POINT EVALUATION FOR POLYNOMIALS ON THE CIRCLE

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Abstract. We study the constant $\mathscr{C}_{d,p}$ defined as the smallest constant C such that $\|P\|_\infty^p \le C\|P\|_p^p$ holds for every polynomial P of degree d, where we consider the L^p norm on the unit circle. We conjecture that $\mathscr{C}_{d,p} \le dp/2+1$ for all $p \ge 2$ and all degrees d. We show that the conjecture holds for all $p \ge 2$ when $d \le 4$ and for all d when $p \ge 6.8$.

1. introduction

Let $\mathbb{T}$ denote the unit circle and let $L^p(\mathbb{T})$ be the set of all functions on $\mathbb{T}$ for which the L^p norm

$$\|f\|_p = \left(\int_{-\pi}^{\pi} |f(e^{i\theta})|^p \frac{d\theta}{2\pi}\right)^{\frac{1}{p}}$$

is finite. In this paper we consider the norm of point evaluation at $z = 1$ for polynomials of degree at most d on the unit circle under the L^p norm. More precisely we study the constant $\mathscr{C}_{d,p}$ defined as the smallest constant C such that

$$\|P\|_\infty^p \le C\|P\|_p^p,$$

for any polynomial of degree at most d. This is a classical problem related to Nikolskii-type inequalities. The case $p = 2$ is particularly favourable. It follows from the Cauchy–Schwarz inequality and Parseval's identity that $\mathscr{C}_{d,2} = d+1$. The classical power trick, see e.g. Timan [5, page 229], tells us that for any integer n it holds that $\mathscr{C}_{d,np} \le \mathscr{C}_{nd,p}$, and consequently it follows from Hölder's inequality that $\mathscr{C}_{d,p} \le d\lceil p/2\rceil + 1$. We propose the following stronger conjecture.

Conjecture. If $d \ge 1$ and $p \ge 2$, then $\mathscr{C}_{d,p} \le dp/2+1$.

Again by applying the power trick, this conjecture holds for all $p \ge 2$ if verified for $2 \le p \le 4$ and all degrees d. We also conjecture that $\mathscr{C}_{d,p}/(dp/2+1)$ is decreasing in p. Our main result reads as follows.

Theorem 1. *If $1 \le d \le 4$ and $2 \le p \le 4$, then*

$$\mathscr{C}_{d,p} \le \frac{dp}{2} + 1.$$

In fact we improve the above bound slightly, as stated in Theorem 14 and illustrated in Figure 1. The problem of determining $\mathscr{C}_{d,p}$ can be equivalently formulated as the extremal problem

$$\frac{1}{\mathscr{C}_{d,p}} = \inf_{P \in \mathscr{P}_d} \{\|P\|_p^p : P(1) = 1\}, \tag{1}$$

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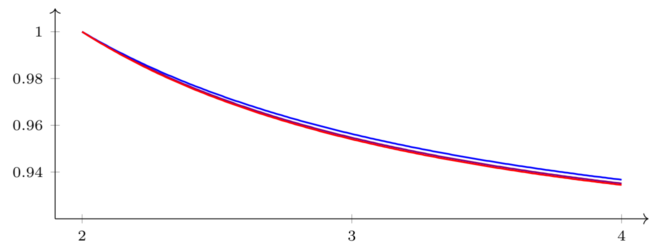

FIGURE 1. The upper bound for $\mathscr{C}_{d,p}/(dp/2+1)$ for $d=2$, $d=3$ and $d=4$ from Theorem 14.

where $\mathscr{P}_d$ consists of all polynomials of degree at most d. For $1<p<\infty$, the extremal problem (1) has a unique solution $\varphi_{d,p}$. The extremal function $\varphi_{d,p}$ is of degree d and has d zeroes on the unit circle $\mathbb{T}$ and $P(e^{i\theta})=\overline{P(e^{-i\theta})}$. Brevig, Chirre, Ortega-Cerdà and Seip [1] have studied the related problem of determining $\mathscr{C}_p$ defined as the smallest constant C such that $|f(0)|^p \le C\|f\|_p^p$ for all functions in the Paley–Wiener space PW^p. Levin and Lubinsky [4] have showed that these two problems are connected via the limit

$$\lim_{d\to\infty}\frac{\mathscr{C}_{d,p}}{d}=\mathscr{C}_p.$$

Our conjecture implies the result $\mathscr{C}_p \le p/2$ from [1]. Similarly to what was done in [1] we provide an extended power trick (see Theorem 8) and use information on the zeroes of the extremal function in order to prove Theorem 1. The obstacle to further progress is the limited information on the zeroes. In [1] information about the zeroes is obtained by considering prolate spheroidal wave functions.

For fixed values d and p we can obtain a good lower bound for $\mathscr{C}_{d,p}$ by numerically optimizing over the zeroes of the polynomial. The lower bound obtained numerically for $d=4$ and $2\le p\le 4$ is seen in Figure 2, together with the corresponding upper bound. In Section 7 we also show that a certain specific polynomial provides a lower bound very close to that obtained by numerical optimization.

We also consider upper bounds for $\mathscr{C}_{d,p}$ for larger values of p and prove the following result. In this paper $B(x,y)$ denotes the beta function.

Theorem 2. *If $d\ge 1$ and $p\ge 1$, then*

$$\mathscr{C}_{d,p}\le\frac{d\pi}{B((p+1)/2,1/2)}.$$

Due to well-known asymptotics for the beta function it follows that

$$\mathscr{C}_{d,p}\le d\left(\sqrt{\frac{\pi p}{2}}+O\left(\frac{1}{\sqrt{p}}\right)\right),$$

as $p\to\infty$. In particular if $p\ge 6.8$ then the conjectured upper bound $\mathscr{C}_{d,p}\le dp/2+1$ holds for all degrees d.

This paper is organized as follows. We start with some preliminary results in Section 2. Next, in Section 3 we prove a generalized power trick, which provides the foundations for proving Theorem 1. In Section 4 we give two results on the zeroes of the extremal functions in (1). Section 5 contains the proof of Theorem 1. In Section 6 we prove Theorem 2. Finally in Section 7 we provide some numerical estimates for the zeroes of $\varphi_{d,p}$ and corresponding lower bounds for $\mathscr{C}_{d,p}$.

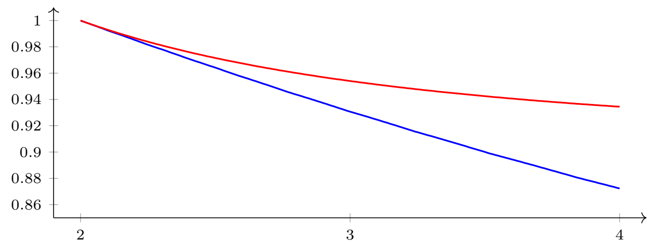

FIGURE 2. The lower and upper bound for $\mathscr{C}_{4,p}/(2p+1)$. The lower bound is obtained by numerical optimization, while the upper bound is from Theorem 14.

2. Preliminaries

We start with some preliminary results on the existence and structure of the solution of (1).

Lemma 3. *Let $1 < p < \infty$. Then there exists a unique solution of (1).*

Proof. Let $(P_n)_n$ be a sequence in $\mathscr{P}_d$ such that $P_n(1) = 1$ and $(\|P_n\|_p^p)_n$ converges to $1/\mathscr{C}_{d,p}$. Since the space $\mathscr{P}_d$ is a finite dimensional vector space there exists a subsequence of polynomials $(P_{n_k})_{n_k}$ that converges to a polynomial P in $\mathscr{P}_d$ such that $\|P\|_p^p = 1/\mathscr{C}_{d,p}$, and so a solution exists. Uniqueness follows from strict convexity of $L^p(\mathbb{T})$. $\square$

As the extremal function $\varphi_{d,p}$ is unique, we can also deduce its structure.

