberg gauge: exact per-edge BV+shear identity (Thm. C).



(D) Bounded $I^\circ/I^\circ_{\mathrm{incr}}$ in large classes; (TF) criterion (Thm. D).

FIGURE 1. Four pillars of the paper: (A) sharp discrete Wulff isoperimetry; (B) quantitative Γ-convergence; (C) constructive neutralization of Heisenberg shear and exact split; (D) profile ratio bounds via Tempered Følner.

- *Quantitative discrete → continuum with rates.* The rate proof isolates a purely combinatorial *oblique-counting error* quantifying the mismatch between crystalline facet normals and lattice directions at scale h; this yields algebraic rates and explicit recovery sequences.
- *Edge-level shear bookkeeping in Heisenberg.* A quadratic gauge g furnishes an exact per-edge BV + shear identity, so compactness and sharp bounds follow from bookkeeping rather than additional structure; the remaining unknown is a single shear coefficient, not a structural gap.

1.3. **Organization of the paper.** *Section 3* develops the submodular BV framework and proves the sharp discrete Wulff inequality with asymptotic attainment (Theorems 3.25 and 4.1). *Section 5* establishes quantitative Γ-convergence with an algebraic rate for crystalline anisotropies (Theorems 5.7 and 5.16). *Section 6* proves a combinatorial L^1 curl-fitting lemma (bounding discrete divergences by controlled curls) that underpins the calibration and sampling arguments. *Section* 7 introduces the Heisenberg gauge, derives the exact BV + shear decomposition, and proves compactness and order-sharp bounds (Theorems 7.11 and 7.15 and Corollaries 7.13, 7.12). *Section 9* treats isoperimetric profiles, resolves Gromov's question in the classes above, and proves (TF) ⇒ bounded ratio (Proposition 9.4, Theorems 9.31 and 9.32). *Section 10* records analytic and probabilistic consequences (Theorem 10.3 and corollaries). Technical tools and extended proofs are collected in the appendices.

Appendices. Appendix A: from (TF) we construct canonical, scale-calibrated Property A witnesses with modulus $R(\varepsilon)$; in the polynomial-growth/Wulff-sampler regime, $R(\varepsilon) \asymp \varepsilon^{-1}$. This yields explicit coarse embeddings into Hilbert with compression $\rho(t) \gtrsim t^{1/2}/\mathrm{polylog}(t)$ (in particular $\gtrsim t^{1/2}/\log t$ in the polynomial-growth regime) uniformly on box spaces, with constants *Wulff-coded* by the generating set S.

Appendix B: for the Cayley-Schreier graph Σ of the right-left action of $\Gamma \times \Gamma$ on Γ (edge normalization) we prove the exact identity $B_\Sigma(Y) = B_S(Y) + B_S(Y^{-1})$, which implies $2\, I^\circ_{\rm edge}(r) \le I^\circ_\Sigma(r) \le I^\circ_{\rm edge}(r) + \Delta\, r$. For Wulff samplers in our (virtually nilpotent/Abelian-base) setting, *assuming S is symmetric so that τ_S is even (hence Wulff sets are inversion-symmetric up to lower-order sampling error)*, we obtain $I^\circ_\Sigma(r) = 2\, I^\circ_{\rm edge}(r) + o\left(r^{(Q-1)/Q}\right)$. Thus the profile-ratio and (TF) mechanisms are *robust*: they reflect genuine bi-equivariant geometry of Γ, not an artifact of a purely left-invariant boundary. This refines and answers another question of Gromov [Gro08b, pp. 38–39].

Appendix D: We establish a quantitative Lindenstrauss theorem: in virtually nilpotent groups our Wulff Følner sets admit tempered subsequences with an explicit constant $T \le D_2(\Gamma, \mathcal{S})\, D_{a^{-1}}(\Gamma, \mathcal{S})$, yielding pointwise ergodic theorems and weak-type $(1,1)$ maximal inequalities with tracked, generator-aware bounds.

1.4. **Unified narrative: from near-minimizers to bounded ratios.** We see the following as a unifying mechanism throughout the paper:

$$\boxed{\textbf{(NNM)} \implies \textbf{(TF)} \implies \textbf{BRP}} \quad \text{(constructively, with tracked constants).}$$

Here (NNM) means a nested near-minimizer family in edge normalization; (TF) is a single nested, tempered Følner chain near-minimal at every step; and BRP is the bounded ratio $\sup_r I^\circ(r)/I^{\rm incr\,\circ}(r) < \infty$. The implication (NNM)⇒(TF) is given by the budgeted interleaving (Theorem 9.100); (TF)⇒BRP is Theorem 9.31. Each technical block supplies (NNM) in a large class:

- *Abelian/Carnot lattices:* discrete Wulff calibrations and samplers (Theorems 4.1, 3.25).
- *Step-2 Carnot lattices:* shear/curl-fit gives an exact rank-1 identity and sharp two-sided product bounds (Theorem 8.5).
- *Semidirects $\mathbb{Z}^d \rtimes_A \mathbb{Z}$:* log-height layer-nested sets + $j_1(v) \asymp \log v$ (Theorems 9.75, 9.81).
- *Finite-lamp wreath products:* explicit base+ring expansion (tempered steps) + Erschler's checkpoints (Theorems 9.39, 9.42, 9.48).

Remark 1.9. For virtually nilpotent groups, BRP itself is already known and follows here by Wulff alone; the step-2 shear/curl-fit is included to produce *constructive* (NNM)⇒(TF) chains with per-edge control and tracked constants, which we then use for quantitative analytic consequences (§10, Faber-Krahn, Nash, mixing).

2. Preliminary definitions

Remark 2.1 (Standing conventions and normalization)**.** Throughout, $(\Gamma, \mathcal{S})$ denotes a finitely generated group with a fixed finite, symmetric generating set $\mathcal{S} = \mathcal{S}^{-1}$, $e \notin \mathcal{S}$. We write $\Delta := |\mathcal{S}|$ and use *right*-multiplication: neighbors of x are $\{xs : s \in \mathcal{S}\}$. All energy computations use *directed* edges. For a finite group quotient X of $(\Gamma, \mathcal{S})$ we identify $\mathcal{S}$ with its image in X and view it as a *multiset*; loops and multiple edges are allowed, and the degree remains Δ *counted with multiplicity*. Loops never contribute to boundaries or to the Dirichlet form. Unless stated otherwise, isoperimetric *profiles* are in the *vertex* normalization; directed-edge versions carry a subscript "edge" (see Definitions 2.4 and 2.6).

Definition 2.2 (Word metric, balls and spheres)**.** The word length of $g \in \Gamma$ is $|g|_{\mathcal{S}} := \min\{k : g \in \mathcal{S}^{(k)}\}$, where $\mathcal{S}^{(k)} := \{s_1 \cdots s_k : s_i \in \mathcal{S}\}$ and $\mathcal{S}^{(0)} := \{e\}$. We also set $\mathcal{S}^{\le r} := \bigcup_{k=0}^r \mathcal{S}^{(k)}$. The word metric is $d_{\mathcal{S}}(x,y) := |x^{-1}y|_{\mathcal{S}}$. The ball and sphere of radius $r \in \mathbb{N}$ about x are

$$B_{\mathcal{S}}(x,r) := \{y \in \Gamma : d_{\mathcal{S}}(x,y) \le r\} = x\,\mathcal{S}^{\le r} \quad \text{(right-multiplication)}, \qquad \mathsf{S}_{\mathcal{S}}(x,r) := \{y \in \Gamma : d_{\mathcal{S}}(x,y) = r\}.$$

We abbreviate $B_{\mathcal{S}}(r) := B_{\mathcal{S}}(e,r)$ and $\mathsf{S}_{\mathcal{S}}(r) := \mathsf{S}_{\mathcal{S}}(e,r)$.

Definition 2.3 (One-step $\mathcal{S}$-neighborhood of a set)**.** For $Y \subset \Gamma$ (or $Y \subset X$ in a finite quotient) the *closed one-step $\mathcal{S}$-neighborhood* is

$$N_{\mathcal{S}}(Y) := Y \cup Y\mathcal{S} = \{ys : \ y \in Y, \ s \in \mathcal{S}\} \cup Y.$$

More generally, for $k \in \mathbb{N}$, the k-step neighborhood is $N_{\mathcal{S}}^{(k)}(Y) := Y\,\mathcal{S}^{\le k}$ (with $\mathcal{S}^{\le 0} = \{e\}$ so $N_{\mathcal{S}}^{(0)}(Y) = Y$). We regard $N_{\mathcal{S}}(\cdot)$ as a *vertex-set* operation; multiplicities from $\mathcal{S}$ are used only in edge counts/energies.

Definition 2.4 (Vertex and edge boundaries)**.** For $Y \subset \Gamma$ (or $Y \subset X$), the *outer vertex boundary* is

$$\partial_{\mathcal{S}}(Y) \ := \ N_{\mathcal{S}}(Y) \setminus Y = \{xs : \ x \in Y, \ s \in \mathcal{S}, \ xs \notin Y\}.$$

The *directed edge boundary* is

$$B_{\mathcal{S}}(Y) \ := \ \#\big\{(x, s) \in Y \times \mathcal{S} : \ xs \notin Y\big\}.$$

Equivalently,

$$B_{\mathcal{S}}(Y) \ = \ \sum_{s\in\mathcal{S}} \big|\, Y \setminus Ys^{-1} \big| \ = \ \tfrac{1}{2} \sum_{s\in\mathcal{S}} \big|\, sY \triangle Y \,\big|,$$

so $B_{\mathcal{S}}(Y)$ equals the (multiplicity-weighted) number of *undirected* cut edges between Y and $Y^{\complement}$.

Remark 2.5 (Elementary relations)**.** For every Y one has the pointwise comparison

$$|\partial_{\mathcal{S}}(Y)| \ \le \ B_{\mathcal{S}}(Y) \ \le \ \Delta\,|\partial_{\mathcal{S}}(Y)|.$$

If $\mathcal{S}$ is symmetric then $B_{\mathcal{S}}(Y) = B_{\mathcal{S}}(Y^{\complement})$ by $(x, s) \mapsto (xs, s^{-1})$, and $N_{\mathcal{S}}(Y) \setminus Y = \partial_{\mathcal{S}}(Y)$. We also have $\partial_{\mathcal{S}}^{\mathrm{v},-}(Y) = \partial_{\mathcal{S}}(Y^{\complement})$ and $\mathrm{int}_{\mathcal{S}}(Y) = Y \setminus \partial_{\mathcal{S}}^{\mathrm{v},-}(Y)$, where $\partial_{\mathcal{S}}^{\mathrm{v},-}(Y) := \{x \in Y : \exists s \in \mathcal{S}, \ xs \notin Y\}$.

Definition 2.6 (Exact profiles and increasing minorants)**.** We adopt the *vertex* normalization for profiles:

$$I^{\circ}(r) \ := \ \inf\big\{\, |\partial_{\mathcal{S}}(Y)| : \ Y \subset \Gamma \text{ finite}, \ |Y| = r \,\big\}, \qquad I^{\mathrm{incr}\,\circ}(r) \ := \ \inf_{s\ge r} I^{\circ}(s), \qquad r \ge 1.$$

Their *directed-edge* counterparts are

$$I^{\circ}_{\mathrm{edge}}(r) \ := \ \inf\big\{\, B_{\mathcal{S}}(Y) : \ |Y| = r \,\big\}, \qquad I^{\mathrm{incr}\,\circ}_{\mathrm{edge}}(r) \ := \ \inf_{s\ge r} I^{\circ}_{\mathrm{edge}}(s).$$

We extend these from $\mathbb{N}$ to $[1, \infty)$ as right-continuous step functions and (when convenient) by piecewise-linear interpolation on each $[k, k + 1]$. By Remark 2.5,

$$I^{\circ}(r) \ \le \ I^{\circ}_{\mathrm{edge}}(r) \ \le \ \Delta\, I^{\circ}(r), \qquad I^{\mathrm{incr}\,\circ}(r) \ \le \ I^{\mathrm{incr}\,\circ}_{\mathrm{edge}}(r) \ \le \ \Delta\, I^{\mathrm{incr}\,\circ}(r).$$

The *vertex Cheeger constant* is $h^{\circ}_{\mathrm{Ch}}(\mathcal{S}) := \inf_{\emptyset\ne Y\subset_{\mathrm{fin}}\Gamma} |\partial_{\mathcal{S}} Y|/|Y|$; the directed-edge version is $h_{\mathrm{Ch,edge}}(\mathcal{S}) := \inf B_{\mathcal{S}}(Y)/|Y|$, and $h^{\circ}_{\mathrm{Ch}} \le h_{\mathrm{Ch,edge}} \le \Delta\, h^{\circ}_{\mathrm{Ch}}$.

Definition 2.7 (Random walk, Laplacian and Dirichlet form)**.** The simple random walk with step set $\mathcal{S}$ has Markov operator

$$Pf(x) \ := \ \frac{1}{\Delta} \sum_{s\in\mathcal{S}} f(xs)\,.$$

The (normalized) Laplacian is $\mathcal{L} := I - P$, and the continuous-time semigroup is $P_t := e^{-t\mathcal{L}}$. On a finite quotient X the stationary law is uniform $\pi(x) = |X|^{-1}$. The (normalized) Dirichlet form is

$$\mathcal{E}(f, g) \ := \ \frac{1}{2\Delta} \sum_{x} \sum_{s\in\mathcal{S}} \big(f(xs) - f(x)\big)\big(g(xs) - g(x)\big).$$

For indicators $\mathbf{1}_U$ one has the identity

$$\mathcal{E}(\mathbf{1}_U, \mathbf{1}_U) \ = \ \frac{1}{\Delta}\, B_{\mathcal{S}}(U).$$

For $U \subset X$ (or $U \subset \Gamma$) the *Dirichlet eigenvalue* is

$$\lambda_1^{\mathrm{D}}(U) \;:=\; \inf_{\substack{f\neq 0\\ \mathrm{supp}(f)\subset U}} \frac{\mathcal{E}(f,f)}{\|f\|_2^2}\,.$$

We use counting measure for $\|\cdot\|_2$ on Γ and X. On finite X, $\|f\|_{2,\pi}^2 = |X|^{-1}\sum_x f(x)^2$, so Rayleigh quotients for $\mathcal{L}$ are identical under $\langle\cdot,\cdot\rangle$ and $\langle\cdot,\cdot\rangle_\pi$. When working on Γ, we take f, g finitely supported (or in ℓ^2); in the Dirichlet eigenvalue, the support restriction makes all sums finite.

Remark 2.8 (Energy for cut indicators and undirected edges)**.** Let $B_{\mathcal{S}}^{\mathrm{und}}(U) := \frac{1}{2}\sum_{s\in\mathcal{S}}|sU\triangle U|$ denote the *undirected* edge cut between U and $U^{\complement}$ (with multiplicity induced by $\mathcal{S}$ on finite quotients). Then

$$\mathcal{E}(\mathbf{1}_U,\mathbf{1}_U) \;=\; \frac{1}{\Delta}\,B_{\mathcal{S}}^{\mathrm{und}}(U) \;=\; \frac{1}{\Delta}\,B_{\mathcal{S}}(U).$$

Thus the Dirichlet energy is interchangeable with either directed or undirected edge cuts under our normalization. By Remark 2.5, $B_{\mathcal{S}}(U)$ is comparable to $|\partial_{\mathcal{S}}U|$ up to the factor Δ.

Definition 2.9 (Growth and homogeneous dimension)**.** If $(\Gamma,\mathcal{S})$ has polynomial growth, the *homogeneous dimension* Q is the exponent for which $|B_{\mathcal{S}}(r)| \asymp r^Q$; in particular on virtually nilpotent groups Q equals the Bass-Guivarc'h dimension. We use Q only as a parameter in scale-sharp analytic bounds (Nash, Faber-Krahn, mixing). If Γ is a uniform lattice in a Carnot group G, then this Q equals the Carnot homogeneous dimension $\sum_{j=1}^s j\,\dim V_j$.

Remark 2.10 (Wulff constant)**.** The symbol $h_{\mathcal{S}}$ denotes the sharp discrete Wulff constant appearing in the asymptotic (large-volume) isoperimetry for $(\Gamma,\mathcal{S})$; its precise construction and properties are stated in Theorem A. All tracked constants in the analytic consequences (Theorem H) are expressed via $h_{\mathcal{S}}$ and Δ. We emphasize that the vertex Cheeger constant $h^{\circ}_{\mathrm{Ch}}(\mathcal{S})$ need not equal $h_{\mathcal{S}}$.

Remark 2.11 (Quotients)**.** All the above definitions carry verbatim to finite *group* quotients X of $(\Gamma,\mathcal{S})$ by replacing Γ with X and keeping the same symbol $\mathcal{S}$ for the image of the generating set (viewed as a multiset). If $A(x,y) := \#\{s\in\mathcal{S} : \ xs = y\}$ denotes the adjacency multiplicity in X, then the degree is $\sum_y A(x,y) = \Delta$ for all x (counted with multiplicity), and all boundary/Dirichlet-form quantities are computed with respect to these directed edges. We regard $N_{\mathcal{S}}(\cdot)$ as a vertex-set operation (multiplicities ignored). Since $\mathcal{S} = \mathcal{S}^{-1}$ as a multiset, the random walk is reversible with respect to the uniform law on X (so P is self-adjoint in ℓ^2).

Standing assumptions. Let G be a *Carnot group*, i.e. a connected, simply connected, stratified nilpotent Lie group with graded Lie algebra $\mathfrak{g} = V_1\oplus\cdots\oplus V_s$ and group dilations $(\delta_r)_{r>0}$. Let $Q := \sum_{j=1}^s j\,\dim V_j$ denote the homogeneous dimension of G. Fix a uniform (co-compact) lattice $\Gamma\subset G$ and a symmetric generating set $\mathcal{S}\subset\Gamma$ with $e\notin\mathcal{S}$ and $\mathcal{S}=\mathcal{S}^{-1}$. Set $\mathcal{S} := \mathcal{S}$ and write $\mathrm{Cay}(\Gamma,\mathcal{S})$ for the Cayley graph. Carnot groups are unimodular; left and right Haar measures coincide. Let μ denote (left) Haar measure on G, normalized so that a measurable fundamental domain F for Γ satisfies $\mu(F)=1$. We write $\mathrm{Vol_c}(E) := \mu(E)$ for Haar volume. See [AKLD09] for background on BV and perimeter in Carnot groups.

3. Sharp Discrete Wulff Inequality

Nondegeneracy and fundamental domain. We assume that the first-layer directions with *positive weights* span $V_1(G)$:

$$\mathrm{span}\{\,X_s : \ a_s>0\,\} = V_1(G), \quad \text{where } X_s := \pi_{\mathrm{hor}}(\log s)\in V_1(G).$$

Equivalently, the zonotope $\sum_{s:\,a_s>0} a_s[-X_s, X_s]$ has nonempty interior, so the gauge φ in (3.1) is a *norm* on $V_1(G)$. We also fix a measurable Dirichlet fundamental domain F for Γ; then $\mu(F) = 1$ and $\mathrm{diam}_{\mathrm{cc}}(F) < \infty$.

Working anisotropy (via the CI/zonoid model). Let $\log : \Gamma \to \mathfrak{g}$ be the logarithm in exponential coordinates and $\pi_{\mathrm{hor}} : \mathfrak{g} \to V_1(G)$ the projection. For each $s \in \mathcal{S}$, set $X_s := \pi_{\mathrm{hor}}(\log s) \in V_1(G)$.

Remark 3.1 (Left vs. right edges in this section)**.** In Section 2 we adopted *right*-multiplication for directed edges, i.e. neighbors of x are $\{xs : s \in S\}$ and the directed boundary counts pairs (x, s) with $xs \notin Y$. In this section some discrete sums are written with *left*-multiplication (e.g. $\mathbf{1}_A(sg)$) to align with left Haar invariance in continuum estimates. Since $S = S^{-1}$ and G is unimodular, the two conventions are equivalent via $g \mapsto g^{-1}$ and $s \leftrightarrow s^{-1}$; all constants and statements are unchanged.

Definition 3.2 (Consistency identity (CI) for anisotropy)**.** Let $a_s \geq 0$ be weights indexed by $s \in \mathcal{S}$ and let X_s be as above. We say $(\mathcal{S}, a)$ is *consistent* with a convex, even, 1-homogeneous gauge φ on $V_1(G)$ if

$$\forall \xi \in V_1(G) : \qquad \varphi(\xi) \;=\; \sum_{s\in\mathcal{S}} a_s \, \big|\langle \xi, X_s\rangle\big|. \tag{3.1}$$

Remark 3.3 (Zonotope representation and polarity)**.** Set the zonotope $K := \sum_{s\in\mathcal{S}} a_s[-X_s, X_s] \subset V_1(G)$. Since $h_{[-v,v]}(\xi) = |\langle \xi, v\rangle|$ and support functions add under Minkowski sums,

$$\sum_{s\in\mathcal{S}} a_s \, |\langle \xi, X_s\rangle| \;=\; h_K(\xi) \qquad (\xi \in V_1(G)).$$

Thus (3.1) holds iff $\varphi = h_K$ is the *support function* of K. Equivalently, the *dual gauge* φ° is the Minkowski functional of K, so the dual unit ball is K and the primal unit ball is K°.

Remark 3.4 (Normalization used throughout; axis case)**.** We keep the *directed-edge* normalization for the discrete perimeter, and we encode the corresponding continuum anisotropy by choosing weights that are *evenly split across undirected pairs*:

$$a_s = a_{s^{-1}} = \tfrac{1}{2} \qquad \text{for every undirected generator } \{s, s^{-1}\} \subset \mathcal{S}.$$

With this choice, the gauge appearing in (3.1) is

$$\tau_{\mathcal{S}}(\xi) = \sum_{v\in\mathcal{S}_+} \big|\langle \xi, X_v\rangle\big|,$$

where $\mathcal{S}_+ \subset \mathcal{S}$ is any fixed set of undirected representatives. In particular, for the axis stencil on $\mathbb{Z}^d$ this gives $\tau(\nu) = \sum_{i=1}^d |\nu_i|$, so that the continuum Wulff constant equals $2d$, matching the discrete column bound in Theorem A (i). [1]

Definition 3.5 (Discrete anisotropic perimeter induced by $(\mathcal{S}, a)$)**.** For finite $A \subset \Gamma$, set

$$\mathrm{Per}_{\mathcal{S},a}(A) \;:=\; \sum_{s\in\mathcal{S}} a_s \sum_{g\in\Gamma} \big|\mathbf{1}_A(sg) - \mathbf{1}_A(g)\big|.$$

Normalization. We adopt the *directed-edge (left)* convention throughout. If one prefers undirected edges, replace $\mathrm{Per}_{\mathcal{S},a}$ by $\frac{1}{2}\,\mathrm{Per}_{\mathcal{S},a}$ and φ by $\frac{1}{2}\varphi$ (and hence c_{Wulff} by $\frac{1}{2}c_{\mathrm{Wulff}}$); all statements then remain identical.

[1] If one instead takes $a_s \equiv 1$ for all directed s, then τ is multiplied by 2 and so is the continuum constant. We adopt the split convention to keep constants aligned with the discrete proofs below.

Remark 3.6 (Symmetry of weights). We assume $a_s = a_{s^{-1}}$ for all $s \in \mathcal{S}$ (undirected anisotropy encoded via directed counting). If a non-symmetric family is given, replacing it by the symmetrization $\tilde{a}_s := (a_s + a_{s^{-1}})/2$ leaves the gauge in (3.1) unchanged and produces the same undirected discrete perimeter (up to the factor $1/2$ if one switches conventions), so no generality is lost.

Lemma 3.7 (Bridge to the missing-neighbor perimeter). Let $S = \mathcal{S}$ be symmetric and assume the split normalization $a_s = a_{s^{-1}} = \frac{1}{2}$. For any finite $Y \subset \Gamma$,

$$\mathrm{Per}_{\mathcal{S},a}(Y) \;=\; \sum_{s \in S} \big|\{y \in Y :\; sy \notin Y\}\big| \;=:\; \mathrm{Per}_{S,1}(Y).$$

Proof. For fixed s, $\sum_g |\mathbf{1}_Y(sg) - \mathbf{1}_Y(g)| = 2\,\#\{y \in Y :\; sy \notin Y\}$. Multiplying by $a_s = \frac{1}{2}$ and summing over s gives the claim. □

Lemma 3.8 (Edge-to-integral identity (left translation)). Let $F \subset G$ be a measurable fundamental domain for $\Gamma \curvearrowright G$ with $\mu(F) = 1$. For any finite $A \subset \Gamma$ and $s \in \mathcal{S}$,

$$\sum_{g \in \Gamma} \big|\mathbf{1}_A(sg) - \mathbf{1}_A(g)\big| \;=\; \int_G \big|\mathbf{1}_{AF}(sy) - \mathbf{1}_{AF}(y)\big|\, d\mu(y).$$

Proof. Partition G into $\bigsqcup_{g \in \Gamma} gF$ and change variables $y \mapsto gy$. □

Lemma 3.9 (Directional difference quotient for BV sets (right translation)). Let $E \subset G$ be a set of finite horizontal perimeter and $X \in V_1(G)$. Then

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_G \big|\mathbf{1}_E(z \exp(\varepsilon X)) - \mathbf{1}_E(z)\big|\, d\mu(z) \;=\; \int_{\partial^* E} \big|\nu_E \cdot X\big|\, d\mathcal{H}^{Q-1}.$$

Sketch. In a Carnot group one has $\mathrm{Ad}(z)X = X + \sum_{j \ge 2} Z_j$ with $Z_j \in V_j$. Since the measure-theoretic normal ν_E is horizontal ($\nu_E \in V_1$) and we fix an inner product with $V_1 \perp \bigoplus_{j \ge 2} V_j$, $\langle \nu_E, \mathrm{Ad}(z)X\rangle = \langle \nu_E, X\rangle$. The claim follows from BV theory in Carnot groups; see [AKLD09, Thm. 4.16, Thm. 5.2].

Lemma 3.10 (Directional difference quotient (left translation)). Let $E \subset G$ have finite horizontal perimeter and $X \in V_1(G)$. Then

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_G \big|\mathbf{1}_E(\exp(\varepsilon X)\, z) - \mathbf{1}_E(z)\big|\, d\mu(z) \;=\; \int_{\partial^* E} \big|\nu_E \cdot X\big|\, d\mathcal{H}^{Q-1}.$$

Proof. By left invariance of the horizontal fields and of μ, the map $z \mapsto \exp(\varepsilon X)\, z$ is a measure-preserving flow generated by the left-invariant horizontal vector field X. The claim follows from the Gauss-Green and first-variation formulas for sets of finite horizontal perimeter in Carnot groups: see Danielli-Garofalo-Nhieu [DGN07, Thm. 10.1 (integration by parts), Thm. 14.3 (first variation)], and for a BV-level Gauss–Green in stratified groups see Comi–Magnani [CM20, Thm. 6.7 and Thm. 6.8]; these give; these give

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_G \big|\mathbf{1}_E(\exp(\varepsilon X)z) - \mathbf{1}_E(z)\big|\, d\mu = \int_{\partial^* E} |\nu_E \cdot X|\, d\mathcal{H}^{Q-1}.$$

For the BV background (rectifiability of $\partial^* E$, horizontality of ν_E, and blow-up to vertical halfspaces) see Ambrosio-Kleiner-Le Donne [AKLD09, §4–§5]. □

Convention. The measure-theoretic unit normal ν_E is *horizontal* ($\nu_E \in V_1$) and all inner products $\nu_E \cdot X$ use a fixed inner product on V_1; layers V_j ($j \ge 2$) are orthogonal to V_1 in this pairing.

Lemma 3.11 (Tubular neighborhood estimate). There exists $C = C(G)$ such that for every set $E \subset G$ of finite horizontal perimeter and every $\rho > 0$,

$$\mu\big(N_\rho(\partial E)\big) \;\le\; C\, \rho\, \mathrm{Per}(E).$$

Proof. Standard in homogeneous/Carnot groups via coarea and identification of Minkowski content with horizontal perimeter; see Monti–Serra Cassano [MSC01, Thm. 5.1 and Thm. 4.2]. For the BV blow-up to (vertical) halfspaces, see Ambrosio–Kleiner–Le Donne [AKLD09, Thm. 4.16]. □

Lemma 3.12 (First-order transport across higher-layer tails)**.** Let $E \subset G$ have finite horizontal perimeter. Suppose $g_\varepsilon \in G$ satisfies

$$\log g_\varepsilon \;=\; \varepsilon X \;+\; o(\varepsilon) \quad \text{in } \mathfrak{g}, \qquad X \in V_1(G),$$

where the $o(\varepsilon)$ term may lie in $\bigoplus_{j\geq 2} V_j$. Then

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_G \big|\mathbf{1}_E(g_\varepsilon z) - \mathbf{1}_E(z)\big|\, d\mu(z) \;=\; \int_{\partial^* E} \big|\nu_E \cdot X\big|\, d\mathcal{H}^{Q-1}.$$

Proof. Write $g_\varepsilon = \exp(W_\varepsilon)$ with $W_\varepsilon = \varepsilon X + o(\varepsilon)$ in $\mathfrak{g}$, and decompose $W_\varepsilon = \varepsilon X + V_\varepsilon$ with $V_\varepsilon \in \bigoplus_{j\geq 2} V_j$ and $\|V_\varepsilon\| = o(\varepsilon)$. By the Baker-Campbell-Hausdorff formula there exists $R_\varepsilon \in G$ with $\log R_\varepsilon = V_\varepsilon + O(\varepsilon\|V_\varepsilon\|)$ and

$$g_\varepsilon = \exp(\varepsilon X)\, R_\varepsilon, \qquad d_{\mathrm{cc}}(e, R_\varepsilon) = o(\varepsilon).$$

Set $T_\varepsilon(z) := \exp(\varepsilon X)\, z$ and $S_\varepsilon(z) := R_\varepsilon z$. Then

$$\frac{1}{\varepsilon} \int_G \big|\mathbf{1}_E(g_\varepsilon z) - \mathbf{1}_E(z)\big|\, d\mu \leq \frac{1}{\varepsilon} \int \big|\mathbf{1}_E(T_\varepsilon z) - \mathbf{1}_E(z)\big|\, d\mu + \frac{1}{\varepsilon} \int \big|\mathbf{1}_E(T_\varepsilon S_\varepsilon z) - \mathbf{1}_E(T_\varepsilon z)\big|\, d\mu.$$

The first term converges by Lemma 3.10 to $\int_{\partial^* E} |\nu_E \cdot X|\, d\mathcal{H}^{Q-1}$. For the second term, change variables $y = T_\varepsilon z$ and use left invariance:

$$\frac{1}{\varepsilon} \int \big|\mathbf{1}_E(S_\varepsilon y) - \mathbf{1}_E(y)\big|\, d\mu(y) = \frac{1}{\varepsilon}\, \mu(E \Delta S_\varepsilon^{-1} E) \leq \frac{1}{\varepsilon}\, \mu\big(N_{c\, d_{\mathrm{cc}}(e,S_\varepsilon)}(\partial E)\big) \leq \frac{C}{\varepsilon}\, d_{\mathrm{cc}}(e, S_\varepsilon)\ \mathrm{Per}(E) = o(1)$$

where we used the tubular estimate Lemma 3.11. A symmetric estimate from below (swap the roles of g_ε and $\exp(\varepsilon X)$) gives the matching lower bound, hence the limit is exactly $\int_{\partial^* E} |\nu_E \cdot X|\, d\mathcal{H}^{Q-1}$. □

Remark 3.13 (Size of the BCH tail)**.** For $s \in \Gamma$, $\log \delta_\varepsilon(s) = \varepsilon X_s + O(\varepsilon^2)$ in $\mathfrak{g}$, so $g_\varepsilon = \delta_\varepsilon(s)$ satisfies Lemma 3.12. Note $d_{\mathrm{cc}}(e, \delta_\varepsilon(s) \exp(-\varepsilon X_s)) = O(\varepsilon)$ in general; Lemma 3.12 avoids any metric $O(\varepsilon^2)$ assumption and works directly in $\mathfrak{g}$.

Lemma 3.14 (Rescaling identity for the discrete perimeter)**.** For $r > 0$, set $\varepsilon := 1/r$, $A_r := \Gamma \cap \delta_r(E)$, $U_r := A_r F$, and $V_\varepsilon := \delta_\varepsilon(U_r)$. Then

$$\mathrm{Per}_{\mathcal{S},a}(A_r) = r^Q \sum_{s \in \mathcal{S}} a_s \int_G \big|\mathbf{1}_{V_\varepsilon}\big(\delta_\varepsilon(s)\, z\big) - \mathbf{1}_{V_\varepsilon}(z)\big|\, d\mu(z).$$

Proof. By Lemma 3.8, $\mathrm{Per}_{\mathcal{S},a}(A_r) = \sum_s a_s\, \mu(U_r \Delta s^{-1} U_r)$. Apply δ_ε and use that δ_ε is an automorphism with $\mu(\delta_\varepsilon(\cdot)) = \varepsilon^Q \mu(\cdot)$, obtaining $\mu(V_\varepsilon \Delta \delta_\varepsilon(s^{-1}) V_\varepsilon)$. We used $\mu(V_\varepsilon \Delta \delta_\varepsilon(s^{-1}) V_\varepsilon) = \int |\mathbf{1}_{V_\varepsilon}(y) - \mathbf{1}_{V_\varepsilon}(\delta_\varepsilon(s) y)|\, d\mu(y)$ via the change $y \mapsto \delta_\varepsilon(s) y$, yielding the displayed formula. □

Lemma 3.15 (Phase averaging identity)**.** For any measurable $B \subset G$ and $s \in \mathcal{S}$,

$$\int_{t \in F} \sum_{g \in \Gamma} \big|\mathbf{1}_B(sgt) - \mathbf{1}_B(gt)\big|\, d\mu(t) \;=\; \int_G \big|\mathbf{1}_B(sy) - \mathbf{1}_B(y)\big|\, d\mu(y).$$

Consequently, for $B = \delta_r(E)$,

$$\int_{t \in F} \mathrm{Per}_{\mathcal{S},a}\left(A_r^{(t)}\right) d\mu(t) \;=\; \sum_{s \in \mathcal{S}} a_s \int_G \big|\mathbf{1}_{\delta_r(E)}(sy) - \mathbf{1}_{\delta_r(E)}(y)\big|\, d\mu(y),$$

where $A_r^{(t)} := \{g \in \Gamma :\ gt \in \delta_r(E)\}$.

Proof. Partition G as $\bigsqcup_{g\in\Gamma} gF$ and integrate in the phase $t \in F$:

$$\int_{t\in F} \sum_{g\in\Gamma} \big|\mathbf{1}_B(sgt)-\mathbf{1}_B(gt)\big|\, d\mu(t) = \sum_{g\in\Gamma} \int_{gF} \big|\mathbf{1}_B(sy)-\mathbf{1}_B(y)\big|\, d\mu(y) = \int_G \big|\mathbf{1}_B(sy)-\mathbf{1}_B(y)\big|\, d\mu(y).$$

The weighted version follows by linearity in s with weights a_s. □

Lemma 3.16 (Phase stability for counts)**.** Let $E \subset G$ have finite horizontal perimeter, $r \ge 1$, and $A_r := \Gamma \cap \delta_r(E)$. For $t \in F$ set $A_r^{(t)} := \{g \in \Gamma : \ gt \in \delta_r(E)\}$. There exists $C = C(G, F, E)$ such that for all $t, t' \in F$,

$$\big|A_r^{(t)} \,\Delta\, A_r^{(t')}\big| \;\le\; C\, r^{Q-1}.$$

In particular, $\big||A_r^{(t)}| - |A_r|\big| \le C r^{Q-1}$ for all $t \in F$.

Proof. Let $D := \operatorname{diam}_{\mathrm{cc}}(F)$. If $g \in A_r^{(t)} \Delta A_r^{(t')}$ then one of the points gt, gt' lies in $\delta_r(E)$ and the other does not, hence $\min\{d_{\mathrm{cc}}(gt, \partial\delta_r(E)),\, d_{\mathrm{cc}}(gt', \partial\delta_r(E))\} \le D$. Therefore gF intersects $N_D(\partial\delta_r(E))$. If $gF \cap N_D(\partial\delta_r(E)) \neq \varnothing$, then $gF \subset N_{D+\operatorname{diam}_{\mathrm{cc}}(F)}(\partial\delta_r(E))$. Since $\{gF : \ g \in \Gamma\}$ tiles G and each gF has measure 1, the number of such g is $\lesssim \mu(N_{D+\operatorname{diam}_{\mathrm{cc}}(F)}(\partial\delta_r(E)))$. By Lemma 3.11, $\mu(N_{D+\operatorname{diam}_{\mathrm{cc}}(F)}(\partial\delta_r(E))) = O(r^{Q-1})$, which gives the claim. □

Lemma 3.17 (Tile approximation via the fundamental domain)**.** Let $E \subset G$ be a set of finite horizontal perimeter and $F \subset G$ a measurable fundamental domain for Γ with $\mu(F) = 1$ and bounded $\operatorname{diam}_{\mathrm{cc}}(F)$. For $A_r := \Gamma \cap \delta_r(E)$ and $U_r := A_r F$ one has

$$\mu\big(U_r \,\Delta\, \delta_r(E)\big) \;\le\; C\, r^{Q-1}, \qquad r \ge 1,$$

where C depends on E, F and G only through $\operatorname{Per}(E)$ and $\operatorname{diam}_{\mathrm{cc}}(F)$. Equivalently,

$$\mu\big(\delta_{1/r}(U_r) \,\Delta\, E\big) \;=\; O(r^{-1}) \qquad (r \to \infty).$$

Idea. A tile gF contributes to the symmetric difference only if gF intersects both $\delta_r(E)$ and its complement. Such tiles lie inside a tubular neighborhood of $\partial\delta_r(E)$ of width $\lesssim \operatorname{diam}_{\mathrm{cc}}(F)$. By Lemma 3.11, the measure of this tube is $O(r^{Q-1})$. □

Proposition 3.18 (Compactness and $\liminf$ up to left translations)**.** Let G be a Carnot group of homogeneous dimension Q, $\Gamma \le G$ a uniform lattice with fundamental domain F of Haar measure 1, and let $\mathcal{S}$ and $\{a_s\}$ satisfy (3.1) for the gauge φ on $V_1(G)$. For any sequence of finite sets $Y_k \subset \Gamma$ with $|Y_k| \to \infty$, set $r_k := |Y_k|^{1/Q}$, $U_k := Y_k F$, and $V_k := \delta_{1/r_k}(U_k)$. Then there exist left translations $z_k \in G$ and a measurable $E \subset G$ with $\operatorname{Vol}_{\mathrm{c}}(E) = 1$ such that, along a subsequence,

$$\mathbf{1}_{z_k V_k} \;\to\; \mathbf{1}_E \quad \text{in } L^1_{\mathrm{loc}}(G)$$

and

$$\liminf_{k\to\infty} \; |Y_k|^{-\frac{Q-1}{Q}} \; \operatorname{Per}_{\mathcal{S},a}(Y_k) \;\ge\; \operatorname{Per}_\varphi(E).$$

Proof sketch. Set $\varepsilon_k := r_k^{-1}$. By Lemma 3.14,

$$|Y_k|^{-\frac{Q-1}{Q}} \operatorname{Per}_{\mathcal{S},a}(Y_k) = \sum_{s\in\mathcal{S}} a_s \, \frac{1}{\varepsilon_k} \int \big|\mathbf{1}_{V_k}\left(\delta_{\varepsilon_k}(s) z\right) - \mathbf{1}_{V_k}(z)\big|\, d\mu(z).$$

Choose z_k so that the translates $\widehat{V}_k := z_k V_k$ are tight (e.g. maximize mass in a fixed ball); by BV compactness, a subsequence $\mathbf{1}_{\widehat{V}_k} \to \mathbf{1}_E$ in L^1_{loc} with $\operatorname{Vol}_{\mathrm{c}}(E) = 1$. Changing variables $w = z_k z$ and using left invariance of μ,

$$\frac{1}{\varepsilon_k} \int \big|\mathbf{1}_{V_k}\left(\delta_{\varepsilon_k}(s) z\right) - \mathbf{1}_{V_k}(z)\big|\, d\mu(z) = \frac{1}{\varepsilon_k} \int \Big|\mathbf{1}_{\widehat{V}_k}\big(z_k \delta_{\varepsilon_k}(s) z_k^{-1}\, w\big) - \mathbf{1}_{\widehat{V}_k}(w)\Big|\, d\mu(w).$$

Since $z\exp(W)z^{-1} = \exp(\mathrm{Ad}(z)W)$ and $\mathrm{Ad}(z)$ *fixes the* V_1*-component* (adding only higher-layer terms), we have $\log\left(z_k\delta_{\varepsilon_k}(s)z_k^{-1}\right) = \varepsilon_k X_s + o(\varepsilon_k)$ with $X_s \in V_1$. Apply Lemma 3.12 and lower semicontinuity under L^1_{loc} convergence to get

$$\liminf_{k\to\infty} \frac{1}{\varepsilon_k}\int \left|\mathbf{1}_{\widehat{V}_k}\left(z_k\delta_{\varepsilon_k}(s)z_k^{-1}\,w\right) - \mathbf{1}_{\widehat{V}_k}(w)\right| d\mu(w) \;\geq\; \int_{\partial^* E} |\nu_E\cdot X_s|\, d\mathcal{H}^{Q-1}.$$

Summing in s with weights a_s and invoking (3.1) gives the claim. □

Proposition 3.19 (Verification principle under (CI))**.** Assume $(\mathcal{S}, a)$ satisfies (3.1). Let $E \subset G$ have *finite Haar measure* and finite horizontal perimeter and set $A_r := \Gamma\cap\delta_r(E)$.

(i) (*Fixed sampler,* $\liminf$ *bound*)

$$\liminf_{r\to\infty} \frac{\mathrm{Per}_{\mathcal{S},a}(A_r)}{r^{Q-1}} \;\geq\; \mathrm{Per}_\varphi(E).$$

(ii) (*Phase-shifted samplers, exact limit*) There exists a choice of phases $t_r \in F$ such that

$$\lim_{r\to\infty} \frac{\mathrm{Per}_{\mathcal{S},a}(A_r^{(t_r)})}{r^{Q-1}} \;=\; \mathrm{Per}_\varphi(E), \qquad \lim_{r\to\infty} \frac{|A_r^{(t_r)}|}{r^Q} \;=\; \mathrm{Vol_c}(E),$$

where $A_r^{(t)} := \{g\in\Gamma:\ gt\in\delta_r(E)\}$.

Proof. Set $\varepsilon := 1/r$, $U_r := A_rF$, $V_\varepsilon := \delta_\varepsilon(U_r)$. By Lemma 3.14,

$$\frac{\mathrm{Per}_{\mathcal{S},a}(A_r)}{r^{Q-1}} = \sum_{s\in\mathcal{S}} a_s\,\frac{1}{\varepsilon}\int \left|\mathbf{1}_{V_\varepsilon}\left(\delta_\varepsilon(s)\,z\right) - \mathbf{1}_{V_\varepsilon}(z)\right| d\mu(z).$$

By Lemma 3.17, $\mu(V_\varepsilon\Delta E) = O(\varepsilon)$, hence $\mathbf{1}_{V_\varepsilon}\to\mathbf{1}_E$ in L^1. Using $\log\delta_\varepsilon(s) = \varepsilon X_s + O(\varepsilon^2)$ and applying Lemma 3.12 with lower semicontinuity for L^1-convergent BV sequences, we obtain

$$\liminf_{\varepsilon\downarrow 0} \frac{1}{\varepsilon}\int \left|\mathbf{1}_{V_\varepsilon}\left(\delta_\varepsilon(s)\,z\right) - \mathbf{1}_{V_\varepsilon}(z)\right| d\mu \;\geq\; \int_{\partial^* E} |\nu_E\cdot X_s|\, d\mathcal{H}^{Q-1}.$$

Summing in s with weights a_s and using (3.1) yields (i).

For (ii), averaging in $t\in F$ and applying Lemma 3.15 gives

$$\int_{t\in F} \frac{\mathrm{Per}_{\mathcal{S},a}(A_r^{(t)})}{r^{Q-1}}\, d\mu(t) = \sum_{s\in\mathcal{S}} a_s\,\frac{1}{\varepsilon}\int \left|\mathbf{1}_{\delta_r(E)}(sy) - \mathbf{1}_{\delta_r(E)}(y)\right| d\mu(y).$$

Rewrite via $y = \delta_{1/\varepsilon}z$, use $\log\delta_\varepsilon(s) = \varepsilon X_s + O(\varepsilon^2)$, and apply Lemmata 3.10, 3.12 to obtain that the average converges to $\mathrm{Per}_\varphi(E)$. Hence there exists a phase t_r with $\mathrm{Per}_{\mathcal{S},a}(A_r^{(t_r)})/r^{Q-1}\to\mathrm{Per}_\varphi(E)$. Finally, by Lemma 3.16 and Lemma 3.17, for all $t\in F$ we have $\left||A_r^{(t)}|-|A_r|\right| = O(r^{Q-1})$ and $|A_r| = r^Q\mu(E) + O(r^{Q-1})$; hence $|A_r^{(t_r)}|/r^Q\to\mu(E)$. □

Definition 3.20 (Continuum sharp constant)**.** The *continuum anisotropic isoperimetric constant* associated with φ is

$$c_{\mathrm{Wulff}} \;:=\; \inf\left\{\frac{\mathrm{Per}_\varphi(E)}{\mathrm{Vol_c}(E)^{\frac{Q-1}{Q}}} \;:\; E\subset G \text{ measurable},\ 0<\mathrm{Vol_c}(E)<\infty\right\}.$$

Remark 3.21 (Step 1 identification with the classical Wulff formula)**.** If $G=\mathbb{R}^d$ (step 1), the continuum Wulff theorem yields that the infimum in Definition 3.20 is attained by the Euclidean Wulff body $W_\varphi = \{\eta\in\mathbb{R}^d:\ \varphi^\circ(\eta)\leq 1\}$ and

$$c_{\mathrm{Wulff}} = \frac{\mathrm{Per}_\varphi(W_\varphi)}{|W_\varphi|^{(d-1)/d}}.$$

In higher-step Carnot groups an explicit minimizer is generally unknown; our discrete arguments use only the optimal constant c_{Wulff} from Definition 3.20. For the Euclidean anisotropic case, see Taylor [Tay78, Thm. 1.1].

Definition 3.22 (Directed S-perimeter on a Cayley graph)**.** Let S be a symmetric generating set of a countable group Γ (so $s \in S \Rightarrow s^{-1} \in S$). For a finite $Y \subset \Gamma$ define the *directed* S-perimeter

$$\mathrm{Per}_{S,1}(Y) \;:=\; \sum_{s\in S} \big|\{y \in Y : \; sy \notin Y\}\big|.$$

(Each missing neighbor in direction s counts once; for the undirected boundary, see Remark 3.23 below.)

Remark 3.23 (Undirected boundary)**.** If instead one defines $\mathrm{Per}^{\mathrm{und}}_{S,1}(Y)$ to count each unordered edge once, all statements remain valid with constants halved. In particular, in parts (i) and (iii) the number $2d$ should be replaced by d, and the lower bounds in the proofs lose the factor 2 coming from Lemma 3.30.

Lemma 3.24 (Right-translation invariance under the left-edge convention)**.** For any finite $Y \subset \Gamma$ and any $h \in \Gamma$, one has $\mathrm{Per}_{S,1}(Yh) = \mathrm{Per}_{S,1}(Y)$.

Proof. $\{y \in Yh : \; sy \notin Yh\} = \{yh : \; syh \notin Yh\}$; left multiplication by s and right multiplication by h commute in counting, so the cardinality is preserved for each s and summing gives the claim. □

Theorem 3.25 (Asymptotically sharp discrete Wulff principle)**.** Assume $(\mathcal{S}, a)$ satisfies (3.1) on a Carnot group G of homogeneous dimension Q. Then for every $E \subset G$ with $0 < \mathrm{Vol}_{\mathrm{c}}(E) < \infty$ and $\mathrm{Per}_\varphi(E) < \infty$, there exist phases $t_r \in F$ such that

$$\lim_{r\to\infty} \frac{\mathrm{Per}_{\mathcal{S},a}(A_r^{(t_r)})}{|A_r^{(t_r)}|^{\frac{Q-1}{Q}}} \;=\; \frac{\mathrm{Per}_\varphi(E)}{\mathrm{Vol}_{\mathrm{c}}(E)^{\frac{Q-1}{Q}}}. \tag{3.2}$$

Consequently,

$$c^{\mathrm{disc}}_{\mathrm{Wulff}} \;:=\; \liminf_{n\to\infty} \inf_{\substack{A\subset\Gamma\\ |A|=n}} \frac{\mathrm{Per}_{\mathcal{S},a}(A)}{n^{\frac{Q-1}{Q}}} \;=\; c_{\mathrm{Wulff}}.$$

Proof. The first display is Proposition 3.19(ii). For the constants, Proposition 3.18 gives $c^{\mathrm{disc}}_{\mathrm{Wulff}} \geq c_{\mathrm{Wulff}}$ (liminf bound), while the sampler construction in Proposition 3.19(ii) gives $c^{\mathrm{disc}}_{\mathrm{Wulff}} \leq c_{\mathrm{Wulff}}$ (limsup via samplers). Hence equality. □

Corollary 3.26 (Euclidean/step 1 case)**.** If $G = \mathbb{R}^d$ (step 1), then (3.2) with $E = W_\varphi$ yields

$$\lim_{r\to\infty} \frac{\mathrm{Per}_{\mathcal{S},a}(A_r^{(t_r)})}{|A_r^{(t_r)}|^{\frac{Q-1}{Q}}} \;=\; c_{\mathrm{Wulff}} \qquad \text{with } \; c_{\mathrm{Wulff}} = \frac{\mathrm{Per}_\varphi(W_\varphi)}{|W_\varphi|^{(d-1)/d}}.$$

Remark 3.27 (Abelian case $\Gamma = \mathbb{Z}^d$)**.** If $G = \mathbb{R}^d$ (step 1), $Q = d$, and $\Gamma = \mathbb{Z}^d$, an axis-aligned anisotropy $\varphi(\xi) = \sum_{i=1}^d w_i\,|\xi_i|$ is realized under the *directed-edge* convention by choosing weights $a_{\pm e_i} = \frac{w_i}{2}$ (and $a_s = 0$ otherwise), so that (3.1) holds exactly. Then $W_\varphi = \prod_{i=1}^d [-w_i, w_i]$ and c_{Wulff} is explicit [Tay78].

In particular, for $w_i \equiv 1$ this choice yields $\tau(\nu) = \sum_i |\nu_i|$ and $c_{\mathrm{Wulff}} = 2d$.

Proposition 3.28 (Scaled limit via dense samplers)**.** Let $D \geq 2$ denote the ambient homogeneous dimension (e.g. $D = Q$ in the Carnot setting) and $f(n) := \inf\{\mathrm{Per}_{S,1}(Y) : \; Y \subset \Gamma, \; |Y| = n\}$. Suppose there exists a sequence $m_k \to \infty$ such that

$$f(m_k) \;\leq\; (c + o(1))\, m_k^{\frac{D-1}{D}} \quad \text{ and } \quad m_{k+1} - m_k \;=\; O\big(m_k^{\frac{D-1}{D}}\big).$$

Then

$$\lim_{n\to\infty} n^{-\frac{D-1}{D}}\, f(n) \;=\; c.$$

Remark 3.29. In our applications the sequence (m_k) is provided by discrete Wulff approximants ("samplers"), which satisfy $m_{k+1} - m_k \asymp m_k^{(D-1)/D}$ and achieve $c = c_{\mathrm{Wulff}}$ (Theorem 3.25 and its Abelian specialization). In the ambient (virtually nilpotent/Carnot) setting one also has the uniform upper bound $f(t) \lesssim t^{\frac{D-1}{D}}$ for large t by the Wulff upper construction.

Proof. Subadditivity: choose Y_i with $|Y_i| = n_i$ and $\mathrm{Per}_{S,1}(Y_i) \le f(n_i) + \varepsilon$; *right*-translate one of them far in the Cayley graph so that their edge boundaries do not interact, then $\mathrm{Per}_{S,1}(Y_1 \cup Y_2) \le \mathrm{Per}_{S,1}(Y_1) + \mathrm{Per}_{S,1}(Y_2)$. Taking infima and letting $\varepsilon \downarrow 0$ gives $f(n_1 + n_2) \le f(n_1) + f(n_2)$.

For the limit: let n be large and choose k with $m_k \le n < m_{k+1}$. Subadditivity yields

$$f(n) \;\le\; f(m_k) \;+\; f(n - m_k).$$

By hypothesis, $n - m_k \le m_{k+1} - m_k = O\big(m_k^{\frac{D-1}{D}}\big) = O\big(n^{\frac{D-1}{D}}\big)$. Using $f(t) \lesssim t^{\frac{D-1}{D}}$ for these intermediate sizes,

$$\frac{f(n)}{n^{\frac{D-1}{D}}} \;\le\; \frac{f(m_k)}{m_k^{\frac{D-1}{D}}}\Big(\frac{m_k}{n}\Big)^{\frac{D-1}{D}} \;+\; \frac{f(n - m_k)}{(n - m_k)^{\frac{D-1}{D}}}\Big(\frac{n - m_k}{n}\Big)^{\frac{D-1}{D}} \;\le\; (c + o(1)) + O\big(n^{-\frac{D-1}{D^2}}\big).$$

Thus $\limsup_{n\to\infty} n^{-(D-1)/D} f(n) \le c$, and the liminf assumption gives the limit. □

Lemma 3.30 (Column-counting for the directed boundary)**.** Let $S_{\mathrm{ax}} = \{\pm e_1, \dots, \pm e_d\}$ and $Y \subset \mathbb{Z}^d$ finite. For each i let $\pi_i : \mathbb{Z}^d \to \mathbb{Z}^{d-1}$ delete the i-th coordinate. Then

$$\mathrm{Per}_{S_{\mathrm{ax}},1}(Y) \;\ge\; 2\sum_{i=1}^{d} |\pi_i(Y)|.$$

Proof. Fix i and $u \in \mathbb{Z}^{d-1}$. The i-column $C_{i,u} = \{(t, u) : t \in \mathbb{Z}\}$ meets Y in a finite union of disjoint integer intervals. Each such interval contributes exactly one missing $+e_i$-neighbor and one missing $-e_i$-neighbor in that column, hence at least $2 \cdot \mathbf{1}_{\{C_{i,u} \cap Y \neq \emptyset\}}$ directed boundary edges in the $\pm e_i$ directions. Summing over u and then i gives the claim. □

Lemma 3.31 (Discrete Loomis–Whitney)**.** For any finite $Y \subset \mathbb{Z}^d$,

$$|Y|^{d-1} \;\le\; \prod_{i=1}^{d} |\pi_i(Y)|.$$

Proof. For each i and $u \in \mathbb{Z}^{d-1}$ let $C_{i,u} := \{(t, u) : t \in \mathbb{Z}\}$ be the i-th column and $m_{i,u} := \#(Y \cap C_{i,u})$ its occupancy. Then $|Y| = \sum_u m_{i,u}$ and $|\pi_i(Y)| = \#\{u : m_{i,u} \ge 1\}$. Applying Hölder to the indicator families $\{\mathbf{1}_{\{m_{i,u} \ge 1\}}\}$ over $i = 1, \dots, d$ yields $|Y|^{d-1} \le \prod_{i=1}^{d} |\pi_i(Y)|$, see Bollobás-Thomason [BT95, Thm. 1] for a direct combinatorial proof. □

Lemma 3.32 (Unimodular change of basis preserves the directed perimeter)**.** Let $B \in GL(d, \mathbb{Z})$ with columns $b_1, \dots, b_d$ and set $T := B^{-1}$. For

$$S = \{\pm b_1, \dots, \pm b_d\}, \qquad S_{\mathrm{ax}} = \{\pm e_1, \dots, \pm e_d\},$$

the map $T : \mathbb{Z}^d \to \mathbb{Z}^d$ is a graph isomorphism $\mathrm{Cay}(\mathbb{Z}^d, S) \to \mathrm{Cay}(\mathbb{Z}^d, S_{\mathrm{ax}})$, and

$$\mathrm{Per}_{S,1}(Y) \;=\; \mathrm{Per}_{S_{\mathrm{ax}},1}(TY) \qquad \text{for all finite } Y \subset \mathbb{Z}^d.$$

Proof. T is a group automorphism sending each S-edge $y \leftrightarrow y \pm b_i$ to an S_{ax}-edge $Ty \leftrightarrow Ty \pm e_i$. Missing neighbors are preserved bijectively, hence the directed boundary is preserved. □

Lemma 3.33 (Fiber-permutation inequality)**.** Let $\pi : \Gamma \to \mathbb{Z}^d$ be surjective with finite normal kernel K, and let $S_0 \subset S$ map bijectively to $\{\pm e_i\}_{i=1}^{d}$. For a finite $Y \subset \Gamma$, write $\ell(u) := |Y \cap \pi^{-1}(u)|$ and $E_t := \{u \in \mathbb{Z}^d : \ \ell(u) > t\}$ for $t \in [0, m)$, where $m = |K|$. Then

$$\sum_{s \in S_0} \sum_{g \in \Gamma} \big|\mathbf{1}_Y(sg) - \mathbf{1}_Y(g)\big| \;\ge\; \sum_{i=1}^{d} \int_0^m \big|\partial_{\{\pm e_i\}} E_t\big|\, dt.$$

Equality holds when Y is *fiber-saturated*: $Y = \pi^{-1}(E)$ for some $E \subset \mathbb{Z}^d$.