Lemma 4. *Let $1 < p < \infty$. Let $d \geq 1$, and let $k = \lfloor d/2 \rfloor$. Then the unique solution of (1) is of the form*

$$\varphi_{d,p}(z) = C \prod_{j=1}^{k} (z - e^{it_j})(z - e^{-it_j}),$$

for d even and

$$\varphi_{d,p}(z) = C(z+1) \prod_{j=1}^{k} (z - e^{it_j})(z - e^{-it_j}),$$

for d odd, where $0 \leq t_1 \leq t_2 \leq \cdots \leq t_k \leq \pi$ and C is a normalization constant ensuring that $P(1) = 1$.

Proof. Assume P is a solution of (1) of degree $n < d$ and let $Q(z) = zP(z)$ be a polynomial of degree $\leq d$. Then $\|Q\|_p = \|P\|_p$ and $Q(1) = P(1) = 1$. However, since the extremal function is unique this is not possible, so the extremal function must have degree d.

Next, let $P(z) = a_0 + a_1 z + \ldots a_d z^d$ be a solution of (1). If $a_0 = 0$ then the polynomial $P(z)/z$ is also a solution of (1), and so we must have $a_0 \neq 0$. Define the polynomial Q by $Q(z) = z^d \overline{P(\overline{z})} = \overline{a_d} + \overline{a_{d-1}} z + \cdots + \overline{a_0} z^d$. Then Q is of degree d with $Q(1) = P(1) = 1$ and $\|Q\|_p = \|P\|_p$. Since the extremal function is unique it follows that $Q = P$. In particular if re^{it} is a zero of P then so is $(1/r)e^{it}$. We now show that the zeroes of the extremal function must all be on the unit circle.

Assume re^{it} is a zero of the extremal function P and that $r \neq 1$, so that $(1/r)e^{it}$ also is a zero of P. Pick $\varepsilon > 0$ so small that

$$|(e^{i\theta} - re^{it})(e^{i\theta} - (1/r)e^{it})| > |(e^{i\theta} - re^{it})(e^{i\theta} - (1/r)e^{it}) - \varepsilon(e^{i\theta} - 1)^2| > 0,$$

for any $0 < \theta < 2\pi$. We then define the polynomial

$$R(z) = \frac{(z - re^{it})(z - (1/r)e^{it}) - \varepsilon(z-1)^2}{(z - re^{it})(z - (1/r)e^{it})} P(z),$$

of degree d. Then $|R(z)| < |P(z)|$ for all $z \neq 1$ on the unit circle and $\|R\|_p < \|P\|_p$. It follows that P cannot be an extremal function. This means that all zeroes of the extremal function must have absolute value 1. Finally by considering the polynomial $S(z) = \overline{P(\overline{z})}$ it follows that if P is extremal then so is S, and hence $P = S$. In particular $P(e^{it}) = 0$ implies $P(e^{-it}) = 0$, and the claimed structure follows. □

Note that when d is even, then by a shift the problem (1) corresponds to considering real trigonometric polynomials. Given the structure from Lemma 4, the case $d = 1$ is easily solved. We see that $\varphi_{1,p}(z) = (1+z)/2$, and calculating the norm it follows that $\mathscr{C}_{1,p} = \pi/(B((p+1)/2, 1/2)$. Next we consider the classical power trick.

Lemma 5. *Let $p = nq$ for some $p, q \geq 1$ and an integer $n \geq 1$. Moreover, let $d \geq 1$ be an integer. It then follows that*

$$\mathscr{C}_{d,p} \leq \mathscr{C}_{nd,q}.$$

Proof. Let f be a polynomial of degree d and let g be a polynomial of degree nd defined by $g(z) = (f(z))^n$. Then we have

$$|f(1)|^p = |g(1)|^q \leq \mathscr{C}_{nd,q} \|g\|_{L^q}^q = \mathscr{C}_{nd,q} \|f\|_{L^p}^p,$$

which yields that $\mathscr{C}_{d,p} \leq \mathscr{C}_{nd,q}$. □

We observe from this that if we can show $\mathscr{C}_{d,p} \leq dp/2 + 1$ for $2 \leq p < 4$ and all $d \geq 2$, then this bound can be extended to all $p \geq 2$ by Lemma 5. Since this is out of reach for now we instead observe the following.

Lemma 6. *If $2 < p < \infty$ and $d \geq 1$ then $\mathscr{C}_{d,p} \leq d\lceil p/2 \rceil + 1$.*

Proof. Fix an integer d and let n be such that $2n < p \leq 2(n+1)$, and thus $\lceil p/2 \rceil = n+1$. By Lemma 5 it follows that $\mathscr{C}_{d,p} \leq (n+1)d + 1$. Then using Hölder's inequality it follows that

$$\|P\|_\infty^{2(n+1)} \leq ((n+1)d+1)\|P\|_{2(n+1)}^{2(n+1)} \leq ((n+1)d+1)\|P\|_p^p \|P\|_\infty^{2(n+1)-p}.$$

That is $\|P\|_\infty^p \leq ((n+1)d+1)\|P\|_p^p$, which concludes the proof. □

3. Extending the power trick

In this section we develop the strategy for finding an upper bound for $\mathscr{C}_{d,p}$ by formulating a related extremal problem. The techniques used here are similar to those used in [1, Theorem 2.4 and Corollary 6.1] and [3]. Given an integer $d \geq 2$ we let $k = \lfloor d/2 \rfloor$ and T_d be the set of non-decreasing sequences with $k+2$ elements such that $\tau_0 = 0$ and $\tau_{k+1} = \pi$. That is, T_d is the set of sequences of potential arguments of the zeroes of the extremal function $\varphi_{d,p}$, extended with the element

$\tau_0 = 0$, as well as $\tau_{k+1} = \pi$ if d is even. Given an integer $d \geq 2$ and a sequence τ in T_d we define the function

$$K_{d,p}(\tau;\theta) = \sum_{n=0}^{k} \chi_{(\tau_n,\tau_{n+1})}(\theta) \frac{\sin((\frac{dp}{4}+\frac{1}{2})\theta - \frac{p}{2}\pi n)}{\sin(\frac{\theta}{2})},$$

and let $M_{d,p}(\theta) = \max_{\tau \in T_d}(K_{d,p}(\tau;\theta), 0)$. Then similarly as in [1, Corollary 6.1] we show the following.

Theorem 7. *Let $2 \leq p < \infty$, and let $d \geq 2$ be an integer. Then*

$$\mathscr{C}_{d,p} \leq \frac{1}{\pi} \int_0^{\pi} M_{d,p}^2(\theta)\, d\theta.$$

Note that given information on the zeroes of the extremal function $\varphi_{d,p}$ we can replace the set of sequences T_d with a smaller subset, and consequently improve the upper bound for $\mathscr{C}_{d,p}$. In order to prove Theorem 7 we need the following result, which can be viewed as an analogue of [1, Theorem 2.4].

Theorem 8. *Let P be a polynomial of degree d with d zeroes on the unit circle. Let $k = \lfloor d/2 \rfloor$. Assume $P(e^{i\theta}) = \overline{P(e^{-i\theta})}$. Denote the zeroes of P by e^{it_1}, $e^{it_2}, \ldots, e^{it_k}$ and e^{-it_1}, $e^{-it_2}, \ldots e^{-it_k}$, where $0 < t_1 \leq t_2 \leq \ldots, \leq t_k \leq \pi$. If d is odd then $e^{-it_{k+1}} = -1$ is also a zero and $t_{k+1} = \pi$. For notation purposes we let $t_0 = 0$, and also for d even we let $t_{k+1} = \pi$. Let $0 < q < \infty$. Then*

$$|P(1)|^q = \sum_{n=0}^{k} \int_{t_n}^{t_{n+1}} |P(e^{i\theta})|^q \frac{\sin((\frac{d}{2}q + \frac{1}{2})\theta - \pi q n)}{\sin\frac{\theta}{2}} \frac{d\theta}{\pi}.$$

Proof. By rescaling we assume $P(1) = 1$. First, let d be even. Then

$$P(z) = C \prod_{j=1}^{k} (z - e^{it_j})(z - e^{-it_j}),$$

where $C > 0$ is a constant such that $P(1) = 1$. We fix a number $q > 0$ and define $f_q(z)$ on the split plane $\mathbb{C} \setminus \{e^{ix} : t_1/2 \leq x \leq 2\pi - t_1/2\}$ as the function

$$f_q(z) = C \prod_{j=1}^{k} ((z - e^{it_j})(z - e^{-it_j}))^q.$$

Fix a small number $\varepsilon > 0$, and define Γ_ε to be the curve oriented counterclockwise and connecting the points

$$(1+\varepsilon)e^{it_1/2}, \quad (1-\varepsilon)e^{it_1/2}, \quad (1-\varepsilon)e^{-it_1/2} \quad \text{and} \quad (1+\varepsilon)e^{-it_1/2},$$

as illustrated in Figure 3. From Cauchy's integral formula it follows that

$$\begin{aligned}
\frac{1}{2\pi i} \int_{\Gamma_\varepsilon} \frac{f_q(z)}{z-1}\, dz &= \frac{1}{2\pi} \int_{t_1/2}^{2\pi - t_1/2} \frac{f_q((1-\varepsilon)e^{i\theta})}{(1-\varepsilon)e^{i\theta} - 1} (1-\varepsilon)e^{i\theta}\, d\theta \\
&+ \frac{1}{2\pi} \int_{-t_1/2}^{t_1/2} \frac{f_q((1+\varepsilon)e^{i\theta})}{(1+\varepsilon)e^{i\theta} - 1} (1+\varepsilon)e^{i\theta}\, d\theta \\
&+ \frac{1}{2\pi i} \int_{-\varepsilon}^{\varepsilon} \frac{f_q((1+\theta)e^{-it_1/2})}{(1+\theta)e^{-it_1/2} - 1} e^{-it_1/2} - \frac{f_q((1+\theta)e^{it_1/2})}{(1+\theta)e^{it_1/2} - 1} e^{it_1/2}\, d\theta \\
&= |f_q(1)| = |P(1)|^q.
\end{aligned}$$

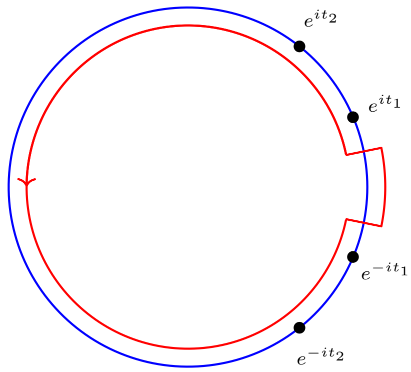

FIGURE 3. The curve Γ_ε in the proof of Theorem 8 and the unit circle. The d zeroes of the polynomial are on the unit circle.

Thus by a substitution and using the symmetry of the zeroes it follows that

$$\begin{aligned}&\frac{1}{\pi}\int_0^{t_1/2}\frac{f_q((1+\varepsilon)e^{i\theta})}{(1+\varepsilon)e^{i\theta}-1}(1+\varepsilon)e^{i\theta}+\frac{f_q((1+\varepsilon)e^{-i\theta})}{(1+\varepsilon)e^{-i\theta}-1}(1+\varepsilon)e^{-i\theta}\,d\theta\\ &+\frac{1}{\pi}\int_{t_1/2}^{\pi}\frac{f_q((1-\varepsilon)e^{i\theta})}{(1-\varepsilon)e^{i\theta}-1}(1-\varepsilon)e^{i\theta}+\frac{f_q((1-\varepsilon)e^{-i\theta})}{(1-\varepsilon)e^{-i\theta}-1}(1-\varepsilon)e^{-i\theta}\,d\theta\\ &+\frac{1}{2\pi i}\int_{-\varepsilon}^{\varepsilon}\frac{f_q((1+\theta)e^{-it_1/2})}{(1+\theta)e^{-it_1/2}-1}e^{-it_1/2}-\frac{f_q((1+\theta)e^{it_1/2})}{(1+\theta)e^{it_1/2}-1}e^{it_1/2}\,d\theta\\ &=|f_q(1)|.\end{aligned}\tag{2}$$

We can discard the last integral since

$$\lim_{\varepsilon\to 0}\frac{1}{2\pi i}\int_{-\varepsilon}^{\varepsilon}\frac{f_q((1+\theta)e^{-it_1/2})}{(1+\theta)e^{-it_1/2}-1}e^{-it_1/2}-\frac{f_q((1+\theta)e^{it_1/2})}{(1+\theta)e^{it_1/2}-1}e^{it_1/2}\,d\theta=0.$$

We now define $f_{q,\pm}(e^{\pm i\theta})=\lim_{\varepsilon\to 0}f_q((1-\varepsilon)e^{\pm i\theta})$ for $t_1/2\le\theta\le\pi$. We observe that

$$(e^{i\theta}-e^{it_n})(e^{i\theta}-e^{-it_n})=2e^{i\theta}(\cos\theta-\cos t_n).$$

Using this observation we see that for $t_n<\theta<t_{n+1}$ we have

$$f_{q,\pm}(e^{\pm i\theta})=|P(e^{i\theta})|^q e^{\pm i\theta qd/2}e^{\mp i\pi qn}.\tag{3}$$

Combining (3) with the observation that

$$\frac{e^{i\theta}}{e^{i\theta}-1}=\frac{e^{i\theta}/2}{2i\sin\frac{\theta}{2}},$$

it follows that as ε tends to 0 we have

$$\begin{aligned}&\frac{1}{\pi}\int_{t_n}^{t_{n+1}}\frac{f_q((1-\varepsilon)e^{i\theta})}{(1-\varepsilon)e^{i\theta}-1}(1-\varepsilon)e^{i\theta}+\frac{f_q((1-\varepsilon)e^{-i\theta})}{(1-\varepsilon)e^{-i\theta}-1}(1-\varepsilon)e^{-i\theta}\,d\theta\\ &\to\frac{1}{\pi}\int_{t_n}^{t_{n+1}}\frac{|P(e^{i\theta})|^q e^{i\theta qd/2}e^{i\theta/2}e^{-i\pi qn}}{2i\sin\frac{\theta}{2}}-\frac{|P(e^{i\theta})|^q e^{-i\theta qd/2}e^{-i\theta/2}e^{i\pi qn}}{2i\sin\frac{\theta}{2}}\,d\theta\\ &=\frac{1}{\pi}\int_{t_n}^{t_{n+1}}|P(e^{i\theta})|^q\frac{\sin\frac{dq+1}{2}\theta-\pi qn}{\sin\frac{\theta}{2}}\,d\theta.\end{aligned}\tag{4}$$