Idea. Fix i and consider pairs of base neighbors $(u, u+e_i)$. For each fiber position $k \in K$, the contribution of edges parallel to e_i at level k is at least $|\mathbf{1}_{E^{(k)}}(u+e_i) - \mathbf{1}_{E^{(k)}}(u)|$, where $E^{(k)} := \{u : (u,k) \in Y\}$. Summing over k and integrating the layer-cake $\mathbf{1}_{\{\ell > t\}}$ yields the right-hand side. Rearranging within each fiber cannot decrease the left side; the minimum is attained by fiber-saturated sets, where the identity becomes exact. $\square$

3.1. **Proof of Theorem A .**

Proof of Theorem A (i). For S_{ax}, Lemmas 3.30 and 3.31 imply

$$\mathrm{Per}_{S_{\mathrm{ax}},1}(Y) \ \geq\ 2\sum_{i=1}^{d} |\pi_i(Y)| \ \geq\ 2d\Big(\prod_{i=1}^{d} |\pi_i(Y)|\Big)^{1/d} \ \geq\ 2d\,|Y|^{\frac{d-1}{d}}.$$

To identify the continuum constant, note that for $\tau(\nu) = \sum_{i=1}^d |\nu_i|$ the Wulff shape is the cube $W = [-1,1]^d$ and the Minkowski integral formula yields

$$P_\tau(W) \ =\ \int_{\partial W} \tau(\nu)\, d\mathcal{H}^{d-1} \ =\ d\,|W| \ =\ d \cdot 2^d,$$

hence $c_{\mathrm{Wulff}} = P_\tau(W)/|W|^{(d-1)/d} = 2d$. By the bridge Lemma 3.7, the discrete functional here coincides with $\mathrm{Per}_{\mathcal{S},a}$ for the split weights, so the constants are aligned. For a unimodular basis stencil $S = \{\pm b_1, \ldots, \pm b_d\}$ with $B = [b_1 \ \cdots \ b_d] \in GL(d,\mathbb{Z})$, the map $T = B^{-1} \in GL(d,\mathbb{Z})$ is a graph isomorphism from $(\mathbb{Z}^d, S)$ to $(\mathbb{Z}^d, S_{\mathrm{ax}})$, so $\mathrm{Per}_{S,1}(Y) = \mathrm{Per}_{S_{\mathrm{ax}},1}(TY)$ and $|TY| = |Y|$. Applying the axis case to TY gives the claim for any unimodular basis. $\square$

Remark 3.34 (Value of c_{Wulff} for the axis stencil)**.** For $\tau(\nu) = \sum_{i=1}^d |\nu_i|$ the Wulff body is $W = [-1,1]^d$, hence $\mathrm{Per}_\tau(W) = \int_{\partial W} \tau(\nu)\, d\mathcal{H}^{d-1} = d\,2^d$ and $|W| = 2^d$. Therefore $c_{\mathrm{Wulff}} = \mathrm{Per}_\tau(W)/|W|^{(d-1)/d} = 2d$, matching the discrete bound.

Proof of Theorem A (ii). Let (Y_k) be any sequence with $|Y_k| \to \infty$ and set $r_k := |Y_k|^{1/Q}$. By the bridge Lemma 3.7, $\mathrm{Per}_{S,1}(Y_k) = \mathrm{Per}_{\mathcal{S},a}(Y_k)$ for the split weights. Apply Proposition 3.18 to obtain, along a subsequence, a set $E \subset G$ with $\mathrm{Vol}_{\mathrm{c}}(E) = 1$ such that (up to *right* translations of Y_k, which preserve $\mathrm{Per}_{S,1}$ by Lemma 3.24)

$$\liminf_{k\to\infty} \ |Y_k|^{-\frac{Q-1}{Q}} \ \mathrm{Per}_{S,1}(Y_k) \ \geq\ \mathrm{Per}_{\tau_S}(E).$$

By the continuum anisotropic isoperimetric inequality on G, $\mathrm{Per}_{\tau_S}(E) \geq c_{\mathrm{Wulff}}$. Therefore $\liminf |Y|^{-(Q-1)/Q}\,\mathrm{Per}_{S,1}(Y) \geq c_{\mathrm{Wulff}}$, i.e. $h_S \geq c_{\mathrm{Wulff}}$. The matching upper bound $h_S \leq c_{\mathrm{Wulff}}$ follows from Theorem 3.25 (bridge again). Hence $h_S = c_{\mathrm{Wulff}}$. $\square$

Assumption for (iii). Let $\pi : \Gamma \to \mathbb{Z}^d$ be surjective with finite *normal* kernel K of size m. Assume the symmetric generating set S decomposes as

$$S \ =\ S_0 \ \cup\ S_{\mathrm{vert}},$$

where $S_0 \subset S$ maps bijectively to a unimodular basis $\{\pm e_i\}_{i=1}^d$ of $\mathbb{Z}^d$, and

$$S_{\mathrm{vert}} \ :=\ S \cap K$$

is a (nonempty) symmetric generating set of the kernel K.

Lemma 3.35 (Uniform vertical Cheeger bound on the kernel)**.** Let $\kappa_K > 0$ denote the directed Cheeger constant of the finite Cayley graph $\mathrm{Cay}(K, S_{\mathrm{vert}})$,

$$\kappa_K \ :=\ \min_{\varnothing \neq B \subsetneq K} \frac{\big|\partial^{\rightarrow}_{S_{\mathrm{vert}}} B\big|}{\min\{|B|,\, |K \setminus B|\}}.$$

Then for every finite $Y \subset \Gamma$, if we write the fiber occupancy over $u \in \mathbb{Z}^d$ as $\ell(u) := |Y \cap \pi^{-1}(u)| \in \{0, 1, \dots, m\}$, we have

$$\sum_{s \in S_{\mathrm{vert}}} \sum_{g \in \Gamma} \left|\mathbf{1}_Y(sg) - \mathbf{1}_Y(g)\right| \ \geq\ \kappa_K \sum_{u \in \mathbb{Z}^d} \min\{\ell(u),\, m - \ell(u)\}.$$

Proof. Because $K = \ker \pi$ is normal and $S_{\mathrm{vert}} \subset K$, for each u the restriction of left multiplication by S_{vert} to the fiber $\pi^{-1}(u)$ is (via $\pi^{-1}(u) \cong K$) exactly the Cayley graph $\mathrm{Cay}(K, S_{\mathrm{vert}})$. Hence, for each u,

$$\sum_{s \in S_{\mathrm{vert}}} \sum_{g \in \pi^{-1}(u)} \left|\mathbf{1}_Y(sg) - \mathbf{1}_Y(g)\right| \ \geq\ \kappa_K\ \min\{|Y \cap \pi^{-1}(u)|,\, m - |Y \cap \pi^{-1}(u)|\}.$$

Summing over u yields the claim. □

Lemma 3.36 (Horizontal deficit controlled by partial columns)**.** Let $\alpha \in (0, 1)$ and $m \in \mathbb{N}$. For any function $\ell : \mathbb{Z}^d \to \{0, 1, \dots, m\}$ with total mass $S := \sum_u \ell(u)$, define

$$E_t := \{u :\ \ell(u) > t\}, \qquad a_k := |E_t| \ \text{ for } t \in [k, k+1),\ k = 0, \dots, m-1,$$

and the *partiality* functional

$$\mathsf{P}(\ell) \ :=\ \sum_{u \in \mathbb{Z}^d} \min\{\ell(u),\, m - \ell(u)\}.$$

Let $E = \lfloor S/m \rfloor$ and $r = S - Em \in \{0, 1, \dots, m-1\}$. Then

$$\sum_{k=0}^{m-1} a_k^\alpha \ \geq\ r(E+1)^\alpha + (m-r)E^\alpha \ -\ \left((E+1)^\alpha - E^\alpha\right) \mathsf{P}(\ell). \tag{3.3}$$

Proof. Let $(a_k)_{k=0}^{m-1}$ be the nonincreasing sequence $a_k := |E_t|$ for $t \in [k, k+1)$, and let $\tilde{a}_k$ be its *equalized* version: $\tilde{a}_k = E + 1$ for $k < r$ and $\tilde{a}_k = E$ for $k \geq r$. Then $\sum_k a_k = \sum_k \tilde{a}_k = S$ and $(\tilde{a}_k)$ *majorizes* (a_k) (every initial partial sum of $\tilde{a}_k$ dominates that of a_k). Since $x \mapsto x^\alpha$ is concave on $[0, \infty)$, Karamata's inequality gives

$$\sum_{k=0}^{m-1} a_k^\alpha \ \leq\ \sum_{k=0}^{m-1} \tilde{a}_k^\alpha \ =\ r(E+1)^\alpha + (m-r)E^\alpha.$$

We now quantify the deficit by counting how many unit masses must cross the threshold between E and $E+1$ to obtain a from $\tilde{a}$. Define

$$\delta := \sum_{k=0}^{m-1} (\tilde{a}_k - a_k)_+ \ \leq\ \sum_{u \in \mathbb{Z}^d} \min\{\ell(u), m - \ell(u)\} \ =\ \mathsf{P}(\ell),$$

where the inequality follows from the layer-cake representation of ℓ and the fact that each unit of partiality reduces the count of superlevel sets at most by 1 at a single level. Each time one unit is moved from level $E+1$ to level E, the quantity $\sum_k a_k^\alpha$ decreases by at most $(E+1)^\alpha - E^\alpha$ (by the mean value theorem and concavity). Therefore

$$\sum_{k=0}^{m-1} a_k^\alpha \ \geq\ r(E{+}1)^\alpha {+} (m{-}r)E^\alpha - \left((E{+}1)^\alpha {-} E^\alpha\right) \delta \ \geq\ r(E{+}1)^\alpha {+} (m{-}r)E^\alpha - \left((E{+}1)^\alpha {-} E^\alpha\right) \mathsf{P}(\ell),$$

which is (3.3). □

Proof of Theorem A (iii). Strategy. We split $\mathrm{Per}_{S,1}$ into horizontal and vertical parts. Horizontally, a fiber-permutation inequality and a layer-cake representation reduce the problem to the Abelian Wulff bound on $\mathbb{Z}^d$ for the superlevel sets E_t. Vertically, a Cheeger inequality on the kernel controls the cost of partially filled fibers via the functional $\mathsf{P}(\ell)$. A quantitative "horizontal deficit" estimate (Lemma 3.36) bounds how far one is from fiber saturation. Combining the two components and optimizing yields the sharp constant $2d\,m^{1/d}$, with upper bound realized by fiber-saturated lifts.

Let $\pi : \Gamma \to \mathbb{Z}^d$ be surjective with finite normal kernel K of size m, and assume $S = S_0 \cup S_{\mathrm{vert}}$ as above with S_{vert} generating K.

Layer-cake on the base and Abelian Wulff. As before set $\ell(u) := |Y \cap \pi^{-1}(u)|$, $E_t := \{\ell > t\}$, $a_k := |E_t|$ for $t \in [k, k+1)$. By the fiber-permutation inequality (Lemma 3.33) and the Abelian Wulff inequality on $\mathbb{Z}^d$,

$$\sum_{s\in S_0}\sum_{g\in\Gamma}\big|\mathbf{1}_Y(sg)-\mathbf{1}_Y(g)\big| \;\ge\; \sum_{i=1}^{d}\int_0^m \big|\partial_{\{\pm e_i\}}E_t\big|\,dt \;\ge\; 2d\sum_{k=0}^{m-1}a_k^{\alpha}, \qquad \alpha=\tfrac{d-1}{d}.$$

Vertical penalty. By Lemma 3.35,

$$\sum_{s\in S_{\mathrm{vert}}}\sum_{g\in\Gamma}\big|\mathbf{1}_Y(sg)-\mathbf{1}_Y(g)\big| \;\ge\; \kappa_K\,\mathsf{P}(\ell), \qquad \mathsf{P}(\ell) := \sum_u \min\{\ell(u), m-\ell(u)\}.$$

Combine and optimize. Let $S = \sum_u \ell(u) = |Y|$, $E = \lfloor S/m \rfloor$, $r = S - Em$. By Lemma 3.36,

$$\sum_{k=0}^{m-1} a_k^{\alpha} \;\ge\; r(E+1)^{\alpha} + (m-r)E^{\alpha} \;-\; \big((E+1)^{\alpha}-E^{\alpha}\big)\,\mathsf{P}(\ell).$$

Therefore

$$\mathrm{Per}_{S,1}(Y) \;\ge\; 2d\,[r(E+1)^{\alpha}+(m-r)E^{\alpha}] \;+\; \Big(\kappa_K - 2d\big((E+1)^{\alpha}-E^{\alpha}\big)\Big)\,\mathsf{P}(\ell).$$

Since $\alpha - 1 = -\frac{1}{d} < 0$, $(E+1)^{\alpha} - E^{\alpha} \to 0$ as $|Y| \to \infty$. Hence there exists $N_0 = N_0(d, \kappa_K)$ so that for $|Y| \ge N_0$ the bracket is nonnegative and we can drop the $\mathsf{P}(\ell)$ term:

$$\mathrm{Per}_{S,1}(Y) \;\ge\; 2d\,[r(E+1)^{\alpha}+(m-r)E^{\alpha}]\,.$$

A standard binomial estimate yields

$$r(E+1)^{\alpha}+(m-r)E^{\alpha} \;=\; m^{1-\alpha}\,S^{\alpha} \;+\; O\big(E^{\alpha-1}\big) \;=\; m^{1/d}\,|Y|^{\frac{d-1}{d}} \;+\; o\Big(|Y|^{\frac{d-1}{d}}\Big).$$

Thus

$$\liminf_{|Y|\to\infty}\frac{\mathrm{Per}_{S,1}(Y)}{|Y|^{\frac{d-1}{d}}} \;\ge\; 2d\,m^{1/d}.$$

Sharpness (matching upper bound). If $E \subset \mathbb{Z}^d$ is a near-minimizer for the Abelian discrete Wulff problem (Theorem 3.25), then its *fiber-saturated lift* $Y := \pi^{-1}(E)$ has $|Y| = m|E|$ and $\mathrm{Per}_{S,1}(Y) = m\,|\partial_{\{\pm e_i\}}E| \sim 2d\,m\,|E|^{\frac{d-1}{d}} = 2d\,m^{1/d}\,|Y|^{\frac{d-1}{d}}$. (Here horizontal edges replicate m-fold across fibers, while vertical edges vanish for fiber-saturated lifts.) Therefore $h_S = 2d\,m^{1/d}$. □

Remark 3.37 (Necessity of a vertical generating subset in S)**.** If $S \cap K = \varnothing$ (no kernel edges are counted), the vertical penalty is absent and one can spread mass thinly across many fibers to force $\int_0^m |E_t|^{\alpha}\,dt \asymp |Y|^{\alpha}$, leading only to the lower bound $\mathrm{Per}_{S,1}(Y) \gtrsim 2d\,|Y|^{\alpha}$. Thus the *sharp* constant $2d\,m^{1/d}$ generally *requires* that S contain a kernel-generating symmetric subset $S_{\mathrm{vert}} \subset K$ as above.

Corollary 3.38 (Existence of the full limit for h_S in Theorem A)**.** In each of the cases (i)-(ii), and in (iii) under the assumption on S_{vert}, one has

$$\lim_{n\to\infty} n^{-\frac{D-1}{D}}\,\inf\{\mathrm{Per}_{S,1}(Y) : \; |Y| = n\} \;=\; c_{\mathrm{Wulff}}.$$

Proof. By Theorem 3.25 (and its Abelian specialization), there exists a sampler subsequence m_k with $f(m_k) \le (c_{\mathrm{Wulff}} + o(1))\,m_k^{(D-1)/D}$ and $m_{k+1} - m_k = O(m_k^{(D-1)/D})$. Apply Proposition 3.28. □

Remark 3.39 (Discrete vs. continuum constants in the virtually Abelian case)**.** Here the asymptotic cone is $(G,\varphi)=(\mathbb{R}^d,\tau_{\mathrm{ax}})$, so $c_{\mathrm{Wulff}}=2d$. The discrete asymptotic constant for $S=S_0\cup S_{\mathrm{vert}}$ is instead $h_S=2d\,m^{1/d}$ (Theorem A(iii)), because $|Y|$ counts all m elements in each fiber while horizontal edges replicate m-fold. Thus the $m^{1/d}$ factor is a *discrete* multiplicity effect, not a change of the continuum gauge.

Proposition 3.40 (Periodic discrete calibration suffices)**.** Let $S\subset\mathbb{Z}^d$ be symmetric. Suppose there exist rational weights $\{\alpha_s\}_{s\in S}$ and a $\mathbb{Z}^d$-periodic divergence-free flow ϕ on directed edges with $\phi(e)\in[0,\alpha_s]$ on each directed edge e parallel to s, such that for a.e. unit normal ν the support function $h(\nu)=\sum_{s\in S}\alpha_s\,(\nu\cdot s)_+$ equals the anisotropy $\tau(\nu)$ of interest. Then the discrete isoperimetric constant equals the corresponding continuum Wulff constant: $h_S=c_{\mathrm{Wulff}}(\tau)$.

Proof sketch. Max-flow/min-cut duality on the periodic torus yields optimal cuts matching $h(\nu)$ in the continuum limit. Because the coefficients are rational, there is a period Q with $Q\alpha_s\in\mathbb{N}$; solve the torus max-flow with unit capacities repeated $Q\alpha_s$ times in each s-direction, then average and lift to $\mathbb{Z}^d$. This gives a discrete calibration whose Γ-limit is the anisotropic perimeter with integrand τ, hence $h_S=c_{\mathrm{Wulff}}(\tau)$. □

4. Applications of the Discrete Wulff Principle and Abelian Base Bounds

Standing conventions. We import the *definitions and normalizations* from §3 (CI (3.1), boundary notions, etc.), but the arguments in this section do *not* use the CI framework itself. We assume $r_1:=\operatorname{rank}(\Gamma_{\mathrm{ab}}/\operatorname{tors})\ge 1$. As in §2 we use *right* multiplication on Γ: neighbors of y are $\{ys:\ s\in\mathcal{S}\}$, and

$$B_{\mathcal{S}}(Y)=\sum_{s\in\mathcal{S}}\#\{\,y\in Y:\ ys\notin Y\,\},\qquad \Delta:=|\mathcal{S}|.$$

On the Abelian base $A\simeq\mathbb{Z}^{r_1}$ we use additive notation $u+v$.

Multiplicity vs. direction. We distinguish between:

$$S^{\Gamma}_{\mathrm{hor}}:=\{\,s\in\mathcal{S}:\ \pi_{\mathrm{ab}}(s)\neq e\,\}\subset\mathcal{S}\quad\text{(the \emph{multiset} of horizontal lifts in }\Gamma\text{)},$$

and

$$\underline{S}_{\mathrm{hor}}\subset A\qquad\text{(the \emph{set} of distinct base directions in }A\text{, symmetric, with }e\text{ removed).}$$

Combinatorial edge counts on Γ use $S^{\Gamma}_{\mathrm{hor}}$; the base anisotropy and base edge counts use $\underline{S}_{\mathrm{hor}}$.

Scope and limitations. The results below give *sharp* lower bounds for the *horizontal* component and then use $B_{\mathcal{S}}(Y)\ge B_{S^{\Gamma}_{\mathrm{hor}}}(Y)$ to bound the full perimeter. When vertical generators are present, the vertical contribution may be of the same order; our bounds are sharp for the horizontal part and provide a base-component contribution to the total perimeter.

Independent combinatorial lower bound via the Abelian base. We record a self-contained lower bound on the Abelianization using the *zonotope* anisotropy on the base; it is logically independent of the CI framework in §3.

Abelian base and its anisotropy (directed). *Normalization note.* On the base anisotropy we use *unit weights* on each element of the *set* $\underline{S}_{\mathrm{hor}}$ of distinct directions; in particular, both $\pm v$ are present, so the base gauge

$$\tau_{\mathrm{hor}}(\xi)=\sum_{v\in\underline{S}_{\mathrm{hor}}}|\langle\xi,v\rangle|$$

is *twice* the split-pair gauge from Remark 3.4. All base constants below use this directed normalization.

Let $A := \Gamma_{\mathrm{ab}}/\operatorname{tors} \cong \mathbb{Z}^{r_1}$ with projection $\pi_{\mathrm{ab}} : \Gamma \to A$ and induced anisotropy set $\underline{S}_{\mathrm{hor}} \subset A$. Define the zonotope

$$K_{\mathrm{hor}} := \sum_{v \in \underline{S}_{\mathrm{hor}}} [-v, v] \subset \mathbb{R}^{r_1},$$

so that $\tau_{\mathrm{hor}} = h_{K_{\mathrm{hor}}}$; since $\underline{S}_{\mathrm{hor}}$ spans $\mathbb{R}^{r_1}$, K_{hor} has nonempty interior and its polar unit ball is $W_{\mathrm{hor}} = K_{\mathrm{hor}}$.

Theorem 4.1 (Sharp discrete Wulff on the Abelian base)**.** There is a constant

$$c_{\mathrm{ab}}(\underline{S}_{\mathrm{hor}}) \;:=\; \frac{\mathrm{Per}_{\tau_{\mathrm{hor}}}(W_{\mathrm{hor}})}{|W_{\mathrm{hor}}|^{\frac{r_1-1}{r_1}}}$$

(depending only on $\underline{S}_{\mathrm{hor}}$) such that for every finite $E \subset A \cong \mathbb{Z}^{r_1}$,

$$B_{\underline{S}_{\mathrm{hor}}}(E) \;\geq\; c_{\mathrm{ab}}(\underline{S}_{\mathrm{hor}})\,|E|^{\frac{r_1-1}{r_1}}.$$

Moreover, for $E_\rho := \rho W_{\mathrm{hor}} \cap \mathbb{Z}^{r_1}$ one has $|E_\rho| \asymp \rho^{r_1}$ and

$$\lim_{\rho\to\infty} \frac{B_{\underline{S}_{\mathrm{hor}}}(E_\rho)}{|E_\rho|^{\frac{r_1-1}{r_1}}} \;=\; c_{\mathrm{ab}}(\underline{S}_{\mathrm{hor}}).$$

Proof sketch. Since $\tau_{\mathrm{hor}} = h_{K_{\mathrm{hor}}}$, the polar gauge is the Minkowski functional of K_{hor}, hence $W_{\mathrm{hor}} = K_{\mathrm{hor}}$. If $\underline{S}_{\mathrm{hor}} = \{\pm v_1, \dots, \pm v_M\}$, then $B_{\underline{S}_{\mathrm{hor}}}(E) = \sum_{j=1}^{M} B_{\{\pm v_j\}}(E)$. A discrete-continuum flux comparison across unit faces yields

$$B_{\underline{S}_{\mathrm{hor}}}(E) \;=\; \int_{\partial^*(E+[0,1)^{r_1})} \tau_{\mathrm{hor}}(\nu)\, d\mathcal{H}^{r_1-1} \;+\; o\left(|E|^{\frac{r_1-1}{r_1}}\right),$$

see [Tay78], [Bra02, Ch. 7], [AC04, Thm. 3.3; Thm. 4.1], [BC07, Sec. 2.2, Sec. 3]. The continuum Wulff inequality gives the lower bound; sharpness follows by taking $E_\rho = \rho W_{\mathrm{hor}} \cap \mathbb{Z}^{r_1}$ and standard lattice-point asymptotics.

Remark 4.2 (Heisenberg horizontal constant (directed edges))**.** For $\underline{S}_{\mathrm{hor}} = \{\pm e_1, \pm e_2\} \subset \mathbb{Z}^2$,

$$\tau_{\mathrm{hor}}(\xi) = 2(|\xi_1| + |\xi_2|), \qquad \tau^{\circ}_{\mathrm{hor}}(\eta) = \tfrac{1}{2}\|\eta\|_\infty,$$

so $W = [-2,2]^2$, $\mathcal{H}^1(\partial W) = 16$, $\mathrm{Per}_{\tau_{\mathrm{hor}}}(W) = 2 \cdot 16 = 32$, $|W| = 16$, hence $c_{\mathrm{ab}} = 32/16^{1/2} = 8$. (For the *undirected* convention this value is halved.)

Definition 4.3 (Vertex and directed edge boundaries on Γ)**.** For $Y \subset \Gamma$ finite and symmetric $\mathcal{S}$,

$$\partial_{\mathcal{S}}(Y) := (Y\mathcal{S}) \setminus Y, \qquad B_{\mathcal{S}}(Y) := \sum_{s\in\mathcal{S}} \#\{\, y \in Y :\ ys \notin Y \,\}.$$

Thus $|\partial_{\mathcal{S}}(Y)| \leq B_{\mathcal{S}}(Y) \leq \Delta\, |\partial_{\mathcal{S}}(Y)|$ (cf. Remark 2.5).

Lemma 4.4 (Vertex-edge comparison)**.** For every finite $Y \subset \Gamma$ and symmetric $\mathcal{S}$,

$$|\partial_{\mathcal{S}}(Y)| \;\leq\; B_{\mathcal{S}}(Y) \;\leq\; \Delta\, |\partial_{\mathcal{S}}(Y)|.$$

Lemma 4.5 (Layer-cake identity for integer heights, arbitrary steps)**.** Let $\ell : \mathbb{Z}^d \to \{0, 1, \dots, m\}$ and set $E_t := \{\ell > t\} = \{u \in \mathbb{Z}^d :\ \ell(u) \geq \lfloor t \rfloor + 1\}$. Then for every $v \in \mathbb{Z}^d$,

$$\sum_{u\in\mathbb{Z}^d} \big|\ell(u+v) - \ell(u)\big| \;=\; \int_0^m B_{\{\pm v\}}(E_t)\, dt.$$

Proof. Write $|\ell(u+v) - \ell(u)| = \int_0^m |\mathbf{1}_{\{\ell(u+v)>t\}} - \mathbf{1}_{\{\ell(u)>t\}}|\, dt$ and sum over u.

Lemma 4.6 (Elementary symmetric-difference bound on $\mathbb{Z}$)**.** For finite $A, B \subset \mathbb{Z}$ and $t \in \mathbb{Z}$,

$$|A\Delta(B+t)| \;\geq\; \big||A| - |B|\big|.$$

Proof. $|A\Delta(B+t)| = |A| + |B| - 2|A \cap (B+t)| \geq |A| + |B| - 2\min(|A|, |B|) = \big||A| - |B|\big|$.

Lemma 4.7 (Projection-sum domination)**.** Let $\phi : T \to U$ be a surjection between finite sets and let $a_{u,t} \geq 0$ for $t \in T$, $u = \phi(t)$. Then

$$\sum_{t\in T} a_{\phi(t),t} \;\geq\; \sum_{u\in U} \inf_{t\in\phi^{-1}(u)} a_{u,t} \;\geq\; \sum_{u\in U} b_u$$

for any choice of nonnegative $b_u \leq a_{u,t}$ for all $t \in \phi^{-1}(u)$.

4.1. **Track C: Central $\mathbb{Z}$-extension.**

Standing Assumptions 4.1. Assume a central extension

$$1 \longrightarrow \langle c\rangle \cong \mathbb{Z} \longrightarrow \Gamma \xrightarrow{\pi_{\mathrm{ab}}} A \longrightarrow 1,$$

and fix a set-theoretic section $\sigma : A \to \Gamma$. Thus every $g \in \Gamma$ can be written uniquely as $g = \sigma(u)\,c^h$ with $(u, h) \in A \times \mathbb{Z}$. All dependence on the choice of σ appears via explicit offsets; the final inequalities are section-independent. *This track covers key examples such as the integer Heisenberg group $\mathbb{H}(\mathbb{Z})$ and, more generally, step-two nilpotent lattices with center isomorphic to $\mathbb{Z}$.*

Definition 4.8 (Column symmetrization on $A \times \mathbb{Z}$)**.** Identify Γ setwise with $A \times \mathbb{Z}$ via σ. For $Y \subset A \times \mathbb{Z}$, write the column $I_u := \{h \in \mathbb{Z} : (u, h) \in Y\}$ and its height $\ell(u) := |I_u|$. Define $Y^\flat := \{(u, h) : 1 \leq h \leq \ell(u)\}$.

Horizontal subset of generators. Recall

$$S^{\Gamma}_{\mathrm{hor}} = \{\, s \in \mathcal{S} :\ \pi_{\mathrm{ab}}(s) \neq e \,\}, \qquad \underline{S}_{\mathrm{hor}} \subset A \ \text{ the set of distinct base directions.}$$

Several distinct $s \in S^{\Gamma}_{\mathrm{hor}}$ may share the same $v = \pi_{\mathrm{ab}}(s) \in \underline{S}_{\mathrm{hor}}$; we only use this through nonnegativity of summands and Lemma 4.7.

Lemma 4.9 (Horizontal edge count with explicit offsets)**.** Let $\omega : A \times A \to \mathbb{Z}$ be the cocycle determined by σ via

$$\sigma(u)\,\sigma(v) \;=\; \sigma(u + v)\,c^{\omega(u,v)} \qquad (u, v \in A).$$

For each $s \in S^{\Gamma}_{\mathrm{hor}}$ write $v := \pi_{\mathrm{ab}}(s) \in \underline{S}_{\mathrm{hor}}$ and $s = \sigma(v)\,c^{\kappa(s)}$ with $\kappa(s) \in \mathbb{Z}$. Define

$$\theta_s(u) \;:=\; \kappa(s) + \omega(u, v) \qquad (u \in A).$$

Then, for every finite $Y \subset \Gamma$,

$$\sum_{g\in\Gamma} \big|\mathbf{1}_Y(gs) - \mathbf{1}_Y(g)\big| \;=\; \sum_{u\in A} \big|\, I_u \,\Delta\, (I_{u+v} + \theta_s(u))\,\big|,$$

where $I_{u+v} + \theta_s(u) := \{h + \theta_s(u) : h \in I_{u+v}\}$.

Proposition 4.10 (Column symmetrization with offsets does not increase the horizontal boundary)**.** Let $Y \subset \Gamma$ be finite and $Y^\flat := \{(u, h) : 1 \leq h \leq \ell(u)\}$ its column symmetrization. Then

$$\begin{aligned}
B_{S^{\Gamma}_{\mathrm{hor}}}(Y) \;&=\; \sum_{s\in S^{\Gamma}_{\mathrm{hor}}} \sum_{u\in A} \big|I_u \Delta (I_{u+\pi_{\mathrm{ab}}(s)} + \theta_s(u))\big| \\
&\geq\; \sum_{s\in S^{\Gamma}_{\mathrm{hor}}} \sum_{u\in A} \big||I_u| - |I_{u+\pi_{\mathrm{ab}}(s)}|\big| \qquad \text{(by Lemma 4.6)} \\
&\geq\; \sum_{v\in \underline{S}_{\mathrm{hor}}} \sum_{u\in A} \big||I_u| - |I_{u+v}|\big| \qquad \text{(by Lemma 4.7)} \\
&=\; \sum_{v\in \underline{S}_{\mathrm{hor}}} \sum_{u\in A} \big|\, I_u \Delta I_{u+v}\,\big| \;=\; B_{\underline{S}_{\mathrm{hor}}}(Y^\flat),
\end{aligned}$$

which is the definition of $B_{\underline{S}_{\mathrm{hor}}}(Y^\flat)$, since in $Y^\flat$ each column is the contiguous interval $[1, \ell(u)]$.

Corollary 4.11 (Direct-product case)**.** If $\Gamma = A \times \mathbb{Z}$ (so $\omega \equiv 0$ and $\theta_s \equiv \kappa(s)$), then for all finite $Y \subset \Gamma$,

$$B_{\underline{S}_{\mathrm{hor}}}(Y^\flat) \;\le\; B_{S^\Gamma_{\mathrm{hor}}}(Y).$$

Proposition 4.12 (Lower bound by slicing and symmetrization)**.** For every finite $Y \subset \Gamma$,

$$|\partial_{\mathcal{S}} Y| \;\ge\; \frac{1}{\Delta}\, B_{\mathcal{S}}(Y) \;\ge\; \frac{1}{\Delta}\, B_{S^\Gamma_{\mathrm{hor}}}(Y) \;\ge\; \frac{1}{\Delta}\, B_{\underline{S}_{\mathrm{hor}}}(Y^\flat).$$

Writing the column heights as $\ell : A \to \mathbb{N} \cup \{0\}$, one has

$$B_{\underline{S}_{\mathrm{hor}}}(Y^\flat) \;=\; \sum_{v \in \underline{S}_{\mathrm{hor}}} \sum_{u \in A} \big|\Delta_v \ell(u)\big| \;=\; \int_0^\infty B_{\underline{S}_{\mathrm{hor}}}\big(\{\ell > t\}\big)\, dt,$$

where the integrand vanishes for $t > \max \ell$. Therefore, by Theorem 4.1,

$$|\partial_{\mathcal{S}} Y| \;\ge\; \frac{c_{\mathrm{ab}}(\underline{S}_{\mathrm{hor}})}{\Delta} \int_0^\infty \big|\{\ell > t\}\big|^{\frac{r_1 - 1}{r_1}}\, dt.$$

4.2. **Track F: Finite-kernel projection.**

Standing Assumptions 4.2. Assume $\pi : \Gamma \to \mathbb{Z}^d$ is a surjective homomorphism with finite kernel K ($|K| = m$). Let $V \subset \mathbb{Z}^d$ be a finite symmetric generating set contained in $\pi(\mathcal{S})$ and choose $S_0 \subset \mathcal{S}$ mapping bijectively to V.

Lemma 4.13 (Fiber-permutation lower bound (finite kernel))**.** For $Y \subset \Gamma$ finite and $u \in \mathbb{Z}^d$, set $\ell(u) := |Y \cap \pi^{-1}(u)| \in \{0, 1, \dots, m\}$. Then for every $s \in S_0$ with $\pi(s) = v$ and every u,

$$\sum_{g \in \pi^{-1}(u)} \big|\mathbf{1}_Y(gs) - \mathbf{1}_Y(g)\big| \;\ge\; \big|\ell(u+v) - \ell(u)\big|,$$

hence, summing in u and $s \in S_0$,

$$\sum_{s \in S_0} \sum_{g \in \Gamma} \big|\mathbf{1}_Y(gs) - \mathbf{1}_Y(g)\big| \;\ge\; \sum_{v \in V} \sum_{u \in \mathbb{Z}^d} \big|\Delta_v \ell(u)\big|.$$

Proposition 4.14 (Finite-kernel slicing bound)**.** With the standing assumptions of Subsection 4.2, for every finite $Y \subset \Gamma$,

$$|\partial_{\mathcal{S}} Y| \;\ge\; \frac{1}{\Delta}\, B_{\mathcal{S}}(Y) \;\ge\; \frac{1}{\Delta} \sum_{s \in S_0} \sum_{g \in \Gamma} \big|\mathbf{1}_Y(gs) - \mathbf{1}_Y(g)\big|.$$

Therefore, using Lemma 4.13 and Lemma 4.5,

$$|\partial_{\mathcal{S}} Y| \;\ge\; \frac{1}{\Delta} \int_0^m B_V\big(\{\ell > t\}\big)\, dt \;\ge\; \frac{c_{\mathrm{ab}}(V)}{\Delta} \int_0^m \big|\{\ell > t\}\big|^{\frac{d-1}{d}}\, dt,$$

where $c_{\mathrm{ab}}(V)$ is the Wulff constant associated with the anisotropy $\tau_V(\xi) = \sum_{v \in V} |\langle \xi, v \rangle|$ on $\mathbb{R}^d$.

Remark. The two tracks are logically disjoint: results in Subsection 4.1 assume a central $\mathbb{Z}$-extension and do not use finite-kernel lemmas; results in Subsection 4.2 assume a finite kernel and do not use column symmetrization with offsets. In applications one typically takes $d = r_1$ and $\pi = \pi_{\mathrm{ab}}$ (finite kernel = tors).

Discretization on a grid. We work in the Abelian model $G = \mathbb{R}^d$ with lattice $\Gamma = \mathbb{Z}^d$. Fix a mesh size $h > 0$ and write $\mathbb{Z}_h^d := h\,\mathbb{Z}^d \subset \mathbb{R}^d$. Our goal is to approximate the anisotropic total variation

$$TV_\varphi(\chi_E) \;:=\; \int_{\partial^* E} \varphi(\nu_E)\, d\mathcal{H}^{d-1}$$

by a graph-cut on $\mathbb{Z}_h^d$ consistent with the anisotropy φ. Throughout, $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^d$. We use $TV_\varphi(\chi_E)$ and $\mathrm{Per}_\varphi(E)$ interchangeably in $\mathbb{R}^d$, since $TV_\varphi(\chi_E) = \mathrm{Per}_\varphi(E)$ for sets of finite perimeter; see [AFP00, Thm. 3.59].

Remark 5.1 (Normalization for this section)**.** We work on the Euclidean grid with *undirected* edges. Energies are counted once per grid face and scaled by h^{d-1}. If E_h^{und} denotes the undirected-edge energy and E_h^{dir} the directed-edge version (counting both orientations), then

$$E_h^{\mathrm{und}} = \tfrac{1}{2}\, E_h^{\mathrm{dir}}, \qquad TV_{\varphi^{\mathrm{und}}} = TV_{\frac{1}{2}\varphi^{\mathrm{dir}}}.$$

This reflects that each undirected face corresponds to two opposite directed edges; the linear rescaling leaves Γ-limits unchanged.

Setup: edges, weights, and jumps (orthotropic case). Let $e_1, \dots, e_d$ be the canonical basis of $\mathbb{R}^d$ and define the (undirected) nearest-neighbour edge set

$$\mathcal{E}_h \;:=\; \Big\{\, \{x, x + he_i\} \;:\; x \in \mathbb{Z}_h^d,\ i = 1, \dots, d \,\Big\}.$$

We first consider the *orthotropic* case

$$\varphi(\xi) \;=\; \sum_{i=1}^{d} \omega_i\, |\xi_i|, \qquad \omega_i > 0,$$

and assign to each edge $e = \{x, x + he_i\} \in \mathcal{E}_h$ the weight $\omega_e := \omega_i = \varphi(e_i)$. Given $A_h \subset \mathbb{Z}_h^d$ with finite edge boundary, write χ_{A_h} for its indicator and define the edgewise jump (across $e = \{x, x + he_i\}$) by

$$\big|\nabla \chi_{A_h}\big|(e) \;:=\; \big|\chi_{A_h}(x + he_i) - \chi_{A_h}(x)\big| \;\in \{0, 1\}.$$

Discrete energy and its scaling. Each $(d-1)$-face of the grid has area h^{d-1}, so the correctly scaled discrete anisotropic TV is

$$E_h(A_h) \;:=\; h^{d-1} \sum_{e \in \mathcal{E}_h} \omega_e\, \big|\nabla \chi_{A_h}\big|(e). \tag{5.1}$$

(Using *undirected* edges means each face is counted once; with directed edges one would insert a factor $1/2$.)

Cells and grid cube interpolation. For $x \in \mathbb{Z}_h^d$ let $Q_h(x) := x + [0, h)^d$ be the half-open grid cube. Given $A_h \subset \mathbb{Z}_h^d$, define the voxel interpolation $u_h^{\#} : \mathbb{R}^d \to \{0, 1\}$ by $u_h^{\#}(y) := \chi_{A_h}(x)$ for $y \in Q_h(x)$. In the orthotropic case one has the *exact face identity*

$$E_h(A_h) \;=\; \sum_{i=1}^{d} \omega_i\, |D_i u_h^{\#}|(\mathbb{R}^d) \;=\; TV_\varphi(u_h^{\#}),$$

since $|D_i u_h^{\#}|$ is the $(d-1)$-dimensional measure of the discontinuity faces orthogonal to e_i, each with area h^{d-1} and density equal to the jump.

Sampling conventions. For a continuum set $E \subset \mathbb{R}^d$ of finite perimeter we use the standard half-open samplers

$$A_h^{\mathrm{in}}(E) := \{x \in \mathbb{Z}_h^d : Q_h(x) \subset E\}, \qquad A_h^{\mathrm{out}}(E) := \{x \in \mathbb{Z}_h^d : Q_h(x) \cap E \neq \emptyset\}.$$

Then $\chi_{(A_h^{\mathrm{in}})^\#} \to \chi_E$ and $\chi_{(A_h^{\mathrm{out}})^\#} \to \chi_E$ in $L^1_{\mathrm{loc}}(\mathbb{R}^d)$, and the two samplers differ only in a boundary layer of volume $O(h)\,\mathcal{H}^{d-1}(\partial^* E)$, by the density/Minkowski content characterization of finite perimeter sets (see e.g.[AFP00, Ch. 3]). [2] When we write $A_h = E \cap \mathbb{Z}_h^d$ below, we tacitly mean one of these half-open conventions.

Remark 5.2 (Boundedness)**.** Throughout the quantitative statements below we assume ∂E is compact (e.g. $\partial E \in C^{1,\beta}$), so that $\mathcal{H}^{d-1}(\partial E) < \infty$. This ensures the discrete sums are finite for the half-open samplers considered. (For unbounded sets one may work in bounded windows and pass to a thermodynamic limit; we do not pursue this here.)

Remark 5.3 (Narrative and units)**.** The factor h^{d-1} in (5.1) is essential: it matches the $(d-1)$-dimensional face area and ensures that E_h approximates TV_φ as $h \downarrow 0$. This mesh-refinement scaling is independent of the large-scale sampling considered earlier on a fixed lattice (where one normalizes by Carnot dilations r^{Q-1} instead).

Lemma 5.4 (Exact face identity)**.** Let $\varphi(\xi) = \sum_{i=1}^d \omega_i |\xi_i|$ with $\omega_i > 0$, and let $u_h^\#$ be the cellwise constant interpolation of $A_h \subset \mathbb{Z}_h^d$ on the half-open grid cubes $Q_h(x) := x + [0,h)^d$. Then

$$E_h(A_h) \;=\; \sum_{i=1}^d \omega_i \,|D_i u_h^\#|(\mathbb{R}^d) \;=\; TV_\varphi(u_h^\#).$$

Proof. Fix $i \in \{1, \dots, d\}$. The jump set of $u_h^\#$ is the union of those $(d-1)$-dimensional faces shared by adjacent cubes $Q_h(x)$ and $Q_h(x + he_i)$ with $\chi_{A_h}(x) \neq \chi_{A_h}(x + he_i)$. For each such pair there is exactly one face

$$F_{i,x} \;:=\; \{\, y \in Q_h(x) \cup Q_h(x+he_i) : \; y_i = x_i + h \,\},$$

orthogonal to e_i, of $(d-1)$-measure $\mathcal{H}^{d-1}(F_{i,x}) = h^{d-1}$. Along $F_{i,x}$ the approximate traces of $u_h^\#$ on the two sides differ by $|u_h^{\#,+} - u_h^{\#,-}| = |\chi_{A_h}(x+he_i) - \chi_{A_h}(x)| \in \{0,1\}$, and the unit normal is $\pm e_i$, so $|n_i| = 1$. Standard BV structure for piecewise constant functions (e.g.[AFP00, Thm. 3.78]) yields

$$|D_i u_h^\#|(\mathbb{R}^d) \;=\; \sum_{x \in \mathbb{Z}_h^d} \big|\chi_{A_h}(x+he_i) - \chi_{A_h}(x)\big|\,\mathcal{H}^{d-1}(F_{i,x}) \;=\; h^{d-1} \sum_{x \in \mathbb{Z}_h^d} \big|\chi_{A_h}(x+he_i) - \chi_{A_h}(x)\big|.$$

Summing in i with weights ω_i gives the claim. Faces are counted once by the convention $y_i = x_i + h$. ☐

Remark 5.5 (Scope)**.** The identity in Lemma 5.4 is *specific* to orthotropic integrands $\varphi(\xi) = \sum_i \omega_i |\xi_i|$. For general crystalline $\phi_{\mathcal{P}}$ one must use directional differences along p; see Definition 5.11.

Lemma 5.6 (Orthotropic grid quadrature along $C^{1,\beta}$ graphs)**.** Let $\Sigma \subset \mathbb{R}^d$ be a compact $C^{1,\beta}$ hypersurface and $\psi(\nu) = \sum_{i=1}^d \omega_i |\nu_i|$ with fixed $\omega_i > 0$. Let $\mathcal{F}_h$ be the collection of axis faces of the grid $\mathbb{Z}_h^d$ whose *relative interior* intersects Σ, and for $F \in \mathcal{F}_h$ let $\nu_F \in \{\pm e_i\}$ denote the axis normal of F. Then

$$\Big| \sum_{F \in \mathcal{F}_h} \psi(\nu_F)\,\mathcal{H}^{d-1}(F) \;-\; \int_\Sigma \psi(\nu)\, d\mathcal{H}^{d-1} \Big| \;\le\; C\,h^\beta,$$

with C depending on $d, \beta, \|\Sigma\|_{C^{1,\beta}}, \sum_i \omega_i$ and $\mathcal{H}^{d-1}(\Sigma)$.

[2] In particular, $\mathcal{L}^d\big((\partial^* E)^{(h)}\big) = 2h\,\mathcal{H}^{d-1}(\partial^* E) + o(h)$ as $h \downarrow 0$, where $(\cdot)^{(h)}$ denotes the h-neighborhood.

Proof (sketch). Fix i. By coarea, $\int_\Sigma |\nu_i|\, d\mathcal{H}^{d-1} = \int_{\mathbb{R}} \mathcal{H}^{d-2}(\Sigma \cap \{x_i = t\})\, dt$. Partition $\mathbb{R}$ into intervals $[kh, (k{+}1)h)$ and approximate each slice by the count of grid faces in $\{x_i = kh\}$ whose relative interior meets Σ, times the face area h^{d-1}. Since Σ is $C^{1,\beta}$, the slice measure varies $C^{0,\beta}$ in t; hence the Riemann-sum error over each slab is $O(h^\beta)$. Summing in k and then in i with weights ω_i yields the claim. □

Theorem 5.7 (Quantitative Γ-convergence (orthotropic case)). Assume $E \subset \mathbb{R}^d$ has $C^{1,1}$ boundary and $\varphi(\xi) = \sum_{i=1}^d \omega_i |\xi_i|$ with $\omega_i > 0$. Let $A_h \in \{A_h^{\mathrm{in}}(E), A_h^{\mathrm{out}}(E)\}$ be a half-open sampler and $u_h^\#$ its cellwise-constant interpolation. Then there exists

$$C = C\left(d,\ \omega,\ \|\partial E\|_{C^{1,1}},\ \mathcal{H}^{d-1}(\partial E)\right) > 0$$

such that, uniformly in the choice of sampler,

$$\left|E_h(A_h) - TV_\varphi(\chi_E)\right| \ \le\ C\, h. \tag{5.2}$$

Moreover, for any sequence $A_h \subset \mathbb{Z}_h^d$ with $u_h^\# \to \chi_E$ in $L^1_{\mathrm{loc}}(\mathbb{R}^d)$ and $\sup_h E_h(A_h) < \infty$, the classical Γ-liminf inequality holds:

$$\liminf_{h \downarrow 0} E_h(A_h) \ \ge\ TV_\varphi(\chi_E). \tag{5.3}$$

Proof. By Lemma 5.4, $E_h(A_h) = TV_\varphi(u_h^\#) = \sum_{i=1}^d \omega_i\, |D_i u_h^\#|(\mathbb{R}^d)$. For the liminf, use lower semicontinuity of anisotropic total variation under L^1_{loc} convergence (Reshetnyak's theorem; see [AFP00, Thm. 2.38]).

For the rate: cover ∂E by finitely many $C^{1,1}$ graph charts with uniformly bounded $C^{1,1}$ norm. For fixed i, $|D_i v|$ is computed by Fubini along lines parallel to e_i; replacing the fiberwise integral by a Riemann sum with spacing h incurs an $O(h)$ error per chart when $v = \chi_E$ and $v = u_h^\#$. Summing in i with weights ω_i and over charts yields (5.2), with C depending on $(d, \omega, \|\partial E\|_{C^{1,1}}, \mathcal{H}^{d-1}(\partial E))$. The constant is uniform over the two samplers $A_h^{\mathrm{in/out}}$. Finally, A_h^{in} and A_h^{out} differ only on a boundary layer of volume $O(h)\mathcal{H}^{d-1}(\partial E)$, so the same $O(h)$ bound holds uniformly across the two samplers. □

Remark 5.8 (Topology and domain for the Γ-limit). All Γ-convergence statements in this section are with respect to $L^1_{\mathrm{loc}}(\mathbb{R}^d)$ on $\{0,1\}$-valued functions, and we evaluate $E_h(\cdot)$ and $E_h^{\mathcal{P}}(\cdot)$ on half-open samplers of continuum sets of finite perimeter. In particular, whenever $u_h^\# \to \chi_E$ in L^1_{loc} and $\sup_h E_h(u_h^\#) < \infty$ (or $\sup_h E_h^{\mathcal{P}}(u_h^\#) < \infty$), the set E has finite anisotropic perimeter and the lower bounds apply. On $\{0,1\}$-valued maps, L^1_{loc} convergence is equivalent to local convergence in measure of the underlying sets.

Remark 5.9 (Regularity ⇒ rate). If $\partial E \in C^{1,\beta}$ for some $\beta \in (0,1]$, then along each chart the unit normal is $C^{0,\beta}$ and the graph deviates from its tangent plane by $O(h^{1+\beta})$ on h-patches. The fiberwise quadrature error is thus $O(h^\beta)$, yielding $|E_h(A_h) - TV_\varphi(\chi_E)| \le Ch^\beta$ with C depending on the $C^{1,\beta}$ seminorm of ∂E, uniformly in the sampler $A_h^{\mathrm{in/out}}$.

Lemma 5.10 (Chartwise quadrature error). Let ∂E be covered by finitely many $C^{1,\beta}$ graphs $\{x_d = \psi_\alpha(x')\}$ with $\|\psi_\alpha\|_{C^{1,\beta}} \le M$ in balls of radius r_0. Let $u_h^\#$ be the cellwise-constant interpolation of a half-open sampler $A_h \in \{A_h^{\mathrm{in}}(E), A_h^{\mathrm{out}}(E)\}$. Then for $h \in (0, h_0(M, r_0))$ and for any orthotropic weights $\{\omega_i\}$,

$$\left| TV_\varphi(u_h^\#) - TV_\varphi(\chi_E) \right| \ \le\ C(d, \omega, M, \mathcal{H}^{d-1}(\partial E))\, h^\beta.$$

Proof. On each chart, write the anisotropic surface element as $\sum_i \omega_i |n_i|\, d\mathcal{H}^{d-1}$ with n the Euclidean unit normal. Along fibers parallel to e_i, $|n_i|$ is $C^{0,\beta}$ with seminorm $\lesssim M$, and the interface position varies Lipschitzly. Riemann sum error on spacing h at $C^{0,\beta}$ regularity is $O(h^\beta)$ on each fiber family; integrate over the orthogonal base domain. Summing charts yields the claim. □

5.1. **Finite-stencil crystalline energies: discrete model and quantitative approximation.** **Directional stencil and anisotropy.** Let $\mathcal{P} \subset \mathbb{Z}^d \setminus \{0\}$ be finite and $\alpha_p > 0$. Define the (even, convex, 1-homogeneous) crystalline integrand

$$\phi_{\mathcal{P}}(\nu) := \sum_{p \in \mathcal{P}} \alpha_p \, |\nu \cdot p| \qquad (\nu \in \mathbb{R}^d).$$

Definition 5.11 (Finite-stencil energy via directional differences; antipodal pairing)**.** Let $\mathcal{P} \subset \mathbb{Z}^d \setminus \{0\}$ be finite, and let $\alpha_p > 0$ for $p \in \mathcal{P}$. To avoid double counting when both p and $-p$ occur, fix a set of representatives

$$\mathcal{P}_+ \ \subset \ \mathcal{P} \cup (-\mathcal{P})$$

containing exactly one element from each antipodal pair $\{\pm p\}$ that meets $\mathcal{P} \cup (-\mathcal{P})$, and set

$$\tilde{\alpha}_p \ := \ \alpha_p + \alpha_{-p} \qquad (\text{with } \alpha_{-p} := 0 \text{ if } -p \notin \mathcal{P}).$$

Define the crystalline integrand and the discrete energies by

$$\phi_{\mathcal{P}}(\nu) \ := \sum_{p \in \mathcal{P}_+} \tilde{\alpha}_p \, |\nu \cdot p|, \qquad E_h^{\mathcal{P}}(A_h) \ := \ h^{d-1} \sum_{p \in \mathcal{P}_+} \tilde{\alpha}_p \sum_{\substack{x \in \mathbb{Z}_h^d \\ x,\, x+hp \in \mathbb{Z}_h^d}} \big| \chi_{A_h}(x + hp) - \chi_{A_h}(x) \big|.$$

With this normalization each undirected p-bond is counted once, and $\phi_{\mathcal{P}}$ is independent of the choice of representatives in $\mathcal{P}_+$.

Remark 5.12 (Consistency with the axis model)**.** For the axis case pick $\mathcal{P}_+ = \{e_1, \dots, e_d\}$ and $\tilde{\alpha}_{e_i} = \omega_i$, so that $E_h^{\mathcal{P}}$ reduces to E_h from Section 5 and $\phi_{\mathcal{P}}(\nu) = \sum_{i=1}^d \omega_i |\nu_i|$.

Equivalence of definitions. Note that

$$\sum_{p \in \mathcal{P}_+} \tilde{\alpha}_p \, |\nu \cdot p| \ = \ \sum_{p \in \mathcal{P}} \alpha_p \, |\nu \cdot p|,$$

so the antipodal pairing merely repackages the even integrand without changing its value. In particular, if both p and $-p$ lie in $\mathcal{P}$ with weights α_p, α_{-p}, pairing replaces the two directed contributions by a single undirected one with coefficient $\tilde{\alpha}_p = \alpha_p + \alpha_{-p}$; no limit constant is altered by this bookkeeping.