for $n \geq 1$. Finally, for the case $n = 0$ we see that (4) is valid with $t_1/2$ and t_1 as the lower and upper bounds of integration. For $0 < \theta < t_1/2$ the function $f_q(e^{\pm i\theta})$ is defined and $f_q(e^{\pm i\theta}) = |P(e^{i\theta})|^q e^{\pm i\theta q d/2}$. In particular, when ε tends to 0 we have

$$\begin{aligned}
&\frac{1}{\pi}\int_0^{t_1/2} \frac{f_q((1+\varepsilon)e^{i\theta})}{(1+\varepsilon)e^{i\theta}-1}(1+\varepsilon)e^{i\theta} + \frac{f_q((1+\varepsilon)e^{-i\theta})}{(1+\varepsilon)e^{-i\theta}-1}(1+\varepsilon)e^{-i\theta}\,d\theta \\
&+\frac{1}{\pi}\int_{t_1/2}^{t_1} \frac{f_q((1-\varepsilon)e^{i\theta})}{(1-\varepsilon)e^{i\theta}-1}(1-\varepsilon)e^{i\theta} + \frac{f_q((1-\varepsilon)e^{-i\theta})}{(1-\varepsilon)e^{-i\theta}-1}(1-\varepsilon)e^{-i\theta}\,d\theta \qquad (5)\\
&\to\frac{1}{\pi}\int_0^{t_1} |P(e^{i\theta})|^q \frac{\sin\frac{dq+1}{2}\theta}{\sin\frac{\theta}{2}}\,d\theta.
\end{aligned}$$

Finally combining (2), (4) and (5) concludes the proof for d even.

Next assume d is odd. Then

$$P(z) = C(z+1)\prod_{j=1}^{k}(z-e^{it_j})(z-e^{-it_j}).$$

We observe that $1 + e^{ix} = \sqrt{2(1+\cos x)}e^{ix/2}$. Thus in this case we define

$$f_q(z) = C(z+1)^q\prod_{j=1}^{k}((z-e^{it_j})(z-e^{-it_j}))^q,$$

and $f_{q,\pm}(e^{\pm i\theta}) = \lim_{\varepsilon\to 0} f_q((1-\varepsilon)e^{\pm i\theta})$ for $t_1/2 \leq \theta \leq \pi$. Thus also for d odd and $t_j < \theta < t_{j+1}$ we have

$$f_{q,\pm}(e^{\pm i\theta}) = |P(e^{i\theta})|^q e^{\pm i\theta q d/2}e^{\mp i\pi q n},$$

and we can repeat the proof given above.

□

We are now ready to prove Theorem 7. Recall that given a sequence τ in T_d we define the function

$$K_{d,p}(\tau;\theta) = \sum_{n=0}^{k}\chi_{(\tau_n,\tau_{n+1})}(\theta)\frac{\sin((\frac{dp}{4}+\frac{1}{2})\theta - \frac{p}{2}\pi n)}{\sin(\frac{\theta}{2})}\,d\theta,$$

and let $M_{d,p}(\tau;\theta) = \max(K_{d,p}(\tau;\theta), 0)$. In the proof below we replace $K_{d,p}(\tau;\theta)$ by its positive part and we also apply the Cauchy–Schwarz inequality. It follows that the upper bound obtained with this technique cannot be optimal for $p \neq 2$. The proof resembles that of [1, Corollary 6.1]

Proof of Theorem 7. Applying Theorem 8 with $P(z) = \varphi_{d,p}(z)$ and $q = p/2$, we get

$$|\varphi_{d,p}(1)|^{p/2} = \frac{1}{\pi}\int_0^{\pi}|\varphi_{d,p}(e^{i\theta})|^{p/2}K_{d,p}(t;\theta)\,d\theta,$$

where $t = (t_j)_j$ is the sequence of arguments of zeroes of $\varphi_{d,p}$ and $t_0 = 0$. Then clearly it also holds that

$$|\varphi_{d,p}(1)|^{p/2} \leq \frac{1}{\pi}\int_0^{\pi}|\varphi_{d,p}(e^{i\theta})|^{p/2}M_{d,p}(t;\theta)\,d\theta. \qquad (6)$$

Squaring both sides of (6) and using the Cauchy–Schwarz inequality we obtain

$$|\varphi_{d,p}(1)|^{p/2} \leq \|\varphi_{d,p}\|_p^p\frac{1}{\pi}\int_0^{\pi}M_{d,p}^2(t;\theta)\,d\theta.$$

Thus by the definition of $\mathscr{C}_{d,p}$ we conclude that

$$\mathscr{C}_{d,p} \leq \frac{1}{\pi}\int_0^\pi M_{d,p}^2(t;\theta)\,d\theta. \qquad \square$$

4. The zeroes of the extremal function

In this section we extract information about the arguments of the zeroes t_n of the extremal function $\varphi_{d,p}$. We begin by stating a result of Turán [6, Theorem 3] which yields a lower bound for the argument t_1 of the first zero.

Lemma 9 (Turán)**.** *Let $1 < p < \infty$, and let t_1 denote the smallest argument of a zero of the function $\varphi_{d,p}$. Then*

$$t_1 \geq \frac{\pi}{d}.$$

We also have the following simple lemma which is key in proving $\mathscr{C}_{4,p} \leq 2p+1$.

Lemma 10. *Let $d = 4$ and assume $2 \leq p \leq 4$. Let $\varphi_{d,p}$ be the extremal function and denote its zeroes by $e^{it_1}, e^{it_2}, e^{-it_1}$ and e^{-it_2}, where $0 < t_1 \leq t_2 \leq \pi$. Then $t_2 \geq \pi/2$.*

Proof. Let $P(z)$ be a polynomial of degree 4 such that $P(1) = 1$ and let the zeroes of P be given by $e^{i\tau_1}$, $e^{-i\tau_1}$, $e^{i\tau_2}$ and $e^{-i\tau_2}$, where $\tau_1 \leq \tau_2 < \pi/2$. Then

$$\begin{aligned} P(e^{i\theta}) &= \frac{(e^{i\theta}-e^{i\tau_1})(e^{i\theta}-e^{-i\tau_1})(e^{i\theta}-e^{i\tau_2})(e^{i\theta}-e^{-i\tau_2})}{(1-e^{i\tau_1})(1-e^{-i\tau_1})(1-e^{i\tau_2})(1-e^{-i\tau_2})} \\ &= e^{2i\theta}\frac{(\cos\theta-\cos\tau_1)(\cos\theta-\cos\tau_2)}{(1-\cos\tau_1)(1-\cos\tau_2)}. \end{aligned}$$

Next we let $\gamma_1 = \pi - \tau_1$ and $\gamma_2 = \pi - \tau_2$ and Q be the polynomial given by

$$\begin{aligned} Q(e^{i\theta}) &= \frac{(e^{i\theta}-e^{i\gamma_1})(e^{i\theta}-e^{-i\gamma_1})(e^{i\theta}-e^{i\gamma_2})(e^{i\theta}-e^{-i\gamma_2})}{(1-e^{i\gamma_1})(1-e^{-i\gamma_1})(1-e^{i\gamma_2})(1-e^{-i\gamma_2})} \\ &= e^{2i\theta}\frac{(\cos\theta-\cos\gamma_1)(\cos\theta-\cos\gamma_2)}{(1-\cos\gamma_1)(1-\cos\gamma_2)}. \end{aligned}$$