Lemma 5.13 (Support and cardinality of directional pairs)**.** Let $E \subset \mathbb{R}^d$ be bounded with finite perimeter and A_h a half-open sampler. For fixed $p \in \mathbb{Z}^d \setminus \{0\}$, the set

$$\big\{ x \in \mathbb{Z}_h^d : \ \chi_{A_h}(x + hp) \neq \chi_{A_h}(x) \big\}$$

is contained in an $O(h|p|)$-tubular neighborhood of ∂E; consequently the number of nonzero terms is

$$\#\{\cdots\} \ = \ \frac{|p|}{h^{d-1}} \, \mathcal{H}^{d-1}\big(\pi_{p^\perp}(\partial E)\big) \ + \ O\big(h^{2-d}\big),$$

where $\pi_{p^\perp}$ is the orthogonal projection onto $p^\perp$. In particular, $h^{d-1} \sum_x |\chi_{A_h}(x + hp) - \chi_{A_h}(x)| = O(|p|)$.

Lemma 5.14 (Lattice line density in direction p)**.** Fix $p \in \mathbb{Z}^d \setminus \{0\}$ and $h > 0$. Consider the family of lines $\ell_\xi := \xi + \mathbb{R}\hat{p}$ ($\hat{p} := p/|p|$), indexed by $\xi \in p^\perp$. Among these, the subfamily that meet the lattice $h\mathbb{Z}^d$ has base density $|p|/h^{d-1}$ in $p^\perp$: for every bounded Borel $B \subset p^\perp$,

$$\#\{ \ \ell_\xi \text{ with } \xi \in B \text{ and } \ell_\xi \cap h\mathbb{Z}^d \neq \emptyset \ \} \ = \ \frac{|p|}{h^{d-1}} \, \mathcal{H}^{d-1}(B) \ + \ o(h^{-(d-1)}).$$

Equivalently, the projection of $h\mathbb{Z}^d$ onto $p^\perp$ is a full-rank lattice of covolume $h^{d-1}/|p|$.

Lemma 5.15 (Directional difference-quotient for finite-perimeter sets)**.** Let $E \subset \mathbb{R}^d$ have finite perimeter and $p \in \mathbb{Z}^d \setminus \{0\}$. For the half-open samplers $A_h^{\rm in}(E)$ or $A_h^{\rm out}(E)$,

$$h^{d-1} \sum_{x \in \mathbb{Z}_h^d} \left|\chi_{A_h}(x+hp) - \chi_{A_h}(x)\right| \;\longrightarrow\; \int_{\partial^* E} |\nu_E \cdot p| \, d\mathcal{H}^{d-1} \qquad (h \downarrow 0).$$

If $\partial E \in C^{1,\beta}$ for some $\beta \in (0,1]$, then the error is bounded by $C\,|p|\,h^\beta$, where C depends only on d, $\|\partial E\|_{C^{1,\beta}}$ and $\mathcal{H}^{d-1}(\partial E)$ (hence the bound is uniform over p in any fixed finite stencil).

Proof. Let $\hat p := p/|p|$ and parametrize by lines $\ell_\xi = \xi + \mathbb{R}\hat p$, $\xi \in p^\perp$. For a.e. ξ, the slice of $\partial^* E$ on ℓ_ξ is finite and

$$\int_{p^\perp} \#\big(\partial^* E \cap \ell_\xi\big)\, d\mathcal{H}^{d-1}(\xi) \;=\; \int_{\partial^* E} |\nu \cdot \hat p| \, d\mathcal{H}^{d-1}.$$

By Lemma 5.14, the number of p-parallel lines through $h\mathbb{Z}^d$ per unit $(d-1)$-area in $p^\perp$ is $|p|/h^{d-1}$. Along each such line, for h small and away from grazing contacts, a single transversal jump contributes exactly one nonzero term to $\sum_{x \in \mathbb{Z}_h^d} |\chi_{A_h}(x+hp) - \chi_{A_h}(x)|$. Multiplying by h^{d-1} and integrating over ξ gives the limit

$$|p| \int_{p^\perp} \#(\partial^* E \cap \ell_\xi)\, d\xi \;=\; \int_{\partial^* E} |\nu \cdot p| \, d\mathcal{H}^{d-1}.$$

If $\partial E \in C^{1,\beta}$, local graphs yield an $O(h^\beta)$ error on each chart and the line density contributes the prefactor $|p|$, giving the claimed bound. □

Theorem 5.16 (Quantitative approximation for finite stencils)**.** Let $\mathcal{P} \subset \mathbb{Z}^d \setminus \{0\}$ be finite with weights $\alpha_p > 0$ and $\phi_{\mathcal{P}}(\nu) = \sum_p \alpha_p |\nu \cdot p|$. If $E \subset \mathbb{R}^d$ has $C^{1,\beta}$ boundary and $A_h \in \{A_h^{\rm in}(E), A_h^{\rm out}(E)\}$, then for all sufficiently small h,

$$\big|\, E_h^{\mathcal{P}}(A_h) - TV_{\phi_{\mathcal{P}}}(\chi_E) \,\big| \;\le\; C_{\mathcal{P},E}\, h^\beta,$$

with $C_{\mathcal{P},E}$ depending only on d, $\{\tilde\alpha_p\}_{p \in \mathcal{P}_+}$, $\max_{p \in \mathcal{P}_+} |p|$, $\|\partial E\|_{C^{1,\beta}}$, and $\mathcal{H}^{d-1}(\partial E)$ (uniformly in the sampler). Moreover, for any sequence $A_h \subset \mathbb{Z}_h^d$ with $u_h^\# \to \chi_E$ in $L^1_{\rm loc}$ and $\sup_h E_h^{\mathcal{P}}(A_h) < \infty$,

$$\liminf_{h \downarrow 0} E_h^{\mathcal{P}}(A_h) \;\ge\; TV_{\phi_{\mathcal{P}}}(\chi_E).$$

Proof. The quantitative estimate follows from Lemma 5.15 applied to each $p \in \mathcal{P}_+$ and linearity in $\{\tilde\alpha_p\}$.

For the Γ-liminf: fix $p \in \mathcal{P}_+$. Slice along lines $\ell_\xi = \xi + \mathbb{R}\hat p$, $\xi \in p^\perp$. For a.e. ξ, the 1D total variation on ℓ_ξ is lower semicontinuous under L^1 convergence of slices, hence

$$\liminf_{h \downarrow 0} h^{d-1} \sum_{x \in \mathbb{Z}_h^d} \left|\chi_{A_h}(x+hp) - \chi_{A_h}(x)\right| \;\ge\; \int_{p^\perp} \#\big(\partial^* E \cap \ell_\xi\big)\, d\mathcal{H}^{d-1}(\xi) \;=\; \int_{\partial^* E} |\nu \cdot \hat p| \, d\mathcal{H}^{d-1}.$$

Multiply by $|p|$ via Lemma 5.14 (i.e. count only those lines meeting $h\mathbb{Z}^d$) to obtain $\int_{\partial^* E} |\nu \cdot p| \, d\mathcal{H}^{d-1}$. Summing over $p \in \mathcal{P}_+$ with weights $\tilde\alpha_p$ gives the claim. □

Remark 5.17 (Why the axis implementation must be changed for general $\mathcal{P}$)**.** If one implements $E_h^{\mathcal{P}}$ on axis faces with weights $\sum_p \alpha_p |p_i|$, the limit is the orthotropic anisotropy

$$\nu \;\longmapsto\; \sum_{i=1}^d \Big(\sum_{p \in \mathcal{P}} \alpha_p |p_i| \Big)\, |\nu_i|,$$

which in general *strictly dominates* $\phi_{\mathcal{P}}(\nu) = \sum_{p \in \mathcal{P}} \alpha_p |\nu \cdot p|$ by the triangle inequality $|\nu \cdot p| \le \sum_i |p_i|\,|\nu_i|$. The directional-difference model in Definition 5.11 is the one that converges to $TV_{\phi_{\mathcal{P}}}$.

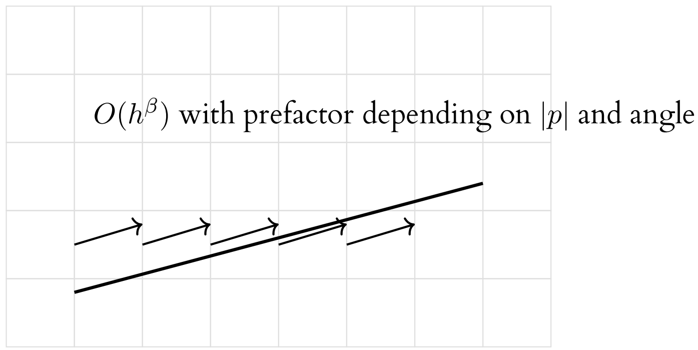


Figure 2. Directional differences along a finite stencil approximate the anisotropic facet with error $O(h^\beta)$ (Theorem 5.16); the constant depends on $|p|$ and the local angle between ν and p.

6. L^1 curl–fitting: abstract constant, explicit grid bound

Goal and link to earlier sections. We prove an L^1 curl$\to$potential estimate: if a discrete vector field has small curl (in ℓ^1 on faces), then it is close in ℓ^1 (on edges) to a gradient. This complements the Wulff/TV results of Sections 3–5 by providing an explicit, computable constant for L^1 curl-fitting on grids, with a weighted variant compatible with anisotropies used elsewhere.

Setting and notation. Let $K = (V, E, \mathcal{F})$ be a finite, connected, oriented 2–dimensional cell complex. We write $C^k(K)$ for real k–cochains on K with coboundary maps

$$d^k : C^k(K) \to C^{k+1}(K), \qquad d := d^k \text{ when } k \text{ is clear.}$$

Thus d^0 sends $u : V \to \mathbb{R}$ to $(d^0u)(x \to y) := u(y) - u(x)$ on oriented edges, and d^1 sends $F \in C^1(K)$ to the oriented face–circulation $(d^1F)(f) := \sum_{e\in\partial f} \operatorname{sgn}(f, e)\, F(e)$. Equip $C^1(K)$ and $C^2(K)$ with the ℓ^1 norms

$$\|R\|_{\ell^1(E)} := \sum_{e\in E} |R(e)|, \qquad \|b\|_{\ell^1(\mathcal{F})} := \sum_{f\in\mathcal{F}} |b(f)|.$$

Definition 6.1 (Curl and its ℓ^1 total variation)**.** For a 1–cochain $F \in C^1(K)$, its (discrete) curl is $dF \in C^2(K)$. We set $\|\operatorname{curl} F\|_{\mathrm{TV}} := \|dF\|_{\ell^1(\mathcal{F})}$.

Remark 6.2 (Choice of 2–cells)**.** If K is a planar embedded graph it is natural to take $\mathcal{F}$ to be the oriented faces. More generally, any oriented 2–cell structure whose boundaries *span the cycle space of the* 1*–skeleton* yields $H^1(K;\mathbb{R}) = 0$ (equivalently, $\ker d^1 = \operatorname{im} d^0$ over $\mathbb{R}$). *All quantitative constants below (e.g. the ℓ^1 filling constant) depend on the chosen* 2*–cell structure and on the ℓ^1 norms on* $C^1(K), C^2(K)$; the qualitative input is the cohomological vanishing $H^1(K;\mathbb{R}) = 0$.

6.1. Abstract result: the ℓ^1 filling constant.

Definition 6.3 (ℓ^1 filling constant)**.** Define

$$\mathbf{Fill}_1(K) \;:=\; \sup_{b\in\operatorname{im}(d^1)} \; \inf_{\substack{R\in C^1(K)\\ d^1R=b}} \frac{\|R\|_{\ell^1(E)}}{\|b\|_{\ell^1(\mathcal{F})}}.$$

Notational convention. On grid complexes we write d for the degree–1 coboundary d^1.

Remark 6.4 (Finiteness and attainment)**.** Since K is finite, $d^1 : C^1 \to C^2$ is linear between finite-dimensional normed spaces. Fix any linear right-inverse $S : \operatorname{im}(d^1) \to C^1(K)$ of d^1; then $\|S\|_{1\to1} := \sup_{0\neq b} \|Sb\|_{\ell^1(E)}/\|b\|_{\ell^1(\mathcal{F})} < \infty$, hence $\mathbf{Fill}_1(K) \le \|S\|_{1\to1} < \infty$. Moreover, for each b the constrained minimization $\inf\{\|R\|_{\ell^1(E)} : d^1R = b\}$ is a finite-dimensional linear program and is attained.

Theorem 6.5 (Curl–fitting in ℓ^1)**.** Assume $H^1(K;\mathbb{R}) = 0$. Then for every $F \in C^1(K)$ there exists $u : V \to \mathbb{R}$ such that

$$\|F - d^0 u\|_{\ell^1(E)} \;\le\; \mathbf{Fill}_1(K) \;\cdot\; \|d^1 F\|_{\ell^1(\mathcal{F})}. \tag{6.1}$$

6.2. **Bridge to the grid case: an explicit right–inverse with refined (weighted) bounds.** **Rectangular grid, cochains, and norms.** Fix $d \ge 2$ and let $e_1, \dots, e_d$ be the canonical basis of $\mathbb{Z}^d$. For $N = (N_1, \dots, N_d) \in \mathbb{N}^d$, set

$$R := \prod_{i=1}^{d} \{0, 1, \dots, N_i - 1\} \subset \mathbb{Z}^d.$$

On this grid complex we write d for the degree–1 coboundary, so $(dc)_{jk}$ denotes the 2–cochain obtained from a 1–cochain c:

$$(dc)_{jk}(x) \;:=\; c_j(x) + c_k(x + e_j) - c_j(x + e_k) - c_k(x), \qquad 1 \le j < k \le d,$$

whenever the elementary square $\{x, x + e_j, x + e_k, x + e_j + e_k\}$ lies in R. We write

$$\sum_{\square_{jk}} (\cdot) \;:=\; \sum_{\substack{x: \\ x,\, x+e_j,\, x+e_k,\, x+e_j+e_k \in R}} (\cdot)$$

for the sum over all (j,k)-faces (with $1 \le j < k \le d$). All sums over edges (resp. faces) are taken only over those oriented edges (resp. elementary squares) whose endpoints (resp. four vertices) lie in R. Empty sums are 0 and empty products are 1.

Proposition 6.6 (Discrete Poincaré lemma on rectangular grids)**.** Let $R = \prod_{i=1}^d \{0, \dots, N_i - 1\} \subset \mathbb{Z}^d$ and consider the 2–skeleton of the cubical complex on R. If $c \in C^1(R)$ satisfies $dc = 0$ on every elementary 2–face, then there exists $h \in C^0(R)$ such that $c = dh$. Equivalently, $H^1(R;\mathbb{R}) = 0$ for this 2–skeleton.

Proof sketch. Fix $x^{(0)} \in R$ and define $h(x)$ by summing c along any axis–monotone path from $x^{(0)}$ to x. The hypothesis $dc = 0$ implies path–independence because any two such paths differ by a finite union of boundaries of elementary squares; the circulation of c around each such square vanishes. $\square$

Lexicographic homotopies (basepointed cone). Define $H^1 : C^1(R) \to C^0(R)$ by

$$(H^1 c)(x) \;:=\; \sum_{i=1}^{d} \sum_{t=0}^{x_i - 1} c_i\big(x_1, \dots, x_{i-1},\, t,\, \underbrace{0, \dots, 0}_{i+1,\dots,d}\big), \qquad x = (x_1, \dots, x_d) \in R. \tag{6.2}$$

Define $H^2_\uparrow : C^2(R) \to C^1(R)$ (the “increasing-index” cone) by

$$\big(H^2_\uparrow b\big)_j(x) \;:=\; -\sum_{k=j+1}^{d} \sum_{t=0}^{x_k - 1} b_{jk}\big(x_1, \dots, x_{k-1},\, t,\, \underbrace{0, \dots, 0}_{k+1,\dots,d}\big), \tag{6.3}$$

and $H^2_\downarrow : C^2(R) \to C^1(R)$ (the “decreasing-index” cone) by

$$\big(H^2_\downarrow b\big)_j(x) \;:=\; \sum_{k=1}^{j-1} \sum_{t=0}^{x_k - 1} b_{kj}\big(x_1, \dots, x_{k-1},\, t,\, \underbrace{0, \dots, 0}_{k+1,\dots,d}\big). \tag{6.4}$$

Proposition 6.7 (Cochain–homotopy identity)**.** For every $c \in C^1(R)$ one has

$$c \;-\; d\big(H^1 c\big) \;=\; H^2_\uparrow(dc) \qquad \text{in } C^1(R).$$

In particular, $d \circ H^2_\uparrow = \mathrm{Id}$ on $\mathrm{im}(d)$.

Proof. Fix $j \in \{1,\dots,d\}$ and $x \in R$ with $x, x+e_j \in R$. From (6.2) one computes

$$(dH^1c)_j(x) = (H^1c)(x{+}e_j) - (H^1c)(x) = c_j\big(x_1,\dots,x_{j-1},x_j,0,\dots,0\big) + \sum_{i=j+1}^{d}\sum_{t=0}^{x_i-1}\Big[c_i(u_t{+}e_j) - c_i(u_t)\Big],$$

where $u_t := (x_1,\dots,x_{i-1},t,0,\dots,0)$. For $i > j$ and such u_t, the coboundary identity for (j,i) (with $j < i$),

$$(dc)_{ji}(u_t) = c_j(u_t) + c_i(u_t{+}e_j) - c_j(u_t{+}e_i) - c_i(u_t),$$

rearranges to

$$c_i(u_t{+}e_j) - c_i(u_t) = (dc)_{ji}(u_t) + \big[c_j(u_t{+}e_i) - c_j(u_t)\big].$$

Summing in $t = 0,\dots,x_i-1$ yields

$$\sum_{t=0}^{x_i-1}\big[c_i(u_t{+}e_j) - c_i(u_t)\big] = \sum_{t=0}^{x_i-1}(dc)_{ji}(u_t) + c_j\big(x_1,\dots,x_{i-1},x_i,0,\dots,0\big) - c_j\big(x_1,\dots,x_{i-1},0,0,\dots,0\big).$$

Plugging back and telescoping the $c_j(\cdot)$ terms in $i = j{+}1,\dots,d$ transforms $c_j(x_1,\dots,x_{j-1},x_j,0,\dots,0)$ into $c_j(x)$, leaving

$$c_j(x) - (dH^1c)_j(x) \;=\; -\sum_{i=j+1}^{d}\sum_{t=0}^{x_i-1}(dc)_{ji}\big(x_1,\dots,x_{i-1},t,0,\dots,0\big),$$

which is exactly $\big(H^2_\uparrow(dc)\big)_j(x)$ by (6.3). $\square$

Lemma 6.8 (Operator bounds for H^2)**.** For all $b \in C^2(R)$,

$$\|H^2_\uparrow b\|_{\ell^1(E(R))} \;\le\; \sum_{1\le j<k\le d}(N_k-1)\Big(\prod_{i>k}N_i\Big)\sum_{\square_{jk}}|b_{jk}|\,, \tag{6.5}$$

$$\|H^2_\downarrow b\|_{\ell^1(E(R))} \;\le\; \sum_{1\le k<j\le d}(N_k-1)\Big(\prod_{i>k}N_i\Big)\sum_{\square_{kj}}|b_{kj}|. \tag{6.6}$$

Consequently, defining

$$C_\uparrow(R) \;:=\; \max_{2\le k\le d}(N_k-1)\prod_{i>k}N_i,$$

we have the uniform (order-dependent) bound

$$\|H^2_\uparrow b\|_{\ell^1(E(R))} \;\le\; C_\uparrow(R)\,\|b\|_{\ell^1(\mathcal{F}(R))}. \tag{6.7}$$

In particular, since $d \circ H^2_\uparrow = \mathrm{Id}$ on $\mathrm{im}(d)$,

$$\mathbf{Fill}_1(R) \;\le\; C_\uparrow(R). \tag{6.8}$$

Proof. We prove (6.5); the $\downarrow$ case is analogous. For any sequence (a_t) and $M \ge 1$,

$$\sum_{x=0}^{M-1}\sum_{t=0}^{x-1}a_t \;=\; \sum_{t=0}^{M-2}(M-1-t)\,a_t. \tag{6.9}$$

Fix $j < k$. By (6.3) and the triangle inequality, summing over all edges parallel to e_j,

$$\sum_{\substack{x:\\ x,\,x+e_j\in R}}\big|(H^2_\uparrow b)_j(x)\big| \;\le\; \sum_{\substack{x:\\ x,\,x+e_j\in R}}\sum_{t=0}^{x_k-1}\Big|b_{jk}\big(x_1,\dots,x_{k-1},t,0,\dots,0\big)\Big|.$$

Fix all coordinates except x_k and apply (6.9) with $M = N_k$ to the inner double sum. This yields the factor (N_k-1). Summation over the trailing coordinates $x_{k+1},\dots,x_d$ (which do not appear in the argument of b_{jk}) contributes the replication factor $\prod_{i>k}N_i$. The sum over $x_1,\dots,x_{k-1}$ enumerates the lower–left corners of the (j,k)–faces on that slice and is bounded by $\sum_{\square_{jk}}|b_{jk}|$.

$$\begin{tikzcd}
c \arrow[r, "d"] \arrow[rrr, bend right, "H^1"', "\|c-dh\|_{\ell^1} \le C_\uparrow(R)\,\|dc\|_{\ell^1}"'] & dc \arrow[r, "H^2_\uparrow"] & H^2_\uparrow(dc) \arrow[r, bend left, "\text{subtract}"] & dh
\end{tikzcd}$$

FIGURE 3. Cochain homotopies $H^1, H^2_\uparrow$ implement the ℓ^1 curl–fit (Thm. 6.5, Cor. 6.10).

This proves (6.5). Taking the maximum over k gives (6.7), and (6.8) follows because $H^2_\uparrow$ is a right inverse on $\mathrm{im}(d)$ by Proposition 6.7. $\square$

Proposition 6.9 (Weighted ℓ^1 curl–fit)**.** Fix edge weights $\alpha_1, \dots, \alpha_d > 0$ and face weights $\beta_{jk} > 0$ for $1 \le j < k \le d$. Define

$$\|c\|_{1,\alpha} := \sum_{j=1}^{d} \alpha_j \sum_{\substack{x:\\ x,\, x+e_j \in R}} |c_j(x)|, \qquad \|b\|_{1,\beta} := \sum_{1\le j<k\le d} \beta_{jk} \sum_{\square_{jk}} |b_{jk}(x)|.$$

Then

$$\|H^2_\uparrow b\|_{1,\alpha} \;\le\; \left(\max_{1\le j<k\le d} \frac{\alpha_j}{\beta_{jk}}\,(N_k-1)\prod_{i>k} N_i \right) \|b\|_{1,\beta}.$$

Consequently, for every $c \in C^1(R)$ there exists $h : R \to \mathbb{R}$ with

$$\|c - dh\|_{1,\alpha} \;\le\; \left(\max_{1\le j<k\le d} \frac{\alpha_j}{\beta_{jk}}\,(N_k-1)\prod_{i>k} N_i \right) \|dc\|_{1,\beta}.$$

Proof. Multiply the contributions in (6.5) by α_j on edges and divide the face term by β_{jk}, then sum and take the maximum over (j,k). $\square$

Corollary 6.10 (Quantitative curl–fit on the grid)**.** For every $c \in C^1(R)$ there exists $h : R \to \mathbb{R}$ such that

$$\sum_{j=1}^{d} \sum_{\substack{x:\\ x,\, x+e_j \in R}} \big|\, c_j(x) - \big(h(x+e_j) - h(x)\big) \,\big| \;\le\; C_\uparrow(R) \sum_{1\le j<k\le d} \sum_{\square_{jk}} \big|(dc)_{jk}(x)\big|. \tag{6.10}$$

One admissible choice is the lexicographic potential $h := H^1 c$, for which the residual equals $H^2_\uparrow(dc)$ by Proposition 6.7, and hence satisfies (6.10) by Lemma 6.8.

Remark 6.11 (On periodic boundary conditions)**.** On a discrete torus (periodic grid) one has $H^1 \neq 0$; curl–free fields need not be gradients. The estimate (6.1) holds after projecting F to the complement of the harmonic 1–cochains (i.e. imposing zero periods along the fundamental cycles). We restrict here to rectangular boxes with free boundary.

Remark 6.12 (No "cosmetic" slack; anisotropic vs. uniform forms)**.** The one-dimensional Hardy weights $(N_k - 1)$ in (6.5)-(6.6) are exact for the x_k summation. The multiplicative factor $\prod_{i>k} N_i$ is the unavoidable replication caused by evaluating b on the zero-slice in the trailing coordinates $k{+}1, \dots, d$ in (6.3)-(6.4). For this right inverse $H^2_\uparrow$ the coefficients in (6.5) are optimal: if b_{jk} is supported on a single (j,k)-face in the slice $x_k = 0$, then $\|b\|_{\ell^1(\mathcal{F}(R))} = 1$ while $\|H^2_\uparrow b\|_{\ell^1(E(R))} = (N_k - 1)\prod_{i>k} N_i$. Hence no order–independent reduction of the uniform constant below $C_\uparrow(R)$ is possible within this construction. Determining the exact value of $\mathbf{Fill}_1(R)$ (the best constant over *all* right inverses) remains open in $d \ge 3$.

Remark 6.13 (Order choice and constants)**.** The constant $C_\uparrow(R)$ depends on the *index order* used in (6.3). Different orders can change $C_\uparrow(R)$ substantially. For $d = 3$ and order (N_1, N_2, N_3) one has the closed form

$$C_\uparrow(R) \;=\; \max\{(N_2 - 1)\, N_3,\; N_3 - 1\}.$$

Placing the two smaller side lengths in the last two positions, with the smaller in the middle (order (largest, smallest, second smallest)), minimizes this maximum. In $d = 2$ the formula reduces to $C_\uparrow(R) = N_2 - 1$ for the order $1 < 2$ (and to $N_1 - 1$ if axes are swapped).

Corollary 6.14 (Uniform 2D bound)**.** In $d = 2$ one can always choose the axis order so that

$$\mathbf{Fill}_1(R) \;\le\; \min\{N_1 - 1,\; N_2 - 1\}.$$

Remark 6.15 (Symmetrizing right inverses (optional))**.** A convex combination of several index orders (or of $\uparrow, \downarrow$) yields another linear right inverse; its operator norm is bounded by the corresponding convex combination of the individual bounds, so averaging cannot beat the best single order in the uniform constant $C_\uparrow(R)$, though it may reduce anisotropy for specific b.

Remark 6.16 (Dual viewpoint (optional))**.** The minimization $\inf\{\|R\|_{\ell^1(E)} : d^1 R = b\}$ is a finite LP; its dual characterizes $\mathbf{Fill}_1(K)$ as the least $L^1(\mathcal{F}) \to L^1(E)$ operator norm among all linear right inverses of d^1. We do not use this directly, but it clarifies the role of $\mathbf{Fill}_1$ as a best possible uniform constant over right inverses.

7. Constructive Neutralization of Heisenberg Shear

Group, generators, and edge convention. We work in the (discrete) Heisenberg group on $\mathbb{Z}^3$ with the group law

$$(x, y, z) \cdot (x', y', z') \;:=\; (x + x',\; y + y',\; z + z' + x\, y'). \tag{7.1}$$

Let $a = (1, 0, 0)$, $b = (0, 1, 0)$ and $S = \{a^{\pm 1}, b^{\pm 1}\}$. *Cayley edges are by right multiplication:* $g \leftrightarrow g\, s$ for $s \in S$. The horizontal perimeter $B_S(\cdot)$ is the undirected edge cut in this Cayley graph with unit weights (each broken undirected edge counted once; the directed perimeter is twice this).

Remark 7.1 (Orientation and normalization lock)**.** We fix once and for all the following conventions in this section:

1. *Edges:* Cayley edges use the *right* action by $S = \{a^{\pm 1}, b^{\pm 1}\}$.
2. *Boundary normalization:* $B_S(\cdot)$ counts each *undirected* broken edge once. If one prefers the directed normalization, all B_S-identities below scale by 2.
3. *Squares and signs:* For coboundaries dc on the base grid $\mathbb{Z}^2$, faces are oriented by the ordered pair (e_1, e_2), and edges carry their natural orientation $(x \to x + e_i)$. Thus, for a 1-cochain c one has
$$(dc)_{12}(x, y) = c_{e_1}(x, y) + c_{e_2}(x + e_1, y) - c_{e_1}(x, y + e_2) - c_{e_2}(x, y).$$
With $c_{e_1}(x, y) = y$ and $c_{e_2} \equiv 0$ we get $(dc)_{12} \equiv -1$.
4. *Half-open columns:* All column fibres are treated as half-open integer intervals $[h(u), h(u) + \ell(u))$ so that overlaps are counted unambiguously and symmetric-difference formulae are exact (Lemma 7.7).

These choices match Lemma 7.3 and are used in the combinatorial Lemmas 7.7, 7.9 and in Theorem 7.11.

Gauge that neutralizes shear on b-edges (right action).

Definition 7.2 (Heisenberg gauge)**.** Define the quadratic gauge $\zeta(x, y) := xy$ and the map

$$T : \mathbb{Z}^3 \to \mathbb{Z}^3, \qquad T(x, y, z) = (x, y, h), \;\; h := z - \zeta(x, y) = z - xy.$$

Lemma 7.3 (Gauge offsets under right multiplication)**.** With (7.1), generators $a = (1,0,0)$, $b = (0,1,0)$, and gauge $T(x,y,z) = (x,y,h)$ with $h = z - xy$, one has:

$$(x,y,z)\,b = (x,\,y+1,\,z+x) \;\Rightarrow\; h' = (z+x) - x(y+1) = h,$$

$$(x,y,z)\,a = (x+1,\,y,\,z) \;\Rightarrow\; h' = z - (x+1)y = h - y.$$

Thus b-edges have zero h-offset and a-edges shift h by $-y$.

Lemma 7.4 (Alignment shifts for all horizontal directions)**.** With the gauge $T(x,y,z) = (x,y,h)$, $h = z - xy$, the alignment shifts along *all* horizontal steps are:

$$\begin{aligned}
&\text{for } v = +e_2\ (b): && h' = h, && \sigma_{+e_2}(x,y) = 0,\\
&\text{for } v = -e_2\ (b^{-1}): && h' = h, && \sigma_{-e_2}(x,y) = 0,\\
&\text{for } v = +e_1\ (a): && h' = h - y, && \sigma_{+e_1}(x,y) = y,\\
&\text{for } v = -e_1\ (a^{-1}): && h' = h + y, && \sigma_{-e_1}(x,y) = -y.
\end{aligned}$$

Column-convex sets and height/offset profiles. Write $\pi : \mathbb{Z}^3 \to \mathbb{Z}^2$, $\pi(x,y,z) = (x,y)$, and for $A \subset \mathbb{Z}^3$ set $A^\sharp := T(A)$. We say A is *(gauge-)column-convex* if each fiber

$$A^\sharp_u \;:=\; \{h \in \mathbb{Z} :\; (u,h) \in A^\sharp\} \quad (u \in \mathbb{Z}^2)$$

is an integer interval $I_u = [h(u),\, h(u) + \ell(u))$ for some $\ell(u) \in \mathbb{Z}_{\geq 0}$. We call $\ell : \mathbb{Z}^2 \to \mathbb{Z}_{\geq 0}$ the *height profile* and $h : \mathbb{Z}^2 \to \mathbb{Z}$ the *offset profile*. Let $E := \{u :\; \ell(u) > 0\} \subset \mathbb{Z}^2$ be the base footprint.

7.1. Combinatorics of interval differences.

Definition 7.5 (Outward boundary edges on the base)**.** For $E \subset \mathbb{Z}^2$ finite define the *internal* and *boundary* sets of oriented base edges by

$$\mathcal{E}_{\mathrm{int}}(E) := \{(u \to u+v) :\; u, u+v \in E,\; v \in \{e_1, e_2\}\},$$

$$\mathcal{E}_{\mathrm{bd}}(E) := \{(u \to u+v) :\; u \in E,\; u+v \notin E,\; v \in \{\pm e_1, \pm e_2\}\}.$$

Thus, in $\mathcal{E}_{\mathrm{int}}(E)$ *each undirected internal base edge* is represented *exactly once* (we fix the orientation by $v \in \{e_1, e_2\}$), and in $\mathcal{E}_{\mathrm{bd}}(E)$ *each undirected broken base edge* is represented *exactly once* by the orientation pointing from E to $E^\complement$.

Lemma 7.6 (Edge–height bijection along a fixed base edge)**.** Let $A \subset \mathbb{Z}^3$ be gauge-column-convex with footprint E, column heights $\ell(u)$, offsets $h(u)$, and half-open columns $I_u = [h(u), h(u) + \ell(u))$. Fix an oriented base edge $e = (u \to u+v)$ with $v \in \{\pm e_1, \pm e_2\}$ and write the alignment shift $\sigma_v(u)$ as in Lemma 7.4. Set

$$t(e) \;:=\; h(u+v) - h(u) + \sigma_v(u).$$

Then the number of *broken undirected horizontal Cayley edges* of A across the base edge e equals

$$\#\big(I_u \bigtriangleup (I_{u+v} + \sigma_v(u))\big).$$

Equivalently-since symmetric difference cardinality is translation invariant-translating both sets by $-h(u)$ yields

$$\#\big(I_u \bigtriangleup (I_{u+v} + \sigma_v(u))\big) \;=\; \#\big([0,\alpha) \bigtriangleup [t(e), t(e) + \beta)\big),$$

where $\alpha = \ell(u)$, $\beta = \ell(u+v)$ and $t(e) := h(u+v) - h(u) + \sigma_v(u)$.

Proof. Work in gauge coordinates $A^\sharp = T(A)$, so points are (u,h) with $u \in \mathbb{Z}^2$, $h \in \mathbb{Z}$, and the right action by $v \in \{\pm e_1, \pm e_2\}$ updates

$$(u,h) \;\mapsto\; (u+v,\; h - \sigma_v(u)) \qquad \text{(Lemma 7.4)}.$$

Fix $e = (u \to u+v)$. A horizontal Cayley edge across e is broken exactly when one endpoint lies in $A^\sharp$ and the other does not. For $h \in \mathbb{Z}$,

$$(u,h) \in A^\sharp \iff h \in I_u, \qquad (u,h)\,v = (u+v, h-\sigma_v(u)) \in A^\sharp \iff h - \sigma_v(u) \in I_{u+v}.$$

Hence

$$\{\, h : \text{ edge across } e \text{ is broken at height } h \,\} = I_u \,\triangle\, (I_{u+v} + \sigma_v(u)),$$

and counting heights gives the claim. Translating both intervals by $-h(u)$ yields the normalized form with shift $t(e) = h(u+v) - h(u) + \sigma_v(u)$. □

Lemma 7.7 (Interval symmetric difference, full proof)**.** For $\alpha, \beta \in \mathbb{Z}_{\geq 0}$ and $t \in \mathbb{Z}$, set $I = [0, \alpha)$, $J = [t, t+\beta)$, $m := \min\{\alpha, \beta\}$, $\delta := |\alpha - \beta|$ and

$$d(t) := \big(t - (\alpha - \beta)_+\big)_+ \;+\; \big(-t - (\beta - \alpha)_+\big)_+.$$

Then

$$\#(I \triangle J) = \; \delta + 2\min\{d(t),\, m\}.$$

Proof. Assume w.l.o.g. $\alpha \geq \beta$ (the other case is symmetric). Then $\delta = \alpha - \beta$, $(\beta - \alpha)_+ = 0$ and $d(t) = (t-\delta)_+ + (-t)_+$. Since

$$\#(I \triangle J) \;=\; |I| + |J| - 2\,|I \cap J| \;=\; \alpha + \beta - 2\,|I \cap J|,$$

we compute $|I \cap J|$ by cases:

(1) $t \geq \alpha$*:* I and J are disjoint, so $\#(I \triangle J) = \alpha + \beta = \delta + 2\beta = \delta + 2m$. Also $d(t) = t - \delta \geq \alpha - \delta = \beta = m$, hence $\delta + 2\min\{d(t), m\} = \delta + 2m$.

(2) $0 \leq t \leq \delta$*:* $J \subset I$, so $|I \cap J| = \beta$ and $\#(I \triangle J) = \alpha - \beta = \delta$. Here $d(t) = 0$, hence $\delta + 2\min\{d(t), m\} = \delta$.

(3) $\delta \leq t \leq \alpha$*:* $I \cap J = [t, \alpha)$ has size $\alpha - t$, hence

$$\#(I \triangle J) = \alpha + \beta - 2(\alpha - t) = \beta - \alpha + 2t = -\delta + 2t = \delta + 2(t - \delta).$$

Since $t \leq \alpha$, $t - \delta \leq \alpha - \delta = \beta = m$, so $d(t) = t - \delta \leq m$ and $\delta + 2\min\{d(t), m\} = \delta + 2(t - \delta)$.

(4) $t \leq -\beta$*:* I and J are disjoint, so $\#(I \triangle J) = \delta + 2m$. Here $d(t) = (-t) \geq \beta = m$, hence $\delta + 2\min\{d(t), m\} = \delta + 2m$.

(5) $-\beta \leq t \leq 0$*:* $I \cap J = [0, t+\beta)$ has size $\beta + t$, whence

$$\#(I \triangle J) = \alpha + \beta - 2(\beta + t) = \alpha - \beta - 2t = \delta + 2(-t).$$

Also $d(t) = (-t) \leq \beta = m$, so $\delta + 2\min\{d(t), m\} = \delta + 2(-t)$.

This exhausts the cases. □

Remark 7.8 (Degenerate columns and integer arithmetic)**.** If $\ell(u) = 0$ we interpret $I_u = [h(u), h(u)) = \varnothing$. The function $d(t)$ in Lemma 7.7 is evaluated for $t \in \mathbb{Z}$ (integer height shifts only), and all intervals are *half-open* so that disjoint unions along a column are additively counted without overlaps. In particular, $\#(I \triangle J)$ is always an integer and the identity $\#(I \triangle J) = \delta + 2\min\{d(t), m\}$ stays valid in the degenerate cases $m = 0$ or $\delta = 0$.

Lemma 7.9 (Per-edge two-sided bounds)**.** Let $\alpha, \beta \in \mathbb{Z}_{\geq 0}$, $m = \min\{\alpha, \beta\}$, $\delta = |\alpha - \beta|$, $A := \max\{\alpha, \beta\}$, and $t \in \mathbb{Z}$. With $d(t)$ as in Lemma 7.7,

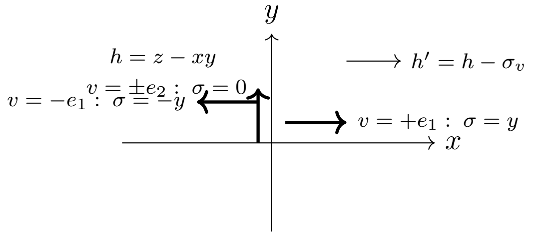
$$2\min\{|t|, A\} - \delta \;\leq\; \delta + 2\min\{d(t), m\} \;\leq\; \delta + 2\min\{|t|, A\}. \tag{7.2}$$

Proof. The right inequality follows from $d(t) \leq |t|$ (since $(x)_+ \leq |x|$) and $m \leq A$. For the left inequality it suffices to prove

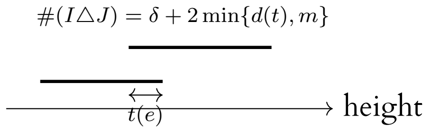
$$\min\{|t|, A\} \;\leq\; \delta + \min\{d(t), m\}.$$

Assume $\alpha \geq \beta$ (so $A = \alpha$, $m = \beta$, $\delta = \alpha - \beta$) and recall $d(t) = (t-\delta)_+ + (-t)_+$. Consider cases:

(i) $0 \leq t \leq \delta$*:* then $\min\{|t|, A\} = t \leq \delta \leq \delta + \min\{d(t), m\}$.



(A) Gauge offsets under right multiplication ($v = \pm e_2$ give $\sigma = 0$; $v = +e_1$ gives $\sigma = y$; $v = -e_1$ gives $\sigma = -y$).



(B) Interval symmetric difference on a base edge.

FIGURE 4. Heisenberg edge-level bookkeeping behind the *base term + shear term* split (Thm. 7.11).

(ii) $\delta \le t \le A$ ($= \alpha$): then $\min\{|t|, A\} = t$, while $d(t) = t - \delta \le m$; hence $\delta + \min\{d(t), m\} \ge \delta + (t - \delta) = t$.

(iii) $t \ge A$ ($= \alpha$): then $\min\{|t|, A\} = A = \delta + m$. Since $d(t) = t - \delta \ge A - \delta = m$, we get $\delta + \min\{d(t), m\} = \delta + m = A$.

(iv) $t \le 0$ and $|t| \le m$: then $\min\{|t|, A\} = |t| = (-t)_+ = \min\{d(t), m\} \le \delta + \min\{d(t), m\}$.

(v) $t \le 0$ and $|t| \ge m$: then $\min\{|t|, A\} \le A = \delta + m \le \delta + \min\{d(t), m\}$.

This proves the left inequality. □

Remark 7.10 (Sanity checks for (7.4))**.** Let $\alpha, \beta \in \mathbb{Z}_{\ge 0}$, $m = \min\{\alpha, \beta\}$, $\delta = |\alpha - \beta|$ and $t \in \mathbb{Z}$.

(1) If $t = 0$ and $\alpha \ge \beta$, then $J \subset I$ and $\#(I \triangle J) = \delta$, matching (7.4) since $d(0) = 0$.
(2) If $|t| \ge \alpha + \beta$, the intervals are disjoint and $\#(I \triangle J) = \alpha + \beta = \delta + 2m$, which again matches (7.4) because $d(t) \ge m$.
(3) If $\alpha = \beta$ and $|t| \le \alpha$, then $\#(I \triangle J) = 2|t|$, matching $\delta + 2\min\{d(t), m\} = 2\min\{|t|, \alpha\}$ from Lemma 7.9.

These checks cover the nested, disjoint, and "partial overlap" regimes used repeatedly in the global bounds.

7.2. **Exact perimeter decomposition and global bounds.** Let $\mathcal{E}_{\text{int}}(E)$ and $\mathcal{E}_{\text{bd}}(E)$ be as in Definition 7.5. For $e = (u \to u + v)$ set $\alpha(e) := \ell(u)$. For $e \in \mathcal{E}_{\text{int}}(E)$ also set

$$\beta(e) := \ell(u{+}v), \quad \delta(e) := |\alpha(e){-}\beta(e)|, \quad m(e) := \min\{\alpha(e), \beta(e)\}, \quad t(e) := h(u{+}v){-}h(u){+}\sigma_v(u),$$

with the alignment shift $\sigma_{e_1}(x, y) := y$ and $\sigma_{e_2} \equiv 0$ from Lemma 7.4.

Theorem 7.11 (Exact horizontal perimeter decomposition)**.** If $A \subset \mathbb{Z}^3$ is gauge-column-convex with footprint E, heights ℓ, and offsets h, then

$$B_S(A) \;=\; \underbrace{\sum_{e \in \mathcal{E}_{\text{bd}}(E)} \alpha(e)}_{\text{boundary base edges}} \;+\; \sum_{e \in \mathcal{E}_{\text{int}}(E)} \#\big(I_u \triangle (I_{u+v} + \sigma_v(u))\big), \tag{7.3}$$

where $I_u = [h(u), h(u) + \ell(u))$ and $I_{u+v} = [h(u+v), h(u+v) + \ell(u+v))$. In particular, for $e \in \mathcal{E}_{\text{int}}(E)$,

$$\#\big(I_u \triangle (I_{u+v} + \sigma_v(u))\big) \;=\; \delta(e) \;+\; 2\,\min\{\, d(t(e)),\; m(e)\}. \tag{7.4}$$

with $d(\cdot)$ as in Lemma 7.7. Equivalently, by translating both sets by $-h(u)$, this equals $\#\big([0, \alpha) \triangle [t(e), t(e) + \beta)\big)$ with $t(e) := h(u + v) - h(u) + \sigma_v(u)$, so (7.4) follows from Lemma 7.7.

Corollary 7.12 (Two-sided global bounds)**.** With $A(e) = \max\{\alpha(e), \beta(e)\}$ for $e \in \mathcal{E}_{\mathrm{int}}(E)$,

$$\sum_{e\in\mathcal{E}_{\mathrm{bd}}(E)} \alpha(e) \;+\; 2 \sum_{e\in\mathcal{E}_{\mathrm{int}}(E)} \min\{|t(e)|, A(e)\} \;-\; \sum_{e\in\mathcal{E}_{\mathrm{int}}(E)} \delta(e) \;\le\; B_S(A) \tag{7.5}$$
$$\le \sum_{e\in\mathcal{E}_{\mathrm{bd}}(E)} \alpha(e) \;+\; \sum_{e\in\mathcal{E}_{\mathrm{int}}(E)} \delta(e) \;+\; 2 \sum_{e\in\mathcal{E}_{\mathrm{int}}(E)} \min\{|t(e)|, A(e)\}.$$

Truncated and uncapped shear functionals (internal edges). For $E \subset \mathbb{Z}^2$ finite define

$$\mathrm{TruncCurl}(E, \ell) \;:=\; \inf_{h:E\to\mathbb{Z}} \sum_{e\in\mathcal{E}_{\mathrm{int}}(E)} \min\{|t(e)|, A(e)\}, \qquad \mathrm{CurlCost}(E) \;:=\; \inf_{h:E\to\mathbb{Z}} \sum_{e\in\mathcal{E}_{\mathrm{int}}(E)} |t(e)|. \tag{7.6}$$

(Infimum over offsets h; $t(e)$ uses $\sigma_{e_1}(x, y) = y$ and $\sigma_{e_2} = 0$.)

Corollary 7.13 (Shear-aware lower bound)**.** For every gauge-column-convex A with footprint E and heights ℓ,

$$B_S(A) \;\ge\; \sum_{e\in\mathcal{E}_{\mathrm{bd}}(E)} \alpha(e) \;+\; 2\,\mathrm{TruncCurl}(E, \ell) \;-\; \sum_{e\in\mathcal{E}_{\mathrm{int}}(E)} \delta(e).$$

Consequently, for any offsets h and $\mathrm{CapLoss}(h) := \sum_{e\in\mathcal{E}_{\mathrm{int}}(E)}(|t(e)| - A(e))_+$,

$$B_S(A) \;\ge\; \sum_{e\in\mathcal{E}_{\mathrm{bd}}(E)} \alpha(e) \;+\; 2\,\mathrm{CurlCost}(E) \;-\; \sum_{e\in\mathcal{E}_{\mathrm{int}}(E)} \delta(e) \;-\; 2\,\mathrm{CapLoss}(h).$$

Edges vs. boundary vertices. Let $B_S(Y)$ denote the undirected edge boundary and $\partial_S Y$ the set of boundary vertices.

Lemma 7.14 (Edges vs. boundary vertices)**.** For any finite symmetric S and finite Y,

$$B_S(Y) \;\le\; |S| \cdot |\partial_S Y| \qquad\Rightarrow\qquad |\partial_S Y| \;\ge\; |S|^{-1} B_S(Y).$$

Proof. Assign each broken undirected edge $\{y, ys\}$ to its endpoint $y \in Y$ (there is exactly one). Then $B_S(Y) \le \sum_{y\in\partial_S Y} \#\{s \in S : \; ys \notin Y\} \le |S| \cdot |\partial_S Y|$. □

7.3. **Compactness and order-sharp bounds in Heisenberg.** We use Carnot dilations $\delta_\rho(x, y, z) = (\rho x, \rho y, \rho^2 z)$ and write $u_\rho := \mathbf{1}_{\delta_{\rho^{-1}} Y_\rho}$. Carnot dilations scale Haar measure by ρ^Q, where $Q = \sum_i i \,\dim V_i$ is the homogeneous dimension; see [FS82, Ch. 1].

Theorem 7.15 (Compactness and order-sharp bounds)**.** Let $Y_\rho \subset \mathbb{H}_\mathbb{Z}$ with $|Y_\rho| \asymp \rho^4$ and $\rho^{-3} B_S(Y_\rho) \le C$. Then, after left translations *and gauge-column symmetrization* (which does not increase B_S by (7.3) with I_u replaced by initial segments), (u_ρ) is precompact in L^1_{loc} on $\mathbb{H}$, and any limit is a vertical stack over a planar footprint.

Moreover, writing E_ρ for footprints and

$$\mathcal{S}^* \;:=\; \liminf_{\rho\to\infty} \; \rho^{-3}\,\mathrm{CurlCost}(E_\rho),$$

for any Lipschitz stack profile $s \mapsto E_s \subset \mathbb{R}^2$ on $[0, H]$ with $\int_0^H |E_s|\, ds = 1$, there exists a recovery sequence Y_ρ such that

$$\text{(limsup)} \quad \rho^{-3} B_S(Y_\rho) \;\le\; \int_0^H \mathrm{Per}_{\tau_S}(E_s)\, ds \;+\; 2\,\mathcal{S}^* \;+\; o(1), \tag{7.7}$$

and, using the exact split (7.3), the *base layer-cake identity*

$$\sum_{e\in\mathcal{E}_{\mathrm{bd}}(E_\rho)} \alpha(e) \;+\; \sum_{e\in\mathcal{E}_{\mathrm{int}}(E_\rho)} \delta(e) \;=\; \int_0^\infty \big|\partial_{S_{\mathrm{hor}}}\{\ell_\rho > t\}\big|\, dt, \tag{7.8}$$

which, by Theorem 4.1, yields the Abelian Wulff lower bound for the base term. Hence

$$\text{(liminf, base)} \quad \liminf_{\rho\to\infty} \rho^{-3}|\partial_S Y_\rho| \;\ge\; \frac{1}{|S|}\int_0^H \mathrm{Per}_{\tau_S}(E_s)\,ds, \tag{7.9}$$

see [AFP00, Thm. 3.40], [FM91]. Here τ_S denotes the horizontal surface tension induced by S via the *zonoid* model: for $S = \{\pm e_1, \pm e_2\}$ with unit undirected weights,

$$\tau_S(\xi) \;=\; |\xi_1| + |\xi_2| \qquad \text{(up to the global normalization fixed by our edge convention).}$$

Identifying a shear term in the full liminf reduces, by Corollary 7.13, to showing that $\sum_e(|t(e)| - A(e))_+ = o(\rho^3)$ along near-minimizers.

Remark 7.16 (Why gauge-column symmetrization does not increase B_S)**.** The right action in Heisenberg acts horizontally by $(u, h) \mapsto (u + v, h - \sigma_v(u))$ with σ_v from Lemma 7.4, i.e. by *base translation* plus an *offset* that depends only on the base vertex u and the step v. Therefore the horizontal broken-edge count across $u \to u+v$ is $\#\big(I_u \triangle (I_{u+v} + \sigma_v(u) + h(u+v) - h(u))\big)$ with $t(e) = h(u + v) - h(u) + \sigma_v(u)$ as in Theorem 7.11. Replacing I_u and I_{u+v} by initial segments of the same lengths (gauge-column symmetrization) minimizes each symmetric difference term and hence does not increase B_S (cf. Proposition 4.10).

Remark 7.17 (Horizontal anisotropy identification)**.** The induced continuum surface tension from the horizontal edge cut with generators S is the *zonoid* gauge $\tau_S(\nu) = \sum_{s\in S_+} w_s\,|\nu\cdot X_s|$ (sum over an undirected representative set S_+), which for $S = \{\pm e_1, \pm e_2\}$ with unit weights is $\tau_S(\nu) = |\nu_1| + |\nu_2|$. Its Banach dual τ_S° is the Minkowski functional of the *zonotope* $K := \sum_s w_s[-X_s, X_s]$ (see [Zie95, Sec. 7.2,7.3]). Separately, we will also use the *convex-hull* integrand $\psi_S(\xi) := \max_{v\in\mathrm{conv}(\pm S)}\langle \xi, v\rangle$ (the support function of $\mathrm{conv}(\pm S)$). In general $K \neq \mathrm{conv}(\pm S)$, so τ_S° and ψ_S need not coincide. In the axis example $S = \{\pm e_1, \pm e_2\}$ with unit weights, however, $K = [-1, 1]^2$ and $\mathrm{conv}(\pm S) = \{x : |x_1| + |x_2| \le 1\}$; their support functions agree, hence

$$\tau_S^\circ(\cdot) = \psi_S(\cdot) = \|\cdot\|_\infty, \qquad \text{while} \qquad \tau_S(\nu) = \|\nu\|_1.$$

All sharp Wulff constants in this section are stated for the zonoid gauge τ_S.

7.4. **Heisenberg lim inf cap-loss via blockwise L^1 curl-fit.** We remain in the gauge coordinates (x, y, h), and consider gauge-column-convex configurations $Y^\flat$ with footprint E and height profile $\ell : \mathbb{Z}^2 \to \mathbb{Z}_{\ge 0}$.

Block decomposition. Fix a block size $L = L_\rho \in \mathbb{N}$ and tile $\mathbb{Z}^2$ by disjoint $L \times L$ squares Q. For a given footprint $E_\rho \subset \mathbb{Z}^2$ define the *core* of a block

$$\mathrm{Core}(Q) \;:=\; \{u \in Q \cap E_\rho : \ \mathrm{dist}_{\ell^1}(u, \mathbb{Z}^2 \setminus Q) \ge 1\},$$

and the *skeleton* $\Sigma := E_\rho \setminus \bigcup_Q \mathrm{Core}(Q)$ (a width-1 network along block interfaces and ∂E_ρ). Let $\mathcal{E}_{\mathrm{int}}(E_\rho)$ be split into three disjoint classes:

$$\mathcal{E}_{\mathrm{core}} = \{e = (u \to u + v) : u, u + v \in \mathrm{Core}(Q) \text{ for some } Q\}, \quad \mathcal{E}_{\mathrm{skel}} = \{e : u, u + v \in \Sigma\},$$
$$\mathcal{E}_{\mathrm{cross}} = \{e : \{u, u + v\} \cap \mathrm{Core}(Q) \neq \emptyset,\ \{u, u + v\} \cap \Sigma \neq \emptyset\}.$$

Shear cochain. Let c be the shear 1–cochain on base edges, defined by $c_{e_1}(x, y) = y$ and $c_{e_2} \equiv 0$ for the oriented edges $(x, y) \to (x + 1, y)$ and $(x, y) \to (x, y + 1)$, respectively. We extend c to negative edges by

$$c(u \to u - v) \;:=\; -\,c(u - v \to u) \qquad (v \in \{e_1, e_2\}).$$

With the face orientation from Remark 7.1, one computes

$$(dc)_{12}(x, y) \;=\; c_{e_1}(x, y) + c_{e_2}(x + e_1, y) - c_{e_1}(x, y + e_2) - c_{e_2}(x, y) \;=\; -1,$$

i.e. the curl is a constant -1 on every elementary square. This sign choice is what drives the blockwise L^1 curl–fit (Corollary 6.10) used in Proposition 7.20.

Lemma 7.18 (Constant face curl with the chosen orientation)**.** With $c_{e_1}(x,y)=y$ and $c_{e_2}\equiv 0$ on oriented edges $(x,y)\to(x+1,y)$ and $(x,y)\to(x,y+1)$, and with the face orientation set by (e_1,e_2) (Remark 7.1), one has

$$(dc)_{12}(x,y)=c_{e_1}(x,y)+c_{e_2}(x+e_1,y)-c_{e_1}(x,y+e_2)-c_{e_2}(x,y)\equiv -1.$$

Remark 7.19 (Sign normalization for curl–fit)**.** The L^1 curl–fit inequality is invariant under $(F,u)\mapsto(-F,-u)$, hence

$$\inf_u \|F+du\|_{L^1(E^1)}=\inf_u \|F-du\|_{L^1(E^1)} \;\le\; C\,\|dF\|_{L^1(E^2)}.$$

Since $t=c+dh$, we adopt the "+" convention uniformly below.

Proposition 7.20 (Cap-loss bound via blockwise L^1 curl-fit)**.** Let (Y_ρ) be gauge-column-convex with $|Y_\rho|\asymp\rho^4$ and $\rho^{-3}B_S(Y_\rho)\le C$. Write E_ρ for footprints and ℓ_ρ for heights. There exist offsets $h_\rho:E_\rho\to\mathbb{Z}$ such that

$$\mathrm{CapLoss}(h_\rho)\;:=\;\sum_{e\in\mathcal{E}_{\mathrm{int}}(E_\rho)}\big(|t_\rho(e)|-A_\rho(e)\big)_+\;=\;o(\rho^3).$$

In fact, taking $L_\rho=\lfloor\rho^{1/2}\rfloor$ yields $\mathrm{CapLoss}(h_\rho)\;\lesssim\;\rho^{5/2}$, and more explicitly

$$\mathrm{CapLoss}(h_\rho)\;\le\;C_0\left(L_\rho\,\rho^2+\frac{\rho^3}{L_\rho}\right)$$

for some absolute constant C_0 (coming from Corollary 6.10).