We observe that Q is of degree 4 and $Q(1) = 1$. From (1) we see that the proof is complete if we can show that $\|Q\|_p^p \leq \|P\|_p^p$, since this shows that P cannot be the extremal function. Calculating the norm and using the substitution $u = x - \pi$ we see that

$$\begin{aligned} \|Q\|_p^p &= \int_{-\pi}^{\pi} |Q(e^{i\theta})|^p\,\frac{d\theta}{2\pi} = \frac{1}{\pi}\int_0^\pi \left|\frac{(\cos\theta-\cos\gamma_1)(\cos\theta-\cos\gamma_2)}{(1-\cos\gamma_1)(1-\cos\gamma_2)}\right|^p d\theta \\ &= \frac{1}{\pi}\int_0^\pi \left|\frac{(\cos\theta+\cos\tau_1)(\cos\theta+\cos\tau_2)}{(1+\cos\tau_1)(1+\cos\tau_2)}\right|^p d\theta \\ &= \frac{1}{\pi}\int_0^\pi \left|\frac{(\cos\theta-\cos\tau_1)(\cos\theta-\cos\tau_2)}{(1+\cos\tau_1)(1+\cos\tau_2)}\right|^p d\theta \\ &\leq \frac{1}{\pi}\int_0^\pi \left|\frac{(\cos\theta-\cos\tau_1)(\cos\theta-\cos\tau_2)}{(1-\cos\tau_1)(1-\cos\tau_2)}\right|^p d\theta \\ &= \|P\|_p^p. \end{aligned}$$

Note that the inequality follows since $0 < \tau_1 \leq \tau_2 < \pi/2$ guarantees that $\cos\tau_1$ and $\cos\tau_2$ both take values in $(0,1)$. $\square$

It is easy to see that we can generalize Lemma 10 to say that if d is even, then at least one zero has an argument greater than or equal to $\pi/2$.

5. Proof of Theorem 1

Let $p \geq 1$, let d be an integer and let $k = \lfloor d/2 \rfloor$. We recall that T_d is the set of all sequences τ with $k+2$ elements such that $0 = \tau_0 \leq \tau_1 \leq \cdots \leq \tau_{k+1} = \pi$. We also recall that

$$K_{d,p}(\tau;\theta) = \sum_{n=0}^{k} \chi_{(\tau_n,\tau_{n+1})}(\theta) \frac{\sin((\frac{dp}{4}+\frac{1}{2})\theta - \frac{p}{2}\pi n)}{\sin(\frac{\theta}{2})},$$

and that $M_{d,p}(\tau;\theta) = \max(K_{d,p}(\tau;\theta), 0)$. Further we define

$$E_{d,p}(\tau) = \frac{1}{\pi}\int_0^\pi M_{d,p}^2(\tau;\theta)\,d\theta.$$

It is an immediate consequence of Theorem 7 that

$$\mathscr{C}_{d,p} \leq \sup_{\tau\in\Lambda} E_{d,p}(\tau), \tag{7}$$

where Λ is the set of sequences of possible arguments of the zeroes of the extremal function $\varphi_{d,p}$. To prove Theorem 1 we will determine the supremum (7) for $2 \leq d \leq 4$ and $2 \leq p \leq 4$, leading to a slightly better upper bound stated in Theorem 14 below. The argument follows the lines of what was done in [3]. In particular, due to Lemma 9, for any integer d we let $\Lambda = \Lambda_d$ where we define

$$\Lambda_d = \{(\tau_n)_{n=0}^{k+1} : 0 = \tau_0 \leq \pi/d \leq \tau_1 \leq \tau_2 \leq \cdots \leq \tau_{k+1} = \pi\}. \tag{8}$$

Because of Lemma 10 we limit this definition further in the special case $d = 4$, where we redefine Λ_4 as

$$\Lambda_4 = \{\tau = (\tau_n)_{n=0}^{3} : \tau_0 = 0,\ \frac{\pi}{d} \leq \tau_1 \text{ and } \frac{\pi}{2} \leq \tau_2 \leq \tau_3 = \pi\}. \tag{9}$$

To determine the supremum in (7) we introduce some terminology similar to that in [1] and [3]. We refer to each connected component of the sets

$$\left\{\theta : \sin\left(\left(\frac{dp}{4}+\frac{1}{2}\right)\theta - \frac{p}{2}\pi n\right) > 0 \text{ and } 0 \leq \theta \leq \pi\right\}$$

as an *interval at level* n. We note that each interval at level n has length $l = 4\pi/(dp+2)$, and that the left endpoint of an interval at level n is always of the form $l(pn+4j)/2$ for some integer j. Furthermore, it is easy to see that when $2 < p < 4$, an interval at level n intersects exactly one interval at level $n+1$. Thus one can think of determining the supremum in (7) as finding the correct "jumps" between intervals at level n and level $n+1$ in order to maximize the corresponding integral $E_{d,p}(\tau)$. See Figure 4 for an illustration. Below we state some lemmas that allows us to exclude certain sequences from Λ_d. In particular we start with a lemma saying that we need not consider sequences where $\tau_n \geq lpn/2$. We recall that $lpn/2$ is a left endpoint of an interval at level n, and that $l = 4\pi/(dp+2)$.

Lemma 11. *Let $2 \leq p \leq 4$ and $d \geq 2$ be an integer. Let γ be a sequence in Λ_d and let τ be the sequence $\tau_n = \min(lpn/2, \gamma_n)$. Then τ is in Λ_d as well and $E_p(\gamma) \leq E_p(\tau)$.*

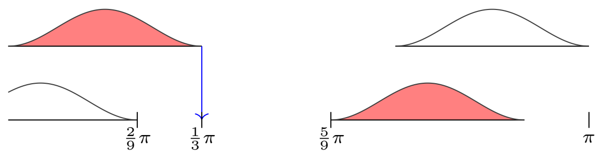

FIGURE 4. An illustration of the intervals at level 0 and level 1 for $d = 3$ and $p = 10/3$. Theorem 14 tells us that the supremum in equation (7) is attained whenever $\pi/3 \leq \tau_1 \leq 5\pi/9$. The shaded area represents $E_p(\tau)$ without considering the integrand factor $1/\sin^2(\theta/2)$.

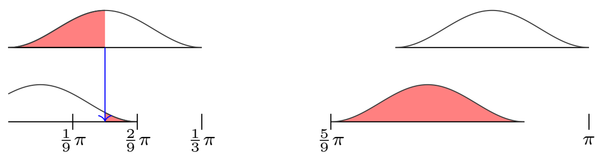

FIGURE 5. The shaded area represents $E_p(\gamma)$ for $\gamma_1 = 3\pi/18$ without considering the integrand factor $1/\sin^2(\theta/2)$ for $d = 3$ and $p = 10/3$. Keep in mind that $\pi/9$ is the midpoint between the first interval at level 0 and the first interval at level 1. Lemma 12 tells us that for any γ_1 in $[\pi/9, \pi/3]$ it will follow that $E_p(\gamma) \leq E_p(\tau)$ where $\tau_1 = \pi/3$ as seen in Figure 4.