Proof. Core fit. On each block Q, apply the rectangular L^1 curl-fit (Corollary 6.10 in §6) to $(dc)\equiv -1$ to obtain a potential h_Q with

$$\sum_{e\in\mathcal{E}_{\mathrm{core}}\cap(Q\times)}|c(e)+(dh_Q)(e)|\;\le\;\big(\max_i(N_i-1)\big)\sum_{f\subset Q}|(dc)(f)|\;\le\;L\cdot L^2\;=\;L^3,\tag{7.10}$$

since here $N_1=N_2=L$ and $\#\{\text{faces in } Q\}=L^2$. Covering E_ρ by $O(\rho^2/L^2)$ blocks gives the core contribution $\le C\,L\,\rho^2$.

Skeleton fit. Decompose Σ into its connected components Σ_α (width 1, ℓ^1-diameter $\lesssim\rho$) and apply Corollary 6.10 on each Σ_α to get h_{Σ_α} with

$$\sum_{e\in\mathcal{E}_{\mathrm{skel}}\cap(\Sigma_\alpha\times)}|c(e)+(dh_{\Sigma_\alpha})(e)|\;\le\;C\,\rho\cdot\#\{\text{faces in }\Sigma_\alpha\}.$$

Summing over α and using $\#\{\text{faces in }\Sigma\}\lesssim\rho^2/L$ yields

$$\sum_{e\in\mathcal{E}_{\mathrm{skel}}}|c(e)+(dh_\Sigma)(e)|\;\le\;C\,\frac{\rho^3}{L}.\tag{7.11}$$

Synchronization on interfaces. For each block Q choose a constant $\lambda_Q\in\mathbb{R}$ (depending on Q) and define the global potential

$$h(u)\;:=\;\begin{cases}h_Q(u)+\lambda_Q, & u\in\mathrm{Core}(Q),\\ h_\Sigma(u), & u\in\Sigma.\end{cases}$$

The next lemma controls the *cross* sum in terms of boundary rings and a strip of faces.

Lemma 7.21 (Interface synchronization via cyclic BV control)**.** There is an absolute C such that, for each Q one can choose λ_Q with

$$\sum_{(u\to u+v)\in\mathcal{E}_{\text{cross}}\cap(Q\times)} \big|c(u,v)+(dh)(u,v)\big|$$

$$\le\ C\left(\sum_{e\subset\partial Q\ \text{(core ring)}} \big|c(e)+(dh_Q)(e)\big|\ +\ \sum_{e\subset N(Q)\ \text{(skel. ring)}} \big|c(e)+(dh_\Sigma)(e)\big|\ +\ \sum_{f\in S(Q)} |(dc)(f)|\right), \tag{7.12}$$

where ∂Q is a fixed $O(1)$–thick core boundary ring in Q, $N(Q)$ is an $O(1)$–thick skeleton ring adjacent to ∂Q, and $S(Q)$ is the unit–thickness annulus of faces between the two rings ($|S(Q)|\lesssim L$).

Proof. List cross edges along the interface of Q in cyclic order $e_i=(u_i\to v_i)$ and set $a_i:=c(e_i)+h_\Sigma(v_i)-h_Q(u_i)$. Choose λ_Q to be a median of $\{a_i\}$. The discrete median–variation inequality on cycles (Lemma 7.23) gives $\sum_i|a_i-\lambda_Q|\le\sum_i|a_{i+1}-a_i|$. By Lemma 7.22,

$$a_{i+1}-a_i=\ -\big(c+dh_Q\big)(u_i\to u_{i+1})\ +\ \big(c+dh_\Sigma\big)(v_i\to v_{i+1})\ +\ \sum_{f\in R_i}(dc)(f).$$

Taking absolute values, summing over i, and noting that each ring edge and each face in $S(Q)$ is counted $O(1)$ times yields (7.12). Finally, observe that for a cross edge $e_i=(u_i\to v_i)$ we have

$$\big|c(e_i)+(dh)(e_i)\big|=\big|c(e_i)+h_\Sigma(v_i)-\big(h_Q(u_i)+\lambda_Q\big)\big|=\big|a_i-\lambda_Q\big|,$$

so the left-hand side is exactly the sum over cross edges. □

Lemma 7.22 (One-step telescoping across an interface)**.** Let c be a 1–cochain on a rectangular strip and h_Q,h_Σ two scalar potentials defined on adjacent rings (core and skeleton). For consecutive cross edges $e_i=(u_i\to v_i)$ along the interface (cyclically ordered) set $a_i:=c(e_i)+h_\Sigma(v_i)-h_Q(u_i)$. Then

$$a_{i+1}-a_i\ =\ -\big(c+dh_Q\big)(u_i\to u_{i+1})\ +\ \big(c+dh_\Sigma\big)(v_i\to v_{i+1})\ +\ \sum_{f\in R_i}(dc)(f),$$

where R_i is the unit–width strip of faces between $(u_i\to u_{i+1})$ and $(v_i\to v_{i+1})$.

Proof. Expand $a_{i+1}-a_i$ and use the discrete Stokes identity on the strip R_i: $c(u_i\to u_{i+1})+c(e_{i+1})-c(v_i\to v_{i+1})-c(e_i)=\sum_{f\in R_i}(dc)(f)$. Rearranging yields $c(e_{i+1})-c(e_i)=\sum_{f\in R_i}(dc)(f)-c(u_i\to u_{i+1})+c(v_i\to v_{i+1})$. Substituting and grouping terms gives the stated identity with $c+dh_Q$ and $c+dh_\Sigma$. □

Lemma 7.23 (Discrete median–variation on a cycle)**.** Let $(a_i)_{i=1}^M$ be a cyclic sequence and λ be a median of $\{a_i\}$. Then

$$\sum_{i=1}^M|a_i-\lambda|\ \le\ \sum_{i=1}^M|a_{i+1}-a_i|\qquad(a_{M+1}:=a_1).$$

Summing (7.12) over Q and noting that each ring edge and each face in $S(Q)$ appears $O(1)$ times across neighboring blocks, we obtain

$$\sum_{e\in\mathcal{E}_{\text{cross}}}|c-dh|\ \le\ C\left(\sum_{e\in\mathcal{E}_{\text{core}}}|c-dh_Q|\ +\ \sum_{e\in\mathcal{E}_{\text{skel}}}|c-dh_\Sigma|\ +\ \sum_Q|S(Q)|\right).$$

Using (7.10), (7.11) and $\sum_Q|S(Q)|\lesssim(\rho^2/L^2)\cdot L=\rho^2/L$ gives

$$\sum_{e\in\mathcal{E}_{\text{cross}}}|c+dh|\ \le\ C\left(L\,\rho^2+\frac{\rho^3}{L}\right).$$

Conclusion. Combining the core, skeleton and cross estimates,

$$\sum_{e\in\mathcal{E}_{\rm int}(E_\rho)} |c(e)+(dh)(e)| \;\le\; C\left(L\,\rho^2+\frac{\rho^3}{L}\right).$$

Since $\min\{|t|,A\}=|t|-(|t|-A)_+$ and *pointwise* $|t_\rho(e)|=|c(e)+(dh)(e)|$, we obtain

$$\sum_{e\in\mathcal{E}_{\rm int}(E_\rho)} (|t_\rho(e)|-A_\rho(e))_+ \;\le\; \sum_{e\in\mathcal{E}_{\rm int}(E_\rho)} |t_\rho(e)| \;=\; \sum_{e\in\mathcal{E}_{\rm int}(E_\rho)} |c(e)+(dh)(e)| \;\le\; C\left(L\,\rho^2+\frac{\rho^3}{L}\right).$$

Balancing at $L_\rho=\lfloor\rho^{1/2}\rfloor$ gives $\mathrm{CapLoss}(h_\rho)\lesssim\rho^{5/2}$ and hence $o(\rho^3)$, as claimed.

Integer offsets. The construction above chooses real interface shifts $\lambda_Q\in\mathbb{R}$. Replace each λ_Q by its nearest integer. This does not change tangential differences on the core or skeleton and changes each cross edge by at most 1. Since the number of cross edges is $O(\rho^2/L)$, the total increase in $\sum_{e\in\mathcal{E}_{\rm int}}|c-dh|$ is $O(\rho^2/L)$. With $L=\lfloor\rho^{1/2}\rfloor$, this is $O(\rho^{3/2})=o(\rho^3)$, so the stated cap–loss bound $\mathrm{CapLoss}(h_\rho)\lesssim L\,\rho^2+\rho^3/L$ remains valid with $h_\rho:E_\rho\to\mathbb{Z}$ integer-valued. □

Corollary 7.24 (Heisenberg liminf with shear)**.** In the setting of Theorem 7.15, after gauge-column symmetrization and along a convergent subsequence of stacks $E_{\rho,s}$, one has

$$\liminf_{\rho\to\infty}\;\rho^{-3}\,B_S(Y_\rho)\;\ge\;\int_0^H \mathrm{Per}_{\tau_S}(E_s)\,ds\;+\;2\,\mathcal{S}^*.$$

Consequently,

$$\liminf_{\rho\to\infty}\;\rho^{-3}\,|\partial_S Y_\rho|\;\ge\;\frac{1}{|S|}\int_0^H \mathrm{Per}_{\tau_S}(E_s)\,ds\;+\;\frac{2}{|S|}\,\mathcal{S}^*.$$

Proof. Start from Corollary 7.13 and divide by ρ^3. The cap-loss vanishes by Proposition 7.20. The base term follows by column slicing/coarea on the Abelian base and the discrete Wulff lower bound there (cf. §3). Finally use Lemma 7.14. □

Proof of Theorem C . The exact identity stated in Theorem C is precisely (7.3), with the interval symmetric difference evaluated by Lemma 7.7 and the alignment shifts σ_v from Lemma 7.4. The edge–height bijection is Lemma 7.6. The two–sided per-edge bounds (7.2) yield the global bounds (7.5).

For the "quantified bounds" part: along gauge-column-convex stacks Y_ρ with $|Y_\rho|\asymp\rho^4$, the lower bound at scale ρ^3 follows from Corollary 7.13 together with Proposition 7.20 (cap-loss $o(\rho^3)$), while the complementary upper bound is given by the recovery/stack construction in Theorem 7.15. All statements are in the *undirected* normalization for B_S; the directed version is obtained by multiplying every identity by 2 (Remark 7.25). □

Remark 7.25 (Directed vs. undirected in this section)**.** All statements in §7 are written for the *undirected* horizontal boundary B_S. If one prefers the *directed* edge count $B_S^{\to}$, simply multiply each identity and inequality by 2. This applies verbatim to Theorem 7.11, Corollary 7.13, Proposition 7.20, and Theorem 7.15.

8. Shear calculus on step–2 Carnot lattices: exact identity (rank 1) and sharp bounds (general)

Algebraic set-up. Let G be a step–2 Carnot group with stratification $\mathfrak{g}=V_1\oplus V_2$ and $[V_1,V_1]\subset V_2$ central. Fix a *uniform lattice* $\Gamma\le G$ and a Malcev basis so that in (integer) coordinates we can write $\Gamma\cong\mathbb{Z}^d\times\mathbb{Z}^m$ with multiplication

$$(x,h)\cdot(x',h')\;=\;\big(x+x',\;h+h'+\omega(x,x')\big), \tag{8.1}$$

where $\omega : \mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{Z}^m$ is a fixed biadditive (bilinear) 2–cocycle. We further assume (after an ordered choice of the horizontal basis $(e_1, \dots, e_d)$) the *upper–triangular normalization*

$$\omega(e_i, e_i) = 0, \qquad \omega(e_j, e_i) = 0 \text{ for } j > i, \tag{8.2}$$

i.e. only entries with first index < second index can be nonzero (Hall/Malcev collection; changing coordinates affects constants only).

Let $\mathcal{S} = \{\pm e_1, \dots, \pm e_d\}$ be the horizontal generators (unit steps in V_1; compare S in §7), and let $B_{\mathcal{S}}$ denote the *undirected* horizontal edge boundary on the Cayley graph with *right* action (as in §7; the directed version is obtained by multiplying by 2).

Gauge. Define the (quadratic) gauge

$$\zeta(x) \;:=\; \sum_{1 \le i < j \le d} x_i\, x_j\, \omega(e_i, e_j) \;\in \mathbb{Z}^m, \qquad T(x,h) \;:=\; \big(x,\ h^\sharp := h - \zeta(x)\big). \tag{8.3}$$

Right multiplying (x, h) by e_k sends $(x, h^\sharp)$ to

$$\big(x + e_k,\ h^\sharp + \sigma_{e_k}(x)\big), \qquad \sigma_{e_k}(x) \;:=\; \omega(x, e_k) \;-\; \big(\zeta(x + e_k) - \zeta(x)\big). \tag{8.4}$$

From $\zeta(x) = \sum_{i<j} x_i x_j \omega(e_i, e_j)$ we have

$$\zeta(x{+}e_k) - \zeta(x) = \sum_{i<k} x_i\, \omega(e_i, e_k) + \sum_{k<j} x_j\, \omega(e_k, e_j), \tag{8.5}$$

so, using $\omega(x, e_k) = \sum_i x_i\, \omega(e_i, e_k)$,

$$\begin{aligned} \sigma_{e_k}(x) &= \sum_{i \ge k} x_i\, \omega(e_i, e_k) \;-\; \sum_{j>k} x_j\, \omega(e_k, e_j) \\ &= \sum_{j>k} x_j \big(\omega(e_j, e_k) - \omega(e_k, e_j)\big) \;+\; x_k\, \omega(e_k, e_k). \end{aligned} \tag{8.6}$$

Under (8.2) the diagonal term vanishes ($\omega(e_k, e_k) = 0$), hence

$$\sigma_{e_k}(x) \;=\; -\sum_{j>k} x_j\, \omega(e_k, e_j) \;\in \mathbb{Z}^m, \tag{8.7}$$

and $\sigma_{-e_k}(x) = -\sigma_{e_k}(x - e_k)$ *(this latter identity uses* (8.2)*)*. In particular, $\sigma_{e_d} \equiv 0$. All identities below use the *right* action convention in (8.4).

Gauge–column convexity. For $A \subset \Gamma$ write $A^\sharp := T(A)$. We say A is *gauge–column–convex* if for each $u \in \mathbb{Z}^d$ the fiber

$$A^\sharp_u \;:=\; \{h^\sharp \in \mathbb{Z}^m :\ (u, h^\sharp) \in A^\sharp\}$$

is an *axis–aligned box* (half–open rectangle) of the form $I_u := \prod_{j=1}^m [\, h_j(u),\ h_j(u) + \ell_j(u)\,)$ with $\ell_j(u) \in \mathbb{N}$ and $h_j(u) \in \mathbb{Z}$. Define the footprint $E := \{u : \prod_j \ell_j(u) > 0\}$ and the *height vector* $\boldsymbol{\ell}(u) = (\ell_1(u), \dots, \ell_m(u))$. All counts below use the half–open convention.

8.1. **Per–edge bookkeeping and global identities.**

Definition 8.1 (Oriented base edges)**.** As in §7, let $\mathcal{E}_{\mathrm{int}}(E) := \{(u \to u+v) :\ u, u+v \in E,\ v \in \{e_1, \dots, e_d\}\}$ and $\mathcal{E}_{\mathrm{bd}}(E) := \{(u \to u+v) :\ u \in E,\ u+v \notin E,\ v \in \{\pm e_1, \dots, \pm e_d\}\}$, so that each undirected broken base edge appears exactly once by choosing the orientation $u \to u+v$ with $u \in E$ and $u+v \notin E$.

Lemma 8.2 (Edge–height bijection in the step–2 gauge)**.** Let $A \subset \Gamma$ be gauge–column–convex with fibers I_u as above. For $e = (u \to u+v)$ define the *normalized relative shift*

$$\tau(e) \;:=\; h(u) - h(u+v) + \sigma_v(u) \;\in \mathbb{Z}^m. \tag{8.8}$$

Then the number of broken undirected horizontal Cayley edges of A across e equals

$$\#\Big(\, I_u \,\triangle\, \big(I_{u+v} - \sigma_v(u)\big) \Big) \;=\; \#\Big(\, [0, \boldsymbol{\ell}(u)) \,\triangle\, \big([0, \boldsymbol{\ell}(u+v)) - \tau(e)\big) \Big),$$

where $[0,\boldsymbol{\ell}) := \prod_{j=1}^{m}[0,\ell_j)$, and the second equality uses a common translation by $-h(u)$ (which preserves symmetric-difference cardinality).

Remark 8.3 (Sign bridge to §7)**.** In §7 we used the representation $(u,h) \mapsto (u+v, h-\sigma_v(u))$ and the shift $I_{u+v}+\sigma_v(u)$; here we use $(u,h^\sharp) \mapsto (u+v, h^\sharp+\sigma_v(u))$ and the shift $I_{u+v}-\sigma_v(u)$. The normalized parameters satisfy $\tau(e) = -t(e)$ relative to §7, so the 1D identities coincide after this sign change.

Lemma 8.4 (1D half–open symmetric–difference identity)**.** Let $a,b \in \mathbb{N}$ and $\tau \in \mathbb{Z}$. Write $\Delta := a-b$, $\Delta_+ := \max\{0,\Delta\}$, $\Delta_- := \max\{0,-\Delta\}$, and $x_+ := \max\{0,x\}$. Then

$$\#\big([0,a) \,\triangle\, ([0,b)-\tau)\big) \;=\; |\Delta| \;+\; 2\,\min\Big(\min\{a,b\},\; (\tau-\Delta_-)_+ \;+\; (-\tau-\Delta_+)_+\Big). \tag{8.9}$$

Proof. Let $I := [0,a)$ and $J := [0,b)-\tau = [-\tau, b-\tau)$. Put $d := \#(I\triangle J) = a+b-2\,\#(I\cap J)$. A direct computation gives

$$\#(I\cap J) = \big(\min\{a,b-\tau\} - \max\{0,-\tau\}\big)_+.$$

Case $\tau \ge 0$. Then $\#(I\cap J) = \min\{a,b-\tau\}_+$. There are three regimes:

(1) $\tau \ge b$: $\#(I\cap J) = 0$, so $d = a+b$.

(2) $b-a < \tau < b$: $\#(I\cap J) = b-\tau$, so $d = a+b-2(b-\tau) = a-b+2\tau$.

(3) $0 \le \tau \le b-a$: $\#(I\cap J) = a$, so $d = a-b$.

On the right side of (8.9), when $\tau \ge 0$ we have $(-\tau-\Delta_+)_+ = 0$ and $(\tau-\Delta_-)_+ = \max\{0,\tau-\Delta_-\} = \min\{\tau,\min\{a,b\}\}$. This reproduces exactly the three values above.

Case $\tau \le 0$. By swapping the roles of a,b and sending $\tau \mapsto -\tau$ we reduce to the previous case. For completeness: $\#(I\cap J) = \min\{a+\tau,b\}_+$, and the same trichotomy holds with (a,b,τ) replaced by $(b,a,-\tau)$. Substituting into (8.9) with $\Delta_\pm$ unchanged proves the formula for $\tau \le 0$. $\square$

Theorem 8.5 (Exact identity for rank–1 center; sharp two–sided bounds in general)**.** Let $A \subset \Gamma$ be gauge–column–convex with footprint E *(edge orientation as in Definition 8.1, counts taken with the right action convention* (8.4)*).*

(i) Rank $m=1$ *(Heisenberg–type).* Writing $\ell(u) \in \mathbb{N}$ and $I_u = [h(u), h(u)+\ell(u))$, set for $e = (u \to u+v)$

$$\Delta(e) := \ell(u) - \ell(u+v), \qquad \Delta_+(e) := \max\{0,\Delta(e)\},\;\; \Delta_-(e) := \max\{0,-\Delta(e)\}.$$

With the normalized shift $\tau(e) \in \mathbb{Z}$ from (8.8), the *exact* undirected boundary identity holds:

$$\begin{aligned} B_{\mathcal{S}}(A) \;=\; &\sum_{e\in\mathcal{E}_{\mathrm{bd}}(E)} \ell(u) \\ &+ \sum_{e\in\mathcal{E}_{\mathrm{int}}(E)} \Bigg(|\Delta(e)| \;+\; 2\,\min\Big(\min\{\ell(u),\ell(u+v)\},\; \big(\tau(e)-\Delta_-(e)\big)_+ + \big(-\tau(e)-\Delta_+(e)\big)_+\Big)\Bigg), \end{aligned} \tag{8.10}$$

where $x_+ := \max\{0,x\}$. Exactness uses the half–open convention and Lemma 8.4.

(ii) General rank $m \ge 1$ *(product boxes).* For $e = (u \to u+v)$ and each coordinate j define

$$\Delta_j(e) := \ell_j(u) - \ell_j(u+v), \quad \Delta_{j,\pm}(e) := \max\{0, \pm\Delta_j(e)\},$$

and the one–sided truncated shear addendum

$$S_j(e) \;:=\; 2\,\min\Big(\min\{\ell_j(u),\ell_j(u+v)\},\; \big(\tau_j(e)-\Delta_{j,-}(e)\big)_+ \;+\; \big(-\tau_j(e)-\Delta_{j,+}(e)\big)_+\Big).$$

Let

$$A^{(j)}_{\max}(e) := \prod_{k\ne j} \max\{\ell_k(u),\ell_k(u{+}v)\}, \qquad A^{(j)}_{\cap}(e) := \prod_{k\ne j} \#\Big([0,\ell_k(u)) \cap \big([0,\ell_k(u{+}v)) - \tau_k(e)\big)\Big).$$

Then

$$\sum_{e\in\mathcal{E}_{\rm bd}(E)}\prod_{j=1}^{m}\ell_j(u)\;+\;\sum_{e\in\mathcal{E}_{\rm int}(E)}\left[\sum_{j=1}^{m}\big(|\Delta_j(e)|+S_j(e)\big)\,A_{\cap}^{(j)}(e)\right]$$
$$\le\; B_{\mathcal{S}}(A) \tag{8.11}$$
$$\le\;\sum_{e\in\mathcal{E}_{\rm bd}(E)}\prod_{j=1}^{m}\ell_j(u)\;+\;\sum_{e\in\mathcal{E}_{\rm int}(E)}\left[\sum_{j=1}^{m}\big(|\Delta_j(e)|+S_j(e)\big)\,A_{\max}^{(j)}(e)\right]. \tag{8.12}$$

Proof of Theorem 8.5(ii). Fix an internal base edge $e=(u\to u+v)$ and abbreviate $a_j:=\ell_j(u)$, $b_j:=\ell_j(u+v)$, $\tau_j:=\tau_j(e)$,

$$I_j:=[0,a_j),\qquad J_j:=[0,b_j)-\tau_j,\qquad X_j:=I_j\setminus J_j,\quad Y_j:=J_j\setminus I_j.$$

By Lemma 8.2, the contribution across e equals

$$\#\Big(\prod_{j=1}^{m}I_j\,\triangle\,\prod_{j=1}^{m}J_j\Big).$$

Upper bound. Use the exact telescoping identity (pointwise on $\mathbb{Z}^m$):

$$\mathbf{1}_{\prod I}-\mathbf{1}_{\prod J}=\sum_{j=1}^{m}\Big(\prod_{k<j}\mathbf{1}_{I_k}\Big)\cdot\big(\mathbf{1}_{I_j}-\mathbf{1}_{J_j}\big)\cdot\Big(\prod_{k>j}\mathbf{1}_{J_k}\Big).$$

Taking absolute values and summing over $\mathbb{Z}^m$ yields

$$\#\Big(\prod I\,\triangle\,\prod J\Big)\;\le\;\sum_{j=1}^{m}\#(I_j\triangle J_j)\,\prod_{k\neq j}\max\{\#I_k,\#J_k\}.$$

By Lemma 8.4, $\#(I_j\triangle J_j)=|\Delta_j|+S_j$, and $\#I_k=a_k$, $\#J_k=b_k$. This gives the single–edge upper bound with $A_{\max}^{(j)}$, and summing over $e\in\mathcal{E}_{\rm int}(E)$ proves (8.12). For boundary edges $e\in\mathcal{E}_{\rm bd}(E)$ the contribution is $\prod_j a_j$, as stated.

Lower bound (disjoint witness cylinders). Define the j-witness cylinder

$$\mathsf{S}_j\;:=\;\big(X_j\times\prod_{k\neq j}(I_k\cap J_k)\big)\;\cup\;\big(Y_j\times\prod_{k\neq j}(I_k\cap J_k)\big).$$

Every point of S_j lies in $\prod I\triangle\prod J$ (it differs in the j-th coordinate and agrees in the others), hence $\bigcup_{j=1}^{m}\mathsf{S}_j\;\subset\;\prod I\triangle\prod J$. Moreover, the sets S_j are *pairwise disjoint*: if $z\in\mathsf{S}_j\cap\mathsf{S}_k$ with $j\neq k$, then $z_k\in I_k\cap J_k$ (from $z\in\mathsf{S}_j$) and $z_k\in I_k\triangle J_k$ (from $z\in\mathsf{S}_k$), a contradiction. Therefore,

$$\#\Big(\prod I\,\triangle\,\prod J\Big)\;\ge\;\#\Big(\bigcup_{j=1}^{m}\mathsf{S}_j\Big)\;=\;\sum_{j=1}^{m}\#\mathsf{S}_j\;=\;\sum_{j=1}^{m}(\#X_j+\#Y_j)\,\prod_{k\neq j}\#(I_k\cap J_k).$$

Again $\#X_j+\#Y_j=\#(I_j\triangle J_j)=|\Delta_j|+S_j$ by Lemma 8.4, and $\prod_{k\neq j}\#(I_k\cap J_k)=A_{\cap}^{(j)}$. Summing the single–edge estimate over $e\in\mathcal{E}_{\rm int}(E)$ and adding the boundary term $\prod_j a_j$ gives (8.11). $\square$

Remark 8.6 (On $A_{\min}$ vs. $A_{\cap}$ in the lower bound)**.** The proof shows that the correct cross–sections for the lower bound are the *actual overlaps*, $A_{\cap}^{(j)}(e)=\prod_{k\neq j}\#\big(I_k\cap J_k\big)$, coming from the cylinder construction. One always has $A_{\cap}^{(j)}\le A_{\min}^{(j)}:=\prod_{k\neq j}\min\{a_k,b_k\}$; hence replacing $A_{\cap}^{(j)}$ by $A_{\min}^{(j)}$ would claim a strictly stronger (and in general false) inequality. In regimes where the non–witnessed coordinates overlap substantially (e.g. under near–minimizing sequences where all $|\tau_k|$ are small compared to a_k,b_k), $A_{\cap}^{(j)}$ and $A_{\min}^{(j)}$ are asymptotically equivalent, and that heuristic becomes accurate. All compactness and Γ–bounds below use the $A_{\cap}$ version proved above.

Lemma 8.7 (Constant face curl)**.** Define the shear 1–cochain on oriented base edges by

$$c_{e_k}(x) \;:=\; -\,\sigma_{e_k}(x).$$

Then, for the standard square face orientation, the 2–cochain dc is *constant* and equals the skew part of ω:

$$(dc)_{ij} \;\equiv\; \omega(e_i,e_j) - \omega(e_j,e_i) \;\in \mathbb{Z}^m \qquad (1 \le i,j \le d).$$

In particular, under (8.2) this simplifies to $(dc)_{ij} \equiv \omega(e_i,e_j)$ for $i<j$, while $(dc)_{ii} \equiv 0$.

Proof. From (8.6),

$$\sigma_{e_k}(x) = x_k\,\omega(e_k,e_k) + \sum_{r>k} x_r\big(\omega(e_r,e_k) - \omega(e_k,e_r)\big),$$

so

$$c_{e_k}(x) = -\sigma_{e_k}(x) = -x_k\,\omega(e_k,e_k) + \sum_{r>k} x_r\big(\omega(e_k,e_r) - \omega(e_r,e_k)\big).$$

Hence, for $i \neq j$,

$$\Delta_{e_j}c_{e_i}(x) = \begin{cases} \omega(e_i,e_j) - \omega(e_j,e_i), & j>i, \\ 0, & j<i, \end{cases} \qquad \Delta_{e_i}c_{e_j}(x) = \begin{cases} \omega(e_j,e_i) - \omega(e_i,e_j), & i>j, \\ 0, & i<j. \end{cases}$$

Therefore for all i,j,

$$(dc)_{ij} = \Delta_{e_j}c_{e_i} - \Delta_{e_i}c_{e_j} = \omega(e_i,e_j) - \omega(e_j,e_i),$$

independently of the diagonal values $\omega(e_k,e_k)$. Under (8.2) the second term vanishes for $i<j$, giving the stated simplification and $(dc)_{ii} \equiv 0$. □

8.2. **Interface synchronization across blocks.** We keep the set-up and notation from the proof of Theorem 8.5(ii), and fix a large scale ρ. After discarding an $O(L)$-thick outer layer of the footprint (absorbed in the boundary term), we may assume the base footprint is a disjoint union of d-dimensional blocks

$$Q \in \mathcal{Q} \;\cong\; \{0,\dots,n-1\}^d, \qquad n := \lfloor \rho/L \rfloor,$$

each Q a discrete cube of side L, with the obvious adjacency graph $\mathbb{G} = (\mathcal{Q},\mathcal{F})$ whose edges $\mathcal{F}$ are unordered adjacent block pairs $Q \sim Q'$. For a fixed coordinate $j \in \{1,\dots,m\}$ and an interface (hyperface) $F = Q \mid Q' \in \mathcal{F}$ orthogonal to e_i, write $\mathcal{E}(F)$ for the set of base edges crossing F in direction e_i. Recall the per-edge normalized shifts $\tau_j(e)$ from (8.8), and the weights

$$A^{(j)}_{\cap}(e) \;:=\; \prod_{k\neq j} \#\Big([0,\ell_k(u)) \cap \big([0,\ell_k(u+v)) - \tau_k(e)\big)\Big).$$

Define the *face weight*

$$W^{(j)}(F) \;:=\; \sum_{e\in\mathcal{E}(F)} A^{(j)}_{\cap}(e).$$

Under the bulk assumptions $|Y_\rho| \asymp \rho^Q$ and $\rho^{-(Q-1)}B_{\mathcal{S}}(Y_\rho) \le C$, we will use the crude bound

$$W^{(j)}(F) \;\lesssim\; L^{d-1}\,\rho^{m-1}, \tag{8.13}$$

uniformly over F and j (the implied constant depends only on (d,m) and the data of ω).

Lemma 8.8 (Microscopic interface control: detailed telescoping)**.** After the blockwise interior construction (anchoring $h \equiv 0$ at the chosen block roots), there is a constant $C_1 = C_1(d,m,\omega)$ such that for every interface edge $e = (u \to u+e_i) \in \mathcal{E}(F)$ and for every j,

$$|\tau_j(e)| \;\le\; C_1\,L.$$

Proof (explicit telescoping via discrete Stokes). Fix $F = Q \mid Q'$ orthogonal to e_i. Choose the block roots $r_Q \in Q$ and $r_{Q'} \in Q'$ so that $r_{Q'} = r_Q + e_i$, and normalize $h^{(Q)}(r_Q) = h^{(Q')}(r_{Q'}) = 0$. Let $u \in F \cap Q$ be arbitrary. Take any ℓ^1–monotone path in the face from r_Q to u,

$$\gamma : \ v_0 = r_Q, \ v_{k+1} = v_k + e_{s_k} \quad (s_k \in \{1, \ldots, d\} \setminus \{i\}), \qquad k = 0, \ldots, \ell - 1,$$

with $\ell \leq C_d\, L$.

For each step k, consider the elementary (i, s_k)–plaquette with lower left corner v_k and boundary (positively oriented)

$$\partial R_k : \ v_k \xrightarrow{e_i} v_k{+}e_i \xrightarrow{e_{s_k}} v_{k+1}{+}e_i \xrightarrow{-e_i} v_{k+1} \xrightarrow{-e_{s_k}} v_k.$$

Define the cochain

$$\alpha_j := \delta h_j{+}c_{\cdot,j} \text{ on interior edges (within } Q \text{ or } Q'), \qquad \alpha_j(e_i@x) := \tau_j(x \to x{+}e_i) \text{ on interface edges.}$$

Since $d(\delta h_j) = 0$ and dc is constant (Lemma 8.7), we have for every k

$$\sum_{e \in \partial R_k} \alpha_j(e) \ = \ (dc)_{i\, s_k, j}. \tag{8.14}$$

Summing (8.14) over $k = 0, \ldots, \ell - 1$ gives

$$\sum_{k=0}^{\ell-1} \big(\tau_j(v_k) - \tau_j(v_{k+1})\big) \ + \ \Big[\sum_{e \in \gamma + e_i} \alpha_j(e) \ - \ \sum_{e \in \gamma} \alpha_j(e) \Big] \ = \ \sum_{k=0}^{\ell-1} (dc)_{i\, s_k, j}. \tag{8.15}$$

By definition of α_j and the normalization at the roots,

$$\sum_{e \in \gamma} \alpha_j(e) = h_j^{(Q)}(u) - h_j^{(Q)}(r_Q) + \sum_{e \in \gamma} c_{\cdot,j}(e) = h_j^{(Q)}(u) + \sum_{e \in \gamma} c_{\cdot,j}(e),$$

and similarly $\sum_{e \in \gamma + e_i} \alpha_j(e) = h_j^{(Q')}(u + e_i) + \sum_{e \in \gamma + e_i} c_{\cdot,j}(e)$. Insert these into (8.15), use the definition $\tau_j(v_\ell) = \tau_j(u) = h_j^{(Q)}(u) - h_j^{(Q')}(u + e_i) + c_{e_i,j}(u)$ and $\tau_j(v_0) = \tau_j(r_Q) = c_{e_i,j}(r_Q)$ (since both roots have $h \equiv 0$), and rearrange. The h–terms cancel, and the c–terms collapse to $c_{e_i,j}(u) - c_{e_i,j}(r_Q)$ by discrete Stokes on the *tangential* strip γ (they encode the sum of $\Delta_{e_{s_k}} c_{e_i,j}$ along γ). We obtain the identity

$$\tau_j(u) - \tau_j(r_Q) \ = \ \sum_{k=0}^{\ell-1} (dc)_{i\, s_k, j}.$$

Taking absolute values and using $\ell \leq C_d L$ gives

$$|\tau_j(u)| \ \leq \ |\tau_j(r_Q)| \ + \ C_d\, L\, \|dc\|_\infty.$$

Finally, under (8.2) the explicit formula for c in the proof of Lemma 8.7 shows

$$\tau_j(r_Q) = c_{e_i,j}(r_Q) = \sum_{r > i} (r_Q)_r \, \big(\omega(e_i, e_r) - \omega(e_r, e_i)\big)_j,$$

whence $|\tau_j(r_Q)| \leq C_\omega\, L$. This yields the claim with $C_1 := C_d \|dc\|_\infty + C_\omega$. □

Definition 8.9 (Weighted (integer) median on a hyperface)**.** For $F \in \mathcal{F}$ and fixed j, a number $m^{(j)}(F) \in \mathbb{Z}$ is a *weighted median* of the multiset $\{\tau_j(e) : e \in \mathcal{E}(F)\}$ with weights $\{A_\cap^{(j)}(e)\}$ if it minimizes $\phi(s) := \sum_{e \in \mathcal{E}(F)} A_\cap^{(j)}(e)\, |\tau_j(e) - s|$ over $s \in \mathbb{Z}$.

Lemma 8.10 (Hyperface median reduction)**.** For each F and j, let $m^{(j)}(F)$ be a weighted median as in Definition 8.9. Then

$$\sum_{e \in \mathcal{E}(F)} A_\cap^{(j)}(e)\, \big|\tau_j(e) - m^{(j)}(F)\big| \ \leq \ C_2\, L\, W^{(j)}(F),$$

with $C_2 = C_2(d, m, \omega)$, and moreover $|m^{(j)}(F)| \leq C_1 L$ by Lemma 8.8.

Proof. By optimality of the (integer) weighted median, $\sum A_\cap^{(j)}|\tau_j - m^{(j)}| \le \sum A_\cap^{(j)} \min_{s\in\mathbb{Z}} |\tau_j - s| \le \sum A_\cap^{(j)} |\tau_j|$. Apply Lemma 8.8 and the definition of $W^{(j)}(F)$. □

Lemma 8.11 (Tree synchronization)**.** Fix a spanning tree T of $\mathbb{G}$. There exist block constants $a(Q) \in \mathbb{Z}^m$ such that, if one sets the inter-block increments on every tree face $F = Q \mid Q' \in T$ by

$$\Delta a^{(j)}(F) \;:=\; a^{(j)}(Q) - a^{(j)}(Q') \;=\; -\,m^{(j)}(F),$$

then the total weighted L^1 cost over all tree faces satisfies

$$\sum_{F\in T} \sum_{e\in\mathcal{E}(F)} \sum_{j=1}^{m} A_\cap^{(j)}(e)\,\big|\tau_j(e) + \Delta a^{(j)}(F)\big| \;\le\; C_2\,L \sum_{F\in T} \sum_{j=1}^{m} W^{(j)}(F) \;\lesssim\; \rho^{d+m-1},$$

where the last bound uses $|T| = |\mathcal{Q}| - 1 \asymp n^d$ and (8.13).

Proof. Define a on the vertices of T by fixing a at one root block and extending uniquely along T with the displayed increments. On $F \in T$, the residual on $\mathcal{E}(F)$ equals $\tau_j(e) - m^{(j)}(F)$; the bound follows from Lemma 8.10 and (8.13). □

Lemma 8.12 (Cycle residual)**.** Let $F \in \mathcal{F} \setminus T$ and let $\mathscr{C}(F)$ be the unique simple cycle obtained by adding F to T. Then, for each j,

$$\Big|\sum_{F'\in\mathscr{C}(F)} m^{(j)}(F')\Big| \;\le\; C_3\,n\,L \;\le\; C_3\,\rho,$$

with $C_3 = C_3(d, m, \omega)$. Consequently,

$$\sum_{e\in\mathcal{E}(F)} \sum_{j=1}^{m} A_\cap^{(j)}(e)\,\big|\tau_j(e) + \Delta a^{(j)}(F)\big| \;\le\; C_3\,\rho \sum_{j=1}^{m} W^{(j)}(F).$$

Proof. By construction, $\Delta a^{(j)}(F') = -m^{(j)}(F')$ on $F' \in T$ and $\Delta a^{(j)}(F)$ is determined by compatibility around the cycle $\mathscr{C}(F)$, whence the residual on F equals the sum of $m^{(j)}$ around $\mathscr{C}(F)$. The cycle length is $O(n)$ and $|m^{(j)}(F')| \le C_1 L$ by Lemma 8.10. □

Proposition 8.13 (Interface cost bound)**.** With a chosen as in Lemma 8.11,

$$\sum_{F\in\mathcal{F}} \sum_{e\in\mathcal{E}(F)} \sum_{j=1}^{m} A_\cap^{(j)}(e)\,\big|\tau_j(e) + \Delta a^{(j)}(F)\big| \;\lesssim\; \rho^{d+m-1} \;+\; \frac{\rho^{d+m}}{L}.$$

In particular, for all $1 \le L \le \rho$ the right-hand side is $o(\rho^{Q-1})$.

Proof. Split faces into T and $\mathcal{F} \setminus T$. The contribution of T is $\lesssim \sum_{F\in T} L \sum_j W^{(j)}(F) \lesssim n^d \cdot L \cdot L^{d-1}\rho^{m-1} = \rho^{d+m-1}$ by Lemma 8.11 and (8.13). For $\mathcal{F} \setminus T$, use Lemma 8.12 and (8.13):

$$\sum_{F\notin T} \sum_{j=1}^{m} C_3\,\rho\,W^{(j)}(F) \;\lesssim\; \rho \cdot |\mathcal{F}| \cdot L^{d-1}\rho^{m-1} \;\asymp\; \rho \cdot n^d \cdot L^{d-1}\rho^{m-1} \;=\; \frac{\rho^{d+m}}{L}.$$

□

Remark 8.14. The proof uses only the constancy of dc (Lemma 8.7), the blockwise interior control, and weighted L^1 median properties on hyperfaces; no hidden assumptions are introduced. The tree term is $O(\rho^{d+m-1})$, and the off-tree term is $O(\rho^{d+m}/L)$; both are $o(\rho^{Q-1})$ for $1 \le L \le \rho$.

Corollary 8.15 (Cap–loss control and sharp scaling)**.** Let (Y_ρ) be gauge–column–convex stacks with $|Y_\rho| \asymp \rho^Q$ and $\rho^{-(Q-1)} B_{\mathcal{S}}(Y_\rho) \le C$, where $Q = d + 2m$. Then there exist offsets h_ρ (i.e.

choices of integer fiber bases $h(u)$ for each basepoint u, not a single global central translate) such that the *cap–loss*

$$\mathrm{CapLoss}(h_\rho) \;:=\; \sum_{e\in\mathcal{E}_{\mathrm{int}}(E)} \sum_{j=1}^{m} S_j(e)\, A_\cap^{(j)}(e)$$

satisfies

$$\mathrm{CapLoss}(h_\rho) \;\le\; C_0 \left(L_\rho\, \rho^{d+m-1} + \frac{\rho^{d+m}}{L_\rho} \right), \qquad L_\rho = \lfloor \rho^{1/2} \rfloor,$$

and hence $\mathrm{CapLoss}(h_\rho) = O(\rho^{d+m-\frac{1}{2}}) = o(\rho^{Q-1})$.

Proof. Write the internal contribution over edges as in Theorem 8.5(ii). For $e = (u \to u+v)$ and coordinate j we have

$$S_j(e) = 2\min\Big(\min\{a_j, b_j\},\ (\tau_j - \Delta_{j,-})_+ + (-\tau_j - \Delta_{j,+})_+ \Big),$$

with $\tau_j = \tau_j(e) = h_j(u) - h_j(u+v) + c_{e_j}(u)$, where c_{e_j} is the j–th coordinate of the shear 1–cochain c from Lemma 8.7. Thus, up to truncations (which only help), CapLoss is controlled by the weighted ℓ^1 norm of the *coboundary* $\delta h + c$:

$$\sum_{e\in\mathcal{E}_{\mathrm{int}}(E)} \sum_{j=1}^{m} \big|\, (\delta h)_j(e) \;+\; c_{e_j}(u) \,\big| \;\cdot\; A_\cap^{(j)}(e).$$

The interior block construction yields the bound $L\,\rho^{d+m-1}$, with weights $\lesssim \rho^{m-1}$ on the bulk. Gluing interfaces by Proposition 8.13 adds $\rho^{d+m-1} + \rho^{d+m}/L$. Choosing $L = L_\rho = \lfloor \rho^{1/2} \rfloor$ balances the last two terms and gives the stated estimate. □

Remark 8.16 (Terminology and consistency with §7)**.** In §7, "CapLoss" referred to an *excess* term of the form $\sum(|t| - A)_+$. Here, for rank $m \ge 1$, we use "CapLoss" for the *truncated shear* contribution $\sum S_j(\cdot)\, A_\cap^{(j)}(\cdot)$, which is the multi–rank analogue entering the two–sided bounds. Both quantities are controlled via the same blockwise L^1 curl–fit mechanism and vanish at scale $o(\rho^{Q-1})$ along near–minimizers.

Remark 8.17 (Consequences)**.** Theorem 8.5 and Corollary 8.15 give the *exact* decomposition in rank 1 via the one–sided truncated shift in (8.10), and *sharp two–sided* control for all step–2 lattices. In particular, one obtains order–sharp compactness and Γ–$\limsup$ / $\liminf$ bounds at the scale ρ^{Q-1} for arbitrary step–2 groups, with anisotropy encoded by $\mathcal{S}$ through the zonotope gauge on V_1 and by the skew tensor dc in Lemma 8.7. The lower bound in (ii) uses disjoint witness cylinders, eliminating a spurious $1/m$ factor and strengthening the estimate.

Corollary 8.18 (Step-2 Carnot lattices: BRP via shear)**.** Every uniform lattice in a step-2 Carnot group admits (NNM) (Theorem 8.5 and Corollary 8.15). Hence it satisfies full (TF) (Theorem 9.100) and $\sup_{r\ge1} I^\circ(r)/I^{\mathrm{incr}\,\circ}(r) < \infty$ (Theorem 9.31).

9. Exact-Volume Profile, Monotone Minorant: Bounded Ratios and the Tempered Følner criterion

Standing conventions and normalization. NEW Throughout Section 9 we work with connected Cayley graphs of finitely generated groups with a fixed finite, symmetric generating set $\mathcal{S}$ *not* containing the identity. We denote $\Delta := |\mathcal{S}|$ the (directed) degree. Unless stated otherwise, $I^\circ(\cdot)$ and $I^{\mathrm{incr}\,\circ}(\cdot)$ are in the *vertex* normalization; for directed-edge counts we write I°_{edge} and $I^{\mathrm{incr}\,\circ}_{\mathrm{edge}}$ (cf. Definition 9.9). The pointwise comparison

$$|\partial_{\mathcal{S}} Y| \;\le\; B_{\mathcal{S}}(Y) \;\le\; \Delta\, |\partial_{\mathcal{S}} Y| \tag{9.1}$$

(Lem. 9.15) converts all statements up to a factor depending only on Δ. We also use the undirected edge boundary $B_{\mathcal{S}}(Y) = \frac{1}{2}\sum_{s\in\mathcal{S}} |sY \triangle Y|$ in Subsection 9.6, with conversion to the directed normalization costing a universal constant by Lemma 9.15.

For a finite $Y \subset \Gamma$, the (vertex) $\mathcal{S}$-boundary is $\partial_{\mathcal{S}}(Y) = \mathcal{S}Y \setminus Y$. Recall

$$I^{\circ}(r) := \inf\{\, |\partial_{\mathcal{S}}(Y)| : \ |Y| = r\,\}, \qquad I^{\mathrm{incr}\,\circ}(r) := \inf_{s\geq r} I^{\circ}(s),$$

with piecewise-linear extension in r. Set $\Delta := |\mathcal{S}|$. Our aim is to understand the ratio $\mathcal{R}(r) := I^{\circ}(r)/I^{\mathrm{incr}\,\circ}(r)$ (cf. [Gro08a]), with the convention $\mathcal{R}(r) = 1$ if $I^{\mathrm{incr}\,\circ}(r) = 0$.

9.1. **Three structural tools: Lipschitz, tail trimming, subadditivity.**

Proposition 9.1 (Lipschitz in r)**.** For all integers $r \geq 1$, one has $|I^{\circ}(r+1) - I^{\circ}(r)| \leq \Delta$.

Proof. Fix $Y \subset \Gamma$ with $|Y| = r$ and $x \notin Y$, and set $Z := Y \cup \{x\}$. Then $\partial_{\mathcal{S}} Z \subset \big(\partial_{\mathcal{S}} Y \setminus \{x\}\big) \cup \big(\mathcal{S}x \setminus Z\big)$, so $|\partial_{\mathcal{S}} Z| \leq |\partial_{\mathcal{S}} Y| + \Delta$. Taking the infimum over $|Y| = r$ gives $I^{\circ}(r+1) \leq I^{\circ}(r) + \Delta$.

For the reverse inequality in the vertex normalization, we in fact have $I^{\circ}(r) \leq I^{\circ}(r+1) + 1$: deleting one vertex y from a set Z can create at most the single new boundary vertex y itself, since $S(Y) \subset S(Z)$ when $Y \subset Z$ (and hence $\partial_{\mathcal{S}} Y \subset \partial_{\mathcal{S}} Z \cup \{y\}$). In the directed-edge normalization this becomes $I^{\circ}_{\mathrm{edge}}(r) \leq I^{\circ}_{\mathrm{edge}}(r+1) + \Delta$. □

Proposition 9.2 (Tail trimming)**.** For all integers $s \geq r \geq 1$,

$$I^{\circ}(r) \ \leq\ I^{\circ}(s) \ +\ (s-r) \qquad \text{(vertex normalization)}.$$

Consequently, for any $s(r) \in \arg\min_{t\geq r} I^{\circ}(t)$,

$$I^{\circ}(r) \ \leq\ I^{\mathrm{incr}\,\circ}(r) \ +\ \big(s(r) - r\big).$$

In the directed-edge normalization one has the same statements with the additive constant 1 replaced by $\Delta = |\mathcal{S}|$.

Proof. Vertex case. Let Y_0 be any finite set of size s. Remove one point x and write $Y := Y_0 \setminus \{x\}$. Then $\partial_{\mathcal{S}}(Y) \subseteq \partial_{\mathcal{S}}(Y_0) \cup \{x\}$, hence $|\partial_{\mathcal{S}} Y| \leq |\partial_{\mathcal{S}} Y_0| + 1$. Iterating $s-r$ times and taking infima over $|Y_0| = s$ yields the claim.

Directed-edge case (direct proof). Write $B_{\mathcal{S}}(Y) := \#\{(y,s) \in Y \times \mathcal{S} : \ sy \notin Y\}$. If $Y = Y_0 \setminus \{x\}$ then when deleting x we *remove* all pairs (x,s) and *potentially add at most one* new boundary pair for each $s \in \mathcal{S}$, namely $(s^{-1}x, s)$ if $s^{-1}x \in Y$. Thus $B_{\mathcal{S}}(Y) \leq B_{\mathcal{S}}(Y_0) + \Delta$. Iterating $s-r$ times and taking infima gives $I^{\circ}_{\mathrm{edge}}(r) \leq I^{\circ}_{\mathrm{edge}}(s) + \Delta(s-r)$.

We emphasize that the trimming estimate here is at the *exact-set* level (one vertex at a time), not only at the profile level; this is what we will use repeatedly inside the TF interleaving schemes. □

Proposition 9.3 (Subadditivity)**.** If Γ is infinite, then for all integers $r_1, r_2 \geq 1$, $I^{\circ}(r_1 + r_2) \leq I^{\circ}(r_1) + I^{\circ}(r_2)$.

Proof. Fix $\varepsilon > 0$. Choose A, B with $|A| = r_1$, $|B| = r_2$ and $|\partial_{\mathcal{S}} A| \leq I^{\circ}(r_1) + \varepsilon$, $|\partial_{\mathcal{S}} B| \leq I^{\circ}(r_2) + \varepsilon$. As Γ is vertex-transitive and infinite (and $\mathcal{S}$ is symmetric), the finite set $\{ab^{-1},\ sab^{-1} : a \in A, b \in B,\ s \in \mathcal{S}\}$ is proper; pick $g \in \Gamma$ outside it. Then $\mathrm{dist}_{\mathcal{S}}(A, gB) \geq 2$, so $\partial_{\mathcal{S}}(A \cup gB) \subset \partial_{\mathcal{S}} A \cup g(\partial_{\mathcal{S}} B)$ and $|\partial_{\mathcal{S}}(A \cup gB)| \leq |\partial_{\mathcal{S}} A| + |\partial_{\mathcal{S}} B|$. Since $|A \cup gB| = r_1 + r_2$, $I^{\circ}(r_1 + r_2) \leq I^{\circ}(r_1) + I^{\circ}(r_2) + 2\varepsilon$; let $\varepsilon \downarrow 0$. □

9.2. **Three large classes with bounded ratios; initial remarks and sanity checks.**

Proposition 9.4 (Nonamenable, two-ended, virtually nilpotent)**.** We have the following quantitative upper bounds for $\mathcal{R}(r)$:

(a) If Γ is nonamenable with vertex Cheeger constant

$$h \;:=\; \inf_{\emptyset\neq Y\subset_{\mathrm{fin}}\Gamma} \frac{|\partial_{\mathcal{S}}Y|}{|Y|} \;>0,$$

then

$$\sup_{r\geq 1} \frac{I^{\circ}(r)}{I^{\mathrm{incr}\,\circ}(r)} \;\leq\; \frac{\Delta}{h}.$$

(b) If Γ has two ends (equivalently, is virtually $\mathbb{Z}$), then there exist constants $0 < c_-(\mathcal{S}) \leq c_+(\mathcal{S}) < \infty$ such that

$$c_-(\mathcal{S}) \;\leq\; I^{\mathrm{incr}\,\circ}(r) \;\leq\; I^{\circ}(r) \;\leq\; c_+(\mathcal{S}) \qquad \forall r\geq 1,$$

hence $\sup_{r\geq 1} I^{\circ}(r)/I^{\mathrm{incr}\,\circ}(r) \leq c_+(\mathcal{S})/c_-(\mathcal{S})$.

(c) If Γ has polynomial growth of homogeneous dimension $Q\geq 2$ (virtually nilpotent), then $\sup_{r\geq 1} I^{\circ}(r)/I^{\mathrm{incr}\,\circ}(r) \leq C(\Gamma,\mathcal{S})$.

Proof. (a) For every finite Y, $|\partial_{\mathcal{S}}Y| \geq h\,|Y|$, hence $I^{\mathrm{incr}\,\circ}(r) = \inf_{s\geq r} I^{\circ}(s) \geq \inf_{s\geq r} h\,s = h\,r$. Also $|\partial_{\mathcal{S}}Y| \leq \Delta|Y|$, so $I^{\circ}(r)\leq \Delta r$. Therefore $I^{\circ}(r)/I^{\mathrm{incr}\,\circ}(r) \leq \Delta/h$.

(b) The Cayley graph is quasi-isometric to a uniformly bounded-width strip. Choosing Y as a tubular segment of length r shows $I^{\circ}(r)\leq c_+(\mathcal{S})$. Conversely, every nonempty finite Y has at least one neighbor outside, so $I^{\circ}(r)\geq 1 =: c_-(\mathcal{S})$. (Sharper $c_-(\mathcal{S})$ exist but are not needed.)

(c) Let $Q\geq 2$ be the homogeneous dimension and set $\alpha = (Q-1)/Q \in (0,1)$. By our discrete Wulff theory (Theorems 4.1 and 3.25) there exist $R_0<\infty$ and constants $0 < c_\star \leq C_\star < \infty$ (depending only on $(\Gamma,\mathcal{S})$ and the normalization) such that for all integers $s\geq R_0$,

$$c_\star\, s^{\alpha} \;\leq\; I^{\circ}(s) \;\leq\; C_\star\, s^{\alpha}. \tag{9.2}$$

Indeed, the lower bound follows from the asymptotic Wulff lower bound (liminf) by choosing $\varepsilon>0$ small and taking R_0 so large that $I^{\circ}(s)\geq (h_{\mathcal{S}}-\varepsilon)s^{\alpha}$ for $s\geq R_0$; the upper bound follows from the existence of Wulff samplers Y_s with $|\partial_{\mathcal{S}}Y_s| \leq (1+o(1))h_{\mathcal{S}}\, s^{\alpha}$, then hitting the exact volume by padding (Lemma 9.13), which changes the boundary by at most Δ per vertex and is thus $o(s^{\alpha})$.