Proof. We see that τ is an increasing sequence and since $lp/2 \geq \pi/d$ it follows that $\tau_1 \geq \pi/d$. Thus if $d \neq 4$, then τ is in Λ_d. Further for $d = 4$ we see that $pl \geq \pi/2$ so τ is in Λ_d also if $d = 4$. If $\gamma_n = \tau_n$ for all n we are done. Thus we fix some integer m such that $\gamma_m > \tau_m$. Let j be the largest integer such that $l(pm + 4j)/2 \leq \gamma_m$. (That is $l(pm + 4j)/2$ is the largest left endpoint of an interval at level m smaller than γ_m.) By a pointwise estimate and using that $\sin(\theta/2)$ is an increasing function for $0 < \theta < \pi$ it follows that replacing γ_m with $l(pm+4j)/2$ will increase the value of $E_p(\gamma)$. Further by periodicity and using that $\sin(\theta/2)$ is an increasing function it follows that replacing γ_m with $mpl/2$ will increase the value of $E_{d,p}(\gamma)$ further. Since this holds for all m such that $\gamma_m > \tau_m$ we conclude that $E_p(\gamma) \leq E_p(\tau)$. $\square$

In particular we note that the above lemma allows us to impose the condition $\tau_n \leq npl/2$ for all n on the set Λ in the supremum (7). Next we state a lemma that essentially tells us that the supremum (7) is not attained if we make a "jump" from level n to level $n+1$ between the right endpoint of an interval at level n and the midpoint between the interval at level n and its intersecting interval at level $n+1$. This is illustrated in Figure 4 and Figure 5.

Lemma 12. *Let $2 \leq p \leq 4$ and $d \geq 2$ be an integer. Let γ be a sequence in Λ_d such that $\gamma_n \leq nlp/2$. Fix an integer r. Assume $M \leq \gamma_{r+1} \leq b$, where (a, b) is an interval at level r and M is the midpoint between (a, b) and the intersecting interval (c, d) at level $r+1$. Let τ be the sequence in T_d defined by $\tau_{r+1} = b$ and $\tau_n = \gamma_n$ for all $n \neq r+1$. Then $E_p(\gamma) \leq E_p(\tau)$. If $d \leq 4$ then it also holds that τ is in Λ_d.*

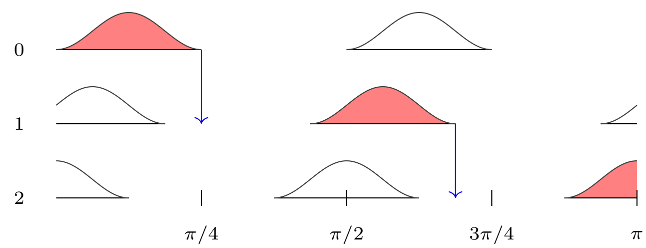

FIGURE 6. Together with Figure 7 this gives an illustration of Lemma 13 for $d = 4$ and $p = 3.5$. Here we see the sequence τ from Lemma 13. The shaded area represents $E_p(\tau)$ without considering the integrand factor $\sin^2(\theta/2)$. The lemma tells us that $E_p(\gamma) \le E_p(\tau)$, where γ is illustrated in Figure 7.

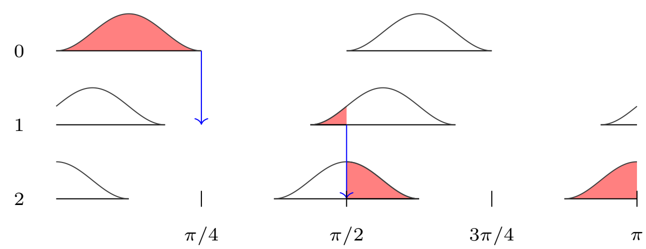

FIGURE 7. Together with Figure 6 this gives an illustration of Lemma 13 for $d = 4$ and $p = 3.5$. Here we see the sequence γ with $\gamma_2 = \pi/2$. The shaded area represents $E_p(\gamma)$ without considering the integrand factor $1/\sin^2(\theta/2)$. Lemma 13 tells us that $E_p(\gamma) \le E_p(\tau)$, where τ is illustrated in Figure 6.

Proof. We need only show

$$\int_M^b \left(\frac{\sin \frac{dp+2}{4}\theta - \frac{p}{2}\pi r}{\sin \frac{\theta}{2}} \right)^2 d\theta \ge \int_M^d \left(\frac{\sin \frac{dp+2}{4}\theta - \frac{p}{2}\pi (r+1)}{\sin \frac{\theta}{2}} \right)^2 d\theta,$$

which follows from the pointwise estimate

$$\left(\frac{\sin \frac{dp+2}{4}\theta - \frac{p}{2}\pi r}{\sin \frac{\theta}{2}} \right)^2 \ge \left(\frac{\sin \frac{dp+2}{4}\theta - \frac{p}{2}\pi (r+1)}{\sin \frac{\theta}{2}} \right)^2$$

for all $M \le \theta \le d$. It remains to show that τ is in Λ_d whenever $d \le 4$. If $d = 2$ or $d = 3$ then $0 = \tau_0 \le \tau_1 \le \tau_2 = \pi$ and clearly $\tau_1 \ge \pi/d$ since $\gamma_1 \ge \pi/d$, so τ is in Λ_d. Next consider $d = 4$. Then $\pi/d \le \tau_1 \le pl/2 \le \pi/2 \le \tau_2$, and so τ is in Λ_d. $\square$

To prove Theorem 1 for $d = 4$ and $3 < p \le 4$ we need the following lemma, which says that if the supremum in (7) is attained by a sequence with $\tau_2 \ge \pi/2$, then necessarily τ_2 is greater than the right endpoint of an interval at level n namely $l(p/2 + 1)$. See Figure 6 and Figure 7 for an illustration.

Lemma 13. *Let $3 < p \leq 4$. Let γ be a sequence in Λ_4 such that $\pi/2 \leq \gamma_2 \leq l(p/2+1)$. Let τ be the sequence in T_d with $\tau_1 = \gamma_1$ and $\tau_2 = l(p/2+1)$. Then τ is in Λ_4 and $E_4(\gamma) \leq E_4(\tau)$.*

Proof. It is clear that τ is in Λ_4 since $\tau_2 \geq \gamma_2$, so we only need to show $E_4(\gamma) \leq E_4(\tau)$. By a pointwise estimate it suffices to consider $\gamma_2 = \pi/2$. Since all other integral parts of $E_p(\tau)$ and $E_p(\gamma)$ will be equal, it suffices to show that

$$\int_{\pi/2}^{l(p/2+1)} \left(\frac{\sin(\frac{2p+1}{2}\theta - \frac{p}{2}\pi)}{\sin\frac{\theta}{2}} \right)^2 d\theta \geq \int_{\pi/2}^{l(p-1)} \left(\frac{\sin(\frac{2p+1}{2}\theta - p\pi)}{\sin\frac{\theta}{2}} \right)^2 d\theta.$$

Since the integrands are positive and $l(p-1) \leq l(p/2+1)$ it suffices to show $f(p) \geq 0$, where

$$f(p) = \int_{\pi/2}^{l(p/2+1)} \left(\frac{\sin(\frac{2p+1}{2}\theta - \frac{p}{2}\pi)}{\sin\frac{\theta}{2}} \right)^2 - \left(\frac{\sin(\frac{2p+1}{2}\theta - p\pi)}{\sin\frac{\theta}{2}} \right)^2 d\theta.$$