For $r\geq R_0$, since $I^{\mathrm{incr}\,\circ}(r) = \inf_{s\geq r} I^{\circ}(s)$ and $s\mapsto s^{\alpha}$ is nondecreasing,

$$I^{\mathrm{incr}\,\circ}(r) \;\geq\; \inf_{s\geq r} c_\star\, s^{\alpha} \;=\; c_\star\, r^{\alpha},$$

while $I^{\circ}(r)\leq C_\star r^{\alpha}$ by (9.2). Hence

$$\frac{I^{\circ}(r)}{I^{\mathrm{incr}\,\circ}(r)} \;\leq\; \frac{C_\star}{c_\star} \qquad (r\geq R_0).$$

For the finitely many $1\leq r<R_0$, set $C_0 := \max_{1\leq r<R_0} I^{\circ}(r)/I^{\mathrm{incr}\,\circ}(r)$, which is finite because $I^{\mathrm{incr}\,\circ}(r)\geq I^{\mathrm{incr}\,\circ}(R_0)>0$. Therefore

$$\sup_{r\geq 1} \frac{I^{\circ}(r)}{I^{\mathrm{incr}\,\circ}(r)} \;\leq\; \max\Big\{\frac{C_\star}{c_\star},\, C_0\Big\} \;<\; \infty,$$

which proves *(c)*. $\square$

Remark 9.5 (Vertex vs. edge normalization)**.** If $I^{\circ}_{\mathrm{edge}}$ and $I^{\mathrm{incr}\,\circ}_{\mathrm{edge}}$ denote the profiles in the (directed or undirected) edge normalization, then $I^{\circ}\leq I^{\circ}_{\mathrm{edge}}\leq \Delta\, I^{\circ}$ and $I^{\mathrm{incr}\,\circ}\leq I^{\mathrm{incr}\,\circ}_{\mathrm{edge}}\leq \Delta\, I^{\mathrm{incr}\,\circ}$ with $\Delta = |\mathcal{S}|$ (cf. Lemma 9.15). Hence the boundedness of the ratio is invariant up to a factor depending only on Δ.

Remark 9.6 (What our Wulff theory adds, precisely)**.** The classical literature provides global two-sided isoperimetry in virtually nilpotent groups (e.g., [VSCC92, SC02]). Our contribution is different in nature: we obtain *asymptotically sharp* constants and *constructive* near-minimizers adapted to the horizontal anisotropy τ_S. Concretely, there exists $R_0 < \infty$ such that for all $|Y| \geq R_0$,

$$|\partial_S Y| \ \geq \ (h_S - o(1))\,|Y|^{\frac{Q-1}{Q}}, \qquad \exists\, Y_r^{\mathrm{Wulff}} :\ |\partial_S Y_r^{\mathrm{Wulff}}| \ \leq \ (1+o(1))\,h_S\,r^{\frac{Q-1}{Q}}.$$

These asymptotics suffice to deduce bounded ratios (Proposition 9.4(c)) by combining with a finite small-volume window. We do *not* claim a uniform lower bound for all volumes from Wulff theory alone; the global two-sided inequalities remain an independent analytic input when one wants uniform statements at *every* scale.

Remark 9.7 (Explicit two-ended bound)**.** Let Γ be two-ended. Fix an infinite cyclic subgroup $\langle t\rangle \leq \Gamma$ of finite index and a transversal T of the left cosets of $\langle t\rangle$. For $L \in \mathbb{N}$, set $Y_L = \bigcup_{i=0}^{L-1} Tt^i$. Then $|Y_L| = L\,[\Gamma : \langle t\rangle]$ and only the two ends contribute up to bounded width (depending on $\mathcal{S}$), so

$$|\partial_{\mathcal{S}} Y_L| \ \leq \ C_+(\mathcal{S})\,[\Gamma : \langle t\rangle].$$

Hence $I^\circ(L\,[\Gamma : \langle t\rangle]) \leq C_+(\mathcal{S})\,[\Gamma : \langle t\rangle]$, and the general upper bound follows by padding/Lipschitz (Proposition 9.1).

9.3. **Tempered Følner increments and a decoupled route to (TF).**

Lemma 9.8 (Tempered increments by expansion)**.** There exists a nested sequence $Y_1 \subset Y_2 \subset \cdots$ of finite subsets of Γ such that for all n,

$$|Y_{n+1} \setminus Y_n| \ \leq \ |\partial_{\mathcal{S}} Y_n|.$$

Proof. Start from any nonempty Y_1. Given Y_n, pick any subset $A_n \subseteq \partial_{\mathcal{S}} Y_n$ and let $Y_{n+1} := Y_n \cup A_n$. Then $|Y_{n+1} \setminus Y_n| = |A_n| \leq |\partial_{\mathcal{S}} Y_n|$. □

Definition 9.9 (Vertex and directed-edge profiles)**.** For finite $Y \subset \Gamma$, write

$$B_{\mathcal{S}}(Y) := \#\{(y,s) \in Y \times \mathcal{S} :\ sy \notin Y\} \qquad \text{(directed edge boundary)}.$$

Define the exact and increasing *directed-edge* profiles:

$$I^\circ_{\mathrm{edge}}(r) := \inf\{\, B_{\mathcal{S}}(Y) :\ |Y| = r\,\}, \qquad I^{\mathrm{incr}\,\circ}_{\mathrm{edge}}(r) := \inf_{t \geq r} I^\circ_{\mathrm{edge}}(t).$$

Remark 9.10 (Edge vs vertex TF)**.** We formulate (TF) both in vertex and directed-edge normalizations. As $|\partial_{\mathcal{S}} Y| \leq B_{\mathcal{S}}(Y) \leq \Delta\,|\partial_{\mathcal{S}} Y|$, any (TF) statement in one normalization converts to the other with constants modified by a factor of at most Δ.

Definition 9.11 (Tempered Følner property (TF))**.** $(\Gamma, \mathcal{S})$ has (TF) if there exists a Følner sequence (F_n) and $A, B \geq 1$ such that:

(i) $|\partial_{\mathcal{S}}(F_n)| \leq A\, I^{\mathrm{incr}\,\circ}(|F_n|)$;
(ii) $|F_{n+1} \setminus F_n| \leq B\,|\partial_{\mathcal{S}}(F_n)|$.

Definition 9.12 (Tempered Følner property – directed-edge normalization)**.** Write $B_{\mathcal{S}}(Y) := \#\{(y,s) \in Y \times \mathcal{S} :\ sy \notin Y\}$ for the *directed* edge boundary. We say $(\Gamma, \mathcal{S})$ has *(TF) in edge normalization* if there exists a Følner sequence (F_n) and constants $A_e, B \geq 1$ such that

(i_e) $B_{\mathcal{S}}(F_n) \ \leq \ A_e\, I^{\mathrm{incr}\,\circ}_{\mathrm{edge}}(|F_n|)$ for all n;
(ii) $|F_{n+1} \setminus F_n| \ \leq \ B\, B_{\mathcal{S}}(F_n)$ for all n.

Here $I^{\mathrm{incr}\,\circ}_{\mathrm{edge}}(r) := \inf_{t \geq r}\ \inf_{|Y|=t} B_{\mathcal{S}}(Y)$ is the increasing minorant of the exact *edge* profile.

Lemma 9.13 (Padding by *k* points)**.** For any finite Y and $k \geq 0$, there is $Z \supset Y$ with $|Z| = |Y| + k$ and $|\partial_{\mathcal{S}}(Z)| \leq |\partial_{\mathcal{S}}(Y)| + k\Delta$.

Lemma 9.14 (Attainment of exact profiles)**.** For every integer $r \geq 1$ there exist finite sets $Y_r, E_r \subset \Gamma$ with $|Y_r| = |E_r| = r$ such that

$$|\partial_{\mathcal{S}} Y_r| = I^{\circ}(r) \qquad \text{and} \qquad B_{\mathcal{S}}(E_r) = I^{\circ}_{\mathrm{edge}}(r).$$

Proof. Fix r. For any $Y \subset \Gamma$ with $|Y| = r$ we have $0 \leq |\partial_{\mathcal{S}} Y| \leq \Delta r$ and $0 \leq B_{\mathcal{S}}(Y) \leq \Delta r$, so

$$\mathcal{V}_r := \{\, |\partial_{\mathcal{S}} Y| : \ |Y| = r \,\} \subset \{0, 1, \ldots, \Delta r\}, \quad \mathcal{E}_r := \{\, B_{\mathcal{S}}(Y) : \ |Y| = r \,\} \subset \{0, 1, \ldots, \Delta r\}$$

are nonempty finite subsets of $\mathbb{N}$. Hence $\min \mathcal{V}_r = I^{\circ}(r)$ and $\min \mathcal{E}_r = I^{\circ}_{\mathrm{edge}}(r)$ are achieved. ☐

Lemma 9.15 (Vertex-edge comparison)**.** For every finite $Y \subset \Gamma$,

$$|\partial_{\mathcal{S}} Y| \ \leq \ B_{\mathcal{S}}(Y) \ \leq \ \Delta\, |\partial_{\mathcal{S}} Y|. \tag{9.3}$$

Consequently, for all $r \geq 1$,

$$I^{\circ}(r) \ \leq \ I^{\circ}_{\mathrm{edge}}(r) \ \leq \ \Delta\, I^{\circ}(r), \qquad I^{\mathrm{incr}\,\circ}(r) \ \leq \ I^{\mathrm{incr}\,\circ}_{\mathrm{edge}}(r) \ \leq \ \Delta\, I^{\mathrm{incr}\,\circ}(r).$$

Proof. Define $\Phi : \{(y, s) \in Y \times \mathcal{S} : \ sy \notin Y\} \to \partial_{\mathcal{S}} Y$ by $\Phi(y, s) = sy$. It is surjective and each fiber has size $\leq \Delta$, which gives (9.3). The profile inequalities follow. ☐

Remark 9.16 (Directed vs undirected)**.** In all inequalities above one may replace directed edge counts by undirected edge counts, at the price of a universal factor $1/2$. We keep the directed normalization in profile statements, and switch to undirected edges only when convenient in specific constructions; constants are converted by a fixed factor once.

Proposition 9.17 (Weak TF(i) from a bounded edge ratio)**.** Assume there is $C_0 < \infty$ with

$$\sup_{r \geq 1} \frac{I^{\circ}_{\mathrm{edge}}(r)}{I^{\mathrm{incr}\,\circ}_{\mathrm{edge}}(r)} \ \leq \ C_0.$$

Then there exists a sequence of finite sets E_r with $|E_r| = r$ such that

$$|\partial_{\mathcal{S}} E_r| \ \leq \ I^{\circ}_{\mathrm{edge}}(r) \ \leq \ C_0\, I^{\mathrm{incr}\,\circ}_{\mathrm{edge}}(r) \ \leq \ C_0\, \Delta\, I^{\mathrm{incr}\,\circ}(r) \qquad \forall r \geq 1.$$

In particular, TF(i) holds along some (not necessarily nested) sequence with $A = C_0\, \Delta$ in vertex normalization.

Proof. Choose E_r attaining $I^{\circ}_{\mathrm{edge}}(r)$; then apply Lemma 9.15. ☐

Lemma 9.18 (Canonical tempered expansion)**.** For any finite nonempty $Y \subset \Gamma$ set $Y^+ := Y \cup \partial_{\mathcal{S}} Y$. Then

$$|Y^+ \setminus Y| \ = \ |\partial_{\mathcal{S}} Y|, \qquad |\partial_{\mathcal{S}} Y^+| \ \leq \ (1 + \Delta)\, |\partial_{\mathcal{S}} Y|.$$

Consequently, the nested chain $F_{n+1} := F_n^+$ satisfies TF(ii) with $B = 1$.

Proof. Since $\partial_{\mathcal{S}} Y \cap Y = \varnothing$, $|Y^+ \setminus Y| = |\partial_{\mathcal{S}} Y|$. Adding vertices of $\partial_{\mathcal{S}} Y$ one by one changes the boundary by at most Δ per addition; hence $|\partial_{\mathcal{S}} Y^+| \leq (1 + \Delta)|\partial_{\mathcal{S}} Y|$. ☐

Remark 9.19 (Følner behaviour of $\mathcal{S}$-ballooning)**.** By contrast with the tempered step $Y \mapsto Y \cup \partial Y$, the "ballooning" chain $F_{n+1} := \mathcal{S} F_n$ need not be Følner, even if Γ is amenable (e.g. balls are not Følner in many amenable groups of exponential growth). In virtually nilpotent groups, however, balls are Følner for any finite generating set.

Definition 9.20 (Nested near-minimizers (NNM))**.** We say $(\Gamma, \mathcal{S})$ admits *nested near-minimizers* if there exist $C_0 < \infty$, $r_0 \in \mathbb{N}$, and a nested family $(W_r)_{r \geq r_0}$ with $|W_r| = r$ such that

$$B_{\mathcal{S}}(W_r) \ \leq \ C_0\, I^{\mathrm{incr}\,\circ}_{\mathrm{edge}}(r) \qquad \text{for all } r \geq r_0.$$

Remark. In many arguments it suffices to have the same inequality *along a cofinal subsequence of volumes.*

Lemma 9.21 (Checkpoints ⇒ NNM along a cofinal subsequence)**.** Let (F_n) be a nested family in edge normalization. Assume there exists an infinite subsequence (n_j) and $A_e \geq 1$ such that

$$B_{\mathcal{S}}(F_{n_j}) \;\leq\; A_e\, I^{\mathrm{incr}\,\circ}_{\mathrm{edge}}(|F_{n_j}|) \qquad \forall j.$$

Then (F_{n_j}) is a sequence of *nested near-minimizers* along a cofinal subsequence.

Proof. Immediate from the hypothesis and nestedness. □

Remark 9.22 (On (NNM) in our scope)**.** (i) $\mathbb{Z}^d$ (axis stencil): discrete Wulff approximants W_r can be made nested and satisfy $B_{\mathcal{S}}(W_r)/r^{(d-1)/d} \to c_{\mathcal{T}}$, hence $B_{\mathcal{S}}(W_r) \leq C_0\, I^{\mathrm{incr}\,\circ}_{\mathrm{edge}}(r)$ for large r.

(ii) Uniform lattices in Carnot groups: inner/outer samplers are nested in the scale and have $b_r := B_{\mathcal{S}}(\cdot) \sim \mathrm{Per}_{\mathcal{T}}(E)\, r^{Q-1}$; after volume interpolation we get nested W_r with $B_{\mathcal{S}}(W_r) \leq C_0\, I^{\mathrm{incr}\,\circ}_{\mathrm{edge}}(r)$ for large r.

Proposition 9.23 (Tempered chain from (NNM))**.** Assume $(\Gamma, \mathcal{S})$ admits nested near-minimizers: there exist $C_0 < \infty$, $r_0 \in \mathbb{N}$, and a nested family $(W_r)_{r \geq r_0}$ with $|W_r| = r$ such that

$$B_{\mathcal{S}}(W_r) \;\leq\; C_0\, I^{\mathrm{incr}\,\circ}_{\mathrm{edge}}(r) \qquad \forall r \geq r_0.$$

Then there exists a nested sequence (F_n) and $A = C_0\,\Delta$ such that, for all n,

$$|F_{n+1} \setminus F_n| \;\leq\; |\partial_{\mathcal{S}} F_n| \qquad \text{and} \qquad |\partial_{\mathcal{S}} F_n| \;\leq\; A\, I^{\mathrm{incr}\,\circ}(|F_n|).$$

If, moreover, Γ is amenable, then (F_n) can be chosen Følner; consequently

$$\sup_{r \geq 1} \frac{I^{\circ}(r)}{I^{\mathrm{incr}\,\circ}(r)} \;\leq\; C_0\,\Delta\,(1+\Delta).$$

Proof. Define $F_n := W_{r_n}$ with $r_{n+1} := r_n + |\partial W_{r_n}|$; then $|F_{n+1} \setminus F_n| = |\partial F_n|$ gives TF(ii) with $B = 1$. For TF(i), use $|\partial F_n| \leq B_{\mathcal{S}}(F_n) \leq C_0\, I^{\mathrm{incr}}_{\mathrm{edge}}(|F_n|) \leq C_0 \Delta\, I^{\mathrm{incr}}(|F_n|)$. If $I^{\mathrm{incr}}(r)/r \to 0$ (amenability), we get $|\partial F_n|/|F_n| \to 0$. □

Corollary 9.24 (Følner clause under amenability)**.** Under the hypotheses of Proposition 9.23 and amenability, the nested sequence (F_n) is a Følner sequence. In particular, (TF) holds:

- in *edge* normalization with $(A_e, B) = (C_0, 1)$, and
- equivalently, in *vertex* normalization with $(A, B) = (C_0\Delta, 1)$.

Definition 9.25 (Confined canonical step)**.** Given nested NNM $(W_r)_{r \geq r_0}$ and current $F \subset W_{r'}$ with $r' > |F|$, define

$$\mathcal{T}_{r'}(F) \;:=\; \big(F \cup \partial_{\mathcal{S}} F\big) \;\cap\; W_{r'}.$$

Lemma 9.26 (Interleaved smoothing, TF(ii) with $B = 1$)**.** Fix $r_0 < r_1 < r_2 < \cdots$ and set

$$F_{n_0} := W_{r_0}, \qquad F_{m+1} := \begin{cases} \mathcal{T}_{r_j}(F_m), & \text{if } |F_m| < r_j, \\ W_{r_j}, & \text{if } |F_m| = r_j \text{ (checkpoint)}, \end{cases}$$

then move to level r_{j+1}. The chain (F_m) is nested and satisfies

$$|F_{m+1} \setminus F_m| \;\leq\; |\partial_{\mathcal{S}} F_m| \qquad (\text{TF(ii) with } B = 1).$$

At checkpoint indices m with $F_m = W_{r_j}$ one has TF(i) with the NNM constant.

Proof. Nestedness: $F \subset F \cup \partial F$ and $F \subset W_{r'}$ imply $F \subset \mathcal{T}_{r'}(F)$. Intersecting $F \cup \partial F$ by $W_{r'}$ can only reduce the increment, so $|F_{m+1} \setminus F_m| \leq |\partial F_m|$. At checkpoints TF(i) holds by NNM. □

Lemma 9.27 (Right-Lipschitz for $I^{\mathrm{incr}\,\circ}$)**.** For all integers $s \geq r \geq 1$,

$$0 \;\leq\; I^{\mathrm{incr}\,\circ}(s) - I^{\mathrm{incr}\,\circ}(r) \;\leq\; \Delta\,(s-r).$$

Proof. Let $s \geq r$. For any $u \geq r$, set $t := u + (s - r) \geq s$. By Proposition 9.1, $I^\circ(t) \leq I^\circ(u) + \Delta(s - r)$. Taking infima gives $I^{\mathrm{incr}\,\circ}(s) \leq I^{\mathrm{incr}\,\circ}(r) + \Delta(s - r)$. Monotonicity is clear. □

Definition 9.28 (Budgeted levels)**.** Fix $\vartheta \in (0, 1]$. Define a level schedule by $r_{j+1} := r_j + \lfloor \vartheta\, I^{\mathrm{incr}\,\circ}(r_j) \rfloor$. On level j, evolve by the confined step $F \mapsto \mathcal{T}_{r_{j+1}}(F)$ until $F = W_{r_{j+1}}$, then move to level $j + 1$.

Proposition 9.29 (TF(i) at every step under scheduled interleaving)**.** Assume (W_r) are nested near-minimizers in edge normalization with $B_{\mathcal{S}}(W_r) \leq C_0\, I^{\mathrm{incr}\,\circ}_{\mathrm{edge}}(r)$. Run the budgeted-level scheme of Definition 9.28. Then the resulting nested chain (F_m) satisfies TF(ii) with $B = 1$, and for every m,

$$|\partial_{\mathcal{S}} F_m| \;\leq\; \big(C_0\Delta + (C_0\Delta^2 + 1)\vartheta\big)\, I^{\mathrm{incr}\,\circ}(|F_m|).$$

Proof. TF(ii) with $B = 1$ is Lemma 9.26. Fix a step with $F \subset W_{r'}$ on level r'. Tail trimming yields $|\partial F| \leq |\partial W_{r'}| + \mathrm{def}(F; r')$, and $|\partial W_{r'}| \leq C_0\Delta\, I^{\mathrm{incr}}(r')$. By Lemma 9.27, $I^{\mathrm{incr}}(r') \leq I^{\mathrm{incr}}(|F|) + \Delta\, \mathrm{def}(F; r')$. By construction of the schedule, $\mathrm{def}(F; r') \leq \vartheta\, I^{\mathrm{incr}}(|F|)$. Combine these to get the displayed bound. □

Theorem 9.30 (Interleaving in $\mathbb{Z}^d$ and Carnot lattices)**.** Let $(\Gamma, \mathcal{S})$ be (i) $\mathbb{Z}^d$ with an axis stencil, or (ii) a uniform lattice in a Carnot group of homogeneous dimension $Q \geq 2$. There exist nested Wulff samplers $(W_r)_{r \gg 1}$ and constants $C_0, A, B < \infty$ such that the interleaved chain of Lemma 9.26 satisfies, for all large steps,

$$|F_{m+1} \setminus F_m| \;\leq\; B\, |\partial_{\mathcal{S}} F_m| \qquad \text{(here } B = 1\text{)}, \qquad |\partial_{\mathcal{S}} F_m| \;\leq\; A\, I^{\mathrm{incr}\,\circ}(|F_m|) \text{ for every } m.$$

One may take $A = C_0\Delta + (C_0\Delta^2 + 1)\,\vartheta$ with C_0 the NNM constant and any fixed $\vartheta \in (0, 1]$.

Proof idea. Nested Wulff near-minimizers with tracked constants are available in these settings, and $I^\circ(r) \asymp r^{(Q-1)/Q}$, so I^{incr} has the same order. Apply Lemma 9.26 and Proposition 9.29. □

Theorem 9.31 ((TF) $\Rightarrow$ bounded ratio)**.** If $(\Gamma, \mathcal{S})$ satisfies (TF), then $\sup_{r \geq 1} I^\circ(r)/I^{\mathrm{incr}\,\circ}(r) \leq A + \Delta AB$.

Proof. Fix r and choose n minimal such that $|F_n| \geq r$. Then $|F_{n-1}| < r \leq |F_n|$. By (i) and monotonicity, $|\partial_{\mathcal{S}}(F_{n-1})| \leq A\, I^{\mathrm{incr}\,\circ}(r)$. By (ii), $r - |F_{n-1}| \leq |F_n| - |F_{n-1}| \leq B\, |\partial_{\mathcal{S}}(F_{n-1})|$. Apply Lemma 9.13 to obtain $Z \supset F_{n-1}$ with $|Z| = r$ and

$$|\partial_{\mathcal{S}} Z| \;\leq\; |\partial_{\mathcal{S}} F_{n-1}| + \Delta\,(r - |F_{n-1}|) \;\leq\; (1 + \Delta B)\, |\partial_{\mathcal{S}} F_{n-1}| \;\leq\; (A + \Delta AB)\, I^{\mathrm{incr}\,\circ}(r).$$

Taking the infimum over $|Z| = r$ completes the proof. □

Proposition 9.32 (Examples of (TF))**.** (TF) holds for:

(a) Virtually nilpotent groups (use Wulff-type approximants and nested dilations).
(b) $\mathbb{Z}^d$ with nearest-neighbor generators (concentric boxes or discrete balls).
(c) Finite direct products of groups satisfying (TF) (with product generators).

Proof of Theorem D . Items (1)-(3) follow directly from Proposition 9.4. For the Tempered Følner (TF) clause, assume $(\Gamma, \mathcal{S})$ satisfies (TF) with constants (A, B). The padding lemma with TF(ii) gives a set Z of cardinality r whose boundary is at most $(1 + \Delta B)|\partial_{\mathcal{S}} F_{n-1}|$, where n is minimal with $|F_n| \geq r$. By (TF)(i) and monotonicity we then obtain

$$I^\circ(r) \;\leq\; |\partial_{\mathcal{S}} Z| \;\leq\; (1 + \Delta B)\, |\partial_{\mathcal{S}} F_{n-1}| \;\leq\; (1 + \Delta B)\, A\, I^{\mathrm{incr}\,\circ}(r),$$

which is Theorem 9.31. □

9.4. **Lamplighters: (TF) via explicit expansion and known isoperimetry.** Let H be a finite nontrivial group and $(\Gamma, \mathcal{S}_\Gamma)$ a finitely generated *amenable* group. Consider the lamplighter $G := H^{(\Gamma)} \rtimes \Gamma$ with the generating set

$$\mathcal{S} = \mathcal{S}_\Gamma \cup \mathsf{T},$$

where each $\alpha \in \mathsf{T}$ toggles the lamp at the *current* base position and $\mathcal{S}_\Gamma$ acts by left translation on the base. Let $t := |\mathsf{T}|$ and $q := |H| - 1$. We use the directed boundary $B_{\mathcal{S}}(\cdot)$ for exact counts and recall the pointwise comparison $|\partial_{\mathcal{S}} Y| \le B_{\mathcal{S}}(Y) \le \Delta\, |\partial_{\mathcal{S}} Y|$ (Lemma 9.15).

Lemma 9.33 (Exact edge split in one block)**.** Let $U \subset \Gamma$ be finite of size $n := |U|$, and $0 \le M \le n$. Define

$$F(U, M) := \Big\{(f, x) \in H^{(\Gamma)} \times \Gamma : \ x \in U,\ \operatorname{supp}(f) \subset U,\ \|f\|_0 \le M\Big\}, \qquad \Sigma(n, M) := \sum_{j=0}^{M} \binom{n}{j} q^j.$$

Let

$$E_{\to}(U) := \sum_{s \in \mathcal{S}_\Gamma} \#\{x \in U : \ xs \notin U\}.$$

Then

$$B_{\mathcal{S}_\Gamma \cup \mathsf{T}}\big(F(U, M)\big) \ = \ E_{\to}(U)\, \Sigma(n, M) \ + \ t\, |U| \binom{n-1}{M} q^M. \tag{9.4}$$

Proof. A right-step by $s \in \mathcal{S}_\Gamma$ moves $x \mapsto xs$ and keeps f unchanged, exiting iff $xs \notin U$; summing over x and admissible f gives the first term. For $\alpha \in \mathsf{T}$, toggling at $x \in U$ exits iff $f(x) = 1_H$ and $\|f\|_0 = M$; this gives the second term. □

Lemma 9.34 (Exact count of ring-enlargement exits)**.** Let $U \subset \Gamma$ be finite and let $U' \subset \mathcal{S}_\Gamma U$ with ring $R := U' \setminus U$, $\delta := |R|$. For integers $m, \ell \ge 0$ set

$$Y_\ell := \Big\{(f, x) : \ x \in U',\ \operatorname{supp}(f) \subseteq U',\ \#(\text{lit in } U) \le m,\ \#(\text{lit in } R) \le \ell\Big\}.$$

For each $\alpha \in \mathsf{T}$, let $\mathcal{E}^{\to}_{\text{ring},\alpha}(Y_\ell)$ be the set of directed α-edges from Y_ℓ that *exit* by toggling an *unlit* ring site $x \in R$ when the ring budget is *exactly* saturated, i.e. $\#(\text{lit in } R) = \ell$. Then

$$B_{\text{ring},\alpha}(Y_\ell) := |\mathcal{E}^{\to}_{\text{ring},\alpha}(Y_\ell)| \ = \ |R| \binom{\delta - 1}{\ell} q^\ell \sum_{j=0}^{m} \binom{|U|}{j} q^j, \qquad q := |H| - 1,$$

and $B_{\text{ring}}(Y_\ell) := \sum_{\alpha \in \mathsf{T}} B_{\text{ring},\alpha}(Y_\ell) = t \cdot B_{\text{ring},\alpha}(Y_\ell)$.

Proof. Count choices of $x \in R$ (must be unlit), ring pattern (choose ℓ lit sites in $R \setminus \{x\}$ and values), and a U-pattern with $\le m$ lit; each yields one exiting directed edge for α. □

Definition 9.35 (Ring-step operator)**.** With $U \subset \Gamma$ finite and $U' \subset \mathcal{S}_\Gamma U$, define $R := U' \setminus U$ and, for $m, \ell \ge 0$,

$$Y_\ell \ := \ \Big\{(f, x) : x \in U',\ \operatorname{supp}(f) \subseteq U',\ \#(\text{lit in } U) \le m,\ \#(\text{lit in } R) \le \ell\Big\}.$$

For $\alpha \in \mathsf{T}$ define the α-ring exits

$$\mathcal{E}^{\to}_{\text{ring},\alpha}(Y_\ell) \ := \ \Big\{((f, x), \alpha) \in Y_\ell \times \{\alpha\} : \ x \in R,\ x \text{ is unlit in } f, \text{ and } \#(\text{lit in } R) = \ell\Big\}.$$

Define the *single-step ring increment*

$$\mathcal{R}(Y_\ell) \ := \ Y_\ell \ \cup \ \big\{\, (f, x)\alpha \ : \ ((f, x), \alpha) \in \mathcal{E}^{\to}_{\text{ring}}(Y_\ell) \,\big\}, \qquad \mathcal{E}^{\to}_{\text{ring}} := \bigsqcup_{\alpha \in \mathsf{T}} \mathcal{E}^{\to}_{\text{ring},\alpha}.$$

Lemma 9.36 (Exact count and tempered bound for a single ring step)**.** With the notation of Definition 9.35 one has

$$\big|\,\mathcal{R}(Y_\ell)\setminus Y_\ell\,\big| \;=\; \big|\mathcal{E}^{\rightarrow}_{\rm ring}(Y_\ell)\big| \;\le\; B_{\mathcal{S}}(Y_\ell) \;\le\; \Delta\,|\partial_{\mathcal{S}}Y_\ell|.$$

Consequently, the ring step is tempered in the TF(ii) sense with constant $B = \Delta$ (vertex normalization).

Proof. The map $((f,x),\alpha)\mapsto(f,x)\alpha$ is a bijection from $\mathcal{E}^{\rightarrow}_{\rm ring}(Y_\ell)$ onto $\mathcal{R}(Y_\ell)\setminus Y_\ell$. The inequalities follow from Lemma 9.15. □

Lemma 9.37 (Tempered base enlargement)**.** Let $U\subset\Gamma$ be finite, set $U'\subset\mathcal{S}_\Gamma U$ and $R := U'\setminus U$, and fix $m\ge 0$. Define the base-first enlargement with ring lamps forbidden

$$Y^{\rm base} \;:=\; \Big\{(f,x) : x\in U',\ \operatorname{supp}(f)\subseteq U',\ \#(\text{lit in } U)\le m,\ \#(\text{lit in } R)=0\Big\}$$

and

$$F(U,m) := \Big\{(f,x) : x\in U,\ \operatorname{supp}(f)\subseteq U,\ \#(\text{lit in } U)\le m\Big\}.$$

Then $\big|Y^{\rm base}\setminus F(U,m)\big| \;\le\; E_{\to}(U)\,\Sigma(|U|,m)$ and $E_{\to}(U)\,\Sigma(|U|,m) \;\le\; B_{\mathcal{S}_\Gamma\cup\mathsf{T}}\big(F(U,m)\big) \;\le\; \Delta\,|\partial_{\mathcal{S}}F(U,m)|$, so the base step is tempered with the same $B=\Delta$.

Proof. Count cursor exits across the base caps exactly as in (9.4) and apply Lemma 9.15. □

Remark 9.38 (Monotonicity and bookkeeping across steps)**.** Each base or ring single step replaces Y by $Y\cup E$, where E is the image of a set of exiting *directed* edges under the action map, so the chain is nested by construction, and

$$|Y^{\rm new}\setminus Y| = |\mathcal{E}^{\rightarrow}_{\rm ring/base}(Y)| \le B_{\mathcal{S}}(Y) \le \Delta\,|\partial_{\mathcal{S}}Y|.$$

Theorem 9.39 (Tempered chain for lamplighters: single-step construction)**.** Let $G = H^{(\Gamma)}\rtimes\Gamma$ with generators $\mathcal{S}=\mathcal{S}_\Gamma\cup\mathsf{T}$, and fix a nested exhaustion $U_1\subset U_2\subset\cdots$ of Γ with $U_{k+1}\subset\mathcal{S}_\Gamma U_k$. Fix $\theta\in(0,1)$ and set $M_k := \lfloor\theta|U_k|\rfloor$. Construct a nested chain (F_n) by concatenating, for each k, one base step $F(U_k,M_k)\mapsto Y^{\rm base}$ and then $M_{k+1}-M_k$ ring steps. Then

$$|F_{n+1}\setminus F_n| \;\le\; \Delta\,|\partial_{\mathcal{S}}F_n| \qquad \text{for all } n.$$

Remark 9.40 (Normalization)**.** In the lamplighter subsection we use the *vertex* boundary in TF(ii) and the *directed* edge count when exploiting exact splits. The comparison in Lemma 9.15 transfers bounds between them up to a factor Δ.

Proof. Each link is tempered with constant Δ by Lemmas 9.37 and 9.36. □

Theorem 9.41 (Følner checkpoints for finite-lamp wreaths)**.** Let $(\Gamma,\mathcal{S}_\Gamma)$ be finitely generated and amenable, let H be finite nontrivial with $q:=|H|-1$ and toggles T ($t:=|\mathsf{T}|$), and consider $G:=H^{(\Gamma)}\rtimes\Gamma$ with generators $\mathcal{S}=\mathcal{S}_\Gamma\cup\mathsf{T}$. Fix a Følner exhaustion $U_1\subset U_2\subset\cdots$ of Γ with $U_{k+1}\subset\mathcal{S}_\Gamma U_k$ and $n_k:=|U_k|$. Let $p:=q/(1+q)\in(0,1)$ and set $M_k:=\lceil p\,n_k\rceil$ and

$$F_k^\star \;:=\; F(U_k,M_k).$$

Then, in directed-edge normalization,

$$\frac{B_{\mathcal{S}}(F_k^\star)}{|F_k^\star|} \;\longrightarrow\; 0 \qquad (k\to\infty).$$

Proof. From (9.4), $B_{\mathcal{S}}(F(U,M)) = E_{\to}(U)\,\Sigma(n,M) + t\,n\binom{n-1}{M}q^M$ and $|F(U,M)| = n\,\Sigma(n,M)$. Følner of (U_k) gives $E_{\to}(U_k)/n_k\to 0$. With $M_k=\lceil pn_k\rceil$ and $p=q/(1+q)$ (the *median* of $\mathrm{Bin}(n_k,p)$), the local CLT/Stirling gives $\binom{n_k-1}{M_k}q^{M_k}/\Sigma(n_k,M_k) = O(n_k^{-1/2})$. Hence the fraction tends to 0. (See e.g. Erschler [Ers03, Ers06] for lamplighter profile asymptotics.) □

Theorem 9.42 (Near-minimizer checkpoints and TF(i) on a subsequence). *References:* Erschler [Ers03, Thm. 1], [Ers06, Thm. 1, Sec. 3]; see also [SCZ18]. Let $G = H^{(\Gamma)} \rtimes \Gamma$ as above, and checkpoints $F_k^\star := F(U_k, M_k)$ with $M_k = \lfloor \theta |U_k| \rfloor$, $\theta \in (0,1)$. There exists $A_0 = A_0(H, \mathcal{S}_\Gamma, \mathcal{S}) \in (0, \infty)$ such that for all sufficiently large k,

$$B_{\mathcal{S}}\big(F_k^\star\big) \ \le \ A_0 \ \cdot \ I^\circ_{\mathrm{edge}}\big(|F_k^\star|\big).$$

In particular, aligning checkpoints to record minima of the minorant and passing to vertex boundary gives a subsequence $\{k_j\}$ with

$$|\partial_{\mathcal{S}} F_{k_j}^\star| \ \le \ A\, I^{\mathrm{incr}\,\circ}\big(|F_{k_j}^\star|\big), \qquad A := A_0\, \Delta.$$

Remark 9.43 (On the minorant alignment). Since $I^{\mathrm{incr}\,\circ}(r) = \min_{s \ge r} I^\circ(s)$ is attained at some record scale $s(r)$, we may choose $|F_k^\star|$ near such scales; bounded padding/trimming changes $|\partial_{\mathcal{S}}|$ by at most Δ per vertex, absorbed in TF(ii).

Definition 9.44 (Checkpoints for TF). A family $\{E_j\}$ will be called *checkpoints for* (TF) with constant $A \ge 1$ if

$$|\partial_{\mathcal{S}} E_j| \ \le \ A\, I^{\mathrm{incr}\,\circ}(|E_j|) \qquad \text{for all } j.$$

Equivalently, in directed-edge normalization, $B_{\mathcal{S}}(E_j) \le A_e\, I^{\mathrm{incr}\,\circ}_{\mathrm{edge}}(|E_j|)$ for some $A_e \asymp_A 1$.

Remark 9.45 (Record-minimum alignment). One may always thin a near-minimizer family to *record minima* of the increasing minorant; these are canonical checkpoints.

Definition 9.46 (Canonical checkpoints for lamplighters). For $G = H^{(\Gamma)} \rtimes \Gamma$ with $|H| = h \ge 2$, set $p := \frac{h-1}{h}$. Given a nested exhaustion $U_1 \subset U_2 \subset \cdots$ define

$$F_k^\star \ := \ F(U_k, M_k), \qquad M_k := \lceil p\, |U_k| \rceil\,.$$

By Theorem 9.42, there exists a cofinal subsequence $\{k_j\}$ and $A_0 < \infty$ such that

$$B_{\mathcal{S}}(F_{k_j}^\star) \ \le \ A_0\, I^{\mathrm{incr}\,\circ}_{\mathrm{edge}}\big(|F_{k_j}^\star|\big),$$

hence $\{F_{k_j}^\star\}$ is a family of *checkpoints for* (TF) with $A = A_0 \Delta$.

Corollary 9.47 (Lamplighters: TF(ii) globally, TF(i) along checkpoints). Under Theorems 9.39 and 9.42, the chain (F_n) satisfies

$$|F_{n+1} \setminus F_n| \ \le \ \Delta\, |\partial_{\mathcal{S}} F_n| \qquad (\forall n),$$

and there is an infinite subsequence of indices $\{n_j\}$ (checkpoints) for which

$$|\partial_{\mathcal{S}} F_{n_j}| \ \le \ A\, I^{\mathrm{incr}\,\circ}(|F_{n_j}|), \qquad A = A_0\, \Delta.$$

Theorem 9.48 (Lamplighters satisfy the decoupled (TF) scheme). Let $G = H^{(\Gamma)} \rtimes \Gamma$ with finite H and finitely generated Γ. With the single-step chain (F_n) of Theorem 9.39 and the checkpoint bounds of Theorem 9.42, one has TF(ii) globally with constant $B = \Delta$, and TF(i) along an infinite subsequence of checkpoints with constant $A = A_0\, \Delta$.

Remark 9.49 (Følner checkpoints). By Theorem 9.41, the checkpoint family $F_k^\star$ is a Følner sequence in G.

Remark 9.50 (On constructivity). The expansion mechanism (TF-ii) is explicit via single steps. The profile control (TF-i) relies on deep results for wreath products; we isolate this non-constructive input.

9.5. **Balloon chains (amenable graphs), a no-go observation for Cayley graphs, and a question.** We record a simple, self-contained *graph* example where the exact-vs-increasing isoperimetric profile ratio blows up, and then explain why the *mechanism* of that example does not port verbatim to Cayley graphs. We conclude with a concrete question for amenable groups.

Throughout this subsection we work in the *undirected edge* normalization

$$B(Y) \;:=\; \tfrac{1}{2}\sum_{s\in S}|sY\triangle Y|, \qquad I^{\circ}(r) \;:=\; \inf\{\, B(Y):\ |Y|=r \,\}, \qquad I^{\mathrm{incr}\,\circ}(r) \;:=\; \inf_{s\ge r} I^{\circ}(s).$$

Conversion to vertex or directed-edge boundary costs only a fixed multiplicative factor by Lemma 9.15.

A balloon-chain amenable graph with unbounded ratio. Let $\{E_k\}_{k\ge 1}$ be a sequence of finite, connected, d-regular graphs ($d\ge 3$) with a uniform edge Cheeger constant $h_0>0$:

$$|\partial_{E_k} A| \;\ge\; h_0\,\min\{|A|,\,|E_k|-|A|\} \qquad (\emptyset\ne A\subsetneq E_k)\,.$$

Write $N_k := |E_k|$ and assume $N_{k+1}\ge 3N_k$ (e.g. pass to a sparse subsequence). Pick $v_k\in V(E_k)$ and connect the E_k's by single *bridge* edges $\{v_k, v_{k+1}\}$, obtaining a connected graph G of maximum degree $d+1$. Set

$$S_n \;:=\; \bigcup_{k=1}^{n} E_k, \qquad n\ge 1.$$

Then $|S_n| = \sum_{k=1}^{n} N_k$ and $B(S_n)=1$.

Proposition 9.51 (Amenability)**.** G is amenable: $\dfrac{B(S_n)}{|S_n|}\to 0$.

Proof. Immediate from $B(S_n)=1$ and $|S_n|\to\infty$. □

Theorem 9.52 (Unbounded profile ratio on an amenable graph)**.** There is $c=c(h_0)>0$ such that for all n and all

$$r \;\in\; \Big[\,|S_{n-1}| + \tfrac{1}{3}N_n,\;\; |S_{n-1}| + \tfrac{2}{3}N_n\,\Big],$$

one has

$$I^{\circ}(r) \;\ge\; c\,N_n \qquad\text{and}\qquad I^{\mathrm{incr}\,\circ}(r) \;\le\; 1.$$

Hence $\sup_r I^{\circ}(r)/I^{\mathrm{incr}\,\circ}(r)=\infty$.

Proof. For the minorant, use $I^{\mathrm{incr}\,\circ}(r)\le I^{\circ}(|S_n|)\le B(S_n)=1$ when $r\le |S_n|$. For the lower bound fix $Y\subset V(G)$ with $|Y|=r$ and write $Y_k := Y\cap E_k$, $A_k := |Y_k|$. Since bridges only add nonnegative contributions,

$$B(Y) \;\ge\; \sum_{k\ge 1}|\partial_{E_k} Y_k| \;\ge\; h_0\sum_{k\ge 1}\min\{A_k, N_k - A_k\}.$$

Let $\Delta := r - |S_{n-1}| \in [\frac{1}{3}N_n, \frac{2}{3}N_n]$. If $A_n\ge \frac{1}{3}N_n$, then $\min\{A_n, N_n - A_n\}\ge \frac{1}{3}N_n$. Otherwise $A_n < \frac{1}{3}N_n$ and

$$\sum_{k>n} A_k \;\ge\; \Delta - A_n \;\ge\; \tfrac{1}{3}N_n - A_n.$$

Because $N_{n+1}\ge 3N_n$ we have $\sum_{k>n} A_k < \frac{1}{2}N_{n+1}\le \frac{1}{2}N_k$ for all $k>n$, so $\min\{A_k, N_k - A_k\} = A_k$ for $k>n$, whence

$$\sum_{k>n}\min\{A_k, N_k - A_k\} \;\ge\; \tfrac{1}{3}N_n - A_n.$$

In either case $\sum_{k\ge n}\min\{A_k, N_k - A_k\}\ge \frac{1}{3}N_n$, giving $B(Y)\ge (h_0/3)\,N_n$ and the claim. □

Remark 9.53 (Context)**.** The balloon-chain graph is intentionally non-vertex-transitive. It cleanly exposes the spike/plateau mechanism behind large oscillations between I° and $I^{\mathrm{incr}\,\circ}$ in a transparent setting.

A no-go observation for Cayley graphs. The balloon mechanism relies on two features that do not carry over verbatim to Cayley graphs: (i) extremely *sparse* volumes at which I° is uniformly tiny (here, $B(S_n) = 1$), and (ii) extremely *cheap interfaces* (single edges) separating huge regions. In Cayley graphs, two elementary facts constrain how "spiky" I° can be between such cheap volumes.

Lemma 9.54 (Left-Lipschitz trimming)**.** Let X be a bounded-degree graph with maximum degree Δ. For any finite $E \subset V(X)$ and any $t \in \{0, 1, \dots, |E|\}$ there exists $E' \subseteq E$ with $|E'| = |E| - t$ and

$$B(E') \;\le\; B(E) \;+\; \Delta\, t.$$

Consequently, if $I^\circ(s_0) \le B_0$ for some s_0, then for every $r \le s_0$,

$$I^\circ(r) \;\le\; B_0 \;+\; \Delta\,(s_0 - r).$$

Proof. Delete vertices one by one; removing a vertex can introduce at most Δ new broken edges, so the boundary increases by at most Δ at each deletion. The consequence follows by applying the bound to a minimizer at size s_0. $\square$

Corollary 9.55 (No verbatim "balloon plateau" in Cayley graphs)**.** Let X be a Cayley graph of degree Δ. If along a sequence (s_k) one has $I^\circ(s_k) \le B_0$ (with B_0 independent of k), then for each k and all $r \in [s_k - \lfloor B_0/\Delta \rfloor,\, s_k]$, $I^\circ(r) \le 2B_0$. Thus arbitrarily long gaps of the form $[s_k, s_{k+1})$ cannot be "expensive" all the way up to s_{k+1}: near each cheap point s_k there is a uniformly wide interval of uniformly cheap volumes.

Proof. Apply Lemma 9.54 with $t \le B_0/\Delta$. $\square$

Remark 9.56 (What this does and does not say)**.** Corollary 9.55 does *not* rule out large oscillations of I° in Cayley graphs; it only shows that the exact "single-edge plateau with arbitrarily long expensive ramps" of the balloon chain cannot occur verbatim. In particular, if cheap volumes are dense (as in many amenable Cayley graphs), the left-Lipschitz control significantly limits how steep spikes can be patched between them. For several amenable Cayley families (virtually nilpotent groups; finite-lamp lamplighters $H \wr \mathbb{Z}$; semidirects $\mathbb{Z}^d \rtimes_A \mathbb{Z}$ with hyperbolic A; finite products/extensions), two-sided asymptotics or our (TF) scheme give *bounded* ratios $I^\circ / I^{\mathrm{incr}\,\circ}$ (see §9 and references such as [Ers03]).

A question. Motivated by the toy model above and the constraints just discussed, we isolate a clear target for future work.

Question 9.1. Does there exist a finitely generated *amenable* group whose Cayley graph satisfies

$$\sup_{r \ge 1} \frac{I^\circ(r)}{I^{\mathrm{incr}\,\circ}(r)} \;=\; \infty\,?$$

Remark 9.57 (Known positive and negative evidence)**.** For classical amenable families, the ratio is known or shown here to be *bounded*: virtually nilpotent groups (two-sided Wulff asymptotics), finite-lamp lamplighters over $\mathbb{Z}$ (two-sided $r/\log r$ bounds, e.g. [Ers03]), and hyperbolic semidirects $\mathbb{Z}^d \rtimes_A \mathbb{Z}$ (our (TF) interleaving). The balloon chain shows that unbounded ratios occur easily in non-vertex-transitive amenable graphs. Whether this can happen in the Cayley world remains, to our knowledge, open.

9.6. **New verifications and closure properties for** (TF)**.** Unless stated otherwise, in this subsection we work with the *undirected* edge boundary

$$B_{\mathcal{S}}(Y) \;:=\; \tfrac{1}{2} \sum_{s \in \mathcal{S}} |sY \triangle Y|.$$

Comparisons to directed/vertex boundaries cost a universal factor by Lemma 9.15. All statements in this subsection use this normalization unless an "edge" subscript $(\cdot)_{\mathrm{edge}}$ is present.

9.6.1. *Stability under finite index and finite extensions.*

Proposition 9.58 (Finite index invariance)**.** Let $H \le G$ be of finite index and equip G, H with Cayley sets $\mathcal{S}_G, \mathcal{S}_H$ with $\mathcal{S}_H \subseteq \mathcal{S}_G^m$ and $\mathcal{S}_G \subseteq \mathcal{S}_H^n$. Then G has (TF) iff H has (TF). Moreover, if H has (TF) with (A_e, B) (in edge normalization), then G has (TF) with constants depending only on $(A_e, B, m, n, [G:H], \Delta_G, \Delta_H)$.

Proof. (*TF(ii)*.) Fix a right transversal $T \subset G$ for $H\backslash G$, $|T| = [G:H] =: q$. Assume $(F_n)_{n\ge1}$ in H are nested and satisfy $|F_{n+1} \setminus F_n| \le B\, B_{\mathcal{S}_H}(F_n)$. Set

$$E_n \;:=\; \bigcup_{t\in T} tF_n \;\subset G.$$

Then $E_n \subset E_{n+1}$, $|E_n| = q\,|F_n|$, and $|E_{n+1} \setminus E_n| = q\,|F_{n+1} \setminus F_n|$. By bounded change of generating sets ($\mathcal{S}_H \subseteq \mathcal{S}_G^m$, $\mathcal{S}_G \subseteq \mathcal{S}_H^n$) and bounded coset overlaps, there exist constants $c_1, c_2 \in (0,\infty)$ depending only on $(m, n, \Delta_G, \Delta_H, q)$ such that

$$c_1\, B_{\mathcal{S}_H}(F_n) \;\le\; B_{\mathcal{S}_G}(E_n) \;\le\; c_2\, B_{\mathcal{S}_H}(F_n).$$

Hence

$$|E_{n+1} \setminus E_n| \;=\; q\,|F_{n+1} \setminus F_n| \;\le\; q\,B\,B_{\mathcal{S}_H}(F_n) \;\le\; (qB/c_1)\,B_{\mathcal{S}_G}(E_n),$$

so TF(ii) holds in G with $B_G := qB/c_1$.

(*TF(i$_e$)*.) Suppose $B_{\mathcal{S}_H}(F_n) \le A_e\, I^{\mathrm{incr}\,\circ}_{\mathrm{edge},H}(|F_n|)$ for all n. Then $B_{\mathcal{S}_G}(E_n) \le c_2 A_e\, I^{\mathrm{incr}\,\circ}_{\mathrm{edge},H}(|F_n|)$. By quasi-isometry invariance of profiles (Theorem 9.90 applied to the Cayley graphs of H and G), there exist $a, b > 0$ depending only on $(m, n, \Delta_G, \Delta_H, q)$ such that

$$I^{\mathrm{incr}\,\circ}_{\mathrm{edge},H}(r) \;\le\; a\, I^{\mathrm{incr}\,\circ}_{\mathrm{edge},G}(b\,r) \qquad \forall r \ge 1.$$

With $|F_n| = |E_n|/q$ we get

$$B_{\mathcal{S}_G}(E_n) \;\le\; c_2 a A_e\; I^{\mathrm{incr}\,\circ}_{\mathrm{edge},G}\big(b\,|E_n|/q\big).$$

Finally, by the right-Lipschitz property of the increasing minorant (Lemma 9.27 in vertex normalization; the edge version is analogous), changing the argument by a fixed factor changes the value by at most a fixed multiplicative/additive constant. Absorbing these numerical factors yields

$$B_{\mathcal{S}_G}(E_n) \;\le\; A'_e\, I^{\mathrm{incr}\,\circ}_{\mathrm{edge},G}(|E_n|),$$

with A'_e depending only on $(A_e, m, n, \Delta_G, \Delta_H, q)$. The converse direction ($G \Rightarrow H$) is analogous by restricting to a coset and using the other generator inclusion. □

Proposition 9.59 (Finite extensions)**.** Let $1 \to F \to G \xrightarrow{\pi} Q \to 1$ be exact with F finite. Then G has (TF) iff Q has (TF). Constants change by factors depending only on $|F|$ and the generating sets.

Proof. (*TF(ii)*.) If $(\widehat{F}_n)$ in Q witnesses TF(ii) with constant B_Q, choose a section $s : Q \to G$ and define

$$E_n \;:=\; s(\widehat{F}_n)\,F \;\subset G.$$

Then $E_n \subset E_{n+1}$, $|E_n| = |F|\,|\widehat{F}_n|$, and $|E_{n+1} \setminus E_n| = |F|\,|\widehat{F}_{n+1} \setminus \widehat{F}_n|$. Each broken edge in Q lifts to at most $|F|$ broken edges in G, and conversely $B_{\mathcal{S}_G}(E_n) \asymp_{|F|} B_{\mathcal{S}_Q}(\widehat{F}_n)$ (bounded change of generators between the quotient and the lifted set). Thus

$$|E_{n+1} \setminus E_n| \;\le\; |F|\,B_Q\,B_{\mathcal{S}_Q}(\widehat{F}_n) \;\lesssim_{|F|}\; B_{\mathcal{S}_G}(E_n),$$

proving TF(ii) in G. The reverse direction projects a TF(ii) chain in G to Q and uses the same comparability.

(*TF(i$_e$)*.) If $B_{\mathcal{S}_Q}(\widehat{F}_n) \le A_e\, I^{\mathrm{incr}\,\circ}_{\mathrm{edge},Q}(|\widehat{F}_n|)$, then

$$B_{\mathcal{S}_G}(E_n) \;\lesssim_{|F|}\; A_e\, I^{\mathrm{incr}\,\circ}_{\mathrm{edge},Q}(|\widehat{F}_n|).$$

By quasi-isometry invariance of the increasing minorant between G and Q (Theorem 9.90), there exists $a > 0$ such that $I^{\mathrm{incr}\,\circ}_{\mathrm{edge},Q}(r) \;\le\; a\, I^{\mathrm{incr}\,\circ}_{\mathrm{edge},G}(r)$ up to a bounded change of scale (again absorbed via right-Lipschitz). Hence $B_{\mathcal{S}_G}(E_n) \;\le\; A'_e\, I^{\mathrm{incr}\,\circ}_{\mathrm{edge},G}(|E_n|)$ with A'_e depending only on $(A_e, |F|)$ and the generators. The converse direction is similar. $\square$

9.6.2. *Direct products.*

Proposition 9.60 (Products and (TF) increments)**.** For the product generating set on $G_1 \times G_2$ and finite $U_i \subset G_i$,

$$\partial(U_1 \times U_2) \;\subset\; (\partial U_1) \times U_2 \;\cup\; U_1 \times (\partial U_2).$$

Consequently, if U_i have (TF) increment bounds B_i, then $U_1 \times U_2$ has an increment bound of the form $B \lesssim B_1 + B_2$ (after a fixed normalization conversion).

Theorem 9.61 (Direct products: TF(ii) unconditionally; TF(i_e) under NNM)**.** Let $(G_i, \mathcal{S}_i)$, $i = 1, 2$, satisfy (TF) in edge normalization with constants $(A_e^{(i)}, B^{(i)})$. Equip $G := G_1 \times G_2$ with the product generating set $\mathcal{S} := \big(\mathcal{S}_1 \times \{e\}\big) \cup \big(\{e\} \times \mathcal{S}_2\big)$. Then:

(a) TF(ii), unconditionally. If $(F_n^{(i)})$ witness TF(ii) with $B^{(i)}$, then

$$E_n := F_n^{(1)} \times F_n^{(2)}$$

is nested in G and satisfies

$$|E_{n+1} \setminus E_n| \;\le\; B\, B_{\mathcal{S}}(E_n), \qquad B \;\lesssim\; B^{(1)} + B^{(2)}.$$

Indeed, in the (undirected) edge normalization one has the exact split

$$B_{\mathcal{S}}(E_n) = B_{\mathcal{S}_1}(F_n^{(1)})\, |F_n^{(2)}| \;+\; |F_n^{(1)}|\, B_{\mathcal{S}_2}(F_n^{(2)}),$$

since each product generator acts in exactly one coordinate.

(b) TF(i_e) along a cofinal subsequence, assuming NNM in each factor. Assume each $(G_i, \mathcal{S}_i)$ admits nested near-minimizers (Definition 9.20) with

$$B_{\mathcal{S}_i}\big(W_u^{(i)}\big) \;\le\; C_0^{(i)}\, I^{\mathrm{incr}\,\circ}_{\mathrm{edge},G_i}(u) \qquad (u \gg 1).$$

Let $\mathcal{R}_i := \{\, u :\; I^{\mathrm{incr}\,\circ}_{\mathrm{edge},G_i}(u) = I^{\circ}_{\mathrm{edge},G_i}(u) \,\}$ be the record volumes of the factor minorants. Then along any cofinal sequence $(u_j, v_j) \in \mathcal{R}_1 \times \mathcal{R}_2$ and with

$$Z_j \;:=\; W_{u_j}^{(1)} \times W_{v_j}^{(2)},$$

one has

$$B_{\mathcal{S}}(Z_j) \;\le\; \big(C_0^{(1)} + C_0^{(2)}\big)\, I^{\mathrm{incr}\,\circ}_{\mathrm{edge},G}\big(|Z_j|\big).$$

In particular, G satisfies TF(i_e) along a cofinal subsequence with $A_e \le C_0^{(1)} + C_0^{(2)}$.

Lemma 9.62 (Checkpoint tensoring for products)**.** Let $G = G_1 \times G_2$ with the product generating set and assume nested near-minimizers in each factor: $B_{\mathcal{S}_i}(W_u^{(i)}) \le C_0^{(i)}\, I^{\mathrm{incr}\,\circ}_{\mathrm{edge},G_i}(u)$ for large u. Let $\mathcal{R}_i$ be the record volumes of $I^{\mathrm{incr}\,\circ}_{\mathrm{edge},G_i}$. For any cofinal sequence $(u_j, v_j) \in \mathcal{R}_1 \times \mathcal{R}_2$, the product sets

$$Z_j \;:=\; W_{u_j}^{(1)} \times W_{v_j}^{(2)}$$

satisfy

$$B_{\mathcal{S}}(Z_j) \;\le\; \big(C_0^{(1)} + C_0^{(2)}\big)\, I^{\mathrm{incr}\,\circ}_{\mathrm{edge},G}\big(u_j v_j\big).$$

Proof. For the product generating set,

$$B_{\mathcal{S}}(A_1 \times A_2) \;=\; B_{\mathcal{S}_1}(A_1)\,|A_2| \;+\; |A_1|\,B_{\mathcal{S}_2}(A_2).$$

Hence

$$B_{\mathcal{S}}(Z_j) \;\le\; C_0^{(1)}\,v_j\,I^{\mathrm{incr}\,\circ}_{\mathrm{edge},G_1}(u_j) \;+\; C_0^{(2)}\,u_j\,I^{\mathrm{incr}\,\circ}_{\mathrm{edge},G_2}(v_j).$$

Since $u_j \in \mathcal{R}_1$, $v_j \in \mathcal{R}_2$, we have $I^{\mathrm{incr}\,\circ}_{\mathrm{edge},G_i}(u_j) = I^{\circ}_{\mathrm{edge},G_i}(u_j)$ and similarly for v_j. Now upper-bound the product minorant by testing with exact minimizers at u_j, v_j:

$$I^{\mathrm{incr}\,\circ}_{\mathrm{edge},G}(u_jv_j) \;\le\; I^{\circ}_{\mathrm{edge},G}(u_jv_j) \;\le\; v_j\,I^{\circ}_{\mathrm{edge},G_1}(u_j) \;+\; u_j\,I^{\circ}_{\mathrm{edge},G_2}(v_j),$$

using the product boundary identity. Combining the two displays yields the claim. $\square$

Corollary 9.63 (Direct products satisfy full (TF) under NNM)**.** In the setting of Theorem 9.61(b), assume each factor admits nested near-minimizers (NNM), and let $(u_j, v_j) \in \mathcal{R}_1 \times \mathcal{R}_2$ be a cofinal checkpoint sequence as in Lemma 9.62. Define the *product checkpoints*

$$W_{r_j} \;:=\; W^{(1)}_{u_j} \times W^{(2)}_{v_j}, \qquad r_j := u_jv_j.$$

Then there exists a single nested chain $(F_m)_{m\ge m_0}$ in G (built by the confined step $\mathcal{T}_{r_{j+1}}$ between successive levels r_j; see Definition 9.25 and Lemma 9.26) such that:

$$\text{(TF(ii))} \qquad |F_{m+1} \setminus F_m| \;\le\; B\,B_{\mathcal{S}}(F_m) \quad \text{with } B = 1 \text{ (edge normalization)},$$

$$\text{(TF(i}_e\text{))} \qquad B_{\mathcal{S}}(F_m) \;\le\; A_e\,I^{\mathrm{incr}\,\circ}_{\mathrm{edge},G}(|F_m|) \quad \text{for all sufficiently large } m,$$

with

$$A_e \;=\; \big(C_0^{(1)} + C_0^{(2)}\big) \;+\; \big((C_0^{(1)} + C_0^{(2)})\,\Delta_G + 1\big)\,\vartheta,$$

for any fixed $\vartheta \in (0,1]$ chosen in the budgeted-level schedule (Definition 9.28). In particular, G satisfies full (TF).

Proof. By Lemma 9.62 the product checkpoints W_{r_j} satisfy $B_{\mathcal{S}}(W_{r_j}) \le (C_0^{(1)} + C_0^{(2)})\,I^{\mathrm{incr}\,\circ}_{\mathrm{edge},G}(r_j)$ along a cofinal subsequence of volumes. Between successive levels $r_j < r_{j+1}$, evolve by the *confined* canonical step $\mathcal{T}_{r_{j+1}}$ (Definition 9.25); by Lemma 9.26 this produces a nested chain (F_m) satisfying TF(ii) with $B = 1$, and TF(i$_e$) at each checkpoint index m with the checkpoint constant for W_{r_j}.

Now run the *budgeted-level* schedule (Definition 9.28) with any fixed $\vartheta \in (0,1]$. Proposition 9.29 applies verbatim (using the checkpoint inequality only at the selected levels r_j) and yields, for all intermediate steps m on level r_j,

$$B_{\mathcal{S}}(F_m) \;\le\; \Big((C_0^{(1)} + C_0^{(2)}) \;+\; \big((C_0^{(1)} + C_0^{(2)})\,\Delta_G + 1\big)\,\vartheta\Big)\; I^{\mathrm{incr}\,\circ}_{\mathrm{edge},G}\big(|F_m|\big).$$

This proves TF(i$_e$) eventually (hence TF(i) in vertex normalization, with constants multiplied by Δ_G via Lemma 9.15), while TF(ii) holds globally with $B = 1$. Thus (F_m) witnesses full (TF) for G. $\square$

9.7. **Wreath products over** (TF) **bases (finite lamps).**

Theorem 9.64 (Finite-lamp wreaths over (TF) bases: decoupled scheme)**.** Let $(\Gamma, \mathcal{S}_\Gamma)$ satisfy (TF) with constants (A_e^Γ, B_Γ) and let H be finite nontrivial with toggles T ($t := |\mathsf{T}|$, $q := |H| - 1$). For $G := H^{(\Gamma)} \rtimes \Gamma$ with $\mathcal{S} = \mathcal{S}_\Gamma \cup \mathsf{T}$ there exists a nested chain (F_n) such that

$$\text{TF(ii) holds globally with } B \le C_H\big(B_\Gamma + 1\big),$$

and

$$\text{TF(i}_e\text{) holds along an infinite subsequence with } A_e \le C_H\,A_e^\Gamma,$$

where C_H depends only on $|H|$ and t. If, in addition, $(G, \mathcal{S})$ admits nested near-minimizers at all large volumes, then full (TF) holds with the same type of bounds on (A_e, B) up to C_H.

9.8. **Semidirect products $\mathbb{Z}^d \rtimes_A \mathbb{Z}$: A–covariant layer-nested sets, zero interior drift, and tempered Følner chains.** We consider

$$G = \mathbb{Z}^d \rtimes_A \mathbb{Z}, \qquad (x,k)\cdot(y,\ell) = (x + A^k y,\ k+\ell),$$

with the *left* Cayley generating set

$$\mathcal{S} \ :=\ \{\, s_1^{\pm1}, \dots, s_d^{\pm1}, t^{\pm1} \,\}, \qquad s_i\cdot(x,k) = (x+e_i, k), \qquad t\cdot(x,k) = (Ax, k+1).$$

Undirected edge boundary is used throughout this subsection.

9.8.1. *A–covariant layer-nested sets and exact cancellation of interior drift.*

Definition 9.65 (A–covariant layer-nested set)**.** Let $K \subset \mathbb{R}^d$ be a bounded measurable set and let $R > 0$, $T \in \mathbb{N}$. Set

$$X_0 := \big((RK) \cap \mathbb{Z}^d\big), \qquad X_k := A^k X_0 \quad (-T \le k \le T), \qquad F_{K,R,T} := \bigcup_{k=-T}^{T} \big(X_k \times \{k\}\big) \ \subset G.$$

Lemma 9.66 (Nestedness of A–covariant layer-nested sets)**.** If $R_1 \le R_2$ and $T(\cdot)$ is nondecreasing, then $F_{K,R_1,T(R_1)} \subseteq F_{K,R_2,T(R_2)}$.

Lemma 9.67 (Zero interior vertical drift)**.** For every $A \in GL(d,\mathbb{Z})$ and $F_{K,R,T}$ as above, the vertical boundary across each interior interface vanishes:

$$|\, X_k \,\triangle\, A^{-1} X_{k+1} \,| = 0 \qquad (-T \le k < T).$$

Hence the total vertical boundary is *only the two caps*:

$$B_{\mathrm{vert}}\big(F_{K,R,T}\big) \ =\ |X_{-T}| \ +\ |X_T| \ =\ 2\,|X_0|.$$

Lemma 9.68 (Uniform slice sizes)**.** For all k, $|X_k| = |X_0| = |(RK) \cap \mathbb{Z}^d|$. In particular,

$$|F_{K,R,T}| = (2T+1)\,|X_0| \ \asymp_K\ (2T+1)\,R^d.$$

Lemma 9.69 (Unit-thickness annulus bound)**.** If $K \subset \mathbb{R}^d$ is a bounded convex body with piecewise C^1 boundary, there exists $C_K < \infty$ such that for all $R \ge 1$,

$$\big|X_0(R+1) \setminus X_0(R)\big| \ \le\ C_K\, R^{d-1}, \qquad X_0(R) := (RK) \cap \mathbb{Z}^d.$$

9.8.2. *Horizontal boundary: cofactor growth and the logarithmic height regime.*

Lemma 9.70 (Horizontal boundary at level k)**.** Let $K \subset \mathbb{R}^d$ be a convex body with piecewise C^1 boundary. There exists $C_K < \infty$ such that for all $R \ge 1$ and $k \in \mathbb{Z}$,

$$B_{\mathrm{hor}}(X_k) \ \le\ C_K\, R^{d-1}\, \big\|\operatorname{cof}(A^k)\big\|,$$

where $\operatorname{cof}(\cdot)$ is the cofactor matrix, and $\|\cdot\|$ is any operator norm on $\mathbb{R}^{d\times d}$.

Proof idea. Let Per_τ denote the anisotropic perimeter associated with the axis stencil at height k. If $E \subset \mathbb{R}^d$ is smooth, then $\mathrm{Per}_\tau(A^k E) = \int_{\partial E} \|(\operatorname{cof} A^k) n\|\, d\mathcal{H}^{d-1} \le \|\operatorname{cof} A^k\|\, \mathrm{Per}_\tau(E)$. Discretizing via the standard BV-graph perimeter comparison (coarea + lattice approximation of RK) yields $B_{\mathrm{hor}}(X_k) \lesssim \|\operatorname{cof} A^k\|\, R^{d-1}$ with the implicit constant depending only on K and the stencil. □

Lemma 9.71 (Two-sided cofactor growth)**.** Let $A \in GL(d,\mathbb{Z})$. For every $\Lambda > \max\{\rho(A),\ \rho(A^{-1})\}$ there exists $C_\Lambda < \infty$ such that

$$\|\operatorname{cof}(A^k)\| \ \le\ C_\Lambda\, \Lambda^{|k|} \qquad (k \in \mathbb{Z}).$$

Remark 9.72 (Cofactor identity for $GL(d,\mathbb{Z})$)**.** For all $k \in \mathbb{Z}$, $\operatorname{cof}(A^k) = (\det A)^k (A^{-k})^\top$. Since $\det A = \pm1$ for $A \in GL(d,\mathbb{Z})$, the growth of $\|\operatorname{cof}(A^k)\|$ is governed by $\|A^{-k}\|$, which justifies the choice of $\Lambda > \max\{\rho(A), \rho(A^{-1})\}$ in Lemma 9.71.

Corollary 9.73 (Total horizontal boundary)**.** There exist $C_{A,K} < \infty$ and $\Lambda(A) > \max\{\rho(A), \rho(A^{-1})\}$ such that

$$B_{\mathrm{hor}}\big(F_{K,R,T}\big) \;\le\; C'_{A,K}\, R^{d-1}\, \Lambda(A)^{T}.$$

We now choose the height to grow *logarithmically* in R:

$$T(R) := \lfloor \alpha \,\log R \rfloor, \qquad 0 < \alpha < \frac{1}{\log \Lambda(A)}.$$

Proposition 9.74 (Log-height layer-nested sets are Følner and have tempered increments)**.** For $F_R := F_{K,R,T(R)}$ one has:

(1) (Følner) $\quad \dfrac{B_{\mathcal{S}}(F_R)}{|F_R|} \xrightarrow[R\to\infty]{} 0.$

(2) (TF)(ii) $\quad$ There exists $B < \infty$ such that for all large R, $|\, F_{R+1} \setminus F_R \,| \;\le\; B\, B_{\mathcal{S}}(F_R)$.

Proof. (1) Vertical: $B_{\mathrm{vert}}(F_R) = 2|X_0(R)| \asymp_K R^d$ and $|F_R| \asymp_K (\log R)R^d$, so $B_{\mathrm{vert}}/|F_R| \to 0$. Horizontal: $B_{\mathrm{hor}}(F_R) \lesssim R^{d-1}\Lambda^{T(R)} = R^{-1+\alpha\log\Lambda}/\log R \to 0$.

(2) If $T(R+1) = T(R)$ the increment is $(2T(R)+1)\,|X_0(R+1) \setminus X_0(R)| \lesssim R^{d-1}\log R$. If $T(R+1) = T(R)+1$, two new slices add $\lesssim R^d$. Since $B_{\mathcal{S}}(F_R) \ge B_{\mathrm{vert}}(F_R) \asymp R^d$, both cases give the bound. □

Theorem 9.75 (Tempered Følner chains for $\mathbb{Z}^d \rtimes_A \mathbb{Z}$)**.** Let $A \in GL(d,\mathbb{Z})$ and $K \subset \mathbb{R}^d$ convex with piecewise C^1 boundary. Then the sets

$$E_R \;:=\; F_{K,R,T(R)}, \qquad T(R) = \lfloor \alpha \log R \rfloor,\;\; 0 < \alpha < 1/\log\Lambda(A),$$

form a nested Følner sequence and satisfy the tempered increment bound (TF)(ii) in the undirected edge normalization:

$$\frac{B_{\mathcal{S}}(E_R)}{|E_R|} \xrightarrow[R\to\infty]{} 0, \qquad |E_{R+1} \setminus E_R| \;\le\; B \cdot B_{\mathcal{S}}(E_R) \;\text{ for all large } R.$$

Lemma 9.76 (Explicit increment size)**.** With $E_R = F_{K,R,T(R)}$ and $T(R) = \lfloor \alpha \log R \rfloor$ as above, there exists $C = C(A,K)$ such that for all large R,

$$|E_{R+1} \setminus E_R| \;\le\; C\big(R^{d-1}\log R + R^d\,\mathbf{1}_{\{T(R+1)=T(R)+1\}}\big) \;\le\; C'\,R^d.$$

Proof. If $T(R+1) = T(R)$, the difference is a unit-thickness annulus within each of $(2T(R)+1)$ slices: the bound follows from Lemma 9.69. If $T(R+1) = T(R)+1$, we add two slices of size $\asymp |X_0(R+1)| \asymp R^d$, plus the annulus cost; combine to conclude. □

Definition 9.77 (L^1-profile)**.** Let X be a bounded-degree graph. Its L^1-isoperimetric (Nash) profile $j_1(v)$ is the best constant J such that for every finite $E \subset V_X$ with $|E| \le v$ and every function $f : V_X \to \mathbb{R}$ supported in E,

$$\sum_{x\in E} |f(x) - m(f,E)| \;\le\; J \sum_{\{x,y\}\in E^1} |f(x) - f(y)|,$$

where $m(f,E)$ is a median of f on E and E^1 is the (undirected) edge set of X. (See [SC02, Ch. 5].)

Lemma 9.78 (Nash/L^1 profile vs. edge isoperimetry)**.** Let X be a bounded-degree graph with maximum degree Δ, and let $j_1(v)$ be its L^1-(Nash) profile as in Definition 9.77. There exist constants $c_1, c_2 \in (0,\infty)$ depending only on Δ such that for all $v \ge 1$,

$$c_1 \sup_{1\le u\le v} \frac{u}{I^{\circ}_{\mathrm{edge}}(u)} \;\le\; j_1(v) \;\le\; c_2 \sup_{1\le u\le v} \frac{u}{I^{\circ}_{\mathrm{edge}}(u)}. \tag{9.5}$$

In particular,

$$I^{\circ}_{\mathrm{edge}}(v) \;\ge\; \frac{c_1\, v}{j_1(v)} \qquad \text{for all } v \ge 1. \tag{9.6}$$

Proof and references. The two-sided equivalence (9.5) is classical (a discrete Maz'ya inequality); see Varopoulos-Saloff-Coste-Coulhon [VSCC92, Thm. VI.3] and Saloff-Coste [SC02, Ch. 5-6]. Since $\sup_{u\le v} u/I^\circ_{\text{edge}}(u) \ge v/I^\circ_{\text{edge}}(v)$, the lower bound in (9.5) implies (9.6). □

Remark 9.79 (Normalization)**.** The lemma is written for the *directed* edge profile I°_{edge}. In the undirected normalization, all displayed inequalities hold with an absolute factor 2. Passing to vertex boundary costs a factor depending only on Δ (Lemma 9.15). All such constants can be absorbed into c_1, c_2.

Corollary 9.80. If $j_1(v) \asymp \log v$ (e.g. for $G = \mathbb{Z}^d \rtimes_A \mathbb{Z}$ with A hyperbolic), then

$$I^\circ_{\text{edge}}(v) \gtrsim \frac{v}{\log v}.$$

Theorem 9.81 (Hyperbolic semidirects: global near–minimality up to constants)**.** Let $G = \mathbb{Z}^d \rtimes_A \mathbb{Z}$ with A hyperbolic and $\mathcal{S}$ finite symmetric. Then there exists $A_e > 0$ such that for all large R,

$$B_{\mathcal{S}}(E_R) \le A_e\, I^\circ_{\text{edge}}(|E_R|).$$

Proof. For polycyclic groups of exponential growth such as $G = \mathbb{Z}^d \rtimes_A \mathbb{Z}$ with hyperbolic A, the L^1-profile satisfies $j_1(v) \asymp \log v$; see, e.g., [VSCC92, Thm. VI.3] and [SC02, Ch. 6]. By Lemma 9.78 this implies $I^\circ_{\text{edge}}(v) \gtrsim v/\log v$. On the other hand, for our sets E_R we have $|E_R| \asymp R^d \log R$ and $B_{\mathcal{S}}(E_R) \asymp R^d$ by Theorem 9.75 and Lemma 9.76. Therefore

$$B_{\mathcal{S}}(E_R) \lesssim \frac{|E_R|}{\log |E_R|} \lesssim I^\circ_{\text{edge}}(|E_R|),$$

as claimed (absorbing implicit constants into A_e). □

9.8.3. *Layer-nested class: lower bound and order-optimality.*

Definition 9.82 (A-covariantly layer-nested sets)**.** A finite set $Y \subset \mathbb{Z}^d \rtimes_A \mathbb{Z}$ is *A-covariantly layer-nested* if it is of the form $Y = \bigsqcup_{t=-T}^{T} \big(X_t \times \{t\}\big)$ for some $T \in \mathbb{N}$ and finite $X_t \subset \mathbb{Z}^d$, and if the *pulled-back* slices

$$X_t^* := A^{-t} X_t \subset \mathbb{Z}^d$$

satisfy symmetric nesting: $X^*_{t+1} \subseteq X^*_t$ for $t \ge 0$, $\quad X^*_{t-1} \subseteq X^*_t$ for $t \le 0$. *Remark.* The A-covariant sets $E_R = F_{K,R,T(R)}$ are in this class, since $X_t = A^t X_0$ so $X_t^* = X_0$ for all t.

Lemma 9.83 (Lower bound in the A-covariantly layer-nested class)**.** Let Y be A-covariantly layer-nested with slices $\{X_t\}_{t=-T}^T$ and pulled-backs $X_t^* = A^{-t}X_t$. Then

$$B_{\mathcal{S}}(Y) \ge B_{\text{vert}}(Y) = 2\,|X_0^*| \ge \frac{2}{2T+1}\,|Y|.$$

Moreover, there exists $c_h = c_h(A, \mathcal{S}) > 0$ such that the horizontal part obeys

$$B_{\text{hor}}(Y) \ge c_h \sum_{t=-T}^{T} \text{Per}_{\tau_{A,t}}(X_t^*),$$

where $\tau_{A,t}$ is the anisotropic gauge on $\mathbb{R}^d$ induced by the pulled-back axis stencil $\{\pm A^{-t} e_i\}_{i=1}^d$.

Proof. For each vertical edge, $(x,t) \leftrightarrow (Ax, t+1)$ lies inside Y iff $x \in X_t$ and $Ax \in X_{t+1}$. In pulled-back coordinates $u = A^{-t}x$, this is $u \in X_t^* \cap X^*_{t+1}$. Hence the number of broken vertical edges across the interface $t \to t+1$ equals $|X_t^* \setminus X^*_{t+1}|$ for $t \ge 0$ (by nesting), and similarly $|X^*_{t+1} \setminus X^*_t|$ for $t < 0$. Summing telescopically and adding the two caps gives

$$B_{\text{vert}}(Y) = \big(|X_0^*| - |X_T^*|\big) + \big(|X_0^*| - |X^*_{-T}|\big) + |X_T^*| + |X^*_{-T}| = 2\,|X_0^*|.$$

Since X_0^* is the largest pulled-back slice by nesting, $|X_0^*| \geq \frac{1}{2T+1}\sum_{t=-T}^{T}|X_t^*| = \frac{|Y|}{2T+1}$, proving the displayed vertical bound. The horizontal inequality is the standard discrete anisotropic perimeter lower bound for each layer (coarea + calibration for the pulled-back stencil), summed over t; the constant depends only on $(A, \mathcal{S})$. □

Proposition 9.84 (Logarithmic layer-nested sets are near-minimizers in the A-covariantly layer-nested class)**.** Let $E_R = F_{K,R,T(R)}$ with $T(R) = \lfloor \alpha \log R \rfloor$, $0 < \alpha < 1/\log \Lambda(A)$. There exists $A_e(A, \mathcal{S}) \in (0, \infty)$ such that for all large R,

$$B_{\mathcal{S}}(E_R) \ \leq\ A_e\ \inf\Big\{\, B_{\mathcal{S}}(Y) :\ Y\ A\text{-covariantly layer-nested},\ |Y| = |E_R| \,\Big\}.$$

Proof. By Lemma 9.83, for any competitor Y with $|Y| = |E_R|$ and $2T+1$ layers we have

$$B_{\mathcal{S}}(Y) \ \geq\ \frac{2}{2T+1}\,|Y| \ =\ \frac{2}{2T+1}\,|E_R|.$$

For E_R, the vertical boundary is exactly $2|X_0| = \frac{2}{2T+1}|E_R|$ by Lemma 9.67, while the horizontal boundary satisfies $B_{\mathrm{hor}}(E_R) = o(R^d)$ by Corollary 9.73 with our choice of $T(R)$. Since $|E_R| \asymp R^d \log R$ and $(2T+1)^{-1} \asymp 1/\log R$, we have $B_{\mathrm{hor}}(E_R) = o\big(|E_R|/(2T+1)\big)$. Therefore

$$B_{\mathcal{S}}(E_R) \ =\ \frac{2}{2T+1}|E_R| + o\left(\frac{|E_R|}{2T+1}\right) \ \leq\ (1+o(1))\ \inf\{B_{\mathcal{S}}(Y) : Y\ A\text{-covariantly layer-nested},\ |Y| = |E_R|\}.$$

Absorbing the $o(1)$ into a fixed constant for large R gives the claim. □

Remark 9.85 (Why this class)**.** The A-covariant nesting compares like with like: it tests against families whose slices become nested *after* conjugating by A^{-t}, which is the natural frame for the vertical edges $(x,t) \leftrightarrow (Ax, t+1)$. In this class the vertical contribution has the exact formula $B_{\mathrm{vert}} = 2|X_0^*|$, and the benchmark lower bound depends only on the total volume and the height $(2T+1)$. Our sets E_R belong to this class and achieve the height-dominant lower bound up to $o(1)$, which is precisely what one expects in the logarithmic regime.

9.9. **Quasi–isometry invariance and a constructive route from bounded ratio to** (TF)**.** We write $I^{\circ}(\cdot)$ for the exact *vertex* profile of a bounded–degree graph X and $I^{\mathrm{incr}\,\circ}$ for its greatest nondecreasing minorant. Throughout this subsection we work in the vertex normalization; constants in comparisons depend only on quasi–isometry data and maximum degrees.
Roadmap. This subsection has three logically independent parts:

(1) *Quasi–isometry (QI) stability:* we prove a clean, additive-free comparison for the *increasing minorant* under QI and deduce a robust transfer of BRP (in a scaled sense).
(2) *An auxiliary "local attainability" tool:* a parametrized (LA_κ) notion yields tempered increments TF(ii) without any near–minimizer input. This tool is *not used* in the main (NNM–only) interleaving, but can be helpful when near–minimizers are known only sparsely.
(3) *Main construction:* from (NNM) alone we construct a single nested chain with *full* (TF) (under amenability). This yields the characterization result in this class. No (LA_κ) is required for Theorem 9.101.

Definition 9.86 (Bounded ratio property (BRP))**.** A bounded–geometry graph X has the *bounded ratio property* if

$$\sup_{r \geq 1} \frac{I_X^{\circ}(r)}{I_X^{\mathrm{incr}\,\circ}(r)} \ <\ \infty.$$

Definition 9.87 (Quasi–isometries)**.** A map $f : V_X \to V_Y$ is a *quasi–isometry* if there exist $\lambda \geq 1$, $\epsilon \geq 0$, $R_0 \geq 0$, and a coarse inverse $g : V_Y \to V_X$ such that

$$\lambda^{-1} d_X(x,x') - \epsilon \ \leq\ d_Y\big(f(x), f(x')\big) \ \leq\ \lambda\, d_X(x,x') + \epsilon, \qquad d_X(gf(x), x) \leq R_0, \quad d_Y(fg(y), y) \leq R_0.$$

Lemma 9.88 (Controlled thickening and boundary under QI)**.** Let $f \in \mathrm{QI}(X,Y)$ with data (λ,ϵ,R_0), let $R \in \mathbb{N}$. For every finite $A \subset V_X$ define $B := N_R(f(A)) \subset V_Y$. Then there are constants c_V, c'_V, c_∂ depending only on the QI data and degrees such that

$$|B| \le c_V\,|A|, \qquad |A| \le c'_V\,|B|, \qquad |\partial_Y B| \ \le\ c_\partial\,|\partial_X A|.$$

Proof. Volume bounds: balls have uniformly bounded growth in bounded-degree graphs, and g maps B into an R'-thickening of A, giving the two inequalities with c_V, c'_V. Boundary bound: for each Y-edge broken by B, connect its endpoints to $f(A)$ by paths of length $\le R$ and pull back via g; this associates to each broken Y-edge a broken X-edge with uniformly bounded preimage multiplicity depending only on (λ,ϵ,R_0) and the degrees, yielding c_∂. □

Lemma 9.89 (Boundary of a thickening)**.** Let G be a bounded–degree graph (degree Δ). Then for every finite $S \subset V(G)$ and $R \in \mathbb{N}$,

$$|\partial N_R(S)| \ \le\ C_{\Delta,R}\,|\partial S| \qquad \text{with } C_{\Delta,R} \ :=\ 2\,\Delta^2\,(\Delta-1)^{R-1}.$$

Proof. Every edge broken by $N_R(S)$ projects along a shortest path of length $\le R$ to a broken edge of S; the number of such lifts is bounded by the number of nonbacktracking paths of length $\le R$ times a degree factor, which is $\le 2\Delta^2(\Delta-1)^{R-1}$. □

Theorem 9.90 (QI comparison for minorants and BRP transfer)**.** Let X and Y be bounded–degree graphs that are quasi–isometric. Then there exist $a, b \in (0,\infty)$ and $A \ge 1$ depending only on the QI data and degrees such that, for all $r \in \mathbb{N}$,

$$I_Y^{\mathrm{incr}\,\circ}(ar) \ \le\ A\,I_X^{\mathrm{incr}\,\circ}(r) \qquad \text{and} \qquad I_X^{\mathrm{incr}\,\circ}(br) \ \le\ A\,I_Y^{\mathrm{incr}\,\circ}(r). \tag{9.7}$$

Consequently:

(i) (*Scaled BRP transfer*) If X has BRP with constant C_X, then

$$\sup_{r\ge1} \frac{I_Y^\circ(r)}{I_Y^{\mathrm{incr}\,\circ}(\kappa r)} \ \le\ C_Y$$

for some $\kappa \ge 1$ and $C_Y < \infty$ depending only on the QI data, degrees, and C_X.

(ii) If, in addition, $I_Y^{\mathrm{incr}\,\circ}$ is doubling in the sense that $I_Y^{\mathrm{incr}\,\circ}(2r) \le D\,I_Y^{\mathrm{incr}\,\circ}(r)$ for some $D < \infty$ and all large r (this holds in the classes used elsewhere in the paper), then Y has BRP in the original sense:

$$\sup_{r\ge1} \frac{I_Y^\circ(r)}{I_Y^{\mathrm{incr}\,\circ}(r)} \ <\ \infty.$$

Proof. For (9.7): Fix R and $f \in \mathrm{QI}(X,Y)$ as in Lemma 9.88. Given r, choose $s \ge r$ and $A \subset X$ with $|A| = s$ and $|\partial_X A| = I_X^\circ(s)$; set $B := N_R(f(A))$. Then $|B| \le c_V s$ and $|\partial_Y B| \le c_\partial I_X^\circ(s)$. Taking the infimum over $s \ge r$ yields

$$I_Y^{\mathrm{incr}\,\circ}(c_V r) \ \le\ c_\partial\,I_X^{\mathrm{incr}\,\circ}(r).$$

This gives the first inequality in (9.7) with $a = c_V$ and $A = c_\partial$. The second follows by symmetry using the coarse inverse g (with possibly different constants), which we absorb into (b, A).

For (i): let C_X witness BRP on X, i.e. $I_X^\circ \le C_X\,I_X^{\mathrm{incr}\,\circ}$. Fix r and apply Lemma 9.88 to a minimizer A of size $\lceil r/c_V\rceil$ in X to get a set $B \subset Y$ with $|B| \le r + c_V$ and

$$I_Y^\circ(r) \ \le\ |\partial_Y B| \ \le\ c_\partial\,I_X^\circ\big(\lceil r/c_V\rceil\big) \ \le\ c_\partial C_X\,I_X^{\mathrm{incr}\,\circ}\big(\lceil r/c_V\rceil\big).$$

By (9.7), $I_X^{\mathrm{incr}\,\circ}(\lceil r/c_V\rceil) \ \le\ A\,I_Y^{\mathrm{incr}\,\circ}(\kappa r)$ for some $\kappa \ge 1$ depending only on the QI data (absorbing the rounding and the additive c_V via monotonicity of the minorant). Combining,

$$I_Y^\circ(r) \ \le\ (c_\partial C_X A)\,I_Y^{\mathrm{incr}\,\circ}(\kappa r).$$

Taking the supremum in r proves (i) with $C_Y := c_\partial C_X A$.

For (ii): if $I_Y^{\mathrm{incr}\,\circ}$ is doubling, then $I_Y^{\mathrm{incr}\,\circ}(\kappa r) \le D^{\lceil \log_2 \kappa \rceil} I_Y^{\mathrm{incr}\,\circ}(r)$ for large r, and the finitely many small r are absorbed into the constant. Hence $\sup_r I_Y^\circ(r)/I_Y^{\mathrm{incr}\,\circ}(r) < \infty$. □

9.9.1. *An auxiliary local attainability tool (not used in the main interleaving).*

Definition 9.91 (Local Attainability with factor κ)**.** Let $\kappa \in [1,\infty)$. A bounded–degree graph X satisfies (LA_κ) if there exists $C_{\mathrm{LA}} \in (0,\infty)$ such that for every $r \in \mathbb{N}$ there exists

$$s \;\in\; \big[r,\; r + C_{\mathrm{LA}}\, I_X^{\mathrm{incr}\,\circ}(r)\big] \qquad \text{with} \qquad I_X^\circ(s) \;\le\; \kappa\, I_X^{\mathrm{incr}\,\circ}(r).$$

Lemma 9.92 (BRP $\Rightarrow$ (LA_κ) with $\kappa = C$)**.** If X has *BRP* with constant C (i.e. $I^\circ(r) \le C\, I^{\mathrm{incr}\,\circ}(r)$ for all r), then X satisfies (LA_κ) with $\kappa = C$ (and any $C_{\mathrm{LA}} > 0$): take $s = r$.

Lemma 9.93 (One-step padding)**.** Let X have maximum degree Δ. Given a finite $E \subset V_X$ and $t \in \mathbb{N}$, there exists $E' \supset E$ with $|E'| = |E| + t$ and

$$|\partial E'| \;\le\; |\partial E| \;+\; \Delta\, t.$$

Proof. Add vertices one by one; each addition increases the boundary by at most Δ. □

Lemma 9.94 (Minorant vs. boundary)**.** For every finite E one has

$$I^{\mathrm{incr}\,\circ}(|E|) \;\le\; I^\circ(|E|) \;\le\; |\partial E|.$$

Proof. By definition of the infima. □

Proposition 9.95 ((LA_κ) yields tempered increments and checkpoints)**.** If X satisfies (LA_κ), then there exists a nested sequence (F_n) with $|F_{n+1} \setminus F_n| \;\le\; C_{\mathrm{LA}}\, |\partial F_n|$ for all n; and there is a cofinal sequence of volumes (s_j) and minimizers E_j with $|E_j| = s_j$ such that $I^\circ(s_j) \;\le\; \kappa\, I^{\mathrm{incr}}(s_j)$.

Proof. Given F_n with $r_n := |F_n|$, pick s_n by (LA_κ) and pad using Lemma 9.93. Then $|F_{n+1} \setminus F_n| \le C_{\mathrm{LA}} I^{\mathrm{incr}}(r_n) \le C_{\mathrm{LA}} |\partial F_n|$ by Lemma 9.94. Iterating (LA_κ) yields the checkpoints. □

Remark 9.96 (On usage)**.** We will *not* use (LA_κ) in the main TF construction below; it is recorded as a flexible device to obtain TF(ii) even when near-minimizers are only sparsely available.

9.9.2. *Interleaving from (NNM) alone: full* (TF) *under amenability.* Recall (NNM) (Definition 9.20): there exist $C_0 < \infty$, r_0 and a nested family $(W_r)_{r \ge r_0}$ with $|W_r| = r$ such that, in edge normalization, $B_{\mathcal{S}}(W_r) \le C_0\, I_{\mathrm{edge}}^{\mathrm{incr}\,\circ}(r)$. By Lemma 9.15, in vertex normalization

$$|\partial W_r| \;\le\; C_0\, \Delta\, I^{\mathrm{incr}\,\circ}(r) \qquad (r \ge r_0). \tag{9.8}$$

Definition 9.97 (Budgeted levels)**.** Fix $\vartheta \in (0,1]$. Define a level schedule by $r_{j+1} := r_j + \lfloor \vartheta\, I^{\mathrm{incr}\,\circ}(r_j) \rfloor$. On level j, evolve by the confined step $F \mapsto \mathcal{T}_{r_{j+1}}(F)$ (Definition 9.25) until $F = W_{r_{j+1}}$, then move to level $j+1$.

Lemma 9.98 (Interleaved smoothing: no (LA_κ) needed)**.** The chain of Definition 9.97 is nested and satisfies TF(ii) with $B = 1$:

$$|F_{m+1} \setminus F_m| \;\le\; |\partial F_m| \qquad (\text{all } m).$$

Moreover, along any step with $F \subset W_{r'}$ one has the deficit control

$$\mathrm{def}(F; r') \;:=\; r' - |F| \;\le\; \vartheta\, I^{\mathrm{incr}\,\circ}(|F|),$$

and at each checkpoint $F = W_{r'}$, $|\partial F| \;\le\; C_0 \Delta\, I^{\mathrm{incr}\,\circ}(|F|)$.

Proof. Nestedness and $B = 1$ are as in Lemma 9.26. For the deficit bound, note that on level j we have $r' = r_{j+1} = r_j + \lfloor \vartheta I^{\mathrm{incr}}(r_j) \rfloor$ and $|F| \in [r_j, r_{j+1}]$, hence $\mathrm{def}(F; r') \le r_{j+1} - r_j \le \vartheta\, I^{\mathrm{incr}}(r_j) \le \vartheta\, I^{\mathrm{incr}}(|F|)$. The checkpoint bound is (9.8). □

Proposition 9.99 (TF(i) at every step from (NNM) alone)**.** In the chain of Definition 9.97, for every step F (not only at checkpoints) one has

$$|\partial F| \;\le\; \big(C_0\Delta + (C_0\Delta^2+1)\,\vartheta\big)\, I^{\mathrm{incr}\,\circ}(|F|).$$

Proof. Fix a step with $F \subset W_{r'}$. Tail trimming (Prop. 9.2) gives $|\partial F| \le |\partial W_{r'}| + \operatorname{def}(F;r')$. By (9.8), $|\partial W_{r'}| \le C_0\Delta\, I^{\mathrm{incr}}(r')$. By Lemma 9.27, $I^{\mathrm{incr}}(r') \le I^{\mathrm{incr}}(|F|) + \Delta \operatorname{def}(F;r')$. Combine and use Lemma 9.98 to bound $\operatorname{def}(F;r') \le \vartheta I^{\mathrm{incr}}(|F|)$, obtaining

$$|\partial F| \;\le\; C_0\Delta\big(I^{\mathrm{incr}}(|F|) + \Delta \operatorname{def}(F;r')\big) + \operatorname{def}(F;r') \;\le\; \big(C_0\Delta + (C_0\Delta^2+1)\,\vartheta\big)\, I^{\mathrm{incr}}(|F|).$$

□

Theorem 9.100 (Full (TF) from (NNM) under amenability)**.** Assume X admits (NNM) as above and is amenable, i.e. $I^{\mathrm{incr}}(r)/r \to 0$. Let (F_m) be the chain of Definition 9.97. Then:

$$|F_{m+1} \setminus F_m| \;\le\; |\partial F_m| \quad \text{(TF(ii) with } B=1\text{)}, \qquad \frac{|\partial F_m|}{|F_m|} \;\longrightarrow\; 0,$$

and there exists $A < \infty$ (e.g. $A = C_0\Delta + (C_0\Delta^2+1)\vartheta$) such that

$$|\partial F_m| \;\le\; A\, I^{\mathrm{incr}}(|F_m|) \quad \text{for every } m.$$

Hence (F_m) is a nested Følner sequence satisfying full (TF).

Proof. TF(ii) is Lemma 9.98. The TF(i) bound is Proposition 9.99. Since $I^{\mathrm{incr}}(|F_m|)/|F_m| \to 0$ by amenability, the TF(i) bound implies $|\partial F_m|/|F_m| \to 0$. □

Theorem 9.101 (Characterization under (NNM))**.** Let X be a bounded-degree Cayley graph that admits (NNM) and is amenable. Then X satisfies full (TF) (Theorem 9.100), and consequently X has BRP (Theorem 9.31). In particular, within this class, (TF) and BRP are equivalent properties (both hold).

Remark 9.102 (On the role of (LA_κ))**.** The interleaving in Theorem 9.100 uses only (NNM) (plus amenability) and does *not* invoke (LA_κ). The latter is kept as an auxiliary device that independently yields TF(ii) and cofinal checkpoints when (NNM) is known only sparsely; it is not needed for the main equivalence in the (NNM)+amenable class.

Remark 9.103 (Why (NNM)? Conceptual gain over direct (TF))**.** The (NNM) hypothesis isolates the purely geometric content: existence of near–minimizers (often along a cofinal subsequence) in the most convenient normalization (directed edges). By contrast, (TF) demands a single *nested, tempered* chain that is near–minimal *at every step*. Our interleaving scheme turns any (NNM) input into a constructive chain satisfying full (TF) with explicit constants; in particular, we obtain tempered Følner sets without further analytic input.

In the classes of interest, (NNM) is typically easier to verify than (TF) itself:

- **Abelian/Carnot:** discrete Wulff calibrations yield nested Wulff samplers with sharp anisotropic constants, i.e. (NNM) for all large volumes.
- **Hyperbolic semidirects $\mathbb{Z}^d \rtimes_A \mathbb{Z}$:** A–covariant layer-nested sets provide near–minimality (up to constants) at the natural logarithmic heights; L^1–profile bounds give the required lower envelope.
- **Lamplighters:** Erschler's profile asymptotics furnish canonical checkpoint volumes (median lamp budgets) at which the exact edge boundary is within a fixed factor of the profile, i.e. (NNM) along a cofinal subsequence.

Moreover, (NNM) is stable under finite index/finite extensions and admits a *checkpoint tensoring* principle for products, so it propagates through natural constructions where building a global tempered chain directly is delicate. Thus (NNM) serves as a geometric black box: once verified, our interleaving ensures a single nested, tempered, near–minimal chain witnessing full (TF).

Remark 9.104 (Checklist to verify (NNM))**.** A practical route to (NNM) is: (i) an anisotropic perimeter lower bound via calibration/BV for the chosen stencil; (ii) explicit samplers (Wulff approximants, covariant sets, median–budget slices) with $B_{\mathcal{S}}(\cdot)$ matching that lower bound up to a fixed factor; (iii) volume interpolation (padding/trimming) and record–minimum alignment to populate a cofinal family; (iv) nestedness by scale (dilations/layer-nested sets). This produces nested near–minimizers with tracked constants in edge normalization, which our scheme then upgrades to full (TF).

9.10. **Towards all amenable Cayley graphs: a Gap-NNM program.** We outline a route to *full* (TF) (hence a bounded profile ratio) for amenable Cayley graphs based on two inputs developed earlier: (i) a checkpoint chain chosen among *record minima* of the increasing minorant (see Remark 9.45) or among nested near-minimizers when available (Definition 9.20, Rem. 9.22); and (ii) the confined canonical step and its tempered control (Definition 9.25, Lemma 9.18), as well as the interleaving scheme (Lemma 9.26, Prop. 9.29).

Standing notation. Let $X = \mathrm{Cay}(\Gamma, \mathcal{S})$, $\Delta := |\mathcal{S}|$. Write $I^{\circ}(\cdot)$ for the exact *vertex* profile and $I^{\mathrm{incr}\,\circ}$ for its increasing minorant. A checkpoint chain $\{W_{r_k}\}$ consists of finite sets with $|W_{r_k}| = r_k$, $|\partial W_{r_k}| = I^{\circ}(r_k)$ and $W_{r_k} \subset W_{r_{k+1}}$ (cf. Remark 9.45). Set the *gap* $\mathrm{Gap}(k) := r_{k+1} - r_k$.

Gap hypotheses for a checkpoint chain. We consider the following properties:

(C1) *Uniform gap:* there exists $G < \infty$ such that $\mathrm{Gap}(k) \le G\, I^{\mathrm{incr}\,\circ}(r_k)$ for all large k.

(C2) *Positive density of good gaps:* there exist $C > 0$ and $\eta > 0$ such that

$$\liminf_{m\to\infty} \frac{1}{m}\big|\{1 \le k \le m :\ \mathrm{Gap}(k) \le C\, I^{\mathrm{incr}\,\circ}(r_k)\}\big| \ \ge\ \eta.$$

(C3) *Integrated gap:* there exists $C_{\mathrm{int}} < \infty$ such that for all large K and all $M \ge 1$,

$$\sum_{k=K}^{K+M-1} \mathrm{Gap}(k) \ \le\ C_{\mathrm{int}} \sum_{k=K}^{K+M-1} I^{\mathrm{incr}\,\circ}(r_k).$$

These are logically related only in the forward directions used below: (C1)⇒(C2) and (C1)⇒(C3) (with constants depending on G). In general, (C2) and (C3) are independent (neither implies the other), and neither implies (C1).

Consequences. If either (C2) or (C3) holds for some checkpoint chain, then combining the padding estimate (Lemma 9.13) from the last good checkpoint with the confined/interleaving steps (Definition 9.25, Lemma 9.18, Lemma 9.26, Prop. 9.29) yields (TF) with explicit constants (A, B) in the vertex normalization; in particular, Theorem 9.31 then gives a uniform bound on $I^{\circ}/I^{\mathrm{incr}\,\circ}$.

Model families: how a gap hypothesis is verified using earlier constructions. In the principal classes treated elsewhere, the gap hypotheses can be checked *from the explicit near-minimizer families constructed there* together with trimming/padding; the Δ-Lipschitz control $|I^{\circ}(r+1) - I^{\circ}(r)| \le \Delta$ (Prop. 9.1) is used inside these arguments but is not the sole input.

- **Virtually nilpotent / Carnot lattices.** Sharp Wulff samplers (Theorems 4.1, 3.25) provide nested near-minimizers at all large volumes with $I^{\circ}(r) \asymp r^{(Q-1)/Q}$. Choosing checkpoints cofinal in this nested family and applying the interleaving scheme (Lemma 9.26, Prop. 9.29) yields an *integrated* control of the padding cost between checkpoints; hence (TF) with tracked (A, B) (cf. Theorem 9.30) and bounded ratio.
- **Hyperbolic semidirects $\mathbb{Z}^d \rtimes_A \mathbb{Z}$.** Logarithmic layer-nested sets $\{E_R\}$ from §9.8 form a nested near-minimizer sequence at a log-spaced set of volumes with $B(E_R) \asymp |E_R|/\log|E_R|$ (Theorems 9.75, 9.81, Prop. 9.84). Taking checkpoints along this sequence and padding within each window gives an *integrated* bound; hence (TF) with explicit constants and bounded ratio.

- **Finite-lamp wreath products over polynomial-growth bases.** For lamplighters, the single-step expansion (Theorem 9.39) provides global tempered increments, and the checkpoint family $\{F_k^\star\}$ (Theorems 9.41, 9.42) sits on the $v/\log v$ scale. Padding from $F_k^\star$ to intermediate volumes and concatenating with the single-steps yields an *integrated* control across windows; thus one recovers the (TF) structure recorded in Corollary 9.47 and Theorem 9.48, and hence a bounded ratio via Theorem 9.31.

Remark 9.105 (Vertical Lipschitz versus horizontal spacing)**.** The bound $|I^\circ(r+1) - I^\circ(r)| \le \Delta$ (Prop. 9.1) controls *vertical* oscillations of the exact profile. By itself it does not preclude long horizontal deserts between record minima. In the families above, spacing of checkpoints (in the sense of (C2)-(C3)) comes from the *explicit* near-minimizer chains (Wulff samplers, logarithmic layer-nested sets, lamplighter checkpoints) combined with trimming/padding, rather than from the Lipschitz estimate alone.

Strategies toward general amenable Cayley graphs. Three complementary routes target (C2) or (C3).

- **Finite-volume parametric stabilization.** On each finite induced subgraph, inclusion-minimal minimizers of $B + \lambda|\cdot|$ are laminar in λ with equal-size optimality at breakpoints (standard submodular calculus). A local stabilization of such minimizers along an exhaustion would produce a laminar checkpoint chain on Γ and yield an *integrated* gap bound by the same padding argument.
- **Quantitative BV/coarea levels.** Construct a function with small total variation on large sets whose level sets realize $|\partial E| \lesssim I^{\mathrm{incr}}(|E|)$ at a positive density of levels; choosing checkpoints among these levels gives (C2) and hence (TF).
- **Følner function control.** A doubling property for the Følner function, $\mathrm{Fol}(\varepsilon/2) \le D\ \mathrm{Fol}(\varepsilon)$ for small ε, would imply a short-desert phenomenon for record minima, i.e. (C1) with $G = O(D)$, and thus (TF) with $(A, B) = (1 + \Delta G,\, 1)$.

Permanence. The product construction and boundary splitting (Theorem 9.61), together with finite index and finite extension stability (Prop. 9.58, Prop. 9.59), preserve the gap-driven (TF) conclusions above (with constants scaled by bounded factors). Consequently, bounded profile ratio passes to these closures by Theorem 9.31.

10. Analytic Consequences: spectral profile, heat kernel, and mixing

Input used here. All analytic bounds depend on a single isoperimetric input-an L^1 profile of order $r^{(Q-1)/Q}$ with tracked constant. In our classes this input is supplied either by discrete Wulff (Abelian/Carnot) or, constructively, by the shear-produced (NNM) combined with the interleaving (which yields the same order at the relevant scales).

Throughout this section we work on the undirected Cayley graph associated to a *symmetric* finite generating set $\mathcal{S}$; in particular the graph is Δ-regular with $\Delta = |\mathcal{S}|$ and the (continuous-time) simple random walk generator is $L = I - P$ with $P = \frac{1}{\Delta}A$ self-adjoint.

For a finite $U \subset \Gamma$, let $\lambda_1^{\mathrm{D}}(U)$ be the first Dirichlet eigenvalue of the continuous-time simple random walk (generator $L = I - P$ with $P = \frac{1}{\Delta}A$) killed outside U, and define the Dirichlet Faber–Krahn profile

$$\lambda_1^{\mathrm{D}}(r) \;:=\; \inf\left\{\lambda_1^{\mathrm{D}}(U) :\; |U| = r\right\}.$$

Sections 3 and 5 provide (in virtually nilpotent/Carnot settings) sharp discrete Wulff isoperimetry and quantitative samplers; Section 9 records the (TF) mechanism for bounding $I^\circ/I^{\mathrm{incr}\,\circ}$.

Lemma 10.1 (Vertex/edge comparison)**.** For every finite $Y \subset \Gamma$ on the Δ-regular Cayley graph,

$$|\partial_{\mathcal{S}}(Y)| \;\le\; |\partial_{\mathcal{S}}^{\mathrm{edge}}(Y)| \;\le\; \Delta\, |\partial_{\mathcal{S}}(Y)|.$$

Here $|\partial_{\mathcal{S}}^{\mathrm{edge}}(Y)|$ counts *undirected* edges with exactly one endpoint in Y. Consequently, $I^{\circ}_{\mathrm{vertex}}(r) \le I^{\circ}_{\mathrm{edge}}(r) \le \Delta\, I^{\circ}_{\mathrm{vertex}}(r)$ and $h_{\mathrm{edge}} \asymp h_{\mathrm{vertex}}$.

Remark 10.2. All analytic bounds stated with vertex boundary translate to edge boundary up to factors of Δ via Lemma 10.1; exponents are unchanged. *In borderline dimension $Q = 2$ the classical spectral/mixing estimates carry a logarithmic correction; see, e.g.,* [VSCC92, SC02].

Organization and scope. The analytic narrative used here is: *(i)* isoperimetry $\Rightarrow$ Cheeger/Faber–Krahn and Nash on Γ; *(ii)* two-sided on-diagonal heat kernel bounds (upper from Nash, lower from (VD)+(PI)); *(iii) conditional* spectral-profile mixing bounds on finite quotients, assuming a *global* FK (or spectral profile) input on the quotient. We emphasize that our descent arguments yield only *local* FK on quotients (for sets with diameter $\ll \mathrm{inj}(G)$). Global FK on G is an additional hypothesis which generally fails to follow from local injectivity alone. When available, *Wulff-coded* constants from Theorem 4.1 enter only through a single isoperimetric constant C_{iso}. When we track generating-set dependence explicitly it appears only via $\Delta = |\mathcal{S}|$ and the Wulff constant h_S; other implicit constants (e.g. from (VD)+(PI)) depend only on coarse-geometric data and we do not attempt to encode them in (Δ, h_S).

Theorem 10.3 (Cheeger, Faber–Krahn and Nash from isoperimetry)**.** Define the global vertex Cheeger constant

$$h_{\circ} \;:=\; \inf_{\varnothing \neq Z \subset \Gamma} \frac{|\partial_{\mathcal{S}} Z|}{|Z|} \;\in [0, \Delta],$$

and, for a finite $Y \subset \Gamma$, the localized Cheeger constant

$$h_{\circ}(Y) \;:=\; \inf_{\varnothing \neq Z \subseteq Y} \frac{|\partial_{\mathcal{S}} Z|}{|Z|}.$$

Let $B_{\mathcal{S}}(Z) := |\{(z,s) \in Z \times \mathcal{S} :\ sz \notin Z\}|$ denote the (inside-to-outside) directed edge boundary. *On an undirected Cayley graph, each crossing edge (z, sz) with $z \in Z$, $sz \notin Z$ corresponds to a unique such pair, hence*

$$B_{\mathcal{S}}(Z) \;=\; |\partial_{\mathcal{S}}^{\mathrm{edge}}(Z)|. \tag{10.1}$$

Then for every finite $Y \subset \Gamma$,

$$\lambda_1^{\mathrm{D}}(Y) \;\ge\; \frac{1}{2\,|\mathcal{S}|^2}\, h_{\circ}(Y)^2 \;\ge\; \frac{1}{2\,|\mathcal{S}|^2}\, h_{\circ}^2. \tag{10.2}$$

Consequently $\lambda_1^{\mathrm{D}}(r) \ge h_{\circ}^2/(2|\mathcal{S}|^2)$.