By substitution and using trigonometric identities we see that

$$\begin{aligned} f(p) &= \frac{1}{2p+1} \int_0^{3\pi/2} \frac{\big(\sin(\frac{x}{2} + \frac{\pi}{4})\big)^2 - \big(\sin(\frac{x}{2} + \frac{\pi}{4} - \frac{\pi}{2}p)\big)^2}{\big(\sin\frac{x}{4p+2} + \frac{\pi}{4}\big)} d\theta \\ &= \frac{2\sin(-p\pi/2)}{2p+1} \int_0^{3\pi/2} \frac{\sin\theta\sin(-p\pi/2) - \cos\theta\cos(p\pi/2)}{1 + \sin\frac{\theta}{2p+1}} d\theta. \end{aligned}$$

It follows that $f(p) \geq 0$ since

$$\int_0^{3/2\pi} \frac{\sin\theta}{1+\sin\frac{\theta}{2p+1}} d\theta \geq \int_0^{\pi} \frac{\sin\theta}{1+\sin\frac{\theta}{2\cdot 3+1}} d\theta + \int_{\pi}^{3/2\pi} \frac{\sin\theta}{1+\sin\frac{\theta}{2\cdot 4+1}} d\theta \geq 0,$$

and

$$\int_0^{3/2\pi} \frac{-\cos\theta}{1+\sin\frac{\theta}{2p+1}} d\theta \geq \int_0^{\pi/2} \frac{-\cos\theta}{1+\sin\frac{\theta}{2\cdot 4+1}} d\theta + \int_{\pi/2}^{3/2\pi} \frac{\sin\theta}{1+\sin\frac{\theta}{2\cdot 3+1}} d\theta \geq 0. \quad \square$$

To prove the upper bound from Theorem 1 we in fact prove a slightly stronger bound given in the theorem below. Figure 1 shows the plot of these upper bounds, and Figures 4 and 6 illustrate the bounds with the corresponding sequences in Λ_d. We recall that $l = 4\pi/(dp+2)$ is the length of an interval at level n. (Below we use the notation τ^d to indicate the dependence on d in the sequence, not to be confused with an exponent.)

Theorem 14. *Let $2 \leq p \leq 4$ and $2 \leq d \leq 4$. Then $\mathscr{C}_{d,p} \leq E_p(\tau^d)$, where*

$$\begin{aligned} \tau^2 &= (0, \tau_1^2, \pi), \\ \tau^3 &= (0, \tau_1^3, \pi), \\ \tau^4 &= (0, \tau_1^4, \tau_2^4, \pi), \end{aligned}$$

and where τ_1^d and τ_2^d are any numbers satisfying $l \leq \tau_1^d \leq lp/2$ and $l(p+2)/2 \leq \tau_2^d \leq pl$.

Proof. In the proof we omit the notation τ^d and instead write τ, but note that τ still depends on l and thus on d since $l = 4\pi/(dp+2)$. By Theorem 7, letting

Λ_d be the set of sequences that may serve as the set of arguments of zeroes of the extremal function, it follows that

$$\mathscr{C}_{d,p} \leq \sup_{\tau \in \Lambda_d} \frac{1}{\pi} E_{d,p}(\tau).$$

In particular, due to Lemma 9 and Lemma 10 we let Λ_d be as defined in (8) and (9) respectively. Assume τ is a sequence in Λ_d such that $E_{d,p}(\lambda) \leq E_{d,p}(\tau)$ for all λ in Λ_d. By Lemma 11, we may assume $\tau_n \leq npl/2$. Furthermore letting $M_1 = l/2$ be the midpoint between $(0, l)$ and its intersecting interval at level $n+1$ we have $M_1 \geq \pi/d$, and so by Lemma 12 we must have $\tau_1 \geq l$. This concludes the proof for $d = 2$ and $d = 3$ since τ_1 is the only variable determining $E_{d,p}(\tau)$ in this case, and clearly $E_{d,p}(\tau)$ obtains the same value for any $l \leq \tau_1 \leq lp/2$.

Now consider $d = 4$. From above we know that $l \leq \tau_1 \leq lp/2$. By assumption $\tau_2 \geq \pi/2$, and by Lemma 11 we have $\tau_2 \leq pl$. Let $M_2 = (3p-2)\pi/(4p+2)$ be the midpoint between the second interval at level 1 and the overlapping interval at level 2. If $2 \leq p \leq 3$ it follows that $\tau_2 \geq \pi/2 \geq M_2$ and so by Lemma 12 it follows that $\tau_2 \geq l(p+2)/2$. For $3 \leq p \leq 4$ it follows from Lemma 13 that $\tau_2 \geq l(p+2)/2$. That is $l \leq \tau_1 \leq lp/2$ and $l(p+2)/2 \leq \tau_2 \leq pl$, and all such sequences give rise to the same value for $E_p(\tau)$, which concludes the proof. □

Due to a numerical analysis we expect the zeroes of the extremal function $\varphi_{d,p}$ to be within the set of sequences τ given in Theorem 14. In particular, this means that further information on the zeroes of the extremal function $\varphi_{d,p}$ will not improve on the upper bound for $\mathscr{C}_{d,p}$ from Theorem 14, given that we use the same technique. Finally, we conclude the proof of Theorem 1 by showing that the upper bounds on $\mathscr{C}_{d,p}$ from Theorem 14 are in fact better than $dp/2 + 1$.

Proof of Theorem 1. Let $2 \leq d \leq 4$ be an integer and let $2 \leq p \leq 4$. The upper bounds from Theorem 14 can be expressed as follows:

$$\begin{aligned}
\mathscr{C}_{2,p} &\leq \frac{1}{\pi}\int_0^l \left(\frac{\sin(\frac{p+1}{2}\theta)}{\sin\frac{\theta}{2}}\right)^2 d\theta + \frac{1}{\pi}\int_{lp/2}^{\pi} \left(\frac{\sin(\frac{p+1}{2}\theta - \frac{p}{2}\pi)}{\sin\frac{\theta}{2}}\right)^2 d\theta \\
\mathscr{C}_{3,p} &\leq \frac{1}{\pi}\int_0^l \left(\frac{\sin(\frac{3p+2}{4}\theta)}{\sin\frac{\theta}{2}}\right)^2 d\theta + \frac{1}{\pi}\int_{lp/2}^{l(p/2+1)} \left(\frac{\sin(\frac{3p+2}{4}\theta - \frac{p}{2}\pi)}{\sin\frac{\theta}{2}}\right)^2 d\theta \\
\mathscr{C}_{4,p} &\leq \frac{1}{\pi}\int_0^l \left(\frac{\sin(\frac{2p+1}{2}\theta)}{\sin\frac{\theta}{2}}\right)^2 d\theta + \frac{1}{\pi}\int_{lp/2}^{l(p/2+1)} \left(\frac{\sin(\frac{2p+1}{2}\theta - \frac{p}{2}\pi)}{\sin\frac{\theta}{2}}\right)^2 d\theta \\
&\quad + \frac{1}{\pi}\int_{lp}^{\pi} \left(\frac{\sin(\frac{2p+1}{2}\theta - p\pi)}{\sin\frac{\theta}{2}}\right)^2 d\theta.
\end{aligned}$$

Using that $\sin x/2$ is increasing for $0 < x < \pi$ it follows that these upper bounds are all less than or equal to $A_{d,p}$ where

$$A_{d,p} = \frac{1}{\pi}\int_0^{2\pi(d+1)/(dp+2)} \left(\frac{\sin\left(\frac{dp+2}{4}\theta\right)}{\sin\frac{\theta}{2}}\right) d\theta.$$