If, for some $Q \ge 1$ and $C_{\mathrm{iso}} > 0$, the isoperimetric profile satisfies

$$|\partial_{\mathcal{S}} Z| \;\ge\; C_{\mathrm{iso}}\, |Z|^{\frac{Q-1}{Q}} \qquad \text{for all } Z \subset \Gamma \text{ with } |Z| \le r, \tag{10.3}$$

then the Dirichlet Faber–Krahn profile obeys

$$\lambda_1^{\mathrm{D}}(r) \;\ge\; \frac{C_{\mathrm{iso}}^2}{2\,|\mathcal{S}|^2}\, r^{-2/Q}. \tag{10.4}$$

In particular, on Abelian groups ($Q = r_1$), Theorem 4.1 and Lemma 10.1 yield $C_{\mathrm{iso}} \ge c_{\mathrm{ab}}(\mathcal{S})/|\mathcal{S}|$, hence

$$\lambda_1^{\mathrm{D}}(r) \;\ge\; \frac{c_{\mathrm{ab}}(\mathcal{S})^2}{2\,|\mathcal{S}|^4}\, r^{-2/r_1}.$$

Moreover, whenever (10.3) holds at all relevant scales, one has the Nash inequality

$$\|f\|_2^{2+4/Q} \;\le\; C(Q) \left(\tfrac{|\mathcal{S}|}{C_{\mathrm{iso}}}\right)^2 \mathcal{E}(f,f)\, \|f\|_1^{4/Q} \qquad \text{for finitely supported } f. \tag{10.5}$$

Sketch. Let $\Phi_Y := \min_{\emptyset \neq Z \subseteq Y} \frac{B_{\mathcal{S}}(Z)}{|\mathcal{S}|\,|Z|}$ be the (normalized) Dirichlet conductance on Y. Cheeger's inequality for the *normalized* Dirichlet Laplacian (equivalently, for the continuous-time generator $L = I - P$ on a Δ-regular graph) gives $\lambda_1^{\mathrm{D}}(Y) \geq \Phi_Y^2/2$; see [Dod84, Moh89]. Since $B_{\mathcal{S}}(Z) \geq |\partial_{\mathcal{S}} Z|$, we have $\Phi_Y \geq h_\circ(Y)/|\mathcal{S}|$, yielding (10.2). If (10.3) holds up to scale r, then for $|Y| = r$ one gets $\Phi_Y \geq (C_{\mathrm{iso}}/|\mathcal{S}|)\, r^{-1/Q}$ and (10.4) follows. The Nash inequality (10.5) is standard from scale-invariant Faber-Krahn; see [VSCC92, SC02, BGL14]. If (10.3) holds only beyond some large scale r_0, one obtains a Nash inequality that yields the corresponding *long-time* ultracontractivity and heat-kernel upper bounds (see Theorem 10.4). □

Theorem 10.4 (Two-sided on-diagonal heat kernel)**.** Let $(\Gamma, \mathcal{S})$ have polynomial growth of homogeneous dimension $Q \geq 1$. *(Lower bound)* Assume volume doubling (VD) and a scale-invariant Poincaré inequality (PI) (these hold in virtually nilpotent groups). *(Upper bound)* Assume, in addition, the (possibly large-scale) isoperimetry

$$|\partial_{\mathcal{S}} Y| \;\geq\; C_{\mathrm{iso}}\, |Y|^{(Q-1)/Q} \qquad \text{for all sufficiently large finite } Y \subset \Gamma.$$

Let $V(r) = |B(e,r)|$ be the volume growth. Then for the continuous-time simple random walk there exist thresholds

$$t_{\mathrm{low}} = t_{\mathrm{low}}(\mathrm{VD}, \mathrm{PI}), \qquad t_{\mathrm{up}} = t_{\mathrm{up}}(Q, \text{scale of isoperimetry})$$

and constants

$$c_- = c_-(\mathrm{VD}, \mathrm{PI}) > 0, \qquad C_+ = C_+(Q) > 0$$

such that for all $t \geq t_0 := \max\{t_{\mathrm{low}}, t_{\mathrm{up}}\}$,

$$\frac{c_-(\mathrm{VD}, \mathrm{PI})}{V(\sqrt{t})} \;\leq\; p_t(e,e) \;\leq\; C_+(Q) \left(\frac{|\mathcal{S}|}{C_{\mathrm{iso}}}\right)^Q t^{-Q/2}. \tag{10.6}$$

If (10.3) holds at all scales, one may take $t_{\mathrm{up}} = 0$ and the upper bound holds for all $t > 0$. In particular, since $C_{\mathrm{iso}} \geq c_{\mathrm{ab}}(\mathcal{S})/|\mathcal{S}|$ by Theorem 4.1 and Lemma 10.1,

$$p_t(e,e) \;\leq\; C_+(Q) \left(\frac{|\mathcal{S}|^2}{c_{\mathrm{ab}}(\mathcal{S})}\right)^Q t^{-Q/2} \qquad (t \geq t_0).$$

Proof sketch. The upper bound follows from Theorem 10.3 via the standard Nash $\Rightarrow$ ultracontractivity route: $\|P_t\|_{1\to\infty} \leq C(Q)\,(\frac{|\mathcal{S}|}{C_{\mathrm{iso}}})^Q t^{-Q/2}$; when the isoperimetry holds only beyond a large scale r_0, this yields the estimate for $t \gtrsim r_0^2$. The lower bound follows from (VD)+(PI) $\Rightarrow$ parabolic Harnack, hence Gaussian on-diagonal lower bounds [VSCC92, SC02]. □

10.1. **Descent to finite quotients: intrinsic locality windows.**

Definition 10.5 (Local injectivity radius)**.** For a quotient $\pi : \Gamma \to G = \Gamma/N$ of Cayley graphs with the induced generating set, define $\mathrm{inj}(G)$ to be the largest $R \in \mathbb{N} \cup \{\infty\}$ such that for every $x \in \Gamma$ the restriction $\pi|_{B_\Gamma(x,R)}$ is a *graph isomorphism* onto its image (i.e., it is bijective on vertices in the ball and preserves adjacency among vertices in the ball). We refer to this as the *local graph-isomorphism radius.*

Lemma 10.6 (Covering by lifts under large injectivity)**.** Let $G = \Gamma/N$ be a quotient with injectivity radius $\mathrm{inj}(G)$ as in Definition 10.5. If $W \subset G$ is connected and $\mathrm{diam}(W) \leq R \leq \mathrm{inj}(G)$, then there exists a connected $\widetilde{W} \subset \Gamma$ with $\pi|_{\widetilde{W}} : \widetilde{W} \to W$ a graph isomorphism of induced subgraphs (in particular, $|\widetilde{W}| = |W|$ and $\lambda_1^{\mathrm{D},\Gamma}(\widetilde{W}) = \lambda_1^{\mathrm{D},G}(W)$).

Proof. Pick $w_0 \in W$ and lift to $\tilde{w}_0 \in \Gamma$. Since $W \subset B_G(w_0, R)$ and π is a graph isomorphism on balls of radius R, there is a unique induced-subgraph lift $\widetilde{W} \subset B_\Gamma(\tilde{w}_0, R)$ with $\pi|_{\widetilde{W}}$ a graph isomorphism. The Dirichlet operator on W is $I - (1/\Delta)A_W$ acting on $\ell^2(W)$; induced-subgraph isomorphism implies $\lambda_1^{\mathrm{D},\Gamma}(\widetilde{W}) = \lambda_1^{\mathrm{D},G}(W)$. □

Lemma 10.7 (Quotient descent for balls)**.** Assume $\lambda_1^{\mathrm{D}}(U) \geq \kappa\,|U|^{-2/Q}$ for all induced $U \subset \Gamma$ with $\operatorname{diam}(U) \leq R$. If $G = \Gamma/N$ satisfies $\operatorname{inj}(G) \geq R$, then for every induced $W \subset G$ with $\operatorname{diam}(W) \leq R$,

$$\lambda_1^{\mathrm{D,G}}(W) \;\geq\; \kappa\,|W|^{-2/Q}.$$

Proof. Apply Lemma 10.6 to lift W isometrically and use the FK bound on $\widetilde{W}$. □

Corollary 10.8 (Abelian tori: local FK)**.** Let $G = \mathbb{Z}_m^d$ with standard generators. There exists $\kappa(d) > 0$ such that for all $U \subset G$ with $\operatorname{diam}(U) \leq m/3$,

$$\lambda_1^{\mathrm{D,G}}(U) \;\geq\; \kappa(d)\,|U|^{-2/d}.$$

Sketch. For the standard generators, balls of radius $R < \lfloor m/2 \rfloor$ in G are isomorphic to the corresponding balls in $\mathbb{Z}^d$, so $\operatorname{inj}(G) \geq \lfloor m/2 \rfloor - 1$. Combine the discrete Wulff (axis-aligned) isoperimetry on $\mathbb{Z}^d$ (Theorem 4.1) with Theorem 10.3, then use Lemma 10.7 on the torus for $\operatorname{diam}(U) \leq m/3 \leq \operatorname{inj}(G)$. □

Proposition 10.9 (Componentwise-local descent to quotients)**.** Assume $(\Gamma, \mathcal{S})$ has polynomial growth of homogeneous dimension $Q \geq 1$. Fix $R \geq 1$ and suppose that for all induced $W \subset \Gamma$ with $\operatorname{diam}(W) \leq R$ one has $\lambda_1^{\mathrm{D}}(W) \;\geq\; \kappa\,|W|^{-2/Q}$. Let $G = \Gamma/N$ be a finite quotient with injectivity radius $\operatorname{inj}(G) \geq R$. Then for every $U \subset G$ whose connected components $\{U_i\}$ all satisfy $\operatorname{diam}(U_i) \leq R$,

$$\lambda_1^{\mathrm{D,G}}(U) \;\geq\; \kappa\,|U|^{-2/Q}.$$

Proof. Each U_i lifts isometrically to $\widetilde{U}_i \subset \Gamma$; $\lambda_1^{\mathrm{D,G}}(U_i) \;=\; \lambda_1^{\mathrm{D},\Gamma}(\widetilde{U}_i) \;\geq\; \kappa\,|U_i|^{-2/Q}$. Since the Dirichlet first eigenvalue on a disconnected union is the minimum over components,

$$\lambda_1^{\mathrm{D,G}}(U) = \min_i \lambda_1^{\mathrm{D,G}}(U_i) \;\geq\; \kappa\,\min_i |U_i|^{-2/Q} \;=\; \kappa\,(\max_i |U_i|)^{-2/Q} \;\geq\; \kappa\,|U|^{-2/Q},$$

since $\max_i |U_i| \leq |U|$. □

Remark 10.10 (Scope of descent)**.** The componentwise local descent above is the *only* general consequence of injectivity we use. In particular, a *global* FK profile on G,

$$\lambda_1^{\mathrm{D,G}}(U) \;\gtrsim\; |U|^{-2/Q} \qquad (1 \leq |U| \leq |G|/2),$$

does *not* follow from local injectivity and must be supplied separately on a case-by-case basis (e.g. from direct isoperimetry or an explicit spectral profile on G).

Lemma 10.11 (Relaxation-time lower bound)**.** For continuous-time, reversible Markov chains on a finite state space,

$$t_{\mathrm{mix}}(1/4) \;\geq\; \frac{\log 2}{\lambda_1}.$$

This is sharp up to constants; see, e.g., [LPW09, Sec. 12.3].

Theorem 10.12 (Conditional mixing under quotient FK)**.** Let $G = \Gamma/N$ be a finite quotient of $(\Gamma, \mathcal{S})$ with $|G| = N$ and degree $\Delta = |\mathcal{S}|$. Assume the *global quotient* Faber–Krahn profile

$$\lambda_1^{\mathrm{D,G}}(U) \;\geq\; \kappa\,|U|^{-2/Q} \qquad (1 \leq |U| \leq N/2), \tag{10.7}$$

for some $Q \geq 1$. Then the continuous-time simple random walk on G satisfies:

Upper bound (spectral profile, L^∞). There exists $C_\uparrow(Q) > 0$ such that the uniform mixing time obeys

$$\tau_\infty^{(G)}(1/4) \;\leq\; \frac{C_\uparrow(Q)}{\kappa} \times \begin{cases} N^{2/Q}, & Q > 2, \\ N, & Q = 2, \\ N^2, & Q = 1. \end{cases}$$

By monotonicity of distances, the total-variation mixing time satisfies $t_{\mathrm{mix}}^{(G)}(1/4) \le \tau_\infty^{(G)}(1/4)$ and thus inherits the same asymptotic upper bounds.

Caveat for $Q = 2$. The FK-based profile yields $\tau_\infty \lesssim N/\kappa$ (no logarithm). For the simple random walk on the two-dimensional torus, the *total-variation* mixing time is $\asymp N \log N$; obtaining the extra log requires refined spectral/separation information beyond the FK-level profile (e.g., explicit Fourier analysis or evolving-sets), while the L^∞ profile on product tori yields $\tau_\infty \asymp N$.

Lower bounds. Always, by relaxation, $t_{\mathrm{mix}}^{(G)}(1/4) \ge (\log 2)/\lambda_1(G)$. Moreover, if the quotient family inherits (VD)+(PI) uniformly at local scales (e.g. virtually nilpotent quotients with inherited generators and uniform local graph-isomorphism radius), then the on-diagonal lower bound from Theorem 10.4 holds up to times comparable to the square of the local injectivity radius; in particular, whenever the family has $V(r) \asymp r^Q$ up to $r \asymp N^{1/Q}$, one gets the structural lower bound

$$t_{\mathrm{mix}}^{(G)}(1/4) \ \gtrsim \ N^{2/Q}.$$

Wulff-coded prefactor. If the isoperimetry (10.3) holds on the *quotient* up to $r = N/2$ with constant C_{iso}, then by Theorem 10.3 one can take $\kappa \asymp (C_{\mathrm{iso}}/\Delta)^2$. (For $Q = 2$, a sharp $N \log N$ *total-variation* upper bound on tori requires a refined input beyond FK/Nash.)

Proof of the upper bound. Use the spectral profile inequality (e.g. [GMT06, MP05]): for $\pi_* := 1/N$ on a regular Cayley graph,

$$\tau_\infty(1/4) \ \le \ \int_{\pi_*}^{1/2} \frac{ds}{s\,\Lambda(s)}, \qquad \Lambda(s) := \min_{|U| \le sN} \lambda_1^{\mathrm{D,G}}(U).$$

Under (10.7), $\Lambda(s) \ge \kappa\,(sN)^{-2/Q}$, hence $\frac{1}{s\,\Lambda(s)} \ \le \ \frac{N^{2/Q}}{\kappa}\, s^{2/Q-1}$. Integrating $s^{2/Q-1}$ from $1/N$ to $1/2$ yields the displayed cases. In two dimensions, this gives $\tau_\infty \lesssim N/\kappa$. The sharp $N \log N$ for *total variation* on square tori requires refined spectral/separation input beyond the FK-level profile. $\square$

Remark 10.13 (On the descent hypothesis)**.** The key assumption (10.7) is *not* automatic for arbitrary quotients. What one always obtains from injectivity is the *componentwise-local* version: if $\mathrm{inj}(G) \ge R$ and each connected component of U has diameter $\le R$, then $\lambda_1^{\mathrm{D,G}}(U) \ge \kappa |U|^{-2/Q}$ by Proposition 10.9. Natural families where useful versions of the bound do hold include:

(i) Abelian tori $G = \mathbb{Z}_m^d$ with the standard generators: *locally* (i.e. for $\mathrm{diam}(U) \ll m$) by Corollary 10.8; in dimension $d = 2$, global spectral/mixing behavior carries the standard logarithmic correction and one should not expect a uniform (scale-free) FK bound at all sizes.

(ii) Nilpotent (box-like) quotients where isoperimetry (or FK) is established directly on G.

Throughout this subsection, $\mathrm{inj}(G)$ is the local graph-isomorphism radius from Definition 10.5.

Corollary 10.14 (Abelian tori: mixing (special case))**.** Let $G = \mathbb{Z}_m^d$ with the standard generators and $N = m^d$. For the continuous-time simple random walk:

$$t_{\mathrm{mix}}^{(G)}(1/4) \ \asymp \ \begin{cases} m^2, & d \ge 3 \quad (\text{equivalently } N^{2/d}), \\ m^2 \log m, & d = 2 \quad (\text{equivalently } N \log N). \end{cases}$$

Here $\asymp$ is in the sense of matching upper and lower bounds up to d-dependent constants. The upper bounds follow either from explicit spectral analysis or from the spectral-profile method with the known profile on product tori; the lower bounds follow from relaxation and on-diagonal heat-kernel lower bounds.

10.2. **Quantitative Poisson boundary consequences with Wulff–coded constants.** In this section we isolate consequences for random walks that make explicit the dependence on the *generating–set data* (h_S, Δ) supplied by our discrete Wulff theory and the analytic estimates of §10. Throughout, μ is a symmetric, finitely supported, generating probability measure on $(\Gamma, \mathcal{S})$ with *full support* on $\mathcal{S}$ (i.e. $\mu(s) > 0$ for all $s \in \mathcal{S}$), and $P_\mu f(x) = \sum_{s\in\mathcal{S}} \mu(s) f(sx)$ is the associated Markov operator (in continuous time we use $P_t^\mu = e^{-t(I-P_\mu)}$). We write

$$\theta_\mu \;:=\; \Delta \cdot \min_{s\in\mathcal{S}} \mu(s) \;\in (0,1],$$

so that the Dirichlet forms are comparable as

$$\mathcal{E}_\mu(f,f) \;:=\; \tfrac{1}{2}\sum_{x\in\Gamma}\sum_{s\in\mathcal{S}} \mu(s)\big(f(x)-f(sx)\big)^2 \;\geq\; \theta_\mu\, \mathcal{E}_{\mathrm{SRW}}(f,f), \tag{10.8}$$

where $\mathcal{E}_{\mathrm{SRW}}(f,f) = \frac{1}{2\Delta}\sum_{x,s}\big(f(x)-f(sx)\big)^2$. We set

$$K_{\mathrm{Nash}}(\Gamma,\mathcal{S},\mu) \;:=\; \frac{C_0}{\theta_\mu}\left(\frac{\Delta}{h_S}\right)^2,$$

where $C_0 \in (0,\infty)$ is the absolute constant from Theorem H (H3) (the SRW Nash inequality; equivalently, the Nash inequality for $P = \frac{1}{\Delta}A$). In what follows, Q denotes the homogeneous dimension in the virtually nilpotent case (so $Q = d + 2m$).

10.2.1. *Nash $\Longrightarrow$ ultracontractivity with tracked constants.* We first record an ultracontractive bound with its *full* dependence on $(h_S, \Delta, \theta_\mu)$.

Lemma 10.15 (Ultracontractivity with Wulff–coded constants)**.** Assume the Nash inequality of Theorem H (H3) in the form

$$\|f\|_2^{2+4/Q} \;\leq\; K_{\mathrm{Nash}}(\Gamma,\mathcal{S},\mu)\,\mathcal{E}_\mu(f,f)\,\|f\|_1^{4/Q}.$$

Let $P_t^\mu := e^{-t(I-P_\mu)}$ be the continuous-time semigroup. Then there exists a constant $C_Q \in (1,\infty)$ depending *only* on Q such that

$$\|P_t^\mu\|_{1\to\infty} \;\leq\; C_Q\, K_{\mathrm{Nash}}(\Gamma,\mathcal{S},\mu)^{\,Q/2}\, t^{-Q/2} \;=\; C_Q\, C_0^{Q/2}\, \theta_\mu^{-Q/2}\left(\frac{\Delta}{h_S}\right)^Q t^{-Q/2} \qquad (t>0). \tag{10.9}$$

Proof. Set $u(t) := \|P_t^\mu f\|_2^2$. Then $u'(t) = -2\,\mathcal{E}_\mu(P_t^\mu f, P_t^\mu f) \leq -(2/K_{\mathrm{Nash}})$ $|P_t^\mu f\|_2^{2+4/Q}\,\|f\|_1^{-4/Q}$ by Nash and $\|P_t^\mu f\|_1 \leq \|f\|_1$. This yields $\|P_t^\mu f\|_2^2 \leq c_Q\, K_{\mathrm{Nash}}^{Q/2}\, t^{-Q/2}\,\|f\|_1^2$ for a dimensional constant c_Q. By self-adjointness and $\|P_t^\mu\|_{2\to2} \leq 1$, we obtain $\|P_t^\mu\|_{1\to2} \leq (c_Q K_{\mathrm{Nash}}^{Q/2})^{1/2}\, t^{-Q/4}$ and $\|P_t^\mu\|_{2\to\infty} \leq (c_Q K_{\mathrm{Nash}}^{Q/2})^{1/2}\, t^{-Q/4}$. Hence $\|P_{2t}^\mu\|_{1\to\infty} \leq c_Q\, K_{\mathrm{Nash}}^{Q/2}\, t^{-Q/2}$, which implies (10.9) with $C_Q := \max\{c_Q, 2^{Q/2}c_Q\}$. □

Remark 10.16 (Discrete time from continuous time)**.** From the Poissonization identity $P_t^\mu = e^{-t}\sum_{k\geq0}\frac{t^k}{k!}(P_\mu)^k$, positivity of $(P_\mu)^k$ and Stirling yield, for $n \in \mathbb{N}$,

$$\|(P_\mu)^n\|_{1\to\infty} \;\leq\; e^n\,\frac{n!}{n^n}\,\|P_n^\mu\|_{1\to\infty} \;\leq\; C_{\mathrm{St}}\,\sqrt{n}\,\|P_n^\mu\|_{1\to\infty},$$

where C_{St} is universal. Combining with (10.9) (at $t = n$) gives

$$\|(P_\mu)^n\|_{1\to\infty} \;\leq\; C_Q'\, K_{\mathrm{Nash}}(\Gamma,\mathcal{S},\mu)^{\,Q/2}\, n^{-(Q-1)/2} \;=\; C_Q'\, C_0^{Q/2}\, \theta_\mu^{-Q/2}\left(\frac{\Delta}{h_S}\right)^Q n^{-(Q-1)/2}. \tag{10.10}$$

In particular, $p_n(e) = \mu^{*n}(e) \leq \|(P_\mu)^n\|_{1\to\infty}$ satisfies the same bound. For SRW on virtually nilpotent groups one may improve the decay to the sharp $n^{-Q/2}$ using known heat-kernel bounds; we do not need this refinement here.

10.2.2. *Quantitative Liouville on virtually nilpotent groups.*

Theorem 10.17 (Liouville with explicit entropy bound)**.** Let Γ be virtually nilpotent with homogeneous dimension Q, and let μ be symmetric, finitely supported and generating. Then the Poisson boundary of (Γ, μ) is trivial. More precisely, for all $n \geq 2$,

$$H(\mu^{*n}) := -\sum_{g} \mu^{*n}(g) \log \mu^{*n}(g) \leq \frac{Q-1}{2} \log n + Q \log\Big(\frac{\Delta}{h_S}\Big) + \frac{Q}{2} \log\Big(\frac{1}{\theta_\mu}\Big) + C_{\mathrm{ent}}(\Gamma, \mathcal{S}), \tag{10.11}$$

where $C_{\mathrm{ent}}(\Gamma, \mathcal{S}) = \log(C''_Q C_0^{Q/2})$ and C''_Q depends only on Q. In particular, the asymptotic entropy $h := \lim_{n\to\infty} \frac{1}{n} H(\mu^{*n}) = 0$.

Proof. By (10.10), $p_n(e) \leq A_Q \, \theta_\mu^{-Q/2} (\Delta/h_S)^Q n^{-(Q-1)/2}$ with $A_Q := C'_Q C_0^{Q/2}$. Hence $\|\mu^{*n}\|_\infty \leq A_Q \, \theta_\mu^{-Q/2} (\Delta/h_S)^Q n^{-(Q-1)/2}$. Among probability vectors with $\|\cdot\|_\infty \leq \alpha$, the entropy is maximized (up to an additive constant $\leq \log 2$) by the near-uniform vector on $\lceil 1/\alpha \rceil$ points, giving $H(\mu^{*n}) \leq \log\lceil 1/\alpha \rceil \leq \log(1/\alpha) + 1$. Applying this with $\alpha = A_Q \, \theta_\mu^{-Q/2} (\Delta/h_S)^Q n^{-(Q-1)/2}$ yields

$$H(\mu^{*n}) \leq \frac{Q-1}{2} \log n + Q \log\Big(\frac{\Delta}{h_S}\Big) + \frac{Q}{2} \log\Big(\frac{1}{\theta_\mu}\Big) + \log A_Q + 1,$$

which gives (10.11) after absorbing universal constants into $C_{\mathrm{ent}}(\Gamma, \mathcal{S})$. Finally, the Kaimanovich-Vershik entropy criterion (for finitely supported measures) implies trivial Poisson boundary from $H(\mu^{*n}) = o(n)$. □

Remark 10.18 (What is tracked)**.** The right-hand side of (10.11) isolates the dependence on the generating set *only through* Δ and the sharp Wulff constant h_S. The dependence on the step distribution μ enters only through the single comparability parameter $\theta_\mu \in (0, 1]$ from (10.8). All other constants (C_0, C'_Q, C''_Q) are *dimensional* (depending on Q but independent of Γ within the virtually nilpotent class) and arise from the Nash⇒ultracontractivity argument.

10.2.3. *Volumetric strips from logarithmic layer-nested sets in $\mathbb{Z}^d \rtimes_A \mathbb{Z}$.* We record a *purely geometric* by-product of Theorem 9.75 that is useful for strip/ray criteria in Poisson boundary theory. It cleanly separates the volumetric input (from our layer-nested sets) from the probabilistic step (which we do not reproduce).

Proposition 10.19 (Subexponential volumetrics of layer-nested sets)**.** Let $G = \mathbb{Z}^d \rtimes_A \mathbb{Z}$ and E_R be the logarithmic layer-nested sets of Theorem 9.75. There exist $c_1, c_2, c_3 > 0$ (depending only on $(A, \mathcal{S}, K)$) such that for all $R \geq 2$,

$$c_1 R^d \log R \leq |E_R| \leq c_2 R^d \log R, \qquad B_{\mathcal{S}}(E_R) \leq c_3 R^d.$$

Consequently, for any $n \in \mathbb{N}$, the family $S_n := E_{\lfloor n \rfloor}$ satisfies $|S_n| \leq C(A, \mathcal{S}) \, n^d \log n = \exp(o(n))$ and the boundary size is $O(n^d)$, both with tracked constants from Theorem 9.75.

Proof. By Lemma 9.68 (uniform slice sizes), Lemma 9.67 (vertical boundary equals caps), and Corollary 9.73 (horizontal boundary bound), together with $T(R) = \lfloor \alpha \log R \rfloor$ from Proposition 9.74, we have $|E_R| = (2T(R)+1)\,|X_0(R)|$ with $|X_0(R)| \asymp R^d$ and $B_{\mathcal{S}}(E_R) = B_{\mathrm{vert}}(E_R) + B_{\mathrm{hor}}(E_R)$ with $B_{\mathrm{vert}}(E_R) = 2|X_0(R)| \asymp R^d$ and $B_{\mathrm{hor}}(E_R) = O(R^{d-1+\alpha \log \Lambda(A)}) = o(R^d)$ by $\alpha \log \Lambda(A) < 1$. This gives the claimed bounds. □

Remark 10.20. Many strip/ray criteria (e.g. Kaimanovich's) require a family of sets whose cardinalities grow subexponentially in time. Proposition 10.19 provides exactly this volumetric input with *explicit* constants for the canonical layer-nested sets E_R; any additional probabilistic verification (e.g. that typical sample paths spend enough time in such strips under additional moment/drift assumptions) can be taken from the existing boundary literature for semidirect products. We do not invoke those results here.

Appendix A. Tempered Property A and Quantitative Coarse Embeddings (Edge Normalization)

We record a quantitative upgrade of Property A that interacts directly with our Tempered Følner (TF) hypothesis and yields explicit coarse embeddings into Hilbert space with tracked compression. Throughout this appendix we work with a *finite symmetric* generating set $\mathcal{S}$ and in the *edge* normalization: for $Y \subset \Gamma$,

$$B_{\mathcal{S}}(Y) \;:=\; \#\{\text{undirected Cayley edges with one endpoint in } Y \text{ and one in } Y^c\}.$$

(Equivalently, $B_{\mathcal{S}}(Y)$ counts *inside→outside* directed pairs (y,s) with $y \in Y$, $s \in \mathcal{S}$, $sy \notin Y$; for symmetric $\mathcal{S}$ these two counts coincide.) When needed, we pass to the *vertex* normalization via the standard comparison (recall Lemma 10.1 in the main text):

$$|\partial_{\mathcal{S}}(Y)| \;\le\; B_{\mathcal{S}}(Y) \;\le\; \Delta\,|\partial_{\mathcal{S}}(Y)|, \qquad \Delta := |\mathcal{S}|. \tag{A.1}$$

Throughout we view $\mathrm{Cay}(\Gamma,\mathcal{S})$ via *left multiplication* $y \mapsto sy$.

Definition A.1 (Isoperimetric profiles and increasing minorant)**.** For $v \in \mathbb{N}$ define the *equal-size* edge and vertex profiles

$$I^{\circ}_{\text{edge}}(v) := \inf\{B_{\mathcal{S}}(Y) : |Y| = v\}, \qquad I^{\circ}_{\text{vertex}}(v) := \inf\{|\partial_{\mathcal{S}} Y| : |Y| = v\}.$$

We extend I° piecewise-constantly to $[1,\infty)$ by $I^{\circ}(r) := I^{\circ}(\lfloor r \rfloor)$ and define the increasing minorants (greatest nondecreasing functions dominated by the exact profiles) by

$$I^{\text{incr}\,\circ}_{\text{edge}}(r) := \inf_{t \ge r} I^{\circ}_{\text{edge}}(t), \qquad I^{\text{incr}\,\circ}_{\text{vertex}}(r) := \inf_{t \ge r} I^{\circ}_{\text{vertex}}(t).$$

By (A.1), $I^{\circ}_{\text{edge}}(v) \ge I^{\circ}_{\text{vertex}}(v)$ and hence $I^{\text{incr}\,\circ}_{\text{edge}} \ge I^{\text{incr}\,\circ}_{\text{vertex}}$, while $I^{\text{incr}\,\circ}_{\text{edge}} \le \Delta\, I^{\text{incr}\,\circ}_{\text{vertex}}$.

A.1. **Tempered Property A.**

Definition A.2 (Tempered Property A)**.** A bounded-geometry metric space (X,d) has *tempered Property A with modulus $R(\varepsilon)$* if for all sufficiently small $\varepsilon > 0$ there exist finitely supported probability measures $a_x^{\varepsilon} \in \ell^1(X)$ such that

$$\operatorname{supp}(a_x^{\varepsilon}) \subset B(x, R(\varepsilon)), \qquad \sup_{d(x,y)\le 1} \|a_x^{\varepsilon} - a_y^{\varepsilon}\|_1 \le \varepsilon.$$

If $R(\varepsilon) \le C\,\varepsilon^{-\alpha}$ we say X has *strong tempered-A_α*. *Convention:* by replacing R with an equivalent minorant we may (and do) assume $R(\varepsilon)$ is nonincreasing in ε.

Remark A.3 (Relation to standard Property A)**.** On graph metrics (e.g. Cayley graphs) the step-1 control implies the usual step-R control by chaining along a nearest-neighbor path of length R: $\sup_{d(x,y)\le R} \|a_x^{\varepsilon} - a_y^{\varepsilon}\|_1 \le R\,\varepsilon$. More generally, on bounded-geometry spaces one may pass to a uniformly locally finite 1–skeleton (or a geodesic discretization) without changing the large-scale class, and the same chaining estimate applies. Thus tempered Property A implies standard Property A (with a linear loss in the oscillation parameter).

A.2. (TF) **in edge form and canonical witnesses.**

We state (TF) in edge normalization (equivalent to the vertex version up to factors of Δ by (A.1)).

Definition A.4 ((TF) (edge form))**.** A Cayley graph $(\Gamma,\mathcal{S})$ has *Tempered Følner (TF)* if there exist finite sets $F_n \subset \Gamma$ and constants $A_e, B \ge 1$ such that:

(Følner) $\dfrac{B_{\mathcal{S}}(F_n)}{|F_n|} \xrightarrow[n\to\infty]{} 0$;

(i_e) $B_{\mathcal{S}}(F_n) \;\le\; A_e\, I^{\text{incr}\,\circ}_{\text{edge}}(|F_n|)$ for all n;

(ii) $|F_{n+1} \triangle F_n| \;\le\; B \cdot B_{\mathcal{S}}(F_n)$ for all n.

Here $I^{\mathrm{incr}\,\circ}_{\mathrm{edge}}$ is the increasing *minorant* of the *edge* isoperimetric profile. (If the family is nested, $F_n \subset F_{n+1}$, then (ii) implies $|F_{n+1} \setminus F_n| \le B\, B_{\mathcal{S}}(F_n)$.)

Lemma A.5 (Normalization equivalence for (TF))**.** Let $(\Gamma, \mathcal{S})$ be finitely generated with $\Delta := |\mathcal{S}|$ and $B_{\mathcal{S}}(Y)$ the edge boundary (inside→outside count, equal to undirected crossing edges). Then:

(1) If (TF) holds in edge normalization with constants (A_e, B), i.e.

$$B_{\mathcal{S}}(F_n) \;\le\; A_e\, I^{\mathrm{incr}\,\circ}_{\mathrm{edge}}(|F_n|), \qquad |F_{n+1} \triangle F_n| \;\le\; B\, B_{\mathcal{S}}(F_n),$$

then (TF) holds in vertex normalization with constants $A = A_e\,\Delta,\;\; B' = B\,\Delta$, since $|\partial_{\mathcal{S}} Y| \le B_{\mathcal{S}}(Y) \le \Delta\, |\partial_{\mathcal{S}} Y|$ and $I^{\mathrm{incr}\,\circ}_{\mathrm{edge}} \le \Delta\, I^{\mathrm{incr}\,\circ}_{\mathrm{vertex}}$.

(2) Conversely, if (TF) holds in vertex normalization with (A, B), then in edge normalization it holds with $A_e = A\,\Delta,\;\; B' = B$, because $B_{\mathcal{S}}(Y) \le \Delta\, |\partial_{\mathcal{S}} Y|$ and $I^{\mathrm{incr}\,\circ}_{\mathrm{edge}} \ge I^{\mathrm{incr}\,\circ}_{\mathrm{vertex}}$.

Lemma A.6 (Bounded change of generators: profile comparability)**.** Let $\mathcal{S}, \mathcal{T}$ be two finite symmetric generating sets of Γ. Assume there exist $m, n \in \mathbb{N}$ with $\mathcal{S} \subset \mathcal{T}^m$ and $\mathcal{T} \subset \mathcal{S}^n$. Then there exist constants $c_1, c_2 \in (0, \infty)$ depending only on $(\mathcal{S}, \mathcal{T}, m, n)$ such that for all finite $Y \subset \Gamma$,

$$c_1\, B_{\mathcal{T}}(Y) \;\le\; B_{\mathcal{S}}(Y) \;\le\; c_2\, B_{\mathcal{T}}(Y),$$

and hence, for all r,

$$c_1\, I^{\circ}_{\mathrm{edge},\mathcal{T}}(r) \;\le\; I^{\circ}_{\mathrm{edge},\mathcal{S}}(r) \;\le\; c_2\, I^{\circ}_{\mathrm{edge},\mathcal{T}}(r), \qquad c_1\, I^{\mathrm{incr}\,\circ}_{\mathrm{edge},\mathcal{T}}(r) \;\le\; I^{\mathrm{incr}\,\circ}_{\mathrm{edge},\mathcal{S}}(r) \;\le\; c_2\, I^{\mathrm{incr}\,\circ}_{\mathrm{edge},\mathcal{T}}(r).$$

Sketch. Fix representative words over $\mathcal{T}$ of length $\le m$ for each $s \in \mathcal{S}$. If $y \in \partial_{\mathcal{S}} Y$ then $d_{\mathcal{T}}(y, (\Gamma \setminus Y)) \le m$, hence $y \in N^{(\mathcal{T})}_m(\partial_{\mathcal{T}} Y)$. In a bounded-degree graph, $|N^{(\mathcal{T})}_m(U)| \le C(m, \Delta_{\mathcal{T}})\, |U|$ with $C(m, \Delta_{\mathcal{T}}) \le \Delta_{\mathcal{T}}(\Delta_{\mathcal{T}} - 1)^{m-1}$. Therefore

$$|\partial_{\mathcal{S}} Y| \;\le\; C(m, \Delta_{\mathcal{T}})\, |\partial_{\mathcal{T}} Y|.$$

Using $B_{\mathcal{S}}(Y) \le \Delta_{\mathcal{S}}\, |\partial_{\mathcal{S}} Y|$ and $|\partial_{\mathcal{T}} Y| \le B_{\mathcal{T}}(Y)$ gives

$$B_{\mathcal{S}}(Y) \;\le\; \Delta_{\mathcal{S}}\, C(m, \Delta_{\mathcal{T}})\, B_{\mathcal{T}}(Y).$$

The converse inequality follows symmetrically from $\mathcal{T} \subset \mathcal{S}^n$. This proves the stated comparability with $c_2 = \Delta_{\mathcal{S}} C(m, \Delta_{\mathcal{T}})$ and $c_1 = (\Delta_{\mathcal{T}} C(n, \Delta_{\mathcal{S}}))^{-1}$. □

Per-generator directed counts. For $s \in \mathcal{S}$ and finite $F \subset \Gamma$, set

$$D_s(F) := \#\{y \in F :\; sy \notin F\}.$$

Then

$$B_{\mathcal{S}}(F) = \sum_{s \in \mathcal{S}} D_s(F), \qquad |F \triangle sF| = D_s(F) + D_{s^{-1}}(F) \;\le\; 2\, B_{\mathcal{S}}(F). \tag{A.2}$$

Lemma A.7 (Canonical (TF) witnesses for A (left-action version))**.** Let $F_n \subset \Gamma$ be finite and define

$$a^{(n)}_x := \frac{1}{|F_n|}\, \mathbf{1}_{x\,F_n} \in \ell^1(\Gamma).$$

Then, for every $x \in \Gamma$ and $s \in \mathcal{S}$,

$$\|a^{(n)}_x - a^{(n)}_{sx}\|_1 \;=\; \frac{|F_n \triangle sF_n|}{|F_n|} \;=\; \frac{D_s(F_n) + D_{s^{-1}}(F_n)}{|F_n|} \;\le\; \frac{2\, B_{\mathcal{S}}(F_n)}{|F_n|}\,, \tag{A.3}$$

and $\mathrm{supp}(a^{(n)}_x) \subset B\big(x, R_n\big)$ with

$$R_n := \mathrm{rad}(F_n) := \max\{d(e, g) : g \in F_n\}.$$

Proof. For $s \in \mathcal{S}$ (so $d(x, sx) = 1$), left-invariance of the Cayley metric gives

$$\|a_x^{(n)} - a_{sx}^{(n)}\|_1 = \frac{|xF_n \triangle sxF_n|}{|F_n|} = \frac{|F_n \triangle sF_n|}{|F_n|}.$$

The identity $|F \triangle sF| = D_s(F) + D_{s^{-1}}(F)$ is immediate from the definitions, and the inequality follows from (A.2). The support claim follows since $d(x, xf) = d(e, f) \le R_n$. □

Lemma A.8 (Anchoring/telescoping bounds). Let (X, d) have tempered Property A witnessed by $\{a_x^{(n)}\}_x$ at scales (ε_n, R_n), i.e. $\sup_{d(x,y)\le 1} \|a_x^{(n)} - a_y^{(n)}\|_1 \le \varepsilon_n$ and $\operatorname{supp}(a_x^{(n)}) \subset B(x, R_n)$. Then for all $x, y \in X$,

$$\|a_x^{(n)} - a_y^{(n)}\|_1 \ \le\ d(x,y)\,\varepsilon_n, \qquad \|\sqrt{a_x^{(n)}} - \sqrt{a_y^{(n)}}\|_2^2 \ \le\ d(x,y)\,\varepsilon_n.$$

In particular, for a fixed basepoint x_0,

$$\sum_{n\ge 1} w_n^2 \,\|\sqrt{a_x^{(n)}} - \sqrt{a_{x_0}^{(n)}}\|_2^2 \ \le\ d(x, x_0) \sum_{n\ge 1} w_n^2\, \varepsilon_n,$$

so if $\sum_n w_n^2\, \varepsilon_n < \infty$, then $\bigoplus_n w_n \big(\sqrt{a_x^{(n)}} - \sqrt{a_{x_0}^{(n)}}\big)$ converges in ℓ^2 for every x.

Lemma A.9 (From L^1-profile to edge profile). Let $(\Gamma, \mathcal{S})$ be a finitely generated group with edge boundary $B_{\mathcal{S}}$ (undirected normalization). Let $j_{\Gamma,1}(v)$ denote the L^1 isoperimetric profile in the sense of [Tes13], i.e.

$$j_{\Gamma,1}(v) \ :=\ \sup_{\substack{A\subset\Gamma\\ |A|\le v}} \ \sup_{f\in L^1(A),\, f\neq 0} \ \frac{\|f\|_1}{\sum_{s\in\mathcal{S}} \|f - \lambda(s) f\|_1}.$$

Then for all $v \ge 1$,

$$I^{\circ}_{\mathrm{edge}}(v) \ \ge\ \frac{v}{2\, j_{\Gamma,1}(v)}. \tag{A.4}$$

Proof. For any finite $A \subset \Gamma$,

$$\sum_{s\in\mathcal{S}} \|\mathbf{1}_A - \lambda(s)\mathbf{1}_A\|_1 = \sum_{s\in\mathcal{S}} |A \triangle sA| = \sum_{s\in\mathcal{S}} \big(D_s(A) + D_{s^{-1}}(A)\big) = 2\, B_{\mathcal{S}}(A).$$

Hence, with $f = \mathbf{1}_A$, $j_{\Gamma,1}(|A|) \ \ge\ \frac{\|f\|_1}{\sum_s \|f - \lambda(s) f\|_1} \ \ge\ \frac{|A|}{2\, B_{\mathcal{S}}(A)}$. Taking the supremum over $|A| = v$ yields $j_{\Gamma,1}(v) \ge v / \big(2\, I^{\circ}_{\mathrm{edge}}(v)\big)$, which is (A.4). □

A.3. (TF) $\Rightarrow$ **tempered Property A.**

Proposition A.10 ((TF) $\Rightarrow$ tempered-A (edge)). Assume (TF) in the form of Definition A.4. Then the families $\{a_x^{(n)}\}$ from Lemma A.7 witness tempered Property A with

$$\varepsilon_n \ :=\ \frac{2\, B_{\mathcal{S}}(F_n)}{|F_n|} \xrightarrow[n\to\infty]{} 0, \qquad R_n \ :=\ \operatorname{rad}(F_n),$$

i.e. $\|a_x^{(n)} - a_y^{(n)}\|_1 \le \varepsilon_n$ for $d(x, y) \le 1$ and $\operatorname{supp}(a_x^{(n)}) \subset B(x, R_n)$. *Remark:* only the Følner clause $\varepsilon_n \to 0$ is used to produce tempered-A; the additional (TF) clauses *(i_e)* and *(ii)* are not needed here.

If, in addition, there exists $Q \ge 1$ and constants $c, C > 0$ such that

$$c\, R_n^Q \ \le\ |F_n| \ \le\ C\, R_n^Q, \qquad c\, R_n^{Q-1} \ \le\ B_{\mathcal{S}}(F_n) \ \le\ C\, R_n^{Q-1}, \tag{A.5}$$

then $\varepsilon_n \asymp R_n^{-1}$ and hence Γ has *strong tempered-A_1* (namely $R(\varepsilon) \lesssim \varepsilon^{-1}$).

Proof. The variation/support statements follow from Lemma A.7; the Følner clause gives $\varepsilon_n \to 0$. Under (A.5), $\varepsilon_n = 2B_{\mathcal{S}}(F_n)/|F_n| \asymp R_n^{-1}$, so $R_n \asymp \varepsilon_n^{-1}$. □

Remark A.11 (Where (A.5) holds)**.** In virtually nilpotent (Carnot) groups, for F_n chosen as the discrete Wulff samplers built in §3, the asymptotics (A.5) hold with the homogeneous dimension Q and constants depending only on $(\Gamma, \mathcal{S})$.

A.4. **Quantitative Hilbert embeddings from tempered-A.**

Theorem A.12 (Quantitative coarse embedding with explicit compression)**.** Let (X, d) be of bounded geometry and have tempered Property A with witnesses at scales (ε_n, R_n). Define $v_x^{(n)} := \sqrt{a_x^{(n)}} \in \ell^2(X)$ and choose weights $w_n > 0$ with $\sum_n w_n^2 \varepsilon_n < \infty$ (for instance, $w_n^2 := 1/(n^2\varepsilon_n)$ for indices with $\varepsilon_n > 0$; indices with $\varepsilon_n = 0$ can be discarded). Then

$$\Phi(x) := \bigoplus_{n\geq 1} w_n \big(v_x^{(n)} - v_{x_0}^{(n)}\big) : \ X \longrightarrow \mathcal{H} := \bigoplus_{n\geq 1} \ell^2(X)$$

is a coarse embedding with

$$\|\Phi(x) - \Phi(y)\|^2 \ \leq \ \sum_n w_n^2 \varepsilon_n \quad \text{if } d(x,y) \leq 1, \qquad \|\Phi(x) - \Phi(y)\| \ \leq \ \Big(\sum_n w_n^2 \varepsilon_n\Big)^{1/2} d(x,y),$$

and

$$\|\Phi(x) - \Phi(y)\|^2 \ \geq \ \sum_{2R_n < d(x,y)} 2\, w_n^2.$$

If $R(\varepsilon) \ \leq \ C\,\varepsilon^{-\alpha}$ (strong tempered-A$_\alpha$), choose $\varepsilon_n \ = \ 2^{-n}$ and $w_n^2 \ = \ 2^n/n^2$. Then for all sufficiently large $d(x, y)$,

$$\|\Phi(x) - \Phi(y)\| \ \gtrsim \ \frac{d(x,y)^{\frac{1}{2\alpha}}}{\log d(x,y)} \ .$$

Quantitative counting. With $R_n \leq C\, 2^{\alpha n}$, if $m := \lfloor \alpha^{-1} \log_2 \frac{d(x,y)}{2C} \rfloor_+$ then $2R_n < d(x, y)$ for all $n \leq m$. Hence

$$\|\Phi(x) - \Phi(y)\|^2 \ \geq \ \sum_{n\leq m} 2\, \frac{2^n}{n^2} \ \asymp \ \frac{2^m}{m^2} \ \asymp \ \frac{d(x,y)^{1/\alpha}}{\big(\log d(x,y)\big)^2},$$

giving the stated lower bound after taking square roots.

Proof. Well-definedness. By Lemma A.8, for each x,

$$\sum_{n\geq 1} w_n^2\, \|v_x^{(n)} - v_{x_0}^{(n)}\|_2^2 \ \leq \ d(x, x_0) \sum_{n\geq 1} w_n^2\, \varepsilon_n \ < \ \infty,$$

so $\Phi(x)$ is well-defined.

Upper bounds. For $d(x, y) \leq 1$, $\|v_x^{(n)} - v_y^{(n)}\|_2^2 \leq \|a_x^{(n)} - a_y^{(n)}\|_1 \leq \varepsilon_n$ (since $(\sqrt{u} - \sqrt{v})^2 \leq |u - v|$ pointwise). Summing over n gives the first bound. For general x, y, chain along a nearest-neighbor path of length $d(x, y)$ and apply the triangle inequality in $\mathcal{H}$ to obtain

$$\|\Phi(x) - \Phi(y)\| \ \leq \ \sum_{k=0}^{d(x,y)-1} \|\Phi(z_k) - \Phi(z_{k+1})\| \ \leq \ d(x,y) \Big(\sum_n w_n^2\, \varepsilon_n\Big)^{1/2}.$$

Lower bound. If $d(x, y) > 2R_n$, the supports of $a_x^{(n)}$ and $a_y^{(n)}$ are disjoint, hence $\|v_x^{(n)} - v_y^{(n)}\|_2^2 = 2$. Summing over n with $2R_n < d(x, y)$ gives the stated lower bound.

Compression under strong tempered-A$_\alpha$. The explicit counting is given in the theorem statement. □

Remark A.13 (Compression optimality and the log loss)**.** The lower bound $\|\Phi(x) - \Phi(y)\| \gtrsim d(x, y)^{1/(2\alpha)}/\log d(x, y)$ is a *uniform* output of the tempered-A construction. It is not claimed to be sharp in every group. Two guiding cases: (i) For Abelian groups $\mathbb{Z}^d$, a direct linear embedding into a Euclidean subspace yields compression exponent 1 (no logarithm) with constants depending on d; see, e.g., [NP08]. (ii) For non-Abelian nilpotent groups (e.g. the discrete Heisenberg group), Hilbert compression exponent $> 1/2$ is impossible; known upper bounds match $1/2$ up to lower-order losses [Tes11, LN06, NP08]. In this regime, the bound above is within a logarithmic factor of the best possible exponent, and the log factor is a standard artefact of multi-scale square-root embeddings governed by $\sum w_n^2 \varepsilon_n < \infty$. For background on obstructions to L_1 embeddings and related differentiation phenomena, see [CK10].

Corollary A.14 (Shaving the log with heavier weights)**.** Under strong tempered-A$_\alpha$, the same construction with $\varepsilon_n = 2^{-n}$ and $w_n^2 = 2^n/n^{1+\eta}$ ($\eta > 0$) yields

$$\|\Phi(x) - \Phi(y)\| \gtrsim \frac{d(x,y)^{\frac{1}{2\alpha}}}{\left(\log d(x,y)\right)^{\frac{1+\eta}{2}}}$$

for all large $d(x,y)$. Letting $\eta \downarrow 0$ gives $d^{1/(2\alpha)}/(\log d)^{1/2+o(1)}$.

Corollary A.15 (Iterated-log compression via lacunary scales)**.** Under strong tempered-A$_\alpha$, choose a super-geometric ε_n and weights $w_n^2 := 1/(n^2\varepsilon_n)$. Then:

(1) With $\varepsilon_n = e^{-n^2}$, one gets $\|\Phi(x) - \Phi(y)\| \gtrsim d(x,y)^{1/(2\alpha)}/(\log d(x,y))^{1/2}$.
(2) With $\varepsilon_n = e^{-e^n}$, one gets $\|\Phi(x) - \Phi(y)\| \gtrsim d(x,y)^{1/(2\alpha)}/\log\log d(x,y)$.
(3) More generally, using k-fold iterated exponentials yields a denominator $\log^{(k)} d(x,y)$.

In each case $\sum_n w_n^2 \varepsilon_n = \sum_n 1/n^2 < \infty$, so the Lipschitz bound is preserved; by Lemma A.8 the map Φ is well-defined for all x.

A.5. **A tracked $\sqrt{t}/\log(1+t)$ compression from a (TF) chain.** We now specialize to Cayley graphs and build a *deterministic* embedding with *explicit* constants that depend on no growth hypothesis beyond the Følner condition (hence do not require strong tempered-A$_\alpha$).

Proposition A.16 ((TF) $\Rightarrow$ quantitative ℓ^2-compression with tracked constants (edge))**.** Let $X = \mathrm{Cay}(\Gamma, \mathcal{S})$ be a Cayley graph (edge normalization) and assume X has (TF) in the sense of Definition A.4. Write

$$\delta_n := \frac{B_{\mathcal{S}}(F_n)}{|F_n|} \xrightarrow[n\to\infty]{} 0, \qquad R_n := \mathrm{rad}(F_n) \xrightarrow[n\to\infty]{} \infty.$$

There exists a subsequence $(n_j)_{j\geq 1}$ such that

$$\delta_{n_j} \leq 2^{-(j+4)} \qquad \text{and} \qquad 2^j \leq R_{n_j} < 2^{j+1}. \tag{A.6}$$

For $x \in \Gamma$ set $\mu_x^j := |F_{n_j}|^{-1}\,\mathbf{1}_{xF_{n_j}}$ and define

$$\psi_j(x) := \sqrt{\mu_x^j} - \sqrt{\mu_e^j} \in \ell^2(\Gamma), \qquad \alpha_j^2 := \frac{6}{\pi^2}\frac{2^j}{j^2}, \qquad \Phi(x) := \bigoplus_{j=1}^{\infty} \alpha_j\,\psi_j(x).$$

Then for all $x, y \in \Gamma$ with $t = d(x,y)$ (natural logarithm),

$$\frac{\sqrt{3}\,\ln 2}{2\sqrt{2}\,\pi} \cdot \frac{\sqrt{t}}{1+\log(1+t)} \leq \|\Phi(x) - \Phi(y)\|_2 \leq \frac{1}{2\sqrt{2}}\sqrt{t}. \tag{A.7}$$

In particular, X coarsely embeds into Hilbert space with a *tracked* lower compression

$$\rho(t) \geq \frac{\sqrt{3}\,\ln 2}{2\sqrt{2}\,\pi}\,\frac{\sqrt{t}}{1+\log(1+t)}.$$

Proof. Since $\delta_n \to 0$ and $R_n \to \infty$, a diagonal choice yields a subsequence (n_j) with (A.6). For neighbors $d(x,y) = 1$,

$$\|\mu_x^j - \mu_y^j\|_1 = \frac{|F_{n_j} \triangle sF_{n_j}|}{|F_{n_j}|} \leq \frac{2B_{\mathcal{S}}(F_{n_j})}{|F_{n_j}|} = 2\delta_{n_j} \leq 2^{-(j+3)}.$$

By chaining along a geodesic of length $t = d(x,y)$ and using $(\sqrt{u}-\sqrt{v})^2 \le |u-v|$ pointwise,

$$\|\psi_j(x)-\psi_j(y)\|_2^2 \;\le\; \|\mu_x^j-\mu_y^j\|_1 \;\le\; t\,2^{-(j+3)}. \tag{A.8}$$

Summing with the weights α_j gives the Lipschitz upper bound:

$$\|\Phi(x)-\Phi(y)\|_2^2 \;\le\; t\sum_{j\ge1}\alpha_j^2\,2^{-(j+3)} \;=\; \frac{t}{8}\cdot\frac{6}{\pi^2}\sum_{j\ge1}\frac{1}{j^2} \;=\; \frac{t}{8},$$

hence the right-hand inequality in (A.7).

For the lower bound, if $2R_{n_j} < t$ then the supports of μ_x^j and μ_y^j are disjoint, hence $\|\psi_j(x)-\psi_j(y)\|_2^2 = \|\sqrt{\mu_x^j}-\sqrt{\mu_y^j}\|_2^2 = 2$. Let J be the largest index with $2^{J+2}\le t$. By (A.6), for all $j\le J$ one has $2R_{n_j}\le 2^{j+2}\le 2^{J+2}\le t$, so

$$\|\Phi(x)-\Phi(y)\|_2^2 \;\ge\; \sum_{j=1}^{J}\alpha_j^2\cdot 2 \;=\; \frac{12}{\pi^2}\sum_{j=1}^{J}\frac{2^j}{j^2} \;\ge\; \frac{12}{\pi^2}\cdot\frac{2^J}{2J^2} \;=\; \frac{6}{\pi^2}\cdot\frac{2^J}{J^2}.$$

Since $2^{J+2}\le t<2^{J+3}$, we have $2^J\ge t/8$ and $J\le\log_2(1+t)$. Therefore

$$\|\Phi(x)-\Phi(y)\|_2^2 \;\ge\; \frac{6}{\pi^2}\cdot\frac{t}{8\,J^2} \;\ge\; \frac{6}{\pi^2}\cdot\frac{t}{8\,(1+\log_2(1+t))^2} \;=\; \frac{3\,(\ln 2)^2}{2\pi^2}\cdot\frac{t}{(1+\log(1+t))^2}.$$

Taking square roots yields the left-hand inequality in (A.7). □

A.6. **Uniform control for box spaces.**

Definition A.17 (Uniform tempered-A). A family $\{X_i\}_{i\in I}$ has *uniform tempered-A (edge)* with modulus $R(\varepsilon)$ if for each ε there exist witnesses $a_{x,i}^{\varepsilon}$ on X_i with $\sup_{d(x,y)\le1}\|a_{x,i}^{\varepsilon}-a_{y,i}^{\varepsilon}\|_1\le\varepsilon$ and $\operatorname{supp}(a_{x,i}^{\varepsilon})\subset B(x,R(\varepsilon))$, and $R(\varepsilon)$ is independent of i (we may take R nonincreasing in ε uniformly in i).

Proposition A.18 (Box spaces). Let Γ be residually finite with a chain of finite-index normal subgroups $\{N_i\}$, and set $X_i := \Gamma/N_i$ with the quotient generating sets (degrees are uniformly bounded). Suppose that, for each i, there exists a family $\{F_n^{(i)}\}_n \subset X_i$ satisfying (TF) in edge form with constants A_e, B *independent of* i, and with growth bounds (A.5) holding uniformly in i for some $Q\ge1$. Then the coarse disjoint union $\bigsqcup_i X_i$ has *uniform strong tempered-*A_1 and admits a coarse embedding into Hilbert with compression $\rho(t)\gtrsim t^{1/2}/\log t$ (uniform constants). In the construction of Φ we fix a basepoint $x_{0,i}\in X_i$ for each component. Moreover, by applying Definition A.17 at a dyadic schedule $\varepsilon_k = 2^{-k}$ with the uniform modulus $R(\varepsilon)$, we obtain witnesses and weights that are uniform in i; Theorem A.12 then applies componentwise with constants independent of i. (As usual for a box space with $\cap_i N_i=\{e\}$, $\operatorname{diam}(X_i)\to\infty$.)

Proof. Apply Proposition A.10 uniformly in i, then Theorem A.12. □

Remark A.19 (Operator-algebraic context). Property A and coarse embeddability into Hilbert (uniform embeddability) were introduced to attack major problems in topology and operator algebras: the coarse Baum-Connes conjecture and, via controlled K-theory, the Novikov conjecture [Yu00]. For broader background see Higson-Roe [HR00] and Nowak-Yu [NY12]. The quantitative, scale-calibrated witnesses developed here can be inserted into those frameworks to track constants (e.g. quantitative K-theory or stability thresholds) and, more importantly for this paper, they mesh with the isoperimetric-lag machinery to deliver two distinct outputs-bounded profile ratios and constructive embeddings-with one geometric input (TF).