By a substitution, we see that the inequality $A_{d,p} \leq dp/2 + 1$ is equivalent to

$$\frac{2}{\pi}\frac{4}{(dp+2)^2}\int_0^{\pi(d+1)/2}\left(\frac{\sin\theta}{\sin\frac{2}{dp+2}\theta}\right)^2 d\theta \leq 1. \tag{10}$$

A straightforward computation shows that (10) is attained for $p = 2$. Thus we conclude that (10) also holds for all $2 < p \leq 4$ by the pointwise estimate

$$\frac{4}{(dp+2)^2}\left(\frac{\sin\theta}{\sin\frac{2}{dp+2}\theta}\right)^2 \leq \frac{4}{(2d+2)^2}\left(\frac{\sin\theta}{\sin\frac{2}{2d+2}\theta}\right)^2.$$

This concludes the proof. □

We do not move beyond $d > 4$ since we do not have sufficient information about the arguments of the zeroes of the extremal function when $d \geq 5$. In particular, we cannot invoke Lemma 12 for all elements as we cannot justify assuming that the arguments are bigger than the related midpoints between intervals at level n and $n + 1$. Note that one can show that our technique is *not* good enough to verify the conjecture $\mathscr{C}_{d,p} \leq dp/2 + 1$ for large d. For example, for $p = 5/2$ and $d = 6$ the supremum (7) with the restrictions $\tau_2 \geq \tau_1 \geq \pi/d$ and $\tau_3 \geq \pi/2$ is at least $8.6 > dp/2 + 1 = 8.5$. With more information about the zeroes this could be improved.

We also note that if for $2 \leq p \leq 4$ and any $d \geq 2$ we knew that the arguments of the zeroes of the extremal function satisfy $t_j \geq (2pj - p - 1)\pi/(dp + 2)$, which is the midpoint between an interval at level $n - 1$ and n, then we could generalize Theorem 14, and in particular prove the conjectured upper bound $\mathscr{C}_{d,p} \leq dp/2+1$. We expect the zeroes of the extremal function to satisfy this condition, since we expect that t_j is somewhat close to γ_j, where $\gamma_j = (2 - p + 2pj)\pi/(dp + 2)$.

6. Proof of Theorem 2

In this section we prove Theorem 2, which gives an upper bound for $\mathscr{C}_{d,p}$ when p is large. We use a technique similar to that in [1, Section 7]. First, however, we need the following lemma by Hyltén-Cavallius [2, Theorem 1].

Lemma 15. *Let $P(e^{i\theta})$ be a polynomial of degree d which is maximized for $\theta = 0$ and where $\|P\|_\infty = P(1) = 1$. Then*

$$|P(e^{i\theta})| \geq \|P\|_\infty \cos(d\theta/2),$$

for any $|\theta| \leq \pi/d$.

Proof of Theorem 2. Let $P(e^{i\theta})$ be a polynomial of degree d. By rotating we may assume the function is maximized for $\theta = 0$, where $\|P\|_\infty = P(1) = 1$. Then from Lemma 15 it follows that

$$|P(e^{i\theta})| \geq \|P\|_\infty \cos(d\theta/2),$$

for any $|\theta| \leq \pi/d$. Exponentiating both sides to the power p and integrating yields that

$$\int_{-\pi/d}^{\pi/d} |P(\theta)|^p \, d\theta \geq \|P\|_\infty^p \int_{-\pi/d}^{\pi/d} (\cos(d\theta/2))^p \, d\theta = \frac{2}{d} B((p+1)/2, 1/2).$$

Hence it follows that

$$\|P\|_p^p \geq \frac{B((p+1)/2, 1/2)}{\pi d},$$

and so by (1) it follows that $\mathscr{C}_{d,p} \leq d\pi/B((p+1)/2, 1/2)$. □

Due to well-known asymptotics for the beta function it follows from Theorem 2 that as $p \to \infty$ we have

$$\mathscr{C}_{d,p} \leq d\left(\sqrt{\frac{\pi p}{2}} + O\left(\frac{1}{\sqrt{p}}\right)\right).$$

Further, using the estimate $B(x, 1/2) \geq \sqrt{\pi(4x+1)}/(2x)$ it also follows that when $p > 6.8$ then $\mathscr{C}_{d,p} \leq dp/2$.

7. Numerical estimates

Consider the polynomial Q of degree d with the zeroes $e^{\pm\gamma_j i}$ where the argument is defined by $\gamma_j = \pi(2-p+2pj)/(dp+2)$, for $1 \leq j \leq \lfloor d/2 \rfloor$, and if d is odd the last zero is $z = -1$. Then $1/\|Q\|_p^p$ provides a good lower bound for $\mathscr{C}_{d,p}$. In particular we have the following

Theorem 16. *Let $d \geq 1$ be an integer and let $\gamma_j = (2-p+2pj)\pi/(dp+2)$. Then $\mathscr{C}_{d,p} \geq 1/C$, where*

$$C = \frac{1}{\pi}\int_0^\pi \prod_{j=1}^{d/2} \left|\frac{\cos x - \cos\gamma_j}{1 - \cos\gamma_j}\right|^p d\theta,$$

if d is even and

$$C = \frac{1}{\pi}\int_0^\pi |\cos x/2|^p \prod_{j=1}^{d/2} \left|\frac{\cos x - \cos\gamma_j}{1 - \cos\gamma_j}\right|^p d\theta,$$

if d is odd.

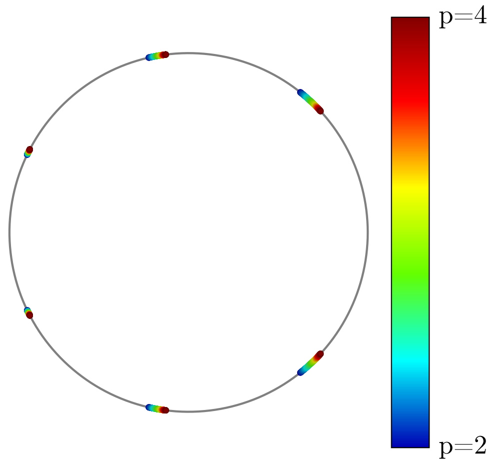

FIGURE 8. The expected zeroes of the extremal function $\varphi_{d,p}$ for $d = 6$ and $2 \leq p \leq 4$ based on numerical optimization.

Observe that for $p = 2$ the polynomial Q is the extremal function for (1), and the lower bound is equal to $\mathscr{C}_{d,2} = d+1$. We also note that for $2 \le p \le 4$ the sequence γ is one of the sequences attaining the supremum $\sup_{\tau\in\Lambda_d} E_{d,p}(\tau)$ and providing the upper bound for $\mathscr{C}_{d,p}$ in Theorem 14. Clearly one can obtain a slightly better lower bound for $\mathscr{C}_{d,p}$ by numerically optimizing over all polynomials. The true bound is still close to the bound stated in Theorem 16.

Based on numerical computations we also expect the arguments of the zeroes of the extremal function to be somewhat close to γ_j. In particular we expect the argument to be greater than $(2pj-p-1)\pi/(dp+2)$, which as mentioned in Section 5 is enough to show the conjecture $\mathscr{C}_{d,p} \le dp/2+1$. Figure 8 illustrates the expected zeroes of the extremal function $\varphi_{6,p}$ for $2 \le p \le 4$ based on numerical optimization. We note that the numerical estimates support both the conjecture $\mathscr{C}_{d,p} \le dp/2+1$, and that $\mathscr{C}_{d,p}/(dp/2+1)$ is decreasing in p.

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