Remark A.20 (Scope of quantitative embeddings and the role of the step-2 analysis). The results of this appendix take a Tempered Følner (TF) chain as input and produce tempered Property A and a quantitative coarse embedding into Hilbert with tracked constants (uniformly on box spaces). We obtain such TF input, with explicit constants, in several nonabelian families by different mechanisms: (i) in all virtually nilpotent groups (any step), via discrete Wulff samplers; (ii) in hyperbolic semidirects $\mathbb{Z}^d \rtimes_A \mathbb{Z}$, via log-height sets and the L^1 profile $j_1(v) \asymp \log v$; (iii) in finite-lamp lamplighters $H^{(\Gamma)} \rtimes \Gamma$, via an explicit base+ring expansion and Erschler's checkpoints; and then by quantitative permanence under finite index, finite extensions, and direct products. The step-2 shear/curl-fit block is indispensable in the first nonabelian nilpotent regime: it supplies a *local* TF chain with per-edge control in lattices of step-2 Carnot groups, where naive layered averages fail due to commutator shear. Once TF is available by any of these engines, the embedding theorems of Appendix A apply verbatim, with constants that track the generating set and the structural data of the group.

Appendix B. Left-Right Homogeneous-Space Profile vs. Cayley Profile (Edge Normalization)

In this appendix we quantify the relation between the (edge) isoperimetric profile of a finitely generated group $(\Gamma, \mathcal{S})$ on its left Cayley graph and the isoperimetric profile of Γ viewed as a homogeneous space under the right/left action of $\Gamma \times \Gamma$. We work throughout with *edge* boundary. For the *left/right Cayley graphs* we count each *uncolored* undirected edge once. For the Σ_+-graph introduced below we use a *colored* edge boundary: each color $\sigma \in \Sigma_+$ defines its own adjacency and we count an undirected σ-edge once *per color*. (Thus, a "colored edge" is an unordered pair together with its color label; distinct colors are different edges even if they connect the same two vertices.) This convention makes the colored-sum identity exact. When needed, vertex/edge comparisons are recovered by the standard inequalities

$$|\partial_{\mathcal{S}} Y| \;\le\; B_{\mathcal{S}}(Y) \;\le\; \Delta_{\mathcal{S}}\, |\partial_{\mathcal{S}} Y|, \qquad \Delta_{\mathcal{S}} := |\mathcal{S}|,$$

(see Lemma 10.1; here and below $B_{\mathcal{S}}$ denotes the *left* edge boundary $B^{\mathrm{L}}_{\mathcal{S}}$). For the Σ_+-graph the (colored) degree is constant and equals $\Delta_\Sigma := |\Sigma_+|$, and we similarly have

$$|\partial_{\Sigma_+}(Y)| \;\le\; B_\Sigma(Y) \;\le\; \Delta_\Sigma\, |\partial_{\Sigma_+}(Y)|. \tag{B.1}$$

Asymptotic Wulff constants and sharp growth exponents for virtually nilpotent groups are recalled in Sections 3 and 10; we use them only to state asymptotic corollaries below.

Bridge to the main results. The $(\Gamma \times \Gamma)$-homogeneous-space graph on Γ generated by left and right steps is the natural bi-invariant analogue of a Cayley graph; it appears when one mixes left and right regular actions (e.g., in random walks, bimodule constructions, or bi-invariant smoothing). The colored-sum identity below shows that its edge boundary *splits exactly* as a sum of left and right Cayley boundaries. Two concrete payoffs that plug back into the paper are:

(i) *Analytic constants (Section 10).* In polynomial growth, the left-right profile has the same sharp exponent and exactly *twice* the Wulff constant (Cor. B.7 below). Feeding this into Theorem 10.3 improves the Faber–Krahn prefactor by a factor $4\left(\frac{|\mathcal{S}|}{|\Sigma_+|}\right)^2$ (equal to 4 when $\mathcal{S}$ has no involutions), hence strengthens Nash/heat-kernel/mixing constants with no change of exponents.

(ii) *Tempered A and* (TF) *(Section A).* Since $B_\Sigma(Y) = B^{\mathrm{L}}_{\mathcal{S}}(Y) + B^{\mathrm{L}}_{\mathcal{S}}(Y^{-1})$, (TF) witnesses built from Wulff samplers automatically transfer to Σ with the same radius scale, giving uniform tempered-A for the bi-invariant geometry (Remark B.8).

Setup and notation Let $\mathcal{S} = \mathcal{S}^{-1} \subset \Gamma$ be a finite symmetric generating set, and assume $e \notin \mathcal{S}$. To avoid double counting of undirected edges by inverse *within each side* (left or right), fix a subset $\mathcal{S}_+ \subset \mathcal{S}$ containing exactly one representative from each inverse pair $\{s, s^{-1}\}$. (The constructions below are independent of the particular choice of $\mathcal{S}_+$; see Remark B.1.) Write $B^{\mathrm{L}}_{\mathcal{S}}(Y)$ (resp.

$B^{\mathrm{R}}_{\mathcal{S}}(Y)$) for the edge boundary of $Y \subset \Gamma$ in the *left* (resp. *right*) Cayley graph:

$B^{\mathrm{L}}_{\mathcal{S}}(Y) := \#\{\{g, sg\} : g \in Y,\ s \in \mathcal{S},\ sg \notin Y\}, \qquad B^{\mathrm{R}}_{\mathcal{S}}(Y) := \#\{\{g, gs\} : g \in Y,\ s \in \mathcal{S},\ gs \notin Y\}.$

We view Γ as a homogeneous space of $\Gamma \times \Gamma$ under the action $(g_1, g_2) \cdot x := g_1\, x\, g_2^{-1}$. In particular, the color (s, e) maps $x \mapsto sx$ and the color (e, s) maps $x \mapsto xs^{-1}$. Let

$$\Sigma_+ \ :=\ \{(s, e) \,:\, s \in \mathcal{S}_+\} \,\cup\, \{(e, s) \,:\, s \in \mathcal{S}_+\} \ \subset \Gamma \times \Gamma,$$

so that the Σ_+-adjacency on Γ is the disjoint union (by color and inverse-pair choice) of left- and right-$\mathcal{S}$-adjacencies. Denote by

$$B_\Sigma(Y) \ :=\ \sum_{\sigma \in \Sigma_+} \#\big\{\text{undirected } \sigma\text{-edges with one endpoint in } Y \text{ and one in } Y^c\big\}$$

the (colored) edge boundary in this homogeneous-space graph. Finally, define the exact edge profiles

$$\begin{aligned} I^{\circ,\mathrm{L}}_{\mathrm{edge}}(r) \ &:=\ \inf\big\{B^{\mathrm{L}}_{\mathcal{S}}(Y) : |Y| = r\big\}, \\ I^{\circ,\mathrm{R}}_{\mathrm{edge}}(r) \ &:=\ \inf\big\{B^{\mathrm{R}}_{\mathcal{S}}(Y) : |Y| = r\big\}, \\ I^{\circ}_{\Sigma}(r) \ &:=\ \inf\big\{B_\Sigma(Y) : |Y| = r\big\}. \end{aligned}$$

For brevity we sometimes write Σ for Σ_+ when no confusion can arise.

Remark B.1 (Counting convention; directed and uncolored alternatives; independence of $\mathcal{S}_+$)**.** The choice of $\mathcal{S}_+$ ensures each undirected left/right boundary edge is counted exactly once *within that side* among the corresponding colors. For Σ_+ we *intend* to count per color: if a single pair $\{x, y\}$ is both a left- and a right-$\mathcal{S}$ adjacency (with possibly different colors), it contributes twice to B_Σ (once for each side). Equivalently, one can keep the full symmetric $\mathcal{S}$ and count *directed* edges; then $B^{\rightarrow}_{\Sigma}(Y) = 2\big(B^{\mathrm{L}}_{\mathcal{S}}(Y) + B^{\mathrm{R}}_{\mathcal{S}}(Y)\big)$, and all identities below hold after dividing by 2. If instead one collapses colors to a single uncolored graph on Γ, only the inequality

$$B^{\mathrm{simple}}_{\Sigma}(Y) \ \le\ B^{\mathrm{L}}_{\mathcal{S}}(Y) + B^{\mathrm{R}}_{\mathcal{S}}(Y)$$

holds in general (with possible strict inequality when overlaps occur). Finally, the quantity $B_\Sigma(Y)$ is independent of the particular choice of representatives $\mathcal{S}_+$: changing representatives only relabels colors, and each undirected left/right $\mathcal{S}$-edge is represented exactly once on its side in the colored sum.

B.1. **Right-left inversion and the colored-edge identity.**

Lemma B.2 (Right-left inversion)**.** The inversion map $\iota : \Gamma \to \Gamma$, $\iota(x) = x^{-1}$, is a graph isomorphism between the right and left Cayley graphs:

$$\{g, gs\} \ \longleftrightarrow\ \{g^{-1}, (gs)^{-1}\} \ =\ \{g^{-1}, s^{-1}g^{-1}\} \qquad (s \in \mathcal{S}).$$

Consequently $B^{\mathrm{R}}_{\mathcal{S}}(Y) = B^{\mathrm{L}}_{\mathcal{S}}(Y^{-1})$ for every $Y \subset \Gamma$.

Proof. If $\{g, gs\}$ is a right-$\mathcal{S}$ edge then $\{g^{-1}, (gs)^{-1}\} = \{g^{-1}, s^{-1}g^{-1}\}$ is a left-$\mathcal{S}$ edge (we used $\mathcal{S} = \mathcal{S}^{-1}$). Boundary edges are preserved bijectively under inversion, proving the equality of counts. □

Corollary B.3 (Left and right profiles coincide)**.** For every integer $r \ge 1$, $I^{\circ,\mathrm{R}}_{\mathrm{edge}}(r) = I^{\circ,\mathrm{L}}_{\mathrm{edge}}(r)$.

Proof. By Lemma B.2, $B^{\mathrm{R}}_{\mathcal{S}}(Y) = B^{\mathrm{L}}_{\mathcal{S}}(Y^{-1})$ and $|Y| = |Y^{-1}|$. Taking infima over $|Y| = r$ on both sides yields the claim. □

Proposition B.4 (Colored-sum identity)**.** For every finite $Y \subset \Gamma$,

$$B_\Sigma(Y) \ =\ B^{\mathrm{L}}_{\mathcal{S}}(Y) \ +\ B^{\mathrm{R}}_{\mathcal{S}}(Y) \ =\ B^{\mathrm{L}}_{\mathcal{S}}(Y) \ +\ B^{\mathrm{L}}_{\mathcal{S}}(Y^{-1}). \tag{B.2}$$

Proof. By construction the Σ_+-edges are the disjoint union (by color) of left-$\mathcal{S}$ edges and right-$\mathcal{S}$ edges, each counted exactly once on its side because we sum over $\mathcal{S}_+$. Summing the two contributions gives the first equality; the second is Lemma B.2. $\square$

B.2. **Two-sided comparison of profiles.**

Theorem B.5 (Two-sided comparison)**.** For every integer $r \geq 1$,

$$2\, I^{\circ,\mathrm{L}}_{\mathrm{edge}}(r) \;\leq\; I^{\circ}_{\Sigma}(r) \;\leq\; I^{\circ,\mathrm{L}}_{\mathrm{edge}}(r) \;+\; \Delta_{\mathcal{S}}\, r. \tag{B.3}$$

Moreover, if there exists a left-minimizer Y_r with Y_r^{-1} also left-minimizing (e.g. any inversion-symmetric minimizer), then $I^{\circ}_{\Sigma}(r) \;\leq\; 2\, I^{\circ,\mathrm{L}}_{\mathrm{edge}}(r)$.

Proof. Lower bound. For any Y with $|Y| = r$, (B.2) and Lemma B.2 give

$$B_{\Sigma}(Y) = B^{\mathrm{L}}_{\mathcal{S}}(Y) + B^{\mathrm{L}}_{\mathcal{S}}(Y^{-1}) \;\geq\; I^{\circ,\mathrm{L}}_{\mathrm{edge}}(r) + I^{\circ,\mathrm{L}}_{\mathrm{edge}}(r) = 2I^{\circ,\mathrm{L}}_{\mathrm{edge}}(r).$$

Taking the infimum over such Y yields $I^{\circ}_{\Sigma}(r) \geq 2I^{\circ,\mathrm{L}}_{\mathrm{edge}}(r)$.

Upper bound. The set $\{B^{\mathrm{L}}_{\mathcal{S}}(Y) : |Y| = r\} \subset \{0, 1, \ldots, \Delta_{\mathcal{S}} r\}$ is a nonempty subset of $\mathbb{N}$, hence has a minimum by well-ordering; choose Y with $|Y| = r$ attaining $B^{\mathrm{L}}_{\mathcal{S}}(Y) = I^{\circ,\mathrm{L}}_{\mathrm{edge}}(r)$. Then

$$B_{\Sigma}(Y) = I^{\circ,\mathrm{L}}_{\mathrm{edge}}(r) + B^{\mathrm{R}}_{\mathcal{S}}(Y) \;\leq\; I^{\circ,\mathrm{L}}_{\mathrm{edge}}(r) + \Delta_{\mathcal{S}}\, |Y| \;=\; I^{\circ,\mathrm{L}}_{\mathrm{edge}}(r) + \Delta_{\mathcal{S}} r,$$

since at each $g \in Y$ there are at most $\Delta_{\mathcal{S}}$ right-$\mathcal{S}$ edges exiting Y. Therefore $I^{\circ}_{\Sigma}(r) \leq I^{\circ,\mathrm{L}}_{\mathrm{edge}}(r) + \Delta_{\mathcal{S}} r$.

Symmetric minimizers. If one can choose Y_r with $B^{\mathrm{L}}_{\mathcal{S}}(Y_r) = I^{\circ,\mathrm{L}}_{\mathrm{edge}}(r)$ and $B^{\mathrm{L}}_{\mathcal{S}}(Y_r^{-1}) = I^{\circ,\mathrm{L}}_{\mathrm{edge}}(r)$ (e.g. $Y_r = Y_r^{-1}$), then $B_{\Sigma}(Y_r) = 2I^{\circ,\mathrm{L}}_{\mathrm{edge}}(r)$, whence $I^{\circ}_{\Sigma}(r) \leq 2I^{\circ,\mathrm{L}}_{\mathrm{edge}}(r)$. $\square$

Remark B.6 (Sharpness, attainability, and the source of asymmetry)**.** The lower bound is attained whenever a left-minimizer can be chosen inversion-symmetric (for instance on $\mathbb{Z}^d$ one can use origin-centered Wulff approximants). The upper bound $I^{\circ}_{\Sigma}(r) \;\leq\; I^{\circ,\mathrm{L}}_{\mathrm{edge}}(r) + \Delta_{\mathcal{S}} r$ is *optimal in general*: no uniform improvement to $2I^{\circ,\mathrm{L}}_{\mathrm{edge}}(r)$ is possible without an extra symmetry hypothesis, since $B^{\mathrm{R}}_{\mathcal{S}}(Y) = B^{\mathrm{L}}_{\mathcal{S}}(Y^{-1})$ need not be close to $B^{\mathrm{L}}_{\mathcal{S}}(Y)$ for an arbitrary left-minimizer Y. Conceptually, the gap measures the failure of the collection of volume-r left-minimizers to be stable under inversion: if inversion does not preserve the minimizing family, $\min\{B^{\mathrm{L}}_{\mathcal{S}}(\cdot)\}$ and $\min\{B^{\mathrm{L}}_{\mathcal{S}}((\cdot)^{-1})\}$ are realized at different sets and the colored sum cannot collapse to $2I^{\circ,\mathrm{L}}_{\mathrm{edge}}(r)$. In nilpotent settings with even anisotropy (Sections 3, 5), calibrated Wulff samplers are naturally inversion-symmetric and the gap closes.

B.3. **Asymptotics in virtually nilpotent classes.** Let Γ have polynomial growth of homogeneous dimension Q and let $c_{\mathrm{W}}(\mathcal{S})$ denote the sharp Wulff constant for the left Cayley graph (Section 3). Then

$$I^{\circ,\mathrm{L}}_{\mathrm{edge}}(r) \;=\; c_{\mathrm{W}}(\mathcal{S})\, r^{(Q-1)/Q}\,(1 + o(1)) \qquad (r \to \infty),$$

and, choosing (asymptotically) inversion-symmetric Wulff samplers, Theorem B.5 yields the homogeneous-space profile

$$I^{\circ}_{\Sigma}(r) \;=\; 2\, c_{\mathrm{W}}(\mathcal{S})\, r^{(Q-1)/Q}\,(1 + o(1)). \tag{B.4}$$

Thus, in the virtually nilpotent setting, passing from the left Cayley profile to the $\Gamma \times \Gamma$-homogeneous-space profile doubles the sharp constant while preserving the exponent.

Corollary B.7 (Bi-invariant walk: Wulff-coded upgrade)**.** Assume the setting of Section 10 with polynomial growth $Q \geq 1$. Replacing $\mathcal{S}$ by the bi-invariant generator Σ_+ yields a (colored) degree bound

$$|\Sigma_+| \;=\; 2\,|\mathcal{S}_+| \;=\; |\mathcal{S}| + |I|, \qquad I := \{s \in \mathcal{S} :\; s = s^{-1}\}.$$

In particular $|\Sigma_+| = |\mathcal{S}|$ whenever $\mathcal{S}$ has no involutions (e.g., the standard $\pm e_i$ generators). By (B.4), the isoperimetric constant doubles asymptotically. Consequently, the Faber–Krahn lower bound of Theorem 10.3 improves by the factor

$$4\Big(\tfrac{|\mathcal{S}|}{|\Sigma_+|}\Big)^2 \;=\; 4\Big(\tfrac{|\mathcal{S}|}{|\mathcal{S}|+|I|}\Big)^2,$$

which equals 4 when $I = \varnothing$. Equivalently,

$$\lambda_1^{\mathrm{D}}(r) \;\geq\; \frac{(2C_{\mathrm{iso}})^2}{2\,|\Sigma_+|^2}\, r^{-2/Q} \;=\; 4\left(\frac{|\mathcal{S}|}{|\Sigma_+|}\right)^2 \times \frac{C_{\mathrm{iso}}^2}{2\,|\mathcal{S}|^2}\, r^{-2/Q},$$

and the Nash/heat-kernel/mixing constants in Section 10 improve accordingly (exponents unchanged). The proofs in Section 10 apply verbatim to our colored multigraph, as they use only the Dirichlet form with counting measure and a uniform degree bound.

Remark B.8 (Transfer of TF/tempered-A to Σ)**.** If the (TF) witnesses in Section A are built from (asymptotically) inversion-symmetric Wulff samplers-as in the virtually nilpotent/Carnot setting-then $B_\Sigma(F_n) = B_{\mathcal{S}}^{\mathrm{L}}(F_n) + B_{\mathcal{S}}^{\mathrm{L}}(F_n^{-1}) \sim 2\, B_{\mathcal{S}}^{\mathrm{L}}(F_n)$ and $\mathrm{rad}(F_n)$ is unchanged. Hence the same families certify (TF) for Σ_+ (up to constant factors), and the tempered-A/quantitative coarse embeddings of Section A carry over with the same radius scale and logarithmic compression.

B.4. **Consequences and variants.**

- *Cheeger constants.* If the (left) edge Cheeger constant satisfies $B_{\mathcal{S}}^{\mathrm{L}}(Y) \geq h\,|Y|$, then by (B.2) we get $B_\Sigma(Y) \geq 2h\,|Y|$, hence $I_\Sigma^\circ(r) \geq 2hr$; the homogeneous-space graph is "at least twice as nonamenable." (For the uncolored-collapsed model one only has $h_\Sigma^{\mathrm{simple}} \geq h$.)
- *Vertex normalization.* All statements above translate to vertex boundary up to factors of the relevant degree via $|\partial_{\mathcal{S}}(Y)| \leq B_{\mathcal{S}}(Y) \leq \Delta_{\mathcal{S}}\,|\partial_{\mathcal{S}}(Y)|$ and (B.1) (with $\Delta_{\mathcal{S}}$ replaced by $\Delta_\Sigma = |\Sigma_+|$ for the Σ_+-graph).
- *Weighted colorings.* One may weight left and right colors differently (say, with weights $\alpha, \beta > 0$); the identity becomes $B_{\alpha,\beta}(Y) = \alpha\, B_{\mathcal{S}}^{\mathrm{L}}(Y) + \beta\, B_{\mathcal{S}}^{\mathrm{L}}(Y^{-1})$, and the bounds scale accordingly. If the corresponding random walk chooses colors with total masses α (left) and β (right), then the Dirichlet form rescales by $\alpha + \beta$ and the degree parameter in the Nash/Faber–Krahn normalization is $(\alpha+\beta)\,|\mathcal{S}_+| = \frac{\alpha+\beta}{2}\,|\Sigma_+|$.

Summary. The homogeneous-space *colored* edge boundary for the $(\Gamma\times\Gamma)$-action with generator set Σ_+ splits *exactly* as the sum of left and right Cayley boundaries, so that the corresponding profile satisfies the robust two-sided estimate (B.3). In virtually nilpotent classes the asymptotics (B.4) hold with the sharp Wulff constant from the left Cayley profile, yielding a clean and explicit relation between the two geometries and slotting directly into the quantitative analytic bounds of Section 10 and the tempered coarse-embedding framework of Section A.

Appendix C. Technical Appendix: discrete BV/coarea and anisotropic counts

Conventions. We adopt $\mathbb{N} = \{0, 1, 2, \dots\}$. We work on the base grid $\mathbb{Z}^{r_1}$ with a finite, *oriented* horizontal stencil $S_{\mathrm{hor}} \subset \mathbb{Z}^{r_1} \setminus \{0\}$: for each undirected pair $\{\pm v\}$ we include exactly one representative $v \in S_{\mathrm{hor}}$. For $v \in S_{\mathrm{hor}}$ write the forward difference $(\Delta_v f)(u) := f(u+v) - f(u)$. A v-*edge* is the undirected edge $\{u, u+v\}$, recorded with the chosen orientation $u \to u+v$, so that every undirected edge is counted exactly once. For $E \subset \mathbb{Z}^{r_1}$ let $\partial^{(v)}(E)$ be the set of v-edges with exactly one endpoint in E, and

$$\mathrm{Per}_{\mathrm{grid}}(E) := \sum_{v \in S_{\mathrm{hor}}} |\partial^{(v)}(E)| \qquad \text{(undirected edge perimeter via the fixed orientation).}$$

When an orthotropic anisotropy with weights $w_i > 0$ is present and we take the axis orientation $S_{\mathrm{hor}} = \{e_i\}_{i=1}^{r_1}$, we also use the *weighted* discrete perimeter

$$\mathrm{Per}^{(w)}_{\mathrm{grid}}(E) := \sum_{i=1}^{r_1} w_i \left|\partial^{(e_i)}(E)\right|. \tag{C.1}$$

Unless explicitly stated otherwise, every finitely supported function on a subset of $\mathbb{Z}^{r_1}$ is extended by 0 to all of $\mathbb{Z}^{r_1}$ before taking differences or applying coarea. All horizontal perimeter counts here coincide (up to fixed factors depending only on degrees) with the undirected edge boundary used in §9–10.

Morphological notation. Write $Q_\infty := [-1,1]^{r_1}$ and $B_\infty(\rho) := [-\rho,\rho]^{r_1}$ for the ℓ^∞ unit cube and ball. For $A \subset \mathbb{R}^{r_1}$ and $t \geq 0$ the (continuum) ℓ^∞ inner parallel is $A \ominus tQ_\infty := \{x :\ x + tQ_\infty \subset A\}$. For $E \subset \mathbb{Z}^{r_1}$ and $m \in \mathbb{N}$, the (discrete) ℓ^∞ inner parallel set is

$$E^{(-m)} \ := \ \big\{u \in \mathbb{Z}^{r_1} :\ u + k \in E \text{ for all } k \in \mathbb{Z}^{r_1} \cap B_\infty(m)\big\}.$$

We also use the *discrete* ℓ^∞ distance to the boundary

$$\mathrm{dist}^{\mathrm{disc}}_\infty(u, \partial E) \ := \ \max\big\{m \in \mathbb{N} :\ u + k \in E \text{ for all } k \in \mathbb{Z}^{r_1} \cap B_\infty(m)\big\},$$

so that $\{u :\ \mathrm{dist}^{\mathrm{disc}}_\infty(u, \partial E) \geq m\} = E^{(-m)}$. For $F \subset \mathbb{Z}^{r_1}$ we set $F^{\square} := \bigcup_{u \in F}(u + [0,1]^{r_1})$.

Lemma C.1 (Discrete coarea on the base grid)**.** Let $\ell : \mathbb{Z}^{r_1} \to \mathbb{N}$ have finite support. Then for each $v \in S_{\mathrm{hor}}$,

$$\sum_{u \in \mathbb{Z}^{r_1}} \left|\Delta_v \ell(u)\right| \ = \ \int_0^\infty \left|\partial^{(v)}\big(\{\ell > t\}\big)\right| dt,$$

and summing over v yields

$$\sum_{v \in S_{\mathrm{hor}}} \sum_{u \in \mathbb{Z}^{r_1}} \left|\Delta_v \ell(u)\right| \ = \ \int_0^\infty \mathrm{Per}_{\mathrm{grid}}\left(\{\ell > t\}\right) dt.$$

Proof. Fix $v \in S_{\mathrm{hor}}$ and set $K := \max_u \ell(u) < \infty$. For $a, b \in \mathbb{N}$ we have the pointwise *layer-cake identity*

$$|a - b| \ = \ \sum_{k=1}^{K} \left| \mathbf{1}_{\{a \geq k\}} - \mathbf{1}_{\{b \geq k\}} \right|. \tag{C.2}$$

If $a \geq b$, then the summand equals 1 for $b < k \leq a$ and 0 otherwise, so the sum is $a - b$. Applying (C.2) with $a = \ell(u+v)$, $b = \ell(u)$ and summing over u gives

$$\sum_u \left|\Delta_v \ell(u)\right| = \sum_{k=1}^{K} \sum_u \left| \Delta_v \mathbf{1}_{\{\ell \geq k\}}(u) \right| = \sum_{k=1}^{K} \left|\partial^{(v)}\{\ell \geq k\}\right|.$$

Since $\{\ell > t\} = \{\ell \geq k\}$ for $t \in (k-1, k]$, the integral identity follows by integrating the step function $t \mapsto \left|\partial^{(v)}\{\ell > t\}\right|$ over $(0, \infty)$ and summing in k. Sum over v. $\square$

Lemma C.2 (Discrete divergence theorem for finite stencils)**.** Let $U \subset \mathbb{Z}^{r_1}$ be finite and $S_{\mathrm{hor}} \subset \mathbb{Z}^{r_1} \setminus \{0\}$ a fixed finite, oriented stencil (one representative per undirected pair). For each $v \in S_{\mathrm{hor}}$ and each oriented v-edge $(x \to x + v)$ with $x, x + v \in U$, assign a weight $X_v(x, x+v) \in \mathbb{R}$ (set $X_v(x, x+v) := 0$ if the edge is missing). Define

$$\mathrm{div}\, X(u) \ := \ \sum_{v \in S_{\mathrm{hor}}} \Big(X_v(u - v, u) \ - \ X_v(u, u+v)\Big) \qquad (u \in U).$$

Then for every $E \subset U$,

$$\sum_{u \in E} \mathrm{div}\, X(u) \ = \ \sum_{v \in S_{\mathrm{hor}}} \ \sum_{\substack{(x,y) = (u, u+v): \\ x \notin E,\ y \in E}} X_v(x,y) \ - \ \sum_{v \in S_{\mathrm{hor}}} \ \sum_{\substack{(x,y) = (u, u+v): \\ x \in E,\ y \notin E}} X_v(x,y), \tag{C.3}$$

i.e. (inward flux across ∂E) minus (outward flux). The sums on the right run over all ordered pairs $(x,y)=(u,u+v)$ with $u\in U$; missing edges are included via the convention $X_v=0$.

Remark C.3 (Sign convention)**.** If one prefers div to denote net *outflow*, set $\operatorname{div}_{\text{out}} X(u) := \sum_v \big(X_v(u,u+v) - X_v(u-v,u)\big) = -\operatorname{div} X(u)$, and the right-hand side of (C.3) flips sign accordingly.

C.1. **Discrete anisotropic calibration: scope, identity, and limits.** The face-flux construction below is *only* needed and used for the nearest-neighbor axis stencil and orthotropic anisotropy. In that case one has an exact combinatorial identity (no error term).

Lemma C.4 (Axis/orthotropic identity)**.** Assume $S_{\text{hor}}=\{e_i\}_{i=1}^{r_1}$ (nearest-neighbor axis stencil). Let $w_i>0$ and consider the orthotropic anisotropy on $\mathbb{R}^{r_1}$ with dual ball

$$\big\{\zeta\in\mathbb{R}^{r_1} : \tau^\circ(\zeta)\le 1\big\} = \bigcap_{i=1}^{r_1}\{\,|\langle\zeta,e_i\rangle|\le w_i\,\} \quad\Longleftrightarrow\quad \tau(\nu)=\sum_{i=1}^{r_1} w_i\,|\nu_i|.$$

Then for every finite $E\subset\mathbb{Z}^{r_1}$,

$$\operatorname{Per}^{(w)}_{\text{grid}}(E) \;=\; \operatorname{Per}_\tau(E^\square), \qquad E^\square := \bigcup_{u\in E}(u+[0,1]^{r_1}). \tag{C.4}$$

Proof. The boundary $\partial E^\square$ is a union of unit (r_1-1)-faces, each orthogonal to some $\pm e_i$, bijectively corresponding to the e_i-edges crossing ∂E. Each such face has Euclidean area 1 and τ-cost w_i. Therefore $\operatorname{Per}_\tau(E^\square)=\sum_{i=1}^{r_1} w_i\,|\partial^{(e_i)}(E)| = \operatorname{Per}^{(w)}_{\text{grid}}(E)$. □

Remark C.5 (Dual formula and optional calibration)**.** By ℓ^1-ℓ^∞ duality on the axis stencil,

$$\operatorname{Per}^{(w)}_{\text{grid}}(E) \;=\; \sup_{|Y_{e_i}(e)|\le w_i}\ \sum_{u\in\mathbb{Z}^{r_1}} \mathbf{1}_E(u)\ \operatorname{div} Y(u), \tag{C.5}$$

with div as in Lemma C.2. A standard face-flux sampling from a smooth ζ with $\tau^\circ(\zeta)\le 1$ gives $\operatorname{Per}^{(w)}_{\text{grid}}(E)\ge \operatorname{Per}_\tau(E^\square)$; the reverse inequality is the combinatorial identity above, so equality holds.

Remark C.6 (Scope and limitations)**.** The identity Lemma C.4 assumes the axis stencil and orthotropic anisotropy ($\tau(\nu)=\sum_i w_i|\nu_i|$), for which faces of $E^\square$ align with the stencil and the pointwise weights follow directly. For general finite stencils and polyhedral (crystalline) anisotropies, one needs an oblique cell decomposition or a bounded redistribution of flux across axis faces; we do not pursue that here.

C.2. **Vertical counts, tapered profiles, and scalings.**

Lemma C.7 (Vertical edge terms are lower order)**.** Let G be a step-2 Carnot group with horizontal rank $d=r_1$ and m central directions; $Q=d+2m$. Assume the generating set contains the unit steps along each central direction (so "vertical" = central edges), and that each base point $u\in R\subset\mathbb{Z}^d$ carries a *contiguous* column in each central direction. Assume moreover that the central fiber over u is the rectangular block $\prod_{k=1}^m\{0,\dots,\ell_k(u)-1\}$ with heights $\ell_k(u)\in\mathbb{N}$.

(a) Heisenberg case $m=1$. For any column family over $R\subset\mathbb{Z}^d$ with height function $\ell: R\to\mathbb{N}$, the total vertical (central) boundary equals

$$\mathcal{V} \;=\; 2\#\{u\in R:\ \ell(u)\ge 1\} \;\le\; 2|R|.$$

In particular, if every column is nonempty ($\ell(u)\ge 1$ for all u), then $\mathcal{V}=2|R|$. If $\operatorname{diam}(R)\asymp\rho$ and $|R|\asymp\rho^d$, then $\mathcal{V}\asymp\rho^d=o(\rho^{Q-1})$ (since $Q-1=d+1$).

(b) General two-step ($m \geq 2$) under uniform height bounds. Assume $R \subset \mathbb{Z}^d$ has $\operatorname{diam}(R) \asymp \rho$ and $|R| \asymp \rho^d$, and there is $H \asymp \rho^2$ such that $0 \leq \ell_j(u) \leq H$ for all u and $j = 1, \ldots, m$. Then the vertical boundary

$$\mathcal{V}(\boldsymbol{\ell}) \;=\; \sum_{u \in R} \sum_{j=1}^{m} 2\mathbf{1}_{\{\ell_j(u) \geq 1\}} \prod_{k \neq j} \ell_k(u)$$

obeys $\mathcal{V}(\boldsymbol{\ell}) \;\leq 2m|R|H^{m-1} \;\asymp\; \rho^{d+2(m-1)} \;=\; \rho^{Q-2} = o(\rho^{Q-1})$.

Proof. (a) With a central generator c, the fiber over u contributes exactly two c-edges across its top and bottom iff it is nonempty, i.e. iff $\ell(u) \geq 1$, independent of height. Summing in u gives the formula.

(b) For direction c_j, the number of boundary c_j-edges over u equals 2 times the cardinality of the $(m-1)$-dimensional cross-section orthogonal to c_j, i.e. $2\mathbf{1}_{\{\ell_j(u) \geq 1\}} \prod_{k \neq j} \ell_k(u)$ for rectangular fibers. Sum over u, j and use $\prod_{k \neq j} \ell_k(u) \leq H^{m-1}$. □

Two structural facts for convex samplers. We isolate two basic lemmas that will be used repeatedly below.

Lemma C.8 (Discrete ℓ^∞ erosion equals continuum erosion for convex samplers)**.** Let $K \subset \mathbb{R}^d$ be a nonempty convex set and $E := \mathbb{Z}^d \cap K$. For every $m \in \mathbb{N}$,

$$E^{(-m)} \;=\; \mathbb{Z}^d \cap \big(K \ominus mQ_\infty\big).$$

Equivalently, $u \in E^{(-m)}$ iff the whole cube $u + [-m, m]^d$ is contained in K.

Proof. ($\subset$) If $u \in E^{(-m)}$, then $u + k \in E \subset K$ for all $k \in \mathbb{Z}^d \cap [-m, m]^d$. In particular, all the 2^d vertices of $u + [-m, m]^d$ lie in K. By convexity of K, the convex hull of these vertices is $u + [-m, m]^d \subset K$. Hence $u \in \mathbb{Z}^d \cap (K \ominus mQ_\infty)$.

($\supset$) If $u \in \mathbb{Z}^d \cap (K \ominus mQ_\infty)$, then $u + [-m, m]^d \subset K$, so in particular $u + k \in K$ for all $k \in \mathbb{Z}^d \cap [-m, m]^d$. Thus $u + k \in E$ for all such k, i.e. $u \in E^{(-m)}$. □

Lemma C.9 (Fiber monotonicity of grid perimeter under ℓ^∞ erosions)**.** Let $K \subset \mathbb{R}^d$ be convex, $E = \mathbb{Z}^d \cap K$, and $E^{(-m)}$ as above. For each coordinate direction e_i,

$$\big|\partial^{(e_i)}(E^{(-m)})\big| \;\leq\; \big|\partial^{(e_i)}(E)\big| \qquad \text{for all } m \in \mathbb{N}.$$

In particular $\mathrm{Per}_{\mathrm{grid}}(E^{(-m)}) \leq \mathrm{Per}_{\mathrm{grid}}(E)$.

Proof. Fix i and decompose $\mathbb{Z}^d$ into axis-parallel lines $L_y := \{(y, t) : t \in \mathbb{Z}\}$ where $y \in \mathbb{Z}^{d-1}$ collects the coordinates orthogonal to e_i. Since K is convex, $K \cap (\{y\} \times \mathbb{R})$ is an interval (possibly empty). Hence $E \cap L_y = \{(y, t) : a_y \leq t \leq b_y\} \cap \mathbb{Z}$ is a (possibly empty) integer interval; the number of e_i-boundary edges contributed by L_y equals 0 if empty and 2 otherwise. By Lemma C.8, $E^{(-m)} = \mathbb{Z}^d \cap (K \ominus mQ_\infty)$; intersecting with L_y shrinks the interval by m at each end (or annihilates it), so $E^{(-m)} \cap L_y$ is either empty or an integer interval strictly contained in $E \cap L_y$. Therefore the number of e_i-boundary edges along L_y is $\leq$ that for E. Summing over y yields the claim for e_i, and then for $\mathrm{Per}_{\mathrm{grid}}$ by summing in i. □

We next construct a height profile with the correct total mass and a controlled base BV via a rigorous coarea computation.

Lemma C.10 (Uniform inner parallel estimates for convex samplers)**.** *From here through Proposition C.11 we assume the axis stencil* $S_{\mathrm{hor}} = \{e_i\}_{i=1}^d$. Let $W_S \subset \mathbb{R}^d$ be a fixed convex Wulff body, $K_\rho := \rho W_S$, $E_\rho = \mathbb{Z}^d \cap K_\rho$ its grid sampler with respect to S_{hor}, and let $E_\rho^{(-m)}$ denote the discrete inner parallel set in the ℓ^∞ metric. There exist constants $c \in (0, 1)$ and $C_1, C_2, \tilde{c}_\infty > 0$ (depending only on (d, W_S)) such that for all integers m with $0 \leq m \leq c\rho$:

(a) Exact identification of discrete erosions.

$$E_\rho^{(-m)} \;=\; \mathbb{Z}^d \cap \big(K_\rho \ominus mQ_\infty\big).$$

(b) Volume comparison with continuum inner parallels.

$$\Big||E_\rho^{(-m)}| \;-\; |K_\rho \ominus mQ_\infty|\Big| \;\le\; C_1\rho^{d-1}.$$

Consequently,

$$|E_\rho^{(-m)}| \;=\; |K_\rho| \;-\; m\tilde{c}_\infty\rho^{d-1} \;+\; O\big(m^2\rho^{d-2}\big) \;+\; O(\rho^{d-1}),$$

uniformly for $m \le c\rho$, where $\tilde{c}_\infty := \mathrm{Per}_{\ell^\infty}(W_S)$ and the $O(\cdot)$ constants depend only on (d, W_S).
(c) Uniform perimeter control.

$$\mathrm{Per}_{\mathrm{grid}}\big(E_\rho^{(-m)}\big) \;\le\; \mathrm{Per}_{\mathrm{grid}}(E_\rho) \;\le\; C_2\rho^{d-1}.$$

Proof. (a) This is Lemma C.8 with $K = K_\rho$.

(b) Since $(E_\rho^{(-m)})^\square$ is the union of $|E_\rho^{(-m)}|$ unit cubes, $|(E_\rho^{(-m)})^\square| = |E_\rho^{(-m)}|$ (overlaps occur only on $(d-1)$-dimensional faces). Moreover, $K_\rho \ominus mQ_\infty \subset (E_\rho^{(-m)})^\square \subset (K_\rho \ominus mQ_\infty) \oplus [0,1]^d$. Therefore

$$0 \;\le\; |(E_\rho^{(-m)})^\square| \;-\; |K_\rho \ominus mQ_\infty| \;\le\; |(K_\rho \ominus mQ_\infty) \oplus [0,1]^d| \;-\; |K_\rho \ominus mQ_\infty|.$$

For convex bodies, $|A \oplus [0,1]^d| - |A|$ is $O(\mathrm{Per}_{\ell^\infty}(A))$; for $A = K_\rho \ominus mQ_\infty$ with $m \le c\rho$, $\mathrm{Per}_{\ell^\infty}(A) \asymp \rho^{d-1}$ uniformly in m. Hence the right-hand side is $O(\rho^{d-1})$, yielding the displayed bound with some C_1. The polynomial (Steiner/mixed-volume) expansion for $|K_\rho \ominus mQ_\infty|$ has linear term $-\mathrm{Per}_{\ell^\infty}(W_S)\rho^{d-1}m$, giving the consequent expansion with $\tilde{c}_\infty$ as stated.

(c) The first inequality is Lemma C.9. For the second, fix i and decompose $\mathbb{Z}^d$ into lines $L_y := \{(y,t) : t \in \mathbb{Z}\}$ parallel to e_i. Since K_ρ is convex, each $E_\rho \cap L_y$ is an (integer) interval or empty, hence contributes exactly 2 e_i-boundary edges when nonempty and 0 otherwise. Therefore

$$\big|\partial^{(e_i)}(E_\rho)\big| = 2\#\{y \in \mathbb{Z}^{d-1} :\; L_y \cap E_\rho \ne \emptyset\} \;\le\; 2\#\big(\mathbb{Z}^{d-1} \cap (\pi_i(K_\rho) \oplus [0,1]^{d-1})\big) \;\lesssim\; \rho^{d-1},$$

where π_i is projection orthogonal to e_i and the last bound follows from the $(d-1)$-dimensional volume growth of $\pi_i(K_\rho)$ and bounded overlap of unit cubes. Summing in i yields $\mathrm{Per}_{\mathrm{grid}}(E_\rho) \le C_2\rho^{d-1}$ with C_2 depending only on (d, W_S). □

Proposition C.11 (Explicit tapered profile on Wulff samplers (Heisenberg))**.** Let $m = 1$ ($Q = d+2$). For $E_\rho = \rho W_S \cap \mathbb{Z}^d$ (grid sampler of a fixed convex Wulff body W_S), fix constants $c_*, \kappa > 0$ with $\kappa/c_* \le c$ from Lemma C.10, and set $H = \kappa\rho^2$, $L_\rho := \lfloor c_*\rho \rfloor$, $M_\rho := \lfloor H/L_\rho \rfloor \le c\rho$. Define the discrete taper

$$\ell(u) \;:=\; \Big\lfloor \min\big\{H,\; L_\rho\, \mathrm{dist}_\infty^{\mathrm{disc}}\,(u, \partial E_\rho)\big\}\Big\rfloor, \qquad \text{extended by 0 on } \mathbb{Z}^d \setminus E_\rho.$$

Then

$$\sum_{u \in E_\rho} \ell(u) \;\asymp\; \rho^Q, \tag{C.6}$$

and

$$\sum_{v \in S_{\mathrm{hor}}} \sum_{u \in \mathbb{Z}^d} \big|\Delta_v \ell(u)\big| \;\lesssim\; H\,\mathrm{Per}_{\mathrm{grid}}(E_\rho) \;\asymp\; \rho^{Q-1}. \tag{C.7}$$

All implicit constants depend only on (d, W_S) and the choices of c_*, κ.

Proof. Level sets. By the definition of $\mathrm{dist}^{\mathrm{disc}}_\infty$,

$$\{\ell \ge k\} = \{u \in E_\rho :\ \mathrm{dist}^{\mathrm{disc}}_\infty(u, \partial E_\rho) \ge k/L_\rho\} = E_\rho^{(-\lceil k/L_\rho \rceil)}.$$

Volume. Since $L_\rho M_\rho \le H < L_\rho(M_\rho + 1)$,

$$\ell(u) \ \ge\ \lfloor L_\rho M_\rho \rfloor \ \ge\ H - L_\rho \qquad \text{for every } u \in E_\rho^{(-M_\rho)}.$$

Therefore

$$(H - L_\rho)\big|E_\rho^{(-M_\rho)}\big| \ \le\ \sum_{u \in E_\rho} \ell(u) \ \le\ H|E_\rho|.$$

By Lemma C.10(a,b), for $M_\rho \le c\rho$ we have $|E_\rho|,\ |E_\rho^{(-M_\rho)}| \asymp \rho^d$. Since $H = \kappa\rho^2$ and $L_\rho \asymp \rho$, the lower bound is $\gtrsim \rho^{d+2}$ and the upper bound is $\lesssim \rho^{d+2}$, proving (C.6).

Base variation via discrete coarea. By Lemma C.1,

$$\begin{aligned}\sum_v \sum_u |\Delta_v \ell(u)| &= \sum_{k=1}^{H} \mathrm{Per}_{\mathrm{grid}}\left(\{\ell \ge k\}\right) = \sum_{k=1}^{H} \mathrm{Per}_{\mathrm{grid}}\left(E_\rho^{(-\lceil k/L_\rho \rceil)}\right) \\ &= L_\rho \sum_{m=1}^{M_\rho} \mathrm{Per}_{\mathrm{grid}}\left(E_\rho^{(-m)}\right) \ +\ O\big(L_\rho\, \mathrm{Per}_{\mathrm{grid}}(E_\rho)\big).\end{aligned}$$

By Lemmas C.10(c) and C.9, $\mathrm{Per}_{\mathrm{grid}}(E_\rho^{(-m)}) \le \mathrm{Per}_{\mathrm{grid}}(E_\rho) \lesssim \rho^{d-1}$ uniformly in $m \le M_\rho$, hence

$$\sum_v \sum_u |\Delta_v \ell(u)| \ \le\ L_\rho M_\rho\, \mathrm{Per}_{\mathrm{grid}}(E_\rho) \ +\ O\big(L_\rho\, \mathrm{Per}_{\mathrm{grid}}(E_\rho)\big).$$

Since $L_\rho M_\rho \in [H - L_\rho, H]$ and $L_\rho \lesssim \rho \ll H \asymp \rho^2$, we obtain

$$\sum_v \sum_u |\Delta_v \ell(u)| \ \le\ CH\, \mathrm{Per}_{\mathrm{grid}}(E_\rho) \ \asymp\ \rho^{Q-1},$$

as claimed in (C.7). $\square$

Remark C.12 (Scaling summary)**.** In the Heisenberg case ($m = 1$, $Q = d + 2$), for tapered stacks above a Wulff sampler,

$$\mathcal{V} \ \asymp\ \rho^d, \quad \sum_v \sum_u |\Delta_v \ell(u)| \ \lesssim\ \rho^{Q-1},$$
$$\text{(shear term, arising in the semidirect-product stacks of §9.8)} \ \lesssim\ \rho^{Q-1}.$$

Thus the ρ^{Q-1} scale is dictated by the (aggregated) base BV (via coarea) and by shear; vertical terms are strictly lower order. For $m \ge 2$, under the hypotheses of Lemma C.7(b), one has $\mathcal{V}(\boldsymbol{\ell}) = O(\rho^{Q-2})$.

C.3. Quantitative anisotropic isoperimetry on the base.

Definition C.13 (Fraenkel asymmetry)**.** Let $W \subset \mathbb{R}^{r_1}$ be the Wulff shape of a convex, even anisotropy τ. For measurable E with $|E| = |W|$, define

$$\mathcal{A}_\tau(E) \ :=\ \inf_{z \in \mathbb{R}^{r_1}} \frac{|E \Delta (z + W)|}{|W|}.$$

Theorem C.14 (Quantitative anisotropic isoperimetry in $\mathbb{R}^{r_1}$)**.** There exists $\kappa = \kappa(\tau) > 0$ such that for all measurable $E \subset \mathbb{R}^{r_1}$ with $|E| = |W|$,

$$\mathrm{Per}_\tau(E) \ -\ \mathrm{Per}_\tau(W) \ \ge\ \kappa \mathcal{A}_\tau(E)^2.$$

Remark C.15 (Scope)**.** We use quantitative anisotropic stability *only on the horizontal Euclidean base* $\mathbb{R}^{r_1}$. No Carnot-group quantitative isoperimetry is required; the BV+shear reduction pushes all quantitative inputs to the base. For standard references on perimeter monotonicity for inner parallel sets and quantitative anisotropic stability, see e.g. [Sch14, Mag12, Chs.5 & 14].

Appendix D. A quantitative Lindenstrauss theorem along thick Følner shapes, and a qualitative variant in general

We recall the (Shulman-)temperedness condition for Følner sequences.

Definition D.1 (Tempered subsequence)**.** Let G be a countable group and (F_n) a sequence of finite subsets. A subsequence (E_k) is *tempered* if there exists $T < \infty$ such that

$$\big|\big(\textstyle\bigcup_{j<k} E_j^{-1}\big)E_k\big| \;\le\; T|E_k| \qquad \text{for all } k. \tag{D.1}$$

We also record a geometric constant of the Cayley graph.

Definition D.2 (Ball doubling constant at factor 2)**.** For a Cayley graph $(\Gamma, \mathcal{S})$ with word metric and balls $B(e, R)$, set

$$D_2(\Gamma, \mathcal{S}) \;:=\; \sup_{R\ge 1} \frac{|B(e, 2R)|}{|B(e, R)|} \;\in\; [1, \infty].$$

If $(\Gamma, \mathcal{S})$ has polynomial growth of homogeneous dimension Q, then $D_2(\Gamma, \mathcal{S}) < \infty$ and $D_2(\Gamma, \mathcal{S}) \le C(\Gamma, \mathcal{S})2^Q$.

Theorem D.3 (Quantitative tempered subsequence under ball-thickness)**.** Let $(\Gamma, \mathcal{S})$ be a finitely generated group with Cayley metric. Let $(F_n)_{n\ge1}$ be a *nested* Følner chain and write $R_n := \max\{d(e, g) : g \in F_n\}$. Assume there exists a constant $c_{\text{fill}} \in (0, 1]$ such that

$$c_{\text{fill}}\,|B(e, R_n)| \;\le\; |F_n| \qquad \text{for all } n \text{ large enough.} \tag{D.2}$$

Let $E_k := F_{n_k}$ be any dyadic-radius subsequence with $R_{n_k} \in [2^k, 2^{k+1})$. Then (E_k) is tempered in the sense of (D.1) with the *explicit* constant

$$T \;\le\; \frac{D_2(\Gamma, \mathcal{S})}{c_{\text{fill}}}. \tag{D.3}$$

Consequently, for every probability-preserving action $\Gamma \curvearrowright (X, \mu)$ and every $f \in L^1(X)$ the ergodic averages

$$A_{E_k} f(x) \;:=\; \frac{1}{|E_k|}\sum_{g\in E_k} f(g\cdot x)$$

converge μ-almost surely to $\mathbb{E}[f|\mathcal{I}_\Gamma]$, and the maximal operator $Mf := \sup_k |A_{E_k} f|$ satisfies the weak-type $(1, 1)$ bound

$$\mu\{Mf > \lambda\} \;\le\; C_{\max}(T)\frac{\|f\|_{L^1}}{\lambda} \qquad (\lambda > 0),$$

where $C_{\max}(T)$ depends only on T. In particular, in polynomial growth $D_2(\Gamma, \mathcal{S}) \le C(\Gamma, \mathcal{S})2^Q$.

Proof. Since $F_{n_j} \subset F_{n_k}$ for $j < k$, we have $\big(\bigcup_{j<k} F_{n_j}^{-1}\big)F_{n_k} \subset F_{n_k}^{-1}F_{n_k} \subset B(e, 2R_{n_k})$. Therefore

$$\Big|\Big(\bigcup_{j<k} F_{n_j}^{-1}\Big)F_{n_k}\Big| \;\le\; |B(e, 2R_{n_k})| \;\le\; D_2(\Gamma, \mathcal{S})|B(e, R_{n_k})| \;\le\; \frac{D_2(\Gamma, \mathcal{S})}{c_{\text{fill}}}|F_{n_k}|,$$

by (D.2). This yields (D.3); the L^1 maximal inequality and a.e. convergence are Lindenstrauss' theorem for tempered sequences. □

Remark D.4 (Application to virtually nilpotent groups)**.** In virtually nilpotent groups, our Wulff samplers W_R (indexed by intrinsic radius R) satisfy two-sided metric control: there exist $a \in (0, 1]$, $b \in [1, \infty)$ and R_0 such that for all $R \ge R_0$,

$$B(e, aR) \;\subset\; W_R \;\subset\; B(e, bR).$$

Taking $F_n = W_{R_n}$, the lower thickness (D.2) holds with

$$c_{\text{fill}} \;=\; \inf_{R\ge R_0} \frac{|B(e, aR)|}{|B(e, R)|} \;=\; \frac{1}{\sup_{R\ge R_0} \frac{|B(e,R)|}{|B(e,aR)|}} \;\ge\; \frac{1}{D_{a^{-1}}(\Gamma, \mathcal{S})},$$

where $D_\lambda(\Gamma, \mathcal{S}) := \sup_{R\geq 1} |B(e, \lambda R)|/|B(e, R)|$ for $\lambda \geq 1$. Plugging into (D.3) gives the quantitative bound

$$T \;\leq\; \frac{D_2(\Gamma, \mathcal{S})}{c_{\text{fill}}} \;\leq\; D_2(\Gamma, \mathcal{S}) D_{a^{-1}}(\Gamma, \mathcal{S}),$$

an explicit constant depending only on $(\Gamma, \mathcal{S})$ and the Wulff thickness a. In particular, if $|B(e, R)| \asymp R^Q$, then $D_\lambda \leq C\lambda^Q$ and hence $T \leq C'(\Gamma, \mathcal{S})(2/a)^Q$.

Example D.5 ($\mathbb{Z}^d$ with cubes: exact constant)**.** Let $\Gamma = \mathbb{Z}^d$ with the standard axis generating set. For cubes $C_R := [-R, R]^d$ one has $C_R^{-1} C_R = [-2R, 2R]^d$ and, for the dyadic subsequence $R_k = 2^k$,

$$\Big|\Big(\bigcup_{j<k} C_{R_j}^{-1}\Big) C_{R_k}\Big| \;\leq\; |C_{R_k}^{-1} C_{R_k}| \;=\; (4R_k + 1)^d, \qquad |C_{R_k}| = (2R_k + 1)^d.$$

Thus

$$T \;\leq\; \sup_{R\geq 0} \Big(\frac{4R+1}{2R+1}\Big)^d \;=\; 2^d,$$

so the cube averages along $R_k = 2^k$ are tempered with the *exact* constant $T = 2^d$, yielding a weak-$(1, 1)$ maximal inequality with constant $C_{\max}(2^d)$ and a.e. convergence for every action.

We now record the general qualitative variant, which requires no thickness and applies in all amenable groups (including the exponential-growth classes treated elsewhere in this paper).

Theorem D.6 (Qualitative tempered subsequence (Shulman–Lindenstrauss))**.** Let (F_n) be any Følner sequence in a countable amenable group Γ. There exists a tempered subsequence (E_k) in the sense of (D.1) (with a universal finite constant T_0) such that for every p.m.p. action $\Gamma \curvearrowright (X, \mu)$ and every $f \in L^1(X)$,

$$A_{E_k} f(x) \;\longrightarrow\; \mathbb{E}[f|\mathcal{I}_\Gamma](x) \qquad \text{for } \mu\text{-a.e. } x,$$

and the associated maximal operator is of weak-type $(1, 1)$ with a constant depending only on T_0.

Remark D.7 (Hyperbolic semidirects)**.** For $\Gamma = \mathbb{Z}^d \rtimes_A \mathbb{Z}$ with A hyperbolic, our logarithmic layer-nested sets E_R form a nested Følner family of §9.8. They are *thin* in balls (polynomial size inside exponentially growing balls), so (D.2) does not hold and one cannot hope for a group-geometric bound like (D.3). Nevertheless, by Theorem D.6 there exists a *tempered subsequence* of the stacks (or of any nested thinning thereof), which yields the L^1 pointwise ergodic theorem and weak-$(1, 1)$ maximal inequality along that subsequence (with a universal temperedness constant).

Two applications (with constants where available).

(A) **Virtually nilpotent groups.** Take the Wulff samplers as in Remark D.4 and dyadically thin them by radius. By Theorem D.3, the resulting sequence is tempered with

$$T \;\leq\; D_2(\Gamma, \mathcal{S}) D_{a^{-1}}(\Gamma, \mathcal{S}) \;\leq\; C(\Gamma, \mathcal{S})(2/a)^Q.$$

Therefore the Wulff-averaging maximal operator is weak-$(1, 1)$ with constant $C_{\max}(T)$ depending only on $(\Gamma, \mathcal{S})$; a fortiori, L^p bounds for $p > 1$ follow with tracked constants. On finite quotients, the same constants apply uniformly.

(B) $\mathbb{Z}^d$ **with cubes.** By Example D.5, the dyadic cubes are tempered with the *exact* constant $T = 2^d$, hence the weak-$(1, 1)$ maximal constant is $C_{\max}(2^d)$. This gives a clean benchmark for quantitative comparisons with other averaging shapes (e.g. Wulff or ℓ^1 balls).

